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SUMMER SCHOOL
on
LOW-DIMENSIONAL QUANTUM SYSTEMS:
Theory and Experiment
(16 - 27 JULY 2001)

PLUS

PRE-TUTORIAL SESSIONS
(11 - 13 JULY 2001)

LOCALIZATION IN QUASI-1D DISORDERED SUPERCONDUCTORS

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Localization in quasi-1D disordered superconductors.

Trieste. July 2001.

Ilya Gruzberg. ITP at Santa Barbara.

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I. Setup.

Interested in quasiparticle states in disordered superconductors – localized vs. delocalized.

Experimental probe – thermal transport.

1. Use mean field ~~for~~ treatment

   ⇒ single-particle problem.

   Hamiltonian H – random matrix

   Ensembles of H classified by symmetries

   E. Wigner, F. Dyson

Main message: A symmetry of your problem should be used to the fullest possible extent.

2. Mean field for superconductors:

BCS or Bogoliubov - de Gennes formalism:

$$H = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix} \quad \text{- BdG Hamiltonian}$$

Particle - hole symmetry

$$\Sigma_1 H \Sigma_1 = -H^T$$

Symmetric spectrum $E \to -E$.

$E=0$ is a special point, may have singularities near $E=0$.

In addition to p/h symmetry can have time-reversal (TR) and/or spin-rotation (SR) symmetries.

Accordingly, there are 4 symmetry classes of disordered BdG Hamiltonians

A. Altland, M. Zirnbauer Ref.[1].

Today concentrate on the class D with

- no TR and no SR present.

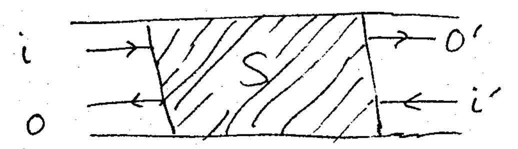
3. Use scattering approach to quantum transport.
Pioneered by Luttinger.
Can construct networks of scatterers
(see A. Ludwig's lecture).
Especially useful in one dimension (1D)
and quasi-1D.

Illustrate the scattering method first
using the standard unitary symmetry class
of Wigner and Dyson.

II. Scattering approach in 1D.

1. Illustrate for standard unitary class.
Full phase coherence.

Ref. [2]: Anderson, Thouless, Abrahams, Fisher.
Landauer



$$\begin{pmatrix} o \\ o' \end{pmatrix} = S \begin{pmatrix} i \\ i' \end{pmatrix} = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix} \begin{pmatrix} i \\ i' \end{pmatrix}$$

scattering matrix S.

Landauer formula: $G = \frac{e^2}{h} |t|^2$.

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2. Derivation:

Bias left lead by $e\delta V$.

Extra population (# of electrons) is $\delta n = \frac{dn}{dE} e\delta V$.

Current in the lead i: $j_i = e v \delta n = e^2 v \frac{dn}{dE} \delta V$

Reflected current $j_o = |r|^2 j_i$

Transmitted current $j_{o'} = |t|^2 j_i$

Conductance $G = \frac{j_{o'}}{\delta V} = e^2 v \frac{dn}{dE} |t|^2$.

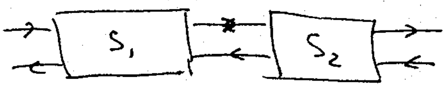
Finally, in 1D the density of states

$$\frac{dn}{dE} = \frac{1}{2\pi\hbar v}.$$

Velocity cancels, and we get $G = \frac{e^2}{h} |t|^2$.

Dimensionless conductance $g = \frac{G}{e^2/h} = |t|^2$

3. Combining scatterers.



Total transmission amplitude:

$$t = t_1 t_2 + t_1 r_2 r_1' t_2 + t_1 (r_2 r_1')^2 t_2 + \dots = t_1 \frac{1}{1 - r_1' r_2} t_2$$

$$g = |t|^2 = \frac{|t_1|^2 \, |t_2|^2}{1 - 2 |r_1' r_2| \cos\varphi + |r_1' r_2|^2}$$

Find an additive quantity.

$$\ln g = \ln g_1 + \ln g_2 + \ln\left(1 - 2|r_1' r_2| \cos\varphi + |r_1' r_2|^2\right)$$

Averaging over φ : $\int_0^{2\pi} d\varphi \, \ln(\ \ ) = 0.$

$$\langle \ln g \rangle_\varphi = \langle \ln g_1 \rangle_\varphi + \langle \ln g_2 \rangle_\varphi$$

Then $\langle \ln g \rangle$ is linear in length of the wire L:

$$\langle \ln g \rangle = -\frac{L}{\xi}$$

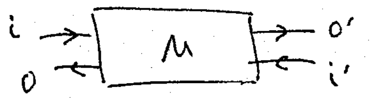
and normally distributed.

$$g_{typ} = e^{\langle \ln g \rangle} = e^{-L/\xi} \quad \text{- localization.}$$

III. Refinement in 1D.

O. Porachov.

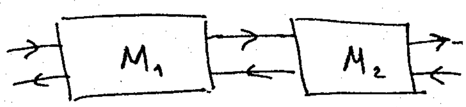
1. Switch from scattering matrix S to transfer matrix M:



$$\begin{pmatrix} o' \\ i' \end{pmatrix} = M \begin{pmatrix} i \\ o \end{pmatrix} = \frac{1}{t'} \begin{pmatrix} tt'-rr' & r' \\ -r & 1 \end{pmatrix} \begin{pmatrix} i \\ o \end{pmatrix}.$$

Conductance $\quad g = \dfrac{1}{|m_{11}|^2}$.

Advantage: multiplicative:



$$M = M_2 M_1$$

Random multiplicative process.

Analog of the central limit theorem (Oseledec):

For $\quad M = \overleftarrow{\prod_{j=1}^{L}} M_j$

the eigenvalues of M^+M are $e^{\pm 2x}$,

where x is "self-averaging" in the limit $L \to \infty$.

"self-averaging" means Gaussian distribution of x with the width growing slower than the mean with L.

x is called "Lyapunov exponent".

2. Current conservation and parameterization

$$|i|^2 + |i'|^2 = |o|^2 + |o'|^2 \Rightarrow S^+ S = 1$$

$\Rightarrow$ $S \in U(2)$ (unitary)

Rewrite:

$$|o'|^2 - |i'|^2 = |i|^2 - |o|^2 \Rightarrow$$

$$\Rightarrow M^+ \sigma_3 M = \sigma_3 \Rightarrow M \in U(1,1) \text{ (pseudo-unitary).}$$

In parti~~c~~cular, $|m_{11}|^2 - |m_{21}|^2 = 1$ (need later)

Parameterization (radial decomposition)

$$M = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \begin{pmatrix} e^{i\varphi_3} & 0 \\ 0 & e^{i\varphi_4} \end{pmatrix} .$$

Conductance $g = \frac{1}{|m_{11}|^2} = \frac{1}{\cosh^2 x}$ ← Lyapunov exponent

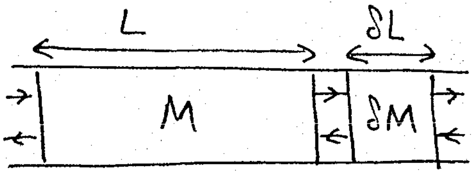
For the product $M = M_2 M_1$

$$m_{11} = \cosh x_1 \cosh x_2 \, e^{i\varphi} + \sinh x_1 \sinh x_2 \, e^{i\chi}$$

Can reproduce the previous results doing $\ln |m_{11}|^2$.

3. Infinitesimal version.

Purpose is to find a Fokker – Planck equation for the distribution function $P(x,L)$ as x performs a random walk. Strategy – find moments of a change in x upon addition of a thin slice δL to the wire.



The slice transfer matrix δM represents weak scattering. Choose it close to $\mathbb{1}$:

$$\delta M = e^{i\sigma_3 V}, \qquad V^{+} = V = \begin{pmatrix} v_1 & v_2 \\ v_2^* & v_3 \end{pmatrix}, \qquad v_1, v_3 \in \mathbb{R}.$$

$$P[V] \propto e^{-\frac{\ell}{2\delta L} \operatorname{tr} V^{+}V} = e^{-\frac{\ell}{2\delta L}\left(v_1^2 + 2|v_2|^2 + v_3^2\right)}$$

ℓ – mean free path.

$$\langle v_i v_j \rangle = 0 \quad \text{for } i \neq j$$

$$\langle |v_2|^2 \rangle = \frac{\delta L}{\ell}.$$

Expand:
$$\delta M = \begin{pmatrix} 1 + i v_1 & i v_2 \\ -i v_2^* & 1 - i v_3 \end{pmatrix} + \dots$$

Distribution $P[V]$ is invariant w.r.t. unitary transformations of V: $\quad V \to u^{+} V u$.

Total transfer matrix

$$\tilde{M} = \delta M \cdot M$$

$$\tilde{m}_{11} = e^{i\varphi}\left(\cosh x \, \delta m_{11} + \sinh x \, \delta m_{12}\right)$$

Using unitarity $\left(|m_{11}|^2 = 1 - |m_{21}|^2 \right)$

and denoting $y = \cosh^2 x$, get

$$\delta y = \cosh^2(x+\delta x) - \cosh^2 x =$$

$$= \cosh^2 x \, |\delta m_{21}|^2 + \sinh^2 x \, |\delta m_{12}|^2 +$$

$$+ \sinh x \cosh x \left(\delta m_{11} \delta m_{12}^* + \delta m_{11}^* \delta m_{12}\right).$$

Thus get

$$\langle \delta y \rangle = \frac{\delta L}{\ell} \cosh 2x \, ; \quad \langle (\delta y)^2 \rangle = \frac{\delta L}{2\ell} \sinh^2 2x .$$

Invert $y = \cosh^2 x$ to get

$$\delta x = \frac{1}{\sinh 2x} \delta y - \frac{\cosh 2x}{\sinh^3 2x} (\delta y)^2 .$$

Then

$$\langle \delta x \rangle = \frac{\delta L}{\ell} \coth 2x - \frac{\delta L}{2\ell} \frac{\cosh 2x}{\sinh^3 2x} \sinh^2 2x = \frac{\delta L}{2\ell} \coth 2x .$$

$$\langle (\delta x)^2 \rangle = \frac{\delta L}{2\ell} .$$

Fokker - Planck equation immediately follows:

$$\delta L \frac{\partial P}{\partial L} = \frac{1}{2} \frac{\partial^2}{\partial x^2}\left(\langle (\delta x)^2 \rangle P \right) - \frac{\partial}{\partial x}\left(\langle \delta x \rangle P \right).$$

Derivation of FP eq. (schematic)

$$P(x,L) = \sum_i \delta(x - x_i(L))$$

$$P(x, L+\delta L) - P(x,L) = \sum_i \left[\delta(x - x_i + \delta x_i) - \delta(x - x_i)\right] =$$

$$= -\sum_i \delta x_i \, \delta'(x-x_i) + \frac{1}{2}\sum_i (\delta x_i)^2 \, \delta''(x - x_i) + \ldots =$$

$$= -\frac{\partial}{\partial x}\left(\delta x \cdot P\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left((\delta x)^2 P\right) + \ldots$$

Rewrite FP as

$$\frac{\partial P}{\partial L} = \frac{1}{4\ell}\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} - 2\coth 2x \cdot P\right) = \frac{1}{4\ell}\frac{\partial}{\partial x} J \frac{\partial}{\partial x} J^{-1} \cdot P .$$

$$J = \sinh 2x$$

Can solve and get the full distribution function of g. Log-normal in the long L limit.

Tracing back, can see that the localization is due to the drift term in the FP, coming from $\langle \delta x \rangle$.

IV. Generalization to N channels: quasi 1D wires.

For a review, see Ref. [3] : C. Beenakker.

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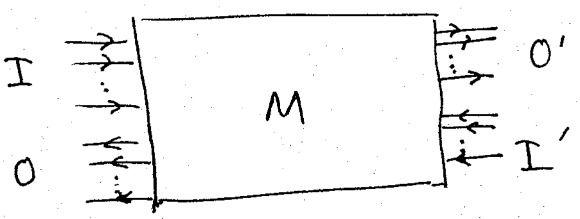


$$\begin{pmatrix} O' \\ I' \end{pmatrix} = M \begin{pmatrix} I \\ O \end{pmatrix}.$$

$$|O'|^2 - |I'|^2 = |I|^2 - |O|^2$$

$M \in U(N,N)$

Parameterization:

$$M = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \cosh X & \sinh X \\ \sinh X & \cosh X \end{pmatrix} \begin{pmatrix} u_3 & 0 \\ 0 & u_4 \end{pmatrix}$$

$X = \mathrm{diag}\,(x_1, \dots, x_N)$ $\qquad x_i \in \mathbb{R}$ – N Lyapunov exponents

$u_i \in U(N)$.

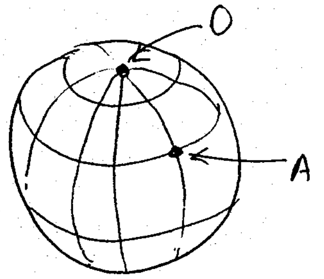
Conductance $\qquad g = \sum_{i=1}^{N} \frac{1}{\cosh^2 x_i}$.

x_i perform random walk on

the coset space $\mathcal{M} = U(N,N) / U(N) \times U(N) = G/H$.

2. Aside on coset spaces.

Example: two-dimensional sphere $S^2 = SO(3)/SO(2)$



$SO(3)$ acts on S^2. Fix the north pole O and associate any other point A with a rotation from $SO(3)$ which takes O into A.

Problem: there are infinitely many such rotations, since O is invariant under $SO(2)$ rotations around the vertical axes. Need to form left cosets of elements from $SO(3)$ with respect to $SO(2)$ subgroup. These cosets are in 1-to-1 correspondence with points on S^2, and form the coset space $SO(3)/SO(2)$.

Point O is called "the origin" on S^2.
Coordinates on S^2 are θ and φ (polar and azimuthal angles). θ increases away from the origin and is called "the radius". φ is "the angle".

Diffusion on S^2 starting from the origin O is governed by "the radial part" of the Laplacian on S^2, that is, the part without $\frac{\partial}{\partial\varphi}$ derivatives:

$$L_{S^2} = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},$$

$$\mathrm{Rad}(L_{S^2}) = \frac{\partial^2}{\partial\theta^2}.$$

3. In general, need the radial part of the Laplace-Beltrami operator on G/H.

These are known, see e.g. Ref. [4] Caselle.

Also, can derive from microscopic models.

Thus obtain a Fokker-Planck equation for $P(x_1, \dots, x_N; L)$. Can write for all 10 symmetry classes:

$$\frac{\partial P}{\partial L} = \frac{1}{2\gamma l} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} J \frac{\partial}{\partial x_i} J^{-1} P$$

$\gamma \propto N$, $\gamma l \propto Nl \propto \xi$ - localization length

$$J = \prod_{i=1}^{N} \sinh^{m_l} 2x_i \prod_{j>k} \left[\sinh(x_j + x_k) \sinh(x_j - x_k)\right]^{m_o}.$$

m_o, m_l - non-negative integers depending on symmetry class

For standard Wigner-Dyson classes this eq. was obtained by Dorokhov, Mello, Pereyra and Kumar (DMPK equation), see Ref. [5, 6].

For chiral classes see Ref. [7] Brouwer et.al.

For BdG classes see Ref. [8] Brouwer et.al.

See also Ref. [10] Titov et.al.

Can analyze the FP equation in two regimes:

1) $L \gg \xi \; (\propto N\ell)$ — "localized" regime.

All x_i and spacings between them are large, $\gg 1$.

g is dominated by x_1, which is Gaussian-distributed with $\langle x_1 \rangle = \frac{m_e L}{\gamma \ell}$ and $\mathrm{var}\, x_1 = \frac{L}{\gamma \ell}$.

Thus, g is log-normal with

$$\langle \ln g \rangle = -\frac{2 m_e L}{\gamma \ell}, \quad \mathrm{var}\, \ln g = \frac{4L}{\gamma \ell}.$$

2) $L \ll \xi$ — diffusive regime.

Ohm's law with weak localization corrections:

$$\langle g \rangle = \frac{N\ell}{L} + O(1).$$

Universal conductance fluctuations.

3) For some classes can solve FP exactly and get the full crossover from diffusive to localized regimes.

V Class D.

1. In this case the BdG Hamiltonian is purely imaginary in some basis: $H^* = -H$.

Then $iH \in so(N)$ (Lie algebra).

$M \in SO_0(N,N)$ (connected component of $\mathbb{1}$).

The appropriate coset space is

$$\mathcal{M} = SO_0(N,N)/SO(N)\times SO(N)$$

For this $\mathcal{M}$, $m_\ell = 0$, and there is <u>no localization!</u>

Instead, g has an extremely broad distribution with

$$\langle g\rangle \propto \sqrt{\frac{\gamma\ell}{L}}\,, \quad \operatorname{var} g \propto \langle g\rangle,$$

$$\langle \ln g\rangle \propto -\sqrt{\frac{L}{\gamma\ell}}\,, \quad \operatorname{var}\ln g \propto \frac{L}{\gamma\ell}.$$

Very surprizing, since usually it is very easy to localize states in 1D.

2. Elementary explanation for this for $N=1$.

M is real $\left(M \in SO(1,1) \right)$:

$$M = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Then for $M = M_2 M_1$

$$m_{11} = \pm \cosh(x_1 \pm x_2).$$

So, x performs a random walk without a bias $\Rightarrow$

$\Rightarrow \langle x \rangle = 0$.

Then, there should be no drift term in the corresponding FP equation.

Indeed for real δM (and real V), see part III.3, we get

$$\delta y = \cosh 2x \, (\delta m_{12})^2 + \sinh 2x \, \delta m_{11} \delta m_{12}$$

$$\langle \delta y \rangle = \frac{\delta L}{\ell} \cosh 2x, \qquad \langle (\delta y)^2 \rangle = \frac{\delta L}{\ell} \sinh^2 2x$$

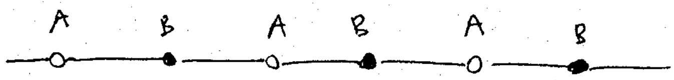
same coefficients $\frac{\delta L}{\ell}$ lead to cancellation of two contributions to $\langle \delta x \rangle$, and $\langle \delta x \rangle = 0 \Rightarrow$ no drift term.

3. This is known to happen in chiral models.

Random hopping between sublattices on a bipartite lattice

Ref. [9]: Dyson, Lifshits, Gade, Wegner.

$$H = \sum_i \left(t_i \, c_i^{+} c_{i+1} + h.c. \right)$$



Here delocalization happens only in a critical point, which is unstable to dimerization $P(t_{even}) \neq P(t_{odd})$.

Geometric interpretation:

The coset $\mathcal{M}$ for this case is a product of two spaces. Each one admits its own FP equation.
This is explained in Ref. [11].
No such thing in class D.

4. SUSY results. J. G., S. Vishveshwara, to be published.

Use a 1D network and a SUSY method (see A. Ludwig's lecture).

Transfer matrix for a node

$$M = \exp\begin{pmatrix} 0 & \theta \\ \theta^T & 0 \end{pmatrix}$$

must be distributed invariantly w.r.t. $SO(N) \times SO(N)$.

There are $\underline{2}$ invariant tensors for $SO(N)$:

Kronecker δ_{ab} and Levi-Civita $\varepsilon_{a_1 \dots a_N}$.

Correspondingly, there are cumulants

$$\langle \theta_{a_1 b_1} \theta_{a_2 b_2} \rangle_c = c_1 \, \delta_{a_1 a_2} \delta_{b_1 b_2} .$$

$$\langle \theta_{a_1 b_1} \theta_{a_2 b_2} \dots \theta_{a_N b_N} \rangle_c = c_2 \, \varepsilon_{a_1 a_2 \dots a_N} \varepsilon_{b_1 b_2 \dots b_N} .$$

The second of these leads to a term in the SUSY Hamiltonian which opens a gap in the SUSY spectrum (localizes the system).

What is the analog in the FP eq.?

Clear that the localization comes from non-Gaussian fluctuations in the disorder (N-th cumulant).

What does it lead to in the scattering approach?

5. Resolution of the paradox.

I.G., N. Read, to be published.

From the geometrical / group-theoretic point of view we need to find the radial parts of all $SO(N)\times SO(N)$ invariant differential operators on $\mathcal{M} = SO_0(N,N)/SO(N)\times SO(N)$.

This is known how to do (some high mathematics): see Ref. [12].

1) Find all invariants of the Weyl group W of the root system of $\mathcal{M}$, constructed of $\frac{\partial}{\partial x_i}$.

For class D,

$W = S_N \ltimes (\mathbb{Z}_2^n)_{even}$

- semidirect product of the permutation group of N objects and an even number of sign flips.

The basis of invariants in this case is given by elementary symmetric polynomials in $\frac{\partial^2}{\partial x_i^2}$ (in particular, $\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$, leading to the Laplace-Beltrami operator), plus $\frac{\partial^N}{\partial x_1 \dots \partial x_N}$, related to the N-th cumulant and the Levi-Civita tensor $\varepsilon_{a_1 \dots a_N}$.

2) Radial parts are obtained by sandwiching with $J^{1/2}$ and $J^{-1/2}$.

Of all the radial parts the new operator

$$L_N = J^{1/2} \frac{\partial^N}{\partial x_1 \dots \partial x_N} J^{-1/2}$$

is invariant w.r.t. $SO(N) \times SO(N)$, but not $O(N) \times O(N)$ (the symmetry of the rest).

When added to the Laplace-Beltrami operator L in the FP equation, it leads to localization.

In-deed, for large L, as we saw, all $x_2, \dots, x_N \gg 1$, and the derivatives w.r.t them can be replaced by constants.

Then $$L_N \to J^{1/2} \frac{\partial}{\partial x_1} J^{-1/2},$$

and this is simply the drift term for x_1.

We see from this argument that delocalization in class D in quasi-1D happens at an isolated critical point, which is in the same universality class as 1D chiral models.

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