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SUMMER SCHOOL
on
LOW-DIMENSIONAL QUANTUM SYSTEMS:
Theory and Experiment
(16 - 27 JULY 2001)

PLUS

PRE-TUTORIAL SESSIONS
(11 - 13 JULY 2001)

NOTES ON "INTRODUCTION TO NON-ABELIAN BOSONIZATION"

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Bosonization table.

Fermions	Bosons
$S = i\bar{\Psi}_\alpha \partial\!\!\!/ \Psi_\alpha$	$S = W[g] + \frac{N}{8\pi}(\partial_\mu \phi)^2$
$R^+_\alpha L_\beta$	$e^{i\phi}\, g_{\alpha\beta}$
$\frac{1}{2} R^+_\alpha \vec{G}_{\alpha\beta} R_\beta = \vec{J}$	$\frac{1}{2\pi} \mathrm{Tr}(\vec{G}\, g^{-1} \partial g)$
$\frac{1}{2} L^+_\alpha \vec{G}_{\alpha\beta} L_\beta = \bar{\vec{J}}$	$\frac{1}{2\pi} \mathrm{Tr}(\vec{G}\, \bar{\partial} g^{-1}\, g)$

Free fermions with $U(1) \otimes SU(N)$ symmetry

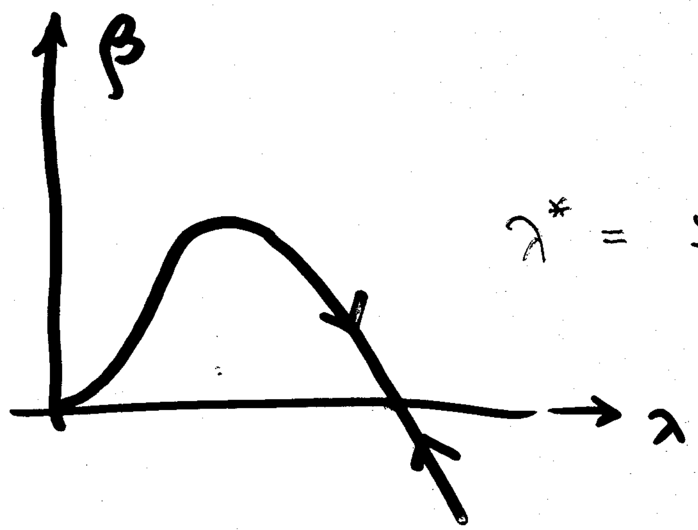
$\backsim$ $U(1)$ field + $SU_1(N)$ WZNW model

$$i\bar{\Psi}_{\alpha,j} \partial\!\!\!/ \Psi_{\alpha j} \qquad \frac{Nk}{8\pi}(\partial_\mu \phi)^2 + k\,W[g : SU(N)] + N\,W[h : SU(k)]$$

Wess - Zumino - Novikov - Witten (WZNW) action:

$$S[g] = \frac{1}{4\lambda^2} \mathrm{Tr}\left(\partial_\mu g^{-1} \partial_\mu g\right) + ik\,\Gamma[g]$$

$$\Gamma[g] = \frac{1}{24\pi} \int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}\left(g^{-1}\partial_\mu g\; g^{-1}\partial_\nu g\; g^{-1}\partial_\rho g\right)$$



$$\lambda^* = \frac{4\pi}{k}$$

At the critical point:

$$S^*[ab] = S^*[a] + S^*[b] + \frac{k}{2\pi} \mathrm{Tr}\left(a^{-1}\bar{\partial} a\; b\,\partial b^{-1}\right)$$

Polyakov - Wiegmann identity

Concrete example : $\underline{g \in SU(2)}$

$$g = e^{i\psi\sigma^3/2}\, e^{i\theta\sigma^x/2}\, e^{i\varphi\sigma^3/2}$$

$$S^*[g] = \frac{k}{16\pi}\Big[(\partial\varphi)(\bar\partial\varphi) + (\partial\psi)(\bar\partial\psi) + (\partial\theta)(\bar\partial\theta) + \cos\theta\, \bar\partial\varphi\, \partial\psi \Big]$$

Conserved currents :

$$J = \frac{k}{2\pi}\, g\partial g^{-1} \qquad \bar\partial J = 0$$

$$\bar J = \frac{k}{2\pi}\, g^{-1}\bar\partial g \qquad \partial \bar J = 0$$

An application of non-Abelian bosonization.

The model :

$$H = H_1 + H_2 + V$$

$$H_j = [-i R^+_{ja\alpha} \partial_x R_{ja\alpha} + i L^+_{ja\alpha} \partial_x L_{ja\alpha}] v_j + g_j \left(\sum_{a=1}^{k} R^+_{ja\alpha} \vec{\tau}_{\alpha\beta} R_{ja\beta}\right)\left(\sum_{b=1}^{k} L^+_{jb\gamma} \vec{\tau}_{\gamma\delta} L_{jb\delta}\right)$$

$$j = 1, 2$$

$$V = c \left(\sum_{a=1} R^+_{a\alpha} L_{a\beta}\right)_1 \left(\sum_b L^+_{b\beta} R_{b\alpha}\right)_2 + h.c.$$

Two $U(1) \otimes SU(k) \otimes SU(N)$ models coupled together by a 4-fermion interaction

In general $(v_1, g_1) \neq (v_2, g_2)$

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The current - current interaction generates a mass gap in each model (M_1, M_2)

At low energies we have :

$$\sum_a R^+_{\alpha a} L_{\beta a} = : e^{i\sqrt{\frac{4\pi}{NK}}\,\phi}\, g_{\alpha\beta} : \Delta$$

The effective action

$$S = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + k W[g_1] + k W[g_2]$$

$$+ c\, \Delta_1 \Delta_2 \left[: \exp\left(\sqrt{\tfrac{4\pi}{kN}}(\phi_1 - \phi_2)\right) g_1 g_2^+ : + \text{c.c.} \right]$$

New variables:

$$\phi^{(+)} = \frac{\phi_1 + \phi_2}{\sqrt{2}}$$

$$\phi^{(-)} = \frac{\phi_1 - \phi_2}{\sqrt{2}}$$

$$g_1 g_2^+ = h \quad \rightsquigarrow \quad g_1 = h g_2$$

Let us consider $v_1 = v_2$ case first.

$$S = \frac{1}{2}(\partial_\mu \phi_+)^2 + k W[g_2] + k W[h g_2]$$

$$+ \frac{1}{2}(\partial_\mu \phi_-)^2 + c \Delta_1 \Delta_2 \left[e^{i\sqrt{\frac{8\pi}{Nk}}\phi_-} \mathrm{Tr}\, h + c.c. \right]$$

The next step : Polyakov - Wiegmann identity

$$W[h g_2] = W[h] + W[g_2] + \frac{2\pi}{k} \mathrm{Tr}\,(J_h \bar{J}_{g_2})$$

$$\downarrow$$

$$S = \frac{1}{2}(\partial_\mu \phi_+)^2 + 2k W[g_2]$$

$$+ \frac{1}{2}(\partial_\mu \phi_-)^2 + k W[h] + c \Delta_1 \Delta_2 \left(e^{i\sqrt{\frac{8\pi}{Nk}}\phi_-} \mathrm{Tr}\, h + c.c. \right)$$

$$+ 2\pi \,\mathrm{Tr}\,(J_h \bar{J}_{g_2})$$

dim : $\quad \dfrac{2}{Nk} + \dfrac{N - 1/N}{N+k} < 2$

relevant perturbation

Claim: $\phi^{(-)}$ and h sector acquires a gap

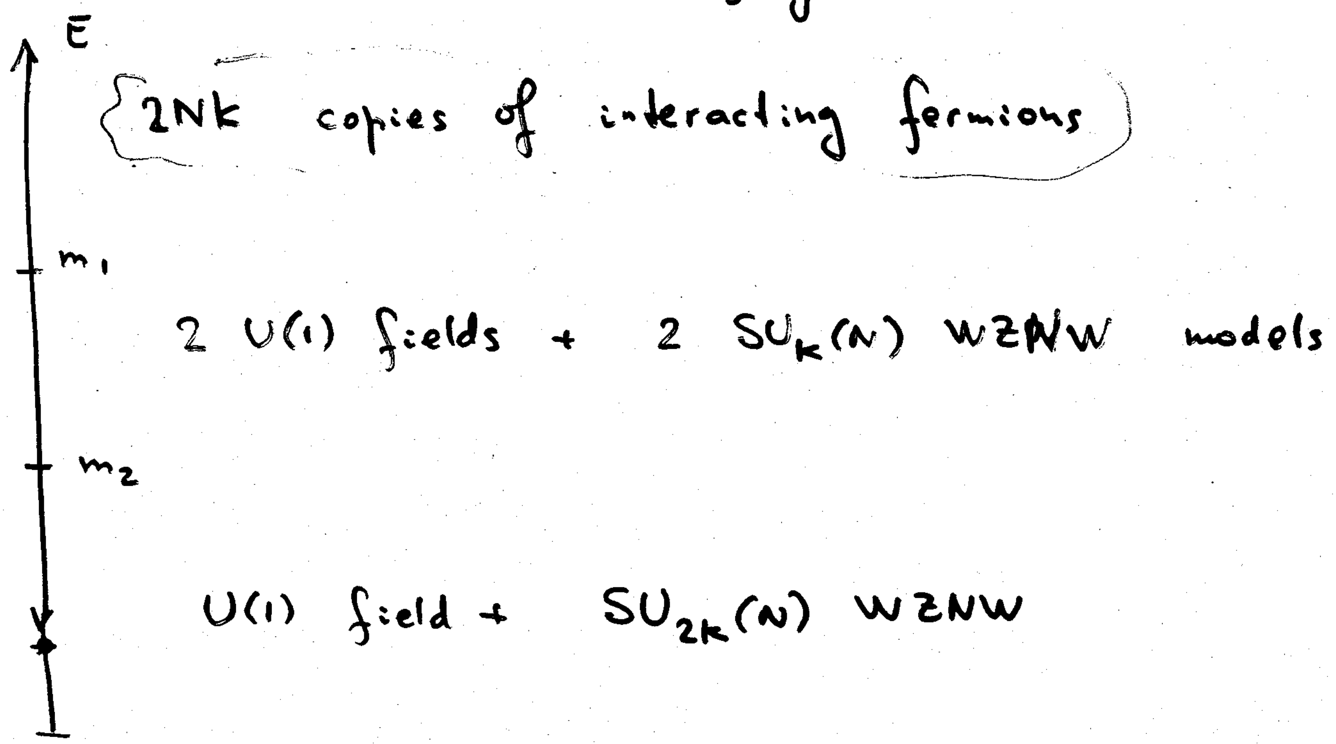
$$\Delta = m_2$$

$$\langle J_h(q)\, J_h(-q) \rangle \sim \frac{q^2}{m_2^2}$$

Integration over h-field gives the contribution to the effective action of g_2:

$$\frac{1}{m_2^2} \left(\bar{\partial} \bar{J}_{g_2} \right)^2$$

– highly irrelevant.



Different velocities.

$$S\left\{\mathrm{Tr}\left[\frac{1}{v}(\partial_\tau g^{-1})(\partial_\tau g) + v(\partial_x g^{-1})(\partial_x g)\right] + i\Gamma(g)\right\}$$

does not depend on v

$$W[v_1, g] + W[v_2, g] =$$

$$S\left\{\left(\frac{1}{v_1} + \frac{1}{v_2}\right)\dot{g}^{-1}\dot{g} + (v_1 + v_2) g_x g_x + 2i\Gamma(g)\right\} dx\, d\tau$$

$x \to a x'$

$\tau \to b\tau'$

$$\frac{a}{b}\left(\frac{1}{v_1} + \frac{1}{v_2}\right) = \frac{b}{a}(v_1 + v_2)$$

$$\frac{1}{2\lambda}\mathrm{Tr}\,(\partial_\mu g^{-1})(\partial_\mu g) + 2ik\Gamma(g)$$

$$\frac{1}{\lambda} = 2k + 2k\left(\sqrt{\frac{v_1}{v_2}} - \sqrt{\frac{v_2}{v_1}}\right)^2$$

Off-critical WZNW — flows towards 2k critical point