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SUMMER SCHOOL
on
LOW-DIMENSIONAL QUANTUM SYSTEMS:
Theory and Experiment
(16 - 27 JULY 2001)

PLUS

PRE-TUTORIAL SESSIONS
(11 - 13 JULY 2001)

DISORDER: BASIC CONCEPTS AND METHODOLOGY

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These are preliminary lecture notes, intended only for distribution to participants

1A

Disorder: Basic concepts and methodology.

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Part I:
Models of disorder

① Impurity potential

$$\boxed{H = \frac{\hat{p}^2}{2m} + U(r)}$$

random Gaussean field

Simplest assumptions about $U(r)$

i) Gaussean distribution $\quad P(U) = e^{-\varkappa^{-1}\int U^2(r)dr}$

ii) zero mean $\quad \langle U(r) \rangle = 0$

iii) white noise correlation

$$\boxed{\langle U(r)\, U(r') \rangle = \varkappa\, \delta(r-r')}$$

Real impurity potential

potential of a single impurity

$$U(r) = \sum_i u(r-r_i) - \left\langle \sum_i u(r-r_i) \right\rangle$$

$$\langle \dots \rangle = \int \frac{dr_i}{V}$$

independent positions of impurity centers

② Anderson model

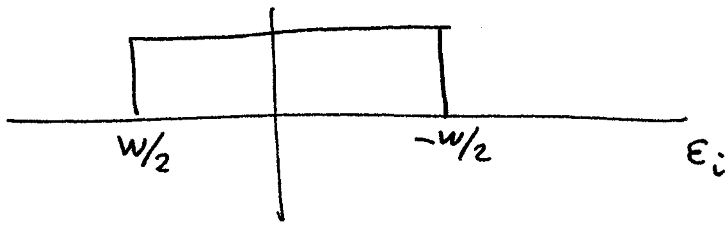
regular hopping — random on-site energies

$$\boxed{H = -\sum_{\langle ij \rangle} c_i^\dagger c_j + \sum_i \varepsilon_i \, c_i^\dagger c_i}$$

Model on the d-dimensional lattice

$$\boxed{\begin{array}{l}\text{diagonal disorder:} \\ \quad \text{only } \varepsilon_i \text{ is random}\end{array}}$$

Distribution of ε_i :



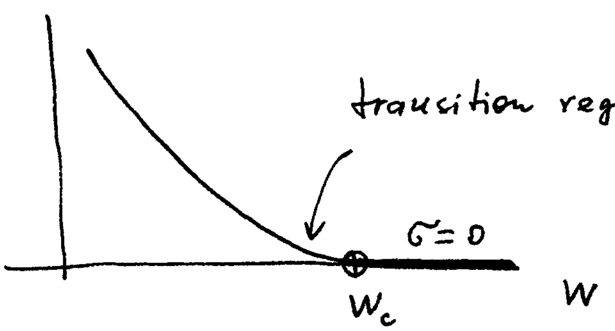
W is the disorder strength

Basic knowedge

$d = 1,2$: all states are localized

$d \geq 3$: there is the Anderson localization - delocalization transition at $W = W_c$

conductivity σ

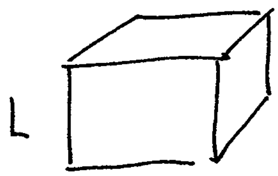


transition region : $\sigma \sim \left| \frac{W - W_c}{W_c} \right|^\beta$

β is the critical exponent that depends on the dimensionality d.

Dimensionless conductance and scaling

Conductance $G = g \dfrac{e^2}{h}$ ← dimensionless conductance



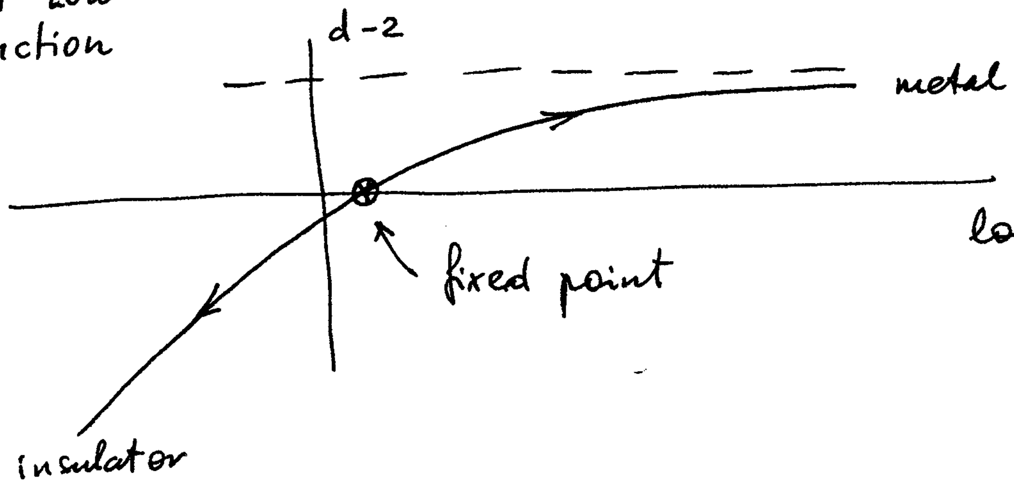
Hypercube sample of the size L

$$g = \begin{cases} \sigma L^{d-2}, & \text{metal} \quad (W < W_c) \\ e^{-L/L_0}, & \text{localization} \quad (W > W_c) \end{cases}$$

$$\boxed{g = g^* = \text{const}} \quad \text{at the transition}$$

$$\beta(g) \equiv \frac{d \log g}{d \log L} = \begin{cases} d-2, & \text{metal} \\ \log g, & \text{insulator} \end{cases}$$

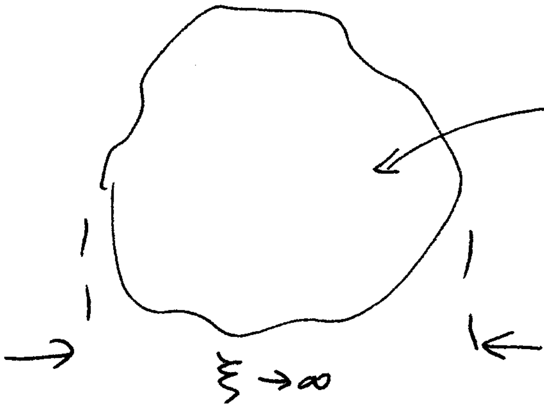
↑
Gell-Mann-Low
β - function



There is a divergent correlation length $\xi = \left|\dfrac{W-W_c}{W_c}\right|^{-\nu}$

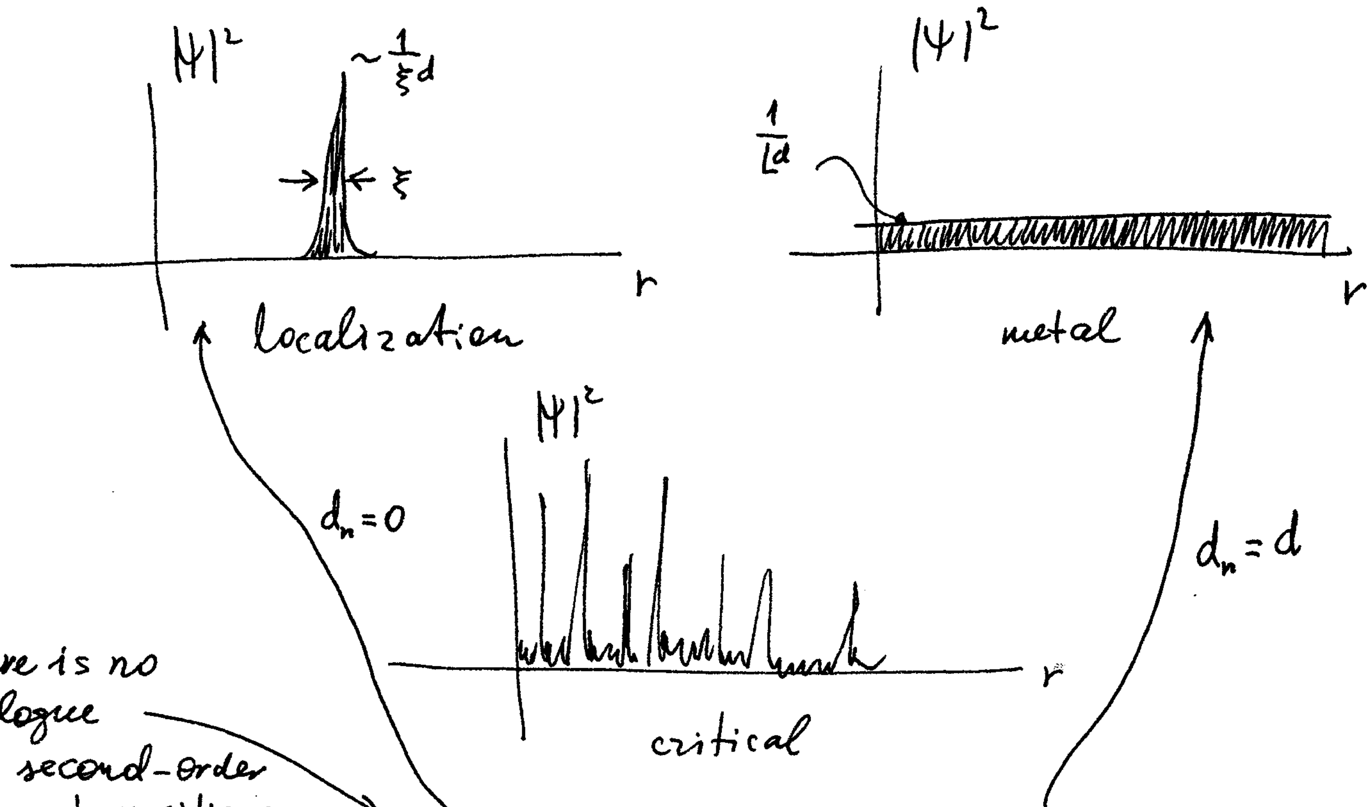
At the critical point $\sigma \sim g^*/\xi^{d-2} = g^* \left|\dfrac{W-W_c}{W_c}\right|^{\nu(d-2)}$

Multifractality of critical wave-functions



what are the properties of wave functions inside the localization volume near the critical point?

Wave functions are extended but irregular



there is no analogue in second-order phase transitions

$$\boxed{\sum_r |\Psi(r)|^{2n} = \frac{1}{L^{d_n(n-1)}}}$$

L = system size

d_n is a set of fractal dimensions depending on n = <u>multifractality</u>

Wave functions correlations

$|r-r'| \gg l$ = elastic scattering length

Insulator: $\langle |\psi(r)|^2 |\psi(r')|^2 \rangle \sim e^{-|r-r'|/\xi}$

Metal: $\langle |\psi(r)|^2 \, |\psi(r')|^2 \rangle = \langle |\psi|^2 \rangle^2$ + weak correlations

Criticality:

$$\langle |\psi(r)|^{2n} |\psi(r')|^{2n} \rangle \sim L^{-2dn} \left(\frac{L}{l} \right)^{2(n-1)(d-d_n)}$$

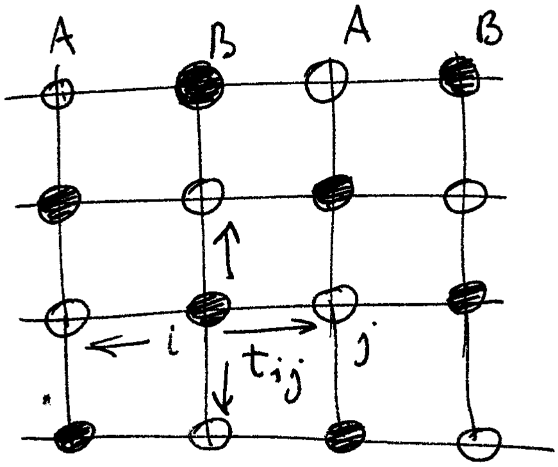
$$\times \underline{\underline{\left(\frac{L}{|r-r'|} \right)^{\zeta_n}}} \gg \langle |\psi|^{2n} \rangle^2$$

$$\underline{\zeta_n = 2n(d_n - d_{2n}) + d_{2n} - 2d_n + d}$$

non-trivial exponents related to fractal dimensions

Role of additional symmetries

off diagonal disorder



Lifshitz model

$$\boxed{H = \sum_{\langle ij \rangle} t_{ij}\, c_i^{+} c_j}$$

There are matrix elements only between sublattice A and B : H_{AB}, H_{BA}

No matrix elements H_{AA}, H_{BB}

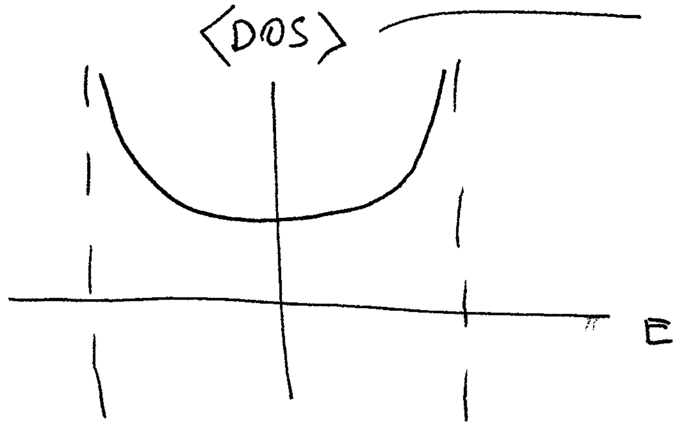
$$H = \left(\begin{array}{c|c} 0 & H_{AB} \\ \hline H_{BA} & 0 \end{array} \right)$$

bloc sublattice representation

The additional symmetry (chiral symmetry)

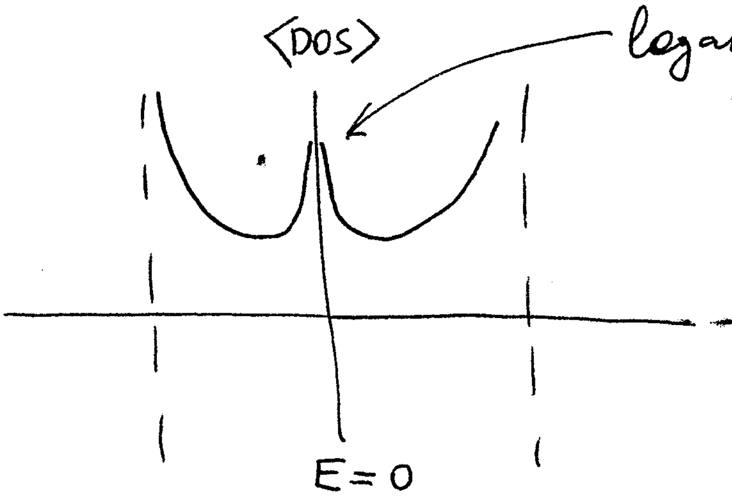
$$\boxed{\Sigma_z H \Sigma_z = -H}, \quad \Sigma_z = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array} \right)$$

One-dimensional case (J.F. Dyson, PR $\underline{92}$, 1331 (1953))



density of states

no disorder



logarithmic divergence at E = 0

with disorder

Disorder-induced singularity!

Recent development: [Altland, Simons, Nucl. Phys. $\underline{B562}$, 445 (1999)]

Gapless superconductor or normal-metal surrounded by a superconductor

Bogolyubov - De Gennes Hamiltonian

$$H = \left(\begin{array}{c|c} \xi + U(r) & \Delta \\ \hline \Delta^* & -\xi^* - U(r) \end{array} \right)$$

Most interesting when $\Delta = 0$

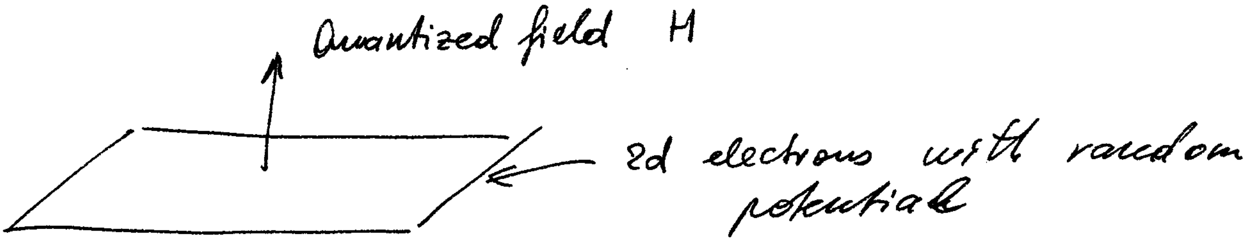
$$\xi_{\hat{p}} = \frac{\left(\hat{p} - \frac{e}{c} A\right)^2}{2m} - \varepsilon_F$$

$$\boxed{\Sigma_2 H \Sigma_2 = -H^*}$$

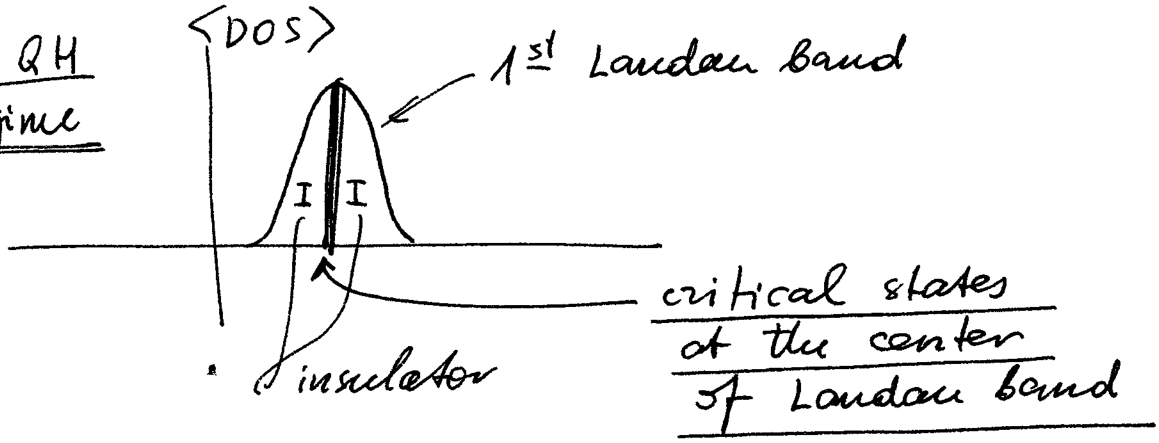
$$\Sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

This symmetry ensures $E \to -E$ symmetry of spectrum

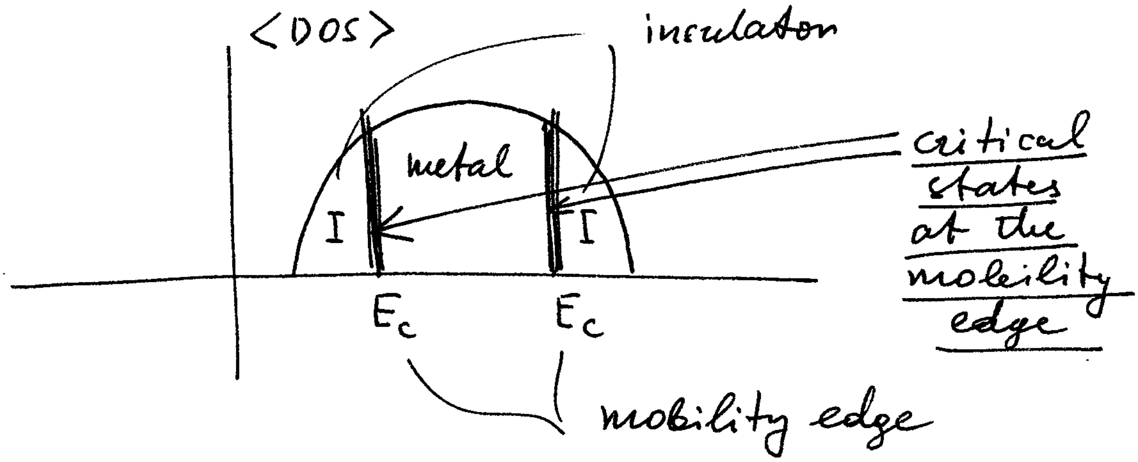
Network models



Integer QH regime



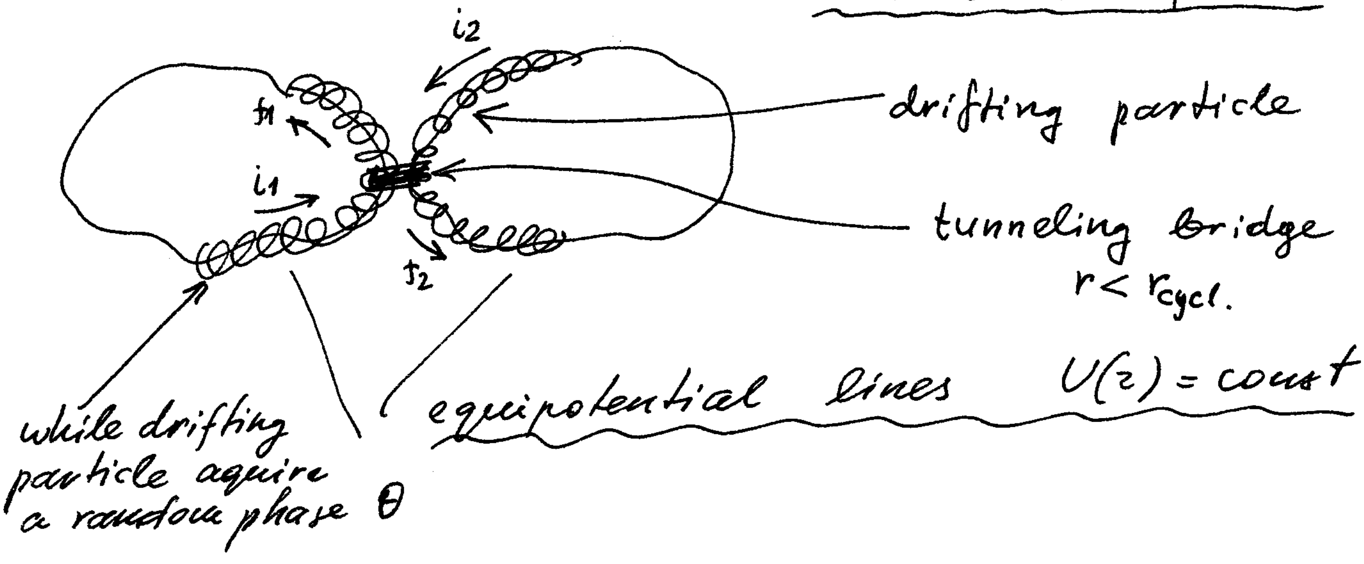
3d Anderson model

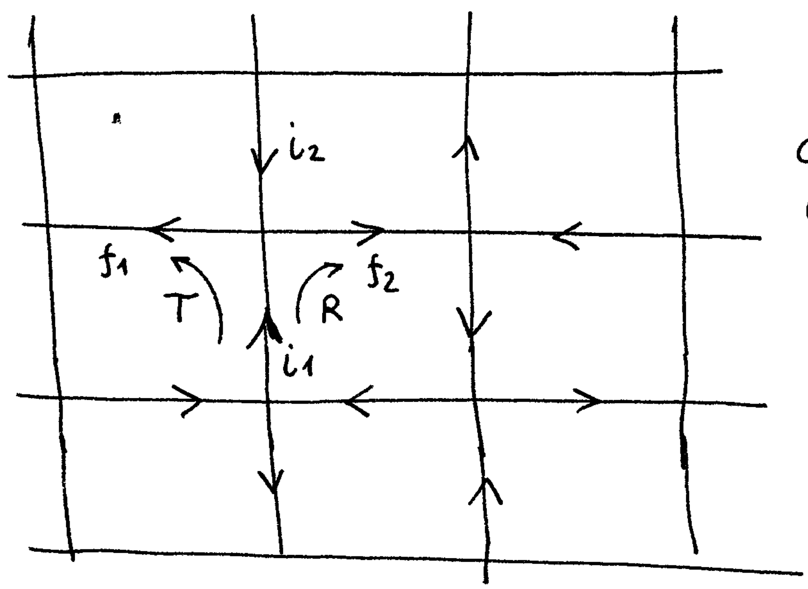


$$\tau = \left| \frac{W - W_c}{W_c} \right| \longleftrightarrow \left| \frac{E - E_c}{E_c} \right|$$

$$\sigma \sim \left| \frac{E - E_c}{E_c} \right|^{\nu(d-2)}$$

Semiclassical picture





chalker-Coddington network

Drifting particle (i_1) can stay on the same equipotential line (f_1) or it can tunnel to another equipotential line (f_2).

This process is described by a 2×2 scattering matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \qquad |S_{11}|^2 = |S_{22}|^2 = T$$

$$|S_{12}|^2 = |S_{21}|^2 = R$$

Integer QH transition at $T = R = 1/2$

Random phases $\arg S_{12} = \theta$

Integer Quantum - Hall system and 2d Dirac Hamiltonian

2x2 matrices $\hat{S}$ amount to the global transfer matrix T_G

stationary state

$$\boxed{T_G \psi = \psi}$$

Relationship between the transfer matrix T_G and effective Hamiltonian $\tilde{H}$:

$$\boxed{T_G = e^{i\tilde{H}} \approx 1 + i\tilde{H}}$$

near criticality $|T-\frac{1}{2}| \ll 1$.

Continuous limit:

$$\tilde{H} = \begin{pmatrix} m+V & p_x - ip_y \\ p_x + ip_y & -m+V \end{pmatrix}$$

2d Diral Hamiltonian

two-component struc. $\Leftrightarrow$ incoming and outgoing links

$$p = -i\nabla + A$$

$\langle m \rangle = 0$ critical point $(T = \frac{1}{2})$

Randomness:
- in A arises from random phases θ_i
- in V arises from random $\sum_{\text{plaquet}} \theta_i$
- in m arises from random T or R

Part II:

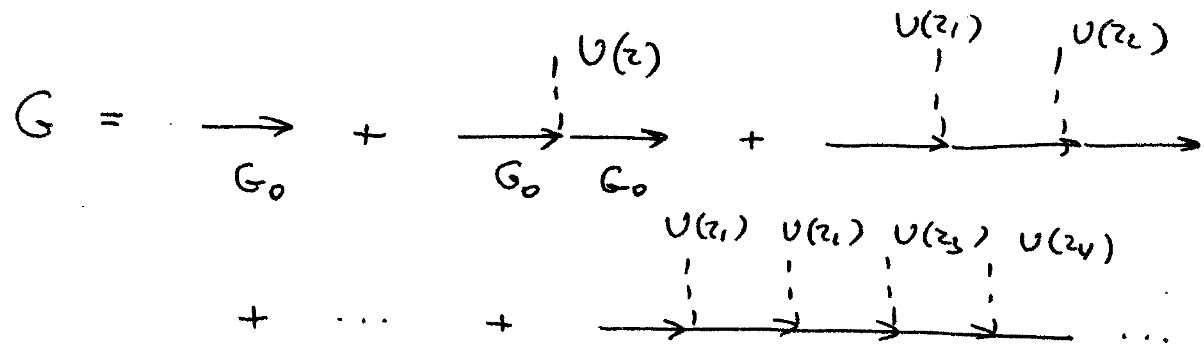
From free particles in a random potential to a deterministic field theory with effective interaction

1. Perturbation series

$$H = \frac{p^2}{2m} + U(r) \Longleftarrow$$ no interaction

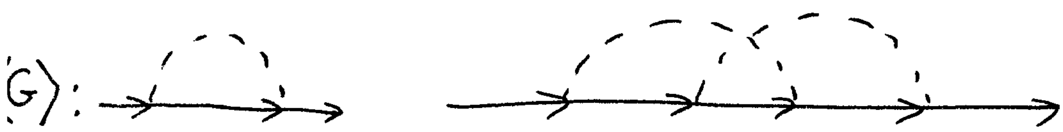
$$\langle U(r)\, U(r') \rangle = \varkappa\, \delta(r-r')$$

Particle Green's function $G(r, r')$



Averaging = connecting dotted lines in pairs

Examples of diagrams

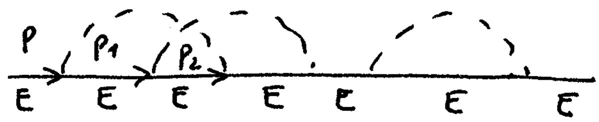


$r \succ - - \prec r' = \varkappa\, \delta(r-r')$ Effective interaction

What is special in the perturbation series?

1). No transfer of energy $\Rightarrow$

no need to consider time, energy is just a parameter



only momentum is transferred but not energy


$= \varkappa\,\delta(r-r')$ is time-independent

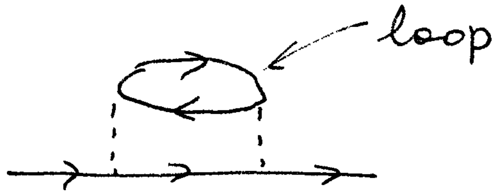
d-dimensional space in non-interacting disordered systems

(d-1) + 1 dimensional spacetime in interacting systems

analogy

② (Crucial) No loops

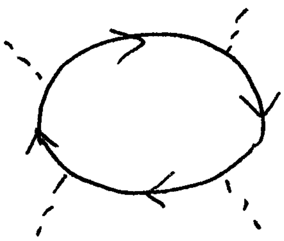

non-interacting disordered system



interacting system

How to kill loops automatically?

1). Replica trick



$= \mathrm{Tr}\, G_0 G_0 G_0 \ldots$

loop means Tr

Consider n species of particles in the same random field $V(z)$

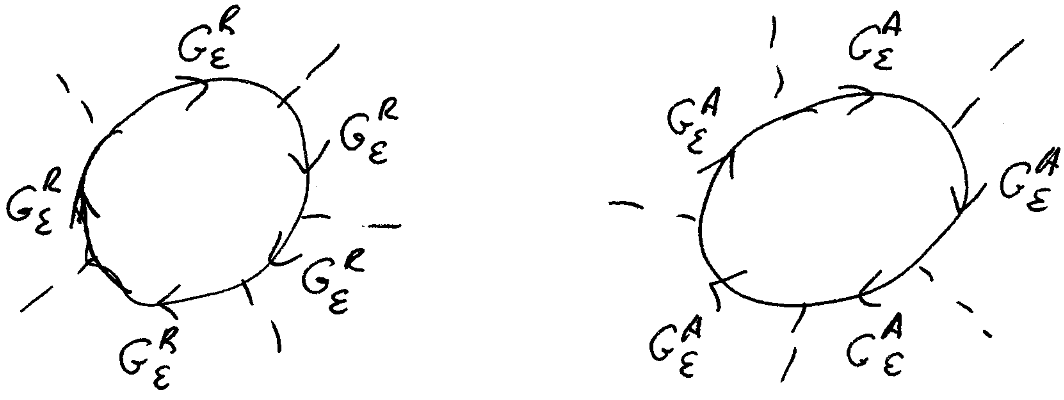
$$G_0^{nm} = \delta_{nm} G_0$$

$$\mathrm{Tr}\, G_0^{nn_1} G_0^{n_1 n_2} \ldots G_0^{n_i n} \propto n \to 0$$

if $n \to 0$.

By making analytial continuation to $n \to 0$ in the final result all loops are killed.

2). time-dependent field theories (Keldysh)



Tr includes $\int_{-\infty}^{+\infty} d\varepsilon$

$$\int_{-\infty}^{+\infty} d\varepsilon \; G^R_\varepsilon \, G^R_\varepsilon \dots G^R_\varepsilon = 0$$

retarded Green functions G^R: all singularities in the upper half-plane

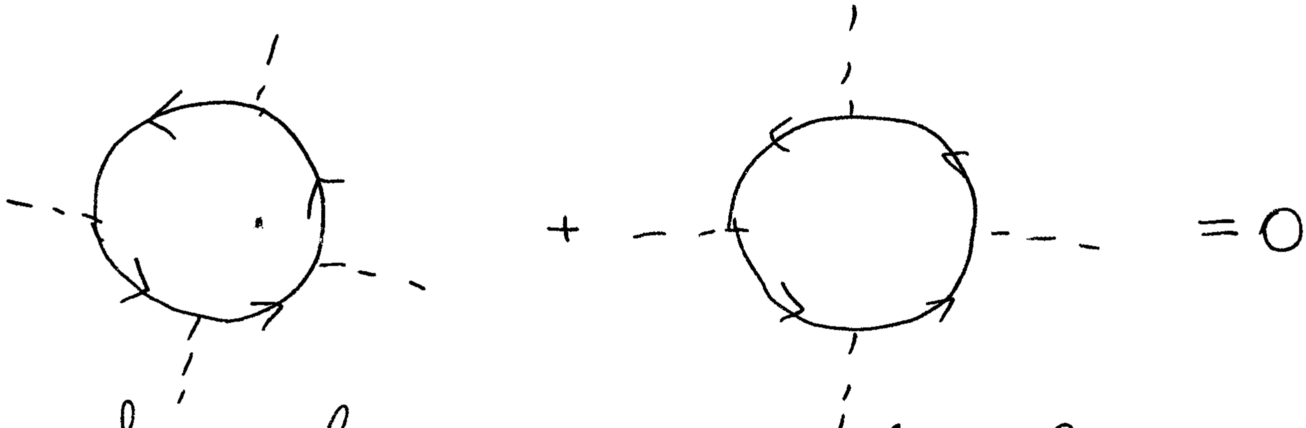
$$\int_{-\infty}^{+\infty} d\varepsilon \; G^A_\varepsilon \, G^A_\varepsilon \dots G^A_\varepsilon = 0$$

advanced Green functions G^A: all singularities in the lower half-plane.

3). <u>Super-symmetry</u>

free particles in the one-particle problem can be both bosons and fermions ⇒

two species insted of (n) in the replica tric. One is bosonic and another is fermionic.



fermion loop + boson loop = 0

Valid only for non-interacting particles in disordered systems where statistics does not matter.

But: <u>allows to obtain exact non-perturbative results</u>

Non-perturbative treatment: supersymmetric approach

Efetov
„Supersymmetry in disorder and chaos", Cambridge Univ. Press

① anti-commuting variables

$\{\chi_i, \chi_j\} = 0$

$\{\chi_i, \chi_j^*\} = 0$ even if $i=j$

$(\chi^*)^* = -\chi$

Rules of integration

$$\int d\chi\, \chi = 1$$
$$\int d\chi = 0$$
$$\int d\chi\, \chi^* = 0$$

$$\int \prod_{i=1}^{N} d\chi_i d\chi_i^* \quad e^{\sum_\mu \chi_\mu^* a_\mu \chi_\mu} = \prod_{i=1}^{N} a_i = \det A$$

only the term containing the „full set" of Grassmann variables makes a non-zero contribution

$$\chi_1^* a_1 \chi_1 \;\; \chi_2^* a_2 \chi_2 \;\; \ldots \;\; \chi_N^* a_N \chi_N$$

$$\boxed{\int \mathcal{D}\varphi_F^* \mathcal{D}\varphi_F \quad e^{\int \varphi_F^* A \varphi} = \det A}$$

No problem of convergence, since „integration" is an expansion of the exponent.

② Averaging over $U(r)$

Electron Green's function

Signs reflect requirement of convergence

$$G^{R(A)} = \frac{\int \mathcal{D}\varphi_B^* \mathcal{D}\varphi_B \;\; \varphi_B \varphi_B^* \;\; e^{\pm i \int \varphi_B^* (E - H \pm i\delta)\varphi_B}}{\underbrace{\int \mathcal{D}\varphi_B^* \mathcal{D}\varphi_B \;\; e^{\pm i \int \varphi_B^* (E - H \pm i\delta)\varphi_B}}_{\equiv Z_B}}$$

$$\boxed{H = H_0 + U(r)}$$

φ_B^*, φ_B - normal (commuting) variables.

The problem: random field $U(r)$ both in the numerator and in the denominator

How to average?

Let us multiply both the numerator and the denominator by

$$Z_F = \int \mathcal{D}\varphi_F^* \mathcal{D}\varphi_F \;\; e^{i \int \varphi_F^* (E - H \pm i\delta)\varphi_F} =$$

$$= i^N \det(E - H \pm i\delta)$$

$$Z_B = (\pm \pi i)^N \left[\det(E - H \pm i\delta)\right]^{-1}$$

no problem of averaging the denominator!

$$\boxed{Z_F Z_B = \text{const}}$$

no dependence on random field $U(r)$

$$G^{R(A)} = \frac{\int \mathcal{D}\varphi_B^* \mathcal{D}\varphi_B \mathcal{D}\varphi_F^* \mathcal{D}\varphi_F \quad \varphi_B \varphi_B^* \quad e^{F[\varphi]}}{Z_B Z_F}$$

$$F[\varphi] = i \int \{ \varphi_B^* \Lambda (E-H) \varphi_B + i\delta\, \varphi_B^* \varphi_B +$$

$$+ \varphi_F^* (E-H) \varphi_F + i\delta\Lambda\, \varphi_F^* \varphi_F \}$$

$$\boxed{\begin{aligned} \overline{\varphi}_B &= \varphi_B^* \Lambda \\ \overline{\varphi}_F &= \varphi_F^* \end{aligned}}$$

$$\Lambda = \begin{cases} +1, & \text{retarded (R)} \\ -1, & \text{advanced (A)} \end{cases}$$

$$F[\varphi] = i \int \{ \overline{\varphi}_B (E-H+i\delta\Lambda) \varphi_B + \overline{\varphi}_F (E-H+i\delta\Lambda) \varphi_F \}$$

$$= i \int \overline{\Psi} (E-H+i\delta\Lambda) \Psi$$

unified description

$$\boxed{\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_B \\ \varphi_F \end{pmatrix}}$$

two-component field

$$\boxed{G^{R(A)} = \frac{\int \mathcal{D}\overline{\Psi} \mathcal{D}\Psi \quad \underbrace{\varphi_B \varphi_B^*} \quad e^{i\int \overline{\Psi}(E-H+i\delta\Lambda)\Psi}}{\int \mathcal{D}\overline{\Psi} \mathcal{D}\Psi \quad e^{i\int \overline{\Psi}(E-H+i\delta\Lambda)\Psi}}}$$

supersymmetry is broken in the pre-exponent (source term)

<u>the action is supersymmetric</u>

Averaging over $U(z)$ can be performed in a general form:

$$\langle G^{R(A)} \rangle = \frac{\int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi \;\; \varphi_B \varphi_B^* \left\langle e^{i\int \bar{\Psi}(E-[redacted]+i\delta\Lambda)\Psi} \right\rangle}{Z_B Z_F}$$

$$\left\langle e^{i\int \bar{\Psi}(E-H+i\delta\Lambda)\Psi} \right\rangle = e^{i\int \bar{\Psi}(E-H_0+i\delta\Lambda)\Psi} \cdot \left\langle e^{-i\int \bar{\Psi} U \Psi} \right\rangle$$

$$\underbrace{\left\langle e^{-i\int \bar{\Psi} U \Psi} \right\rangle = e^{-\frac{\varkappa}{2}\int (\bar{\Psi}\Psi)^2}}$$

<u>Emerging effective interaction</u>

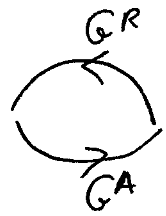
$$\boxed{H_{int} = \frac{\varkappa}{2}\int (\bar{\Psi}\Psi)^2}$$

Effective action <u>after averaging</u>

$$F[\Psi] = i\int \bar{\Psi}(E-H_0+i\delta\Lambda)\Psi \;[redacted]\; \frac{\varkappa}{2}\int (\bar{\Psi}\Psi)^2$$

$$\langle G^{R(A)} \rangle = \frac{\int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi \;\; \varphi_B \varphi_B^* \;\; e^{F[\Psi]}}{Z_B Z_F}$$

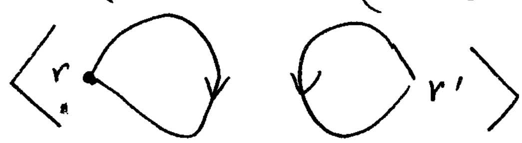
Averaging the product of $G^R G^A$

Conductivity $\quad \sigma \sim$ 

$\rightarrow \langle G^R(r,r')\, G^A(r',r)\rangle$

Correlation function of densities of states

$$(G^R - G^A)_{rr} \quad (G^R - G^A)_{r'r'}$$



$\rightarrow \langle G^R(r,r)\, G^A(r',r')\rangle$

$$\langle G^R G^A \rangle = \frac{\int \mathcal{D}\bar{\Psi}_R \mathcal{D}\Psi_R \mathcal{D}\bar{\Psi}_A \mathcal{D}\Psi_A \overbrace{\varphi_B^R \varphi_B^{R*} \varphi_B^A \varphi_B^{A*}}^{\text{more - complicated pre - exponent}} e^F}{Z_B Z_F}$$

F is of the same form as for $\langle G^{R(A)} \rangle$

but ψ field is further extended

$$\psi = \begin{pmatrix} \psi_R \\ \psi_A \end{pmatrix}$$

$$F[\psi] = i\int \bar{\psi}\,(E - H_0 + i\delta\Lambda)\,\psi \;-\; \frac{\varkappa}{2}\int (\bar{\psi}\psi)^2 \;+ \qquad (1)$$

source terms to generate a particular preexponent

$$\psi = \begin{pmatrix} \psi_R \\ \psi_A \end{pmatrix} \qquad \psi_{R(A)} = \begin{pmatrix} \varphi_B \\ \varphi_F \end{pmatrix}$$

ψ is 4-component vector at least*

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*) At least means if no further symmetry is present in the problem.
The presence of time-reversal symmetry requires further extension

$$\psi \rightarrow \Phi = \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \psi \\ T\psi \end{pmatrix}$$

where T is the operator of time reversal

The action (1) describes particle motion at all length scales

In disordered system particle motion is diffusive at scales larger than elastic scattering length l

what is the effective field theory at scales larger than l ?

Matrix description and Hubbard-Stratonovich transformation

$$(\bar{\psi}\psi)^2 = \mathrm{Tr}\,\hat{q}^2$$

$$\boxed{\hat{q} = \psi \otimes \bar{\psi}}$$

Hubbard - Stratonovich transformation

$$e^{-\frac{\varkappa}{2}\int \mathrm{Tr}\, q^2} = \int \mathcal{D}Q \; e^{-\gamma\int \mathrm{Tr}\, Q^2 + i\alpha\, \mathrm{Tr}\, Qq}$$

new matrix field Q of the same symmetry as $\hat{q} = \psi \otimes \bar{\psi}$

$$\mathrm{Tr}\, Qq \rightarrow \bar{\psi} Q \psi$$

The remaining Gaussian integral

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{i\int \bar{\psi}[E - H_0 + i\alpha Q + i\delta\Lambda]\psi} =$$

$$= e^{+\mathrm{Tr}\log[E - H_0 + i\alpha Q + i\delta\Lambda]}$$

New Q-action :

$$\boxed{F[Q] = \gamma\int \mathrm{Tr}\, Q^2 - \int \mathrm{Tr}\log\left[E - H_0 + i\alpha Q + i\delta\Lambda\right]}$$

The geometric constraint

Saddle-point approximation

$$\frac{\delta F}{\delta Q} = 0 : \qquad \frac{2\gamma Q}{\alpha} = \int \frac{d\vec{p}}{E - H_0 + i\alpha Q}$$

$$E - H_0(p) \equiv \xi_p = E - \frac{p^2}{2m}$$

saddle point equation $\longrightarrow$ $$\boxed{\frac{2\gamma \mathbf{Q}}{\alpha\rho} = \int_{-\infty}^{+\infty} \frac{d\xi}{\xi + i\alpha \mathbf{Q}}}$$

$\rho \equiv$ mean density of states $\qquad d\vec{p} = \rho\, d\xi$

$$\int_{-\infty}^{+\infty} \frac{d\xi}{\xi + \alpha Q} = \int_{-\infty}^{+\infty} \frac{\cancel{\xi} - i\alpha Q}{\xi^2 + \alpha^2 Q^2}\, d\xi \longrightarrow$$

looking for $Q^2 = a\,\mathbb{1}$

$$\rightarrow \quad -\frac{i\pi \,\mathrm{sign}\,\alpha}{\sqrt{Q^2}}\, \mathbf{Q} = \frac{2\gamma}{\alpha\rho}\, \mathbf{Q}$$

By a proper choise of γ the equation can alway be satisfied woth

$$\boxed{Q^2 = 1}$$

$\longleftarrow$ geometric constraint

Longitudinal and transverce fluctuations

Massive (weak) longitudinal fluctuations violate the castraint $Q^2 = 1$

At large $\sqrt{2mE}\, l$ (not very strong disorder) one can neglect them.

Transverse modes are massless

$$Q = \underbrace{\bar{U} \Lambda U}_{\text{transverse modes that may slowly depend on space-coordinate } r}$$

$$\Lambda^2 = 1$$

Gradient expansion and expansion in δ :

$$\boxed{\begin{gathered} F(Q) = g \int \mathrm{Tr}(\nabla Q)^2 + \delta \,\mathrm{Tr}\,\Lambda Q \\ Q^2 = 1 \end{gathered}}$$

non-linear σ-model

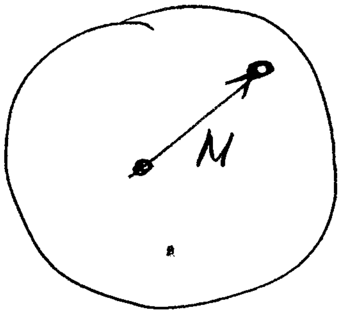
Compare: classical Heisenberg magnetic

$$F(\vec{M}) = g \int (\nabla \vec{M})^2 + h M_z$$

$$\vec{M}^2 = 1$$

What is the main difference between the σ-model for the Heisenberg magnetic and the σ-model for disordered quantum system of free particles?

$\vec{M} = \{M_x, M_y, M_z\}$



M is compact

$M^2 = 1 = M_x^2 + M_y^2 + M_z^2$

$M_i^2 < 1$ bounded

Q-matrix is non-compact

$Q^2 = 1$ allow for arbitrary large components Q_{ij}

The origin is symmetry of Q:

Q has the same symmetry as

$\hat{q} = \psi \otimes \bar{\psi}$

$Q_{FF} = \psi \otimes \psi^* \Longrightarrow$

$Q_{BB} = \psi \otimes \psi^* \Lambda$

$Q_{FF}^+ = Q_{FF}$ fermionic sector

$Q_{BB}^+ = \Lambda Q_{BB} \Lambda$ bosonic sector

In total:

$$\boxed{Q^{+} = KQK}$$

$$K = \left(\begin{array}{c|c} K_{BB} & 0 \\ \hline 0 & K_{FF} \end{array}\right) \qquad K_{BB} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}_{R-A} = \Lambda$$

$$K_{FF} = \mathbb{1}$$

Consider a toy model of 2×2 matrix

(1) $\begin{cases} Q^2 = 1 \\ Q = Q^{+} \end{cases}$ as in fermionic sector

Then $QQ^{+} = 1$

Q is unitary $Q = \begin{pmatrix} \cos\theta & \sin\theta \\ +\sin\theta & -\cos\theta \end{pmatrix}$

$\downarrow$

compact

all components are bounded

(2) $\begin{cases} Q^2 = 1 \\ Q^{+} = \Lambda Q \Lambda \end{cases}$ as in bosonic sector

Then $Q\Lambda Q^{+}\Lambda = 1$

Q is pseudo-unitary $Q = \begin{pmatrix} \cosh\theta & \sinh\theta \\ -\sinh\theta & -\cosh\theta \end{pmatrix}$

non-compact

$\theta \in [-\infty, +\infty]$

Why non-compactness?

Consider the inverse participation ratio

$$\left\langle \sum_n |\psi_n(r)|^4 \, \delta(E-E_n) \right\rangle$$

How to express in terms of electron Green's functions $G^{R(A)}$?

$$G^{R(A)} = \sum_n \frac{\psi_n(r)\,\psi_n^*(r')}{E-E_n \pm i\delta}$$

$$G^R(r,r)\left(G^R-G^A\right)_{rr} =$$

$$= \sum_{n,m} \frac{|\psi_n|^2 |\psi_m|^2}{E-E_n+i\delta} \left(-2\pi i\, \delta(E-E_m)\right)$$

$$\frac{1}{2\pi i}\left(G^R-G^A\right)G^R = \sum_{n,m} \frac{|\psi_n|^2|\psi_m|^2}{(E_m-E_n+i\delta)} \, \delta(E-E_m)$$

multiply by $i\delta$ and do the limit $\delta \to 0$

$$\frac{1}{2\pi} \lim_{\delta\to 0} \delta\left\{\left(G^R-G^A\right)G^R\right\} = \sum_{n,m} \frac{i\delta\, |\psi_n|^2|\psi_m|^2 \, \delta(E-E_m)}{(E_m-E_n+i\delta)}$$

in the limit $\delta \to 0$ only $m=n$ survive
(no symmetry for degeneracy!)

$$\boxed{\frac{1}{2\pi}\lim_{\delta\to 0}\left\{\delta\, G^R(G^R-G^A)\right\} = \sum_n |\psi_n|^4\,\delta(E-E_n)}$$

In terms of σ-model:

$$\langle G^R(G^R-G^A)\rangle = \int \mathcal{D}Q\, \underbrace{Q^{RR}(r)\, Q^{AA}(r)}_{\substack{\text{schematic} \\ \text{pre-exponent}}} \quad e^{-F(Q)}$$

$$F(Q) = g\int \mathrm{Tr}(\nabla Q)^2 + \underbrace{\delta\, \mathrm{Tr}\,\Lambda Q}_{\delta\text{-dependent term}}$$

If all componets of Q were bounded (compactness)

$$\lim_{\delta\to 0} \delta\, \mathrm{Tr}\,\Lambda Q \to 0 \qquad \text{and}$$

$$\boxed{\lim_{\delta\to 0} \delta \langle G^R(G^R-G^A)\rangle \to 0}$$

since $\langle G^R(G^R-G^A)\rangle$ would have a finite limit at $\delta\to 0$

Compact theory would lead to

$$\boxed{\left\langle \sum_n |\psi_n|^4\,\delta(E-E_n)\right\rangle = 0 \quad \text{or} \quad |\psi_n|^2 \equiv 0}$$

Non-compactness is equivalent to existence of normalized wave functions in the initial problem of free particles.

What is the σ-model good for?

$$F(Q) = g\int \mathrm{Tr}(\nabla Q)^2 - i(\omega + i\delta)\rho\int \mathrm{Tr}\,\Lambda Q$$

$Q^2 = 1$

mean DOS

Extension for different frequencies of $G^R_{E+\omega/2}G^A_{E-\omega/2}$

(1) Scaling for the Anderson transition in $2+\varepsilon$ dimensions

No non-trivial renormalization of the term $\propto \mathrm{Tr}\,\Lambda Q$

$\Longrightarrow$ constant DOS at the transition point

Renormalization of the coupling constant g gives scaling for dimensionless conductance

$$\boxed{\frac{d\ln g}{d\ln L} = \epsilon - \frac{1}{g} + \dots}$$

← critical exponents of the Anderson transition in $2+\varepsilon$ dimensions

(2) 0-dimensional σ-model $Q(r) = \mathrm{const} = Q$

$$F(Q) = -i(\omega + i\delta)\rho\int \mathrm{Tr}\,\Lambda Q$$

$$Q^2 = 1$$

is equivalent to Wigner-Dyson RMT

Perturbative consideration of $\mathrm{Tr}(\nabla Q)^2$ allows to compute corrections to RMT

Applications: Quantum dots

③ Quasi-1d σ-model

Rigorous proof of localization in multi-channel wires

④ Nonlinear σ-model on a Bethe lattice

$$F(Q) = -g \sum_{ij} \operatorname{Tr} Q_i Q_j - i(\omega + i\delta)\rho \sum_i \operatorname{Tr} \Lambda Q_i$$

infinite-dimensional Anderson model

⑤ Eigenfunction correlations

$$\left\langle |\psi|^m(r)\; |\psi|^n(r') \right\rangle = \text{certain } \delta \to 0 \text{ limit of}$$

$$\int Q^{m/2}(r)\; Q^{n/2}(r')\; e^{-F(Q)}\; \mathcal{D}Q$$

Multifractality at $d = 2 + \varepsilon$

⑥ Spectral correlations beyond Wigner-Dyson limit

⑦ Quantum chaos ?