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THETA TERMS IN NONLINEAR SIGMA MODELS OR NONLINEAR SIGMA-MODEL WITH HOPF TERM INDUCED BY FERMIONS

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Theta terms in nonlinear sigma models

or

Nonlinear sigma-model with Hopf term induced by fermions.

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Contents

A.G. Abanov and M. Braverman, in preparation
Hopf invariant from fermionic determinant on non-trivial 3-manifolds

A.G. Abanov and P.B. Wiegmann, Phys. Rev. Lett. **86**, 1319-1322 (2001)
Chiral nonlinear σ-models as models for topological superconductivity

A.G. Abanov, Phys. Lett. **B492**, 1-3 (2000)
Hopf term induced by fermions

A.G. Abanov and P.B. Wiegmann, Nucl. Phys. **B570**, 685-698 (2000)
Theta-terms in non-linear sigma-models

Topological terms in action

Topological — "not depending on continuous deformations"

<u>topological term</u> — term of effective action which does not depend on metric

Quantum field theory $Z = \int \mathcal{D}n(x,t)\, e^{iS[n(x,t)]}$

$S[n]$ - action $n(x,t)$ - all fields

We decompose $S[n] = S_0[n] + S_{top}[n]$

regular terms ($S_0[n]$) topological terms ($S_{top}[n]$)

After Wick rotation $\tau = it$

$$\frac{\partial n}{\partial t} = i \frac{\partial n}{\partial \tau} \quad , \quad dt = -i d\tau \qquad \text{etc.}$$

we have $e^{iS[n]} = e^{-S_0^E[n(x,\tau)]}$,

where $S^E[n]$ - Euclidian action

Regular terms are real both in conventional and Euclidian action

Example.

$$S_0[n] = \int dt \; \dot{n}^2 = \int -i d\tau \left(i \frac{\partial n}{\partial \tau} \right)^2 = i \int d\tau \left(\frac{\partial n}{\partial \tau} \right)^2$$

$$i S_0 = -\int d\tau \left(\frac{\partial n}{\partial \tau} \right)^2 = -S_0^E$$

$$S_0^E[n] = \int d\tau \left(\frac{\partial n}{\partial \tau} \right)^2$$

However, $\tau = it$ is just a rescaling of time. Topological terms do not depend on metric and, in particular on scale.

$$S_{top}[n(x,t)] = S_{top}[n(x,\tau)] \qquad i S_{top} = -S_{top}^E$$

$\overset{?}{\Rightarrow}$ Topological terms are always <u>imaginary</u> in Euclidian formulation.

Therefore, topological ~~inter~~ terms describe <u>interference effects</u> even in Euclidian ~~act~~ formulation of the theory. They are of pure <u>quantum</u> origin and distinguish quantum field theory from statistical mechanics.

The effects of topological terms can be drastic even if they are not as big (in absolute value) as other regular terms.

Theta - terms

Different types of topological terms:

- Wess-Zumino terms
- <u>Theta - terms</u>
- Topological current terms
- Chern - Simons terms etc.

Theta-terms :

if S^{d+1} - compactified spacetime

V - target space ($n \in V$)

and $\pi_{d+1}(V) \neq 0$

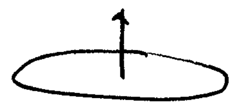
$\Rightarrow$ there exist integer number Q labeling topological classes of spacetime configurations of $n(x,t)$.

$$\boxed{S^E_\theta = i\theta \, Q[n]}$$ - topological theta-term

Example from Quantum Mechanics

Particle on a circle in magnetic field

$$Z = \int \mathcal{D}\varphi(t)\; e^{iS}$$



$$S[\varphi] = \int dt \left(\frac{m\dot{\varphi}^2}{2} + \dot{\varphi}\,\frac{\theta}{2\pi} \right)$$

$$S_0 = \int dt\, \frac{m\dot{\varphi}^2}{2} \qquad S_{top} = \theta \int dt\, \frac{\dot{\varphi}}{2\pi} = \theta\,\frac{\Delta\varphi}{2\pi} = \theta \cdot n$$

n - # of rotations around circle

$$S^E = \int d\tau\, \frac{m\dot{\varphi}^2}{2} - i\theta \int d\tau\, \frac{\dot{\varphi}}{2\pi}$$

$\frac{\theta}{2\pi}$ - flux of magnetic field through the circle

Spectrum $\quad E_\ell = \dfrac{\left(\ell - \frac{\theta}{2\pi}\right)^2}{2m} \quad$ ℓ - integer

$\theta = 0 \qquad E_0 = 0\,, \quad E_{\pm 1} = \frac{1}{2m}\,, \quad \dots$

$\theta = \pi \qquad E_{0,1} = \frac{1}{8m} \qquad E_{-1,2} = \frac{9}{8m}\,, \quad \dots$

At $\theta = \pi$ ground state is degenerate!

Motivation I

1. NLSM - semiclassical description of antiferromagnet.

$$Z = \int D\vec{n}\, e^{-S[\vec{n}]}, \qquad \vec{n}^2 = 1, \;\; \vec{n} \in S^2$$

$$S_{\rm NLSM}[\vec{n}] = \int d^{d+1}x \, \frac{1}{2g}(\partial_\mu \vec{n})^2$$

2. Theta term for 1d NLSM from one-dimensional spin S-chains.

$$S_{\rm theta}[\vec{n}] = i\theta Q[\vec{n}]$$

$$Q[\vec{n}] = \int_{S^2} d^2x \, \frac{1}{8\pi} \epsilon_{\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}]$$

$\theta = 2\pi S$ Haldane (1983)

3. Theta term $\rightarrow$ quantum interference and drastic change in the behavior of the QAFM.

S-integer $\longrightarrow$ gapped spectrum
S-half-integer $\longrightarrow$ gapless spectrum

Motivation II

4. Topology behind S_{theta}.

$$\vec{n}(x,t): \qquad S^2 \to S^2$$
$$\text{(compactified spacetime)} \to \text{(target space of } \vec{n}\text{)}$$

There exist nontrivial classes of mappings labelled by integer Q -winding number: $\pi_2(S^2) = Z$

$$Q[\vec{n}] = \int_{S^2} d^2x \, \frac{1}{8\pi} \epsilon_{\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}]$$

5. Two-dimensional NLSM with Hopf term.

$$S = S_{\mathrm{NLSM}} + i\theta H[\vec{n}]$$

$H[\vec{n}]$ – Hopf invariant – integer labelling classes of mappings: $\pi_3(S^2) = Z$.

$$\vec{n}(x,y,t): \qquad S^3 \to S^2$$

Dzyaloshinskii, Polyakov, Wiegmann (1988):

$$\theta = 2\pi S \qquad \text{for 2d QAFM.}$$

Motivation III

6. No microscopic derivation of NLSM with Hopf term from 2d QAFM analogous to Haldane's semiclassical derivation of NLSM with theta term for quantum spin chains.

   1d: (Berry phases of S_i) $\to$ (theta term)

   2d: (Berry phases of S_i) $\to 0$

   Fradkin & Stone;
   Haldane;
   Wen & Zee; (1988)
   Ioffe & Larkin;
   Dombre & Read.

However, see Khveshchenko & Wiegmann 1989

7. Magnetism occurs in strongly correlated electronic systems.

   Question:

Is there any fundamental fermionic system which produces NLSM with Hopf term as an effective theory?

Results

Yes, this model exists!

$$S = \int_{S^3} d^3x \, \bar{\psi}(i\not{\partial} + im\hat{n})\psi$$

$$\not{\partial} = \gamma^\mu \partial_\mu; \qquad \hat{n} = n^a \tau^a.$$

$\gamma^\mu = \sigma^\mu$ – Pauli matrices acting on spinor index of ψ

τ^a – Pauli matrices acting on isospinor index of ψ

$$\begin{aligned} S_{\text{eff}} &= -\ln\det(i\not{\partial} + im\hat{n}) \\ &= \int d^3x \, \frac{m}{8\pi}(\partial_\mu \vec{n})^2 + i\theta H[\vec{n}] \end{aligned}$$

$$\Downarrow$$

spin $1/2$ and fermionic statistics of solitons

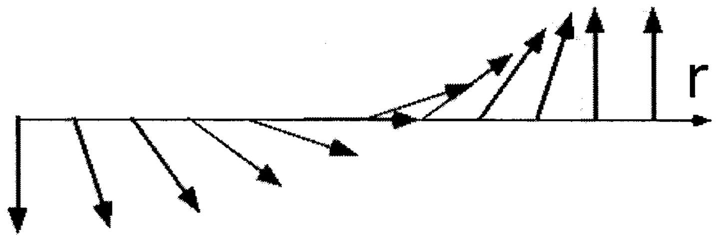
In 1d: $S_{1d} = -\ln\det(i\not{\partial} + im\gamma^5\hat{n})$ Tsvelik (1994)

Hopf invariant

1. Algebraic. Topological current:

$$J^\mu = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\vec{n}[\partial_\nu\vec{n} \times \partial_\lambda\vec{n}]$$

   $Q = \int d^2x\ J^0$ – topological charge (number of skyrmions)



$$\partial_\mu J^\mu \equiv 0 \qquad \longrightarrow \qquad J^\mu \equiv \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$$

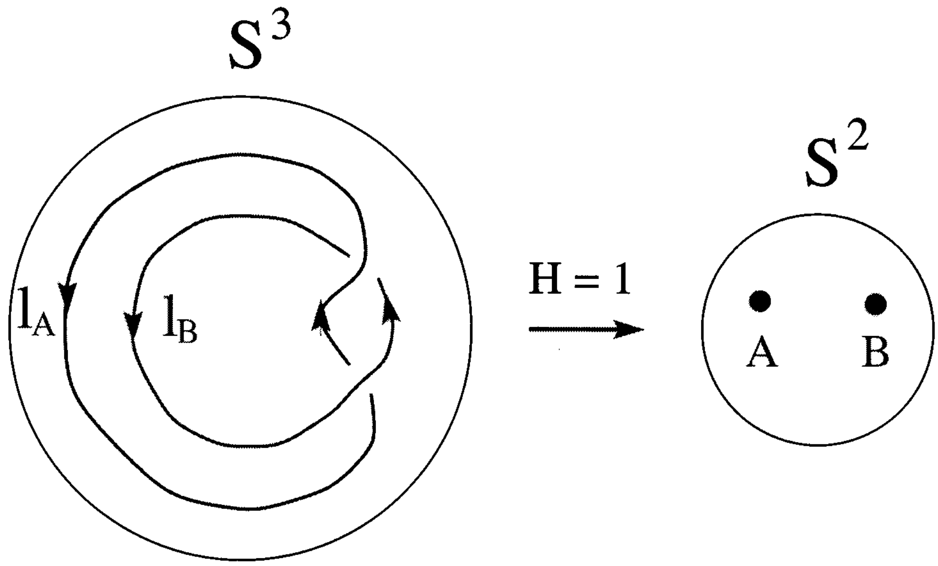
$$H = -\frac{1}{4\pi^2}\int_{S^3} d^3x\ \epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda.$$

2. Geometric. Hopf invariant is a linking number of two world-lines of $\vec{n}$.

3. Physical. Hopf term defines spin and statistics of skyrmions.

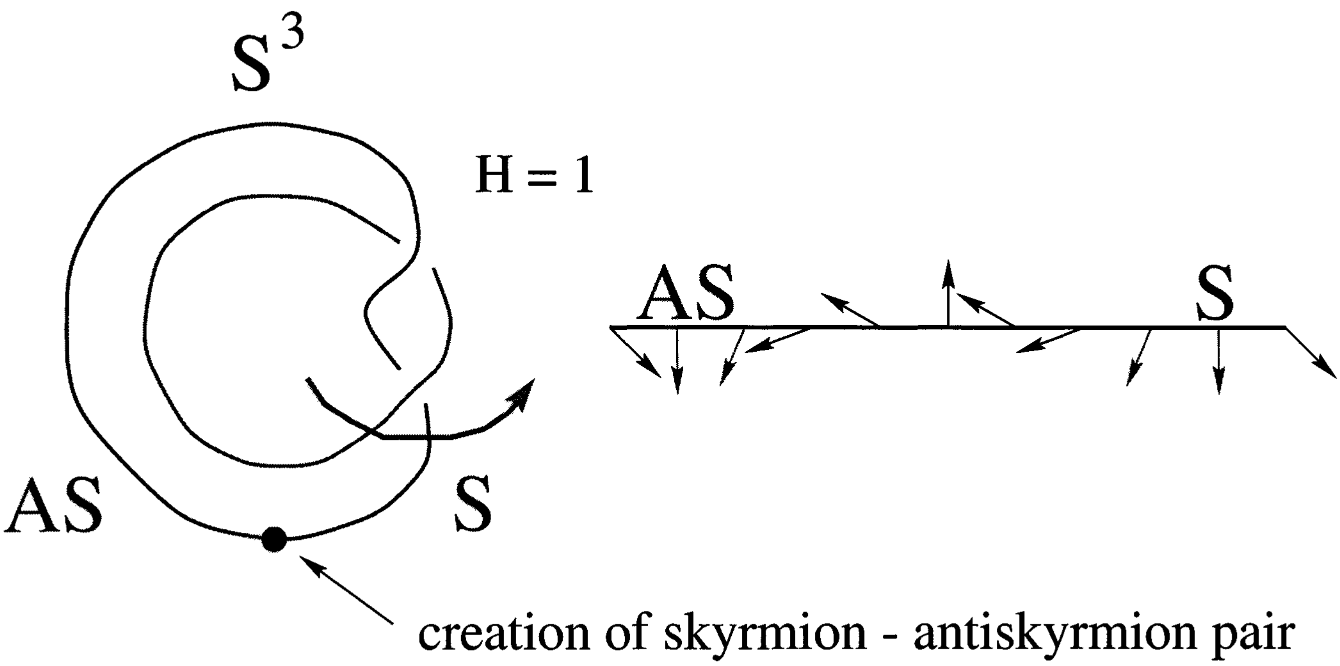
Hopf invariant — linking number of world-lines of $\vec{n}$

$$H = \#(l_A, l_B).$$



Spin of skyrmions

F. Wilczek and A. Zee, (1983)



$H = 1$ for this process $\Rightarrow$ skyrmion rotation contributes $e^{i\theta H} = e^{i\theta} = e^{i2\pi S}$

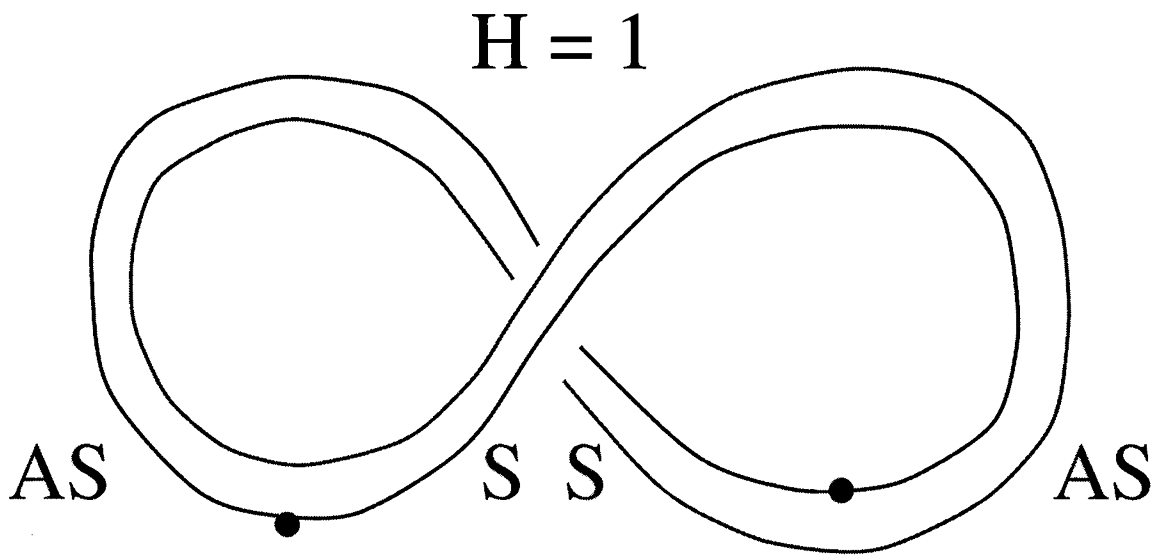
$$S = \frac{\theta}{2\pi}.$$

For $\theta = \pi$

Skyrmion has spin $\frac{1}{2}$!

Statistics of skyrmions

Similarly, the process of creation of two S-AS pairs, the interchange of skyrmions, and then annihilation of pairs has $H = 1$ which leads to a conclusion that statistical angle of skyrmions is θ.



Therefore, for $\theta = \pi$

> Skyrmions are fermions!

Non-linear σ-model induced by fermions

$$S = \int_{S^3} d^3x \; \bar{\psi} \left[i \partial\!\!\!/ + A\!\!\!/ + im\hat{n} \right] \psi$$

Space-time: three-dimensional sphere S^3.

$\partial\!\!\!/ = \gamma^\mu \partial_\mu, \quad \mu = 1, 2, 3,$

γ^μ – 2×2 gamma-matrices

A_μ – external Abelian gauge field

Target space: two-dimensional sphere S^2.

$\hat{n} = \vec{n} \cdot \vec{\tau}, \quad \vec{n}^2 = 1,$

$\vec{\tau}$ – the set of Pauli matrices acting in isospace.

$$e^{-S_{eff}(n)} = \int d\psi \, d\bar{\psi} \, \exp\left(-S \left[\psi, \bar{\psi}, \vec{n} \right] \right)$$

$$\boxed{S_{eff} = -\ln \det \left[i \partial\!\!\!/ + A\!\!\!/ + im\hat{n} \right]}$$

Symmetries

- Parity:

$$\begin{aligned} \psi(x,y,t) &= \gamma^1 \psi'(-x,y,t), \\ \vec{n} &= -\vec{n}'. \end{aligned}$$

- Abelian gauge invariance:

$$\begin{aligned} \psi &= e^{i\alpha}\psi', \\ A_\mu &= A'_\mu + \partial_\mu \alpha. \end{aligned}$$

- $SU(2)$ gauge invariance if one adds non-Abelian gauge field:

$$\begin{aligned} \psi &= U\psi', \\ \hat{n} &= U\hat{n}'U^{-1}, \\ \hat{a}_\mu &= U\hat{a}'_\mu U^{-1} - i\partial_\mu U U^{-1}. \end{aligned}$$

Chiral rotation

$\psi = U\chi$ with $SU(2)$ matrix U: $U^{-1}\hat{n}U = \tau^3$

$$S_{eff} = -\text{Tr}\ln\left[i\partial\!\!\!/ + A\!\!\!/ + \hat{a}\!\!\!/ + im\tau^3\right],$$

where $\hat{a}_\mu = U^{-1}i\partial_\mu U$ is a pure gauge.

Expansion in $\hat{a}_\mu, A_\mu$ gives in effective Lagrangian:

$$\frac{m}{2\pi}\left(\left(a_\mu^1\right)^2 + \left(a_\mu^2\right)^2\right) \longrightarrow \frac{m}{8\pi}(\partial_\mu\vec{n})^2$$

$\frac{i}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda^3 \to iA_\mu J^\mu \to$ **Skyrmion has charge 1!**

but no term

$$\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu^3\partial_\nu a_\lambda^3 \longrightarrow i\pi H$$

Skyrmion has no spin!?

Global chiral rotation

Rotation matrix $U(x)$: $S^3 \to SU(2) = S^3$ is characterized by an integer winding number.

$$\pi_3(SU(2)) = Z$$

This winding number – Hopf invariant of $\vec{n}$.

To unwind $\vec{n}$ with non-zero Hopf invariant one needs globally non-trivial chiral rotation!

$$H[\vec{n}] = \frac{i}{24\pi^2} \int_{S^3} d^3x \; \epsilon^{\mu\nu\lambda} \text{tr} \left(U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\lambda U\right)$$

Transformation $\psi \to U\psi$ is globally non-trivial and can result in non-trivial "Jacobian". In fact,

$$\det\left(U^{-1}DU\right) = (-1)^H \det(D)$$

Witten (1983)

Calculation

$$\begin{aligned}\delta S_{eff} &= -\delta\left(\text{Tr}\ln D\right) = -\text{Tr}\,\delta D D^{-1} \\ &= -\text{Tr}\left[\delta D D^{\dagger}(DD^{\dagger})^{-1}\right],\end{aligned}$$

where

$$\begin{aligned} D &= i\partial\!\!\!/ + im\hat{n}, \\ D^{\dagger} &= i\partial\!\!\!/ - im\hat{n}, \\ \delta D &= im\delta\hat{n}, \\ DD^{\dagger} &= -\partial^2 + m^2 + m\partial\!\!\!/\hat{n}. \end{aligned}$$

Expanding $(DD^{\dagger})^{-1}$ in powers of $m\partial\!\!\!/\hat{n}$ and calculating traces we obtain:

$$\boxed{\delta(\Im S_{eff}) = \frac{\text{sgn}(m)}{32\pi}\int d^3x\,\epsilon^{\mu\nu\lambda}\,\text{tr}\left(\hat{n}\delta\hat{n}\partial_{\mu}\hat{n}\partial_{\nu}\hat{n}\partial_{\lambda}\hat{n}\right).}$$

We have used only the property $\hat{n}^2 = 1$

Calculating trace of τ-matrices we obtain 0!

No surprize! $\quad \delta(i\theta H[\vec{n}]) = 0 \quad$ for any θ.

CP^n generalization of the model

$$S = \int_{S^3} d^3x \; \bar{\psi} \left[i \partial\!\!\!/ + i m \hat{n} \right] \psi,$$

where in CP^1 representation $\vec{n} = z^\dagger \vec{\tau} z$ or

$$\hat{n} = 2zz^\dagger - 1 = \begin{pmatrix} 2z_1 z_1^\star - 1 & 2z_1 z_2^\star \\ 2z_2 z_1^\star & 2z_2 z_2^\star - 1 \end{pmatrix}$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad z^\dagger z = |z_1|^2 + |z_2|^2 = 1.$$

Under $z \longrightarrow e^{i\alpha} z$, $\hat{n} \longrightarrow \hat{n}$ so that target space $S^3/U(1) = CP^1 = S^2$.

We introduce $\hat{n} = 2zz^\dagger - 1$ with:

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n+1} \end{pmatrix}, \qquad z^\dagger z = 1.$$

Target space $S^{2n+1}/U(1) = CP^n$ and still $\hat{n}^2 = 1$.

T. Jaroszewicz (1987)

Reduction

Consider now the particular configuration

$$z = \begin{pmatrix} z_1 & z_2 & 0 & \cdots & 0 \end{pmatrix}^T, \quad z^\dagger z = 1.$$

Then

$$\hat{n} = 2zz^\dagger - 1 = \begin{pmatrix} 2z_1 z_1^\star - 1 & 2z_1 z_2^\star & 0 & \cdots & 0 \\ 2z_2 z_1^\star & 2z_2 z_2^\star - 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

and Lagrangian factorizes into the model we are interested in and $\vec{n}$-independent massive Dirac fermions.

$$L = \bar{\psi}\left[i\partial\!\!\!/ + im\hat{n}\right]\psi + \sum_{k=3}^{n+1} \bar{\psi}_k \left[i\partial\!\!\!/ - im\right]\psi_k.$$

Therefore

$$S_{eff}^{CP^n}[z]\Big|_{z=(z_1,z_2,0,\ldots,0)^T} = S_{eff}^{CP^1}[z]\Big|_{z=(z_1,z_2)} + \text{const}$$

Topology of CP^n

There are solitons in CP^n model:

$$\pi_2(CP^n) = Z$$

and, therefore, topological current can be defined

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda,$$

where $a_\mu = z^\dagger(-i\partial_\mu)z$ is the same (in CP^1 case) as a component of "non-Abelian field" a^3_μ.

However, there is no Hopf invariant for CP^n model with $n > 1$.

$$\pi_3(CP^1) = Z, \quad \rightarrow \quad \text{Hopf invariant,}$$

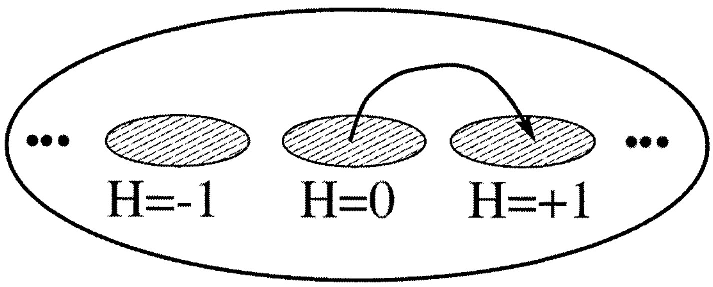
$$\pi_3(CP^{n>1}) = 0, \quad \rightarrow \quad \text{no Hopf invariant.}$$

Therefore, for CP^n model one can restore the effective action from its *variation* δS_{eff} without loosing any information.

Embedding method

Two mappings $\vec{n}(x): \; S^3 \to S^2$ with different Hopf invariants cannot be deformed one into another. It is difficult to compare $\det D(\vec{n})$ for those configurations.

We embedded $S^2 = CP^1$ into the bigger manifold CP^n so that two configurations $z = (z_1, z_2, 0, \ldots, 0)$ and $z' = (z'_1, z'_2, 0, \ldots, 0)$ can always be deformed one into another albeit through configurations with $z_3, z_4, \ldots \neq 0$.



$$\text{Map}\,(S^3 \to CP^1) \subset \text{Map}\,(S^3 \to CP^n)$$

disconnected connected

We can perform this deformation and see that

$$\det D(\vec{n}) \sim (-1)^H .$$

Calculation

$$\delta(\Im S_{eff}) = \frac{\mathrm{sgn}(m)}{32\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} \, \mathrm{tr} \left(\hat{n}\delta\hat{n}\partial_\mu\hat{n}\partial_\nu\hat{n}\partial_\lambda\hat{n} \right).$$

We have used so far only the property $\hat{n}^2 = 1$ which is true for general CP^n field.

To "remove" variation we use $\hat{n} = 2zz^\dagger - 1$ and after some manipulations obtain

$$\delta(\Im S_{eff}) = \frac{\mathrm{sgn}\,(m)}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} \left[\delta \left(a_\mu \, \partial_\nu a_\lambda \right) + \partial_\mu \left(a_\nu \delta a_\lambda + 2iz^\dagger \delta z \partial_\nu a_\lambda \right) \right],$$

On S^3 one can define z globally and full derivative term does not contribute to the integral.

We obtain

$$\Im S_{eff} = \pi \text{sgn}\, m \frac{1}{4\pi^2} \int d^3x\, \epsilon^{\mu\nu\lambda}\, a_\mu\, \partial_\nu a_\lambda .$$

This is correct for general CP^n model.

Let us now make a reduction and consider $a_\mu = z^\dagger(-i\partial_\mu)z$ with $z = (z_1, z_2)$ – CP^1-field. Then

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\vec{n}[\partial_\nu\vec{n} \times \partial_\lambda\vec{n}]$$

and the "Chern-Simons term" is an algebraic representation of Hopf invariant:

$$\boxed{\Im S_{eff} = -\pi \text{sgn}(\, m)\; H.}$$

Regularization by embedding

The question of relative sign of fermionic determinants for configurations with different Hopf invariants is ill-defined. There are several ways to compare those signs:

- Soft constraint. In physical models the constraint $\vec{n}^2 = 1$ is always "soft". Allowing high energy processes with $\vec{n} = 0$ at some points one can deform, say, $H = 0$ into $H = 1$.

- Lattice regularization. All configurations $\vec{n}$ can be connected by smooth transformations.

- Embedding in the bigger space allows to connect configurations with different H.

The advantage of the embedding method — no singular configurations of $\hat{n}$ in the process of deformation.

Hopefully (it is not clear) all these regularizations give the same answer.

Conclusions

- The Hopf term is derived on S^3 for the non-linear σ-model coupled to $2+1$ Dirac fermions through Yukawa coupling. The coefficient in front of Hopf invariant is $\theta = \pi$.

- The value $\theta = \pi$ makes skyrmions spin-$\frac{1}{2}$ fermions. This is in agreement with known electric charge $e = 1$ of skyrmions.

- The calculation of the fermionic determinant is performed by embedding method .

- Hopf term is derived also for the models defined on product spaces $S^2 \times S^1$ and T^3.

Some open problems

- Physical applications: two-dimensional antiferromagnets, quantum Hall ferromagnet, topological superconductivity, superfluid films of $^3He - A$, high energy (high temperature?) physics etc.

- Other types of regularization: lattice and soft constraint.

- Topological terms in the presence of singular configurations of chiral fields.

- Interplay between different types of topological terms.