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DISORDERED DIRAC FERMIONS:
THE MARRIAGE OF THREE DIFFERENT APPROACHES

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Disordered Dirac Fermions: the Marriage of Three Different Approaches

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. We compare the critical multipoint correlation functions for two-dimensional (massless) Dirac fermions in the presence of a random $su(N)$ (non-Abelian) gauge potential, obtained by three different methods. We critically reexamine previous results obtained using the replica approach and in the limit of infinite disorder strength and compare them to new results (presented here) obtained using the supersymmetric approach to the $N = 2$ case. We demonstrate that this *ménage à trois* of different approaches leads to identical results. Remarkable relations between apparently different conformal field theories (CFTs) are thereby obtained. We further establish a connection between the random Dirac fermion problem and the $c = -2$ theory of dense polymers. The presence of the $c = -2$ theory may be seen in all three different treatments of the disorder.

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I. INTRODUCTION

In this paper we use an exactly solvable model of disorder to compare different theoretical approaches used to investigate the critical behaviour of disordered (possibly electronic) systems. A prominent example of a disordered critical point, and a source of many challenging problems, is that governing the plateau transitions in the integer quantum Hall effect (IQHE) — see for example [1]. Despite the numerous field theoretic approaches to this problem based on either Pruisken's non-linear sigma model [2–4] or the Chalker–Coddington network model [5] and its superspin chain descendants [6–10], the nature of the critical point remains elusive and inaccessible by perturbation theory [11, 12]. This is to be contrasted with the Ising model with weak bond disorder [13–15], for example, where the disorder renormalizes to zero and may be treated perturbatively. In order to gain insight into such problems it is imperative to fully exploit, and indeed develop, the methods available for treating disordered systems, and to obtain as many non-perturbative results as possible. In low-dimensional systems we may be aided in this pursuit by the availability of powerful techniques such as conformal field theory [16–18] and the Bethe ansatz [19, 20].

Recent studies of disordered critical points have included Dirac fermions subjected to various random potentials [21–24], the random XY model [25], the plateau transitions occurring in the spin quantum Hall effect and its relation to critical percolation [26–28], and the Nishimori line in the random bond Ising model [29]. In all of these examples the critical points are of a non-perturbative nature and are described by non-trivial field theories. Our knowledge of the critical behaviour in these theories differs somewhat, and the further elucidation of their detailed properties is a valuable enterprise. Indeed, in problems of disorder related to the localization of quantum particles, one needs to depart from the critical point in order to calculate the diffusion propagator or (for problems with a singular density of states) the energy dependence of the density of states; at present this program has been successfully implemented only for the random XY model [25].

We turn our attention now to the problem at the heart of this paper, namely two-dimensional (Euclidean) Dirac fermions in a random non-Abelian gauge potential. This model has the virtue of being amenable to a variety of non-perturbative approaches and was originally introduced in Abelian form as part of an attempt to describe the plateau transitions in the IQHE [21]. The non-Abelian version of the problem appeared in a treatment of disordered d-wave superconductivity [30–32] and has been the subject of deeper investigation and refinement [33–40]. Building on the foundations of these previous studies, we address this problem by means of three independent non-perturbative field-theoretic techniques: these are based upon the commonly used replica [41] and supersymmetric methods [42, 43] together with a third (model specific) approach valid in the limit of strong disorder [39]. In our study of the four-point correlation functions of the local density of states we are able to demonstrate that the three methods yield coincident results. We note that whilst the reliability of the replica approach outside of perturbation theory has been frequently and legitimately questioned — see for example [42–45] — we use it here (in a quite straightforward manner) to reproduce the results of the (mathematically more rigorous) supersymmetric approach. Although the implementation of the replica method remains a delicate issue in general, and in many cases one must adopt various (replica) symmetry breaking schemes — see references [46, 47] for recent examples in random matrix theory — it is noteworthy that

the disordered non-Abelian Dirac fermion problem renders itself here to a textbook (replica) treatment — albeit with the benefit of hindsight.

The structure of this paper is as follows: in section II we provide a brief outline of the disordered non-Abelian Dirac fermion problem. In section III we focus our attention on the four-point correlation functions of the local density of states (LDOS) as encoded in the so-called Q-matrix. This section is subdivided into three main subsections, each of which is devoted to a different non-perturbative field theoretical approach to the disordered Dirac fermion problem; subsection III A deals with the replica approach, III B deals with the so-called strong disorder approach and III C deals with the supersymmetric approach. Within each of these subsections we provide an outline of the theoretical approach and the form of the appropriate Q-matrix, together with a discussion of the relevant conformal dimensions and four-point correlation functions. In order to increase the transparency of these subsections we have relegated many of the important (but arguably involved) technical details on the solution of the relevant WZNW models into a rather substantial appendix. The interested reader will find gathered in this appendix the solutions to the Knizhnik–Zamolodchikov equations which arise in this work. In section IV we continue our study of the supersymmetric approach and provide a remarkably simple free field representation of the disorder averaged theory comprising of a two-component symplectic fermion and a pair of free bosons (one compact and the other non-compact). The two-component symplectic fermion with central charge $c = -2$ arises in the theoretical description of dense polymers and we demonstrate that the twist operators of the latter theory play a fundamental rôle in the non-Abelian Dirac fermion problem. In section V we discuss the convergence of approaches to the disordered non-Abelian Dirac fermion problem. Finally we present concluding remarks and technical appendices. In particular, appendix A is devoted to the $\widehat{su}(N)_k$ WZNW model and is subdivided into our considerations of the $\widehat{u}(0)_k$ WZNW model — arising in the replica approach — and the $\widehat{su}(N)_{-2N}$ WZNW model — arising in the strong disorder approach. Appendix B is devoted to the $\widehat{osp}(2|2)_k$ WZNW model which arises in the supersymmetric approach.

II. NON-ABELIAN DIRAC FERMIONS

We consider N_C colours of two-dimensional massless Dirac fermions minimally coupled to a non-Abelian gauge field $A_\mu \in su(N_C)$:

$$S = \int d^2\xi \sum_{\alpha,\beta=1}^{N_C} \bar\psi^\alpha \not{D}^{\alpha\beta} \psi^\beta, \tag{2.1}$$

where the gauge covariant derivative takes the form

$$\not{D}^{\alpha\beta} = \gamma^\mu (\delta^{\alpha\beta} \partial_\mu - i A_\mu^{\alpha\beta}). \tag{2.2}$$

The γ^μ matrices form a two-dimensional representation of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with Euclidean metric $g^{\mu\nu} = \mathrm{diag}(1,1)$, and the gauge field $A_\mu^{\alpha\beta} = A_\mu^a \tau_a^{\alpha\beta}$ may be expanded in terms of the generators τ_a of $su(N_C)$. Physical quantities are obtained by

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disorder averaging products of Green's functions. We use the distribution functional

$$P[A_\mu] \propto e^{-S[A_\mu]}, \quad S[A_\mu] = \frac{1}{g_A} \int d^2\xi \, \mathrm{Tr} \, A_\mu(\xi) A_\mu(\xi) \tag{2.3}$$

representing the usual choice of δ-correlated Gaussian white noise for the random vector potential. In section III we shall compare three different approaches for evaluating the disorder averaged correlation functions.

III. DISORDER AVERAGED CORRELATION FUNCTIONS

The local density of states (LDOS) of the Dirac fermion problem is given by [30–32]

$$\rho(\mathbf{r}) = \sum_{\alpha=1}^{N_C} \left[R_\alpha^\dagger L_\alpha + L_\alpha^\dagger R_\alpha \right]. \tag{3.1}$$

In sections III A–III C we shall investigate the disorder averaged multipoint correlation functions of the LDOS. In particular, we shall focus our attention on the long distance properties (obtained for example by integrating out the massive modes) encoded in the so-called Q-fields:

$$Q \sim \sum_{\alpha=1}^{N_C} R_\alpha^\dagger L_\alpha, \quad Q^\dagger \sim \sum_{\alpha=1}^{N_C} L_\alpha^\dagger R_\alpha \tag{3.2}$$

We shall study the critical correlation functions of the Q-fields by three different approaches; in each case our Q-fields will be represented by Q-matrices governed by appropriate WZNW models.

A. Replica Approach

In order to average over disorder we introduce N_F flavours (or replicas) of the massless Dirac fermions appearing in equation (2.1):

$$S^{\mathrm{Replica}} = \int d^2\xi \sum_{i=1}^{N_F} \sum_{\alpha,\beta=1}^{N_C} \bar{\psi}^{\alpha,i} \not{D}^{\alpha\beta} \psi^{\beta,i} \tag{3.3}$$

The (replicated) action (3.3) enjoys a global $SU(N_F) \times SU(N_C) \times U(1)$ symmetry and may be recast using Witten's non-Abelian bosonization rules [48] — for reviews of the bosonization approach see for example [49,50]. It is well known that the free Dirac action (in the absence any gauge fields) with the flavour-colour symmetry described above may be represented as the sum of three critical Wess–Zumino–Novikov–Witten (WZNW) models of the form

$$W_k[g] = \frac{k}{16\pi} \int d^2\xi \, \mathrm{Tr}'(\partial^\mu g^{-1} \partial_\mu g) + k\Gamma[g] \tag{3.4}$$

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where (the WZNW term) Γ reads

$$\Gamma[g] = \frac{i}{24\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \, \mathrm{Tr}'(g^{-1}\partial_\mu g \, g^{-1}\partial_\nu g \, g^{-1}\partial_\rho g). \tag{3.5}$$

The celebrated equivalence between the free (flavoured-coloured) Dirac fermion theory and the bosonic WZNW models may be written in the symbolic form

$$\text{Free Dirac Fermions} = \widehat{su}(N_C)_{N_F} \times \widehat{su}(N_F)_{N_C} \times \widehat{u}(1) \tag{3.6}$$

where it is understood that the *chiral* blocks of the free Dirac theory are obtained as products of the *chiral* blocks of the WZNW models; here $\widehat{g}_k$ denotes the group manifold g and the so-called level k of the WZNW action (3.4). That this is indeed an equality between *chiral* blocks is seen most clearly in the work of Fuchs [51]. Of paramount importance in the application of (3.6) to our disordered problem is that the disorder — the $su(N_C)$ gauge potential — couples *only* to (currents from) the $\widehat{su}(N_C)_{N_F}$ sector. Averaging over disorder (equivalently integrating over the $su(N_C)$ gauge potential) generates a (quadratic) interaction between $su(N_C)$ currents only. This interaction scales to the strong coupling regime where a mass gap $M \sim \exp[-2\pi/N_C\lambda]$ is dynamically generated in the $\widehat{su}(N_C)_{N_F}$ sector of the theory. As follows from the decomposition (3.6) the massless degrees of freedom are $SU(N_C)$ singlets and are described by the remaining WZNW models $\widehat{su}(N_F)_{N_C} \times u(1) \sim \widehat{u}(N_F)_{N_C}$. Upon integrating out the massive (colour) degrees of freedom, the fermion bilinears are expressed in terms of the so-called Q-matrix

$$Q_{r\bar{r}} \sim \frac{1}{M} \sum_{\alpha=1}^{N_C} R_{\alpha,r}^\dagger L_{\alpha,\bar{r}} \tag{3.7}$$

whose indices reside in the flavour (replica) space; this is simply a replicated version of the Q-field introduced to describe the LDOS in equations (3.1) and (3.2). The Q-matrix assumes values in the group $U(N_F)$ and is governed by a (critical) effective action of the WZNW form (3.4) with level $k = N_C$. The WZNW model thus plays an analogous rôle to the sigma model in the conventional theory of localization [42] — it is an effective action for the slow degrees of freedom, once the fast degrees of freedom have been integrated out.

1. Conformal Dimensions

In the replica approach, the conformal dimension of the Q-matrix is that of a primary field of the $(N_F \to 0)$ $\widehat{u}(N_F)_{N_C}$ WZNW model transforming in the fundamental representation — see equation (A15) of appendix A:

$$h_Q = \frac{1}{2N_C^2} \tag{3.8}$$

As we shall see in subsections III B 1 and III C 1, the conformal dimension of the Q-matrix (LDOS) is given by (3.8) in all three treatments of the disorder. This ensures the equality of their two-point and three-point correlation functions. It is well known in CFT however,

that the four-point correlation functions are *not* determined solely by scaling dimensions, but in general have a non-trivial dependence on the so-called anharmonic ratios (A3) [52]; in CFTs of the WZNW form, this dependence is obtained by solving the appropriate Knizhnik–Zamolodchikov equations [53]. In order to compare different theoretical approaches to the random Dirac fermion problem it is thus *essential* to study the four-point correlation functions of the Q-matrix.

2. Correlation Functions of the Q-field

The correlation functions of the Q-matrix are those pertaining to the $\widehat{u}(N_F)_{N_C}$ WZNW model in the replica ($N_F \to 0$) limit, and are thus obtained by solving the $\widehat{u}(N_F)_{N_C}$ Knizhnik–Zamolodchikov equations [53]. This fact was recognised in the early replica treatment of the random Dirac fermion problem [30,31]. The approach was subsequently refined to accommodate the logarithmic solutions to the Knizhnik–Zamolodchikov equations which appear in the replica limit [33]. Despite the advances made in these works they suffer (in places) from a naïve implementation of the replica trick; an extraneous trace over replica indices was performed in a number of instances. In order to address these issues, and to demonstrate the convergence of approaches to the random Dirac fermion problem, we rederive a number of results and adopt a uniform notation throughout.

The four-point correlation function of the Q-matrix admits the $U(N_F) \times U(N_F)$ invariant decomposition [33,53] (see appendix A 1):

$$\langle Q_{r_1 \bar{r}_1}(1) Q^\dagger_{r_2 \bar{r}_2}(2) Q^\dagger_{r_3 \bar{r}_3}(3) Q_{r_4 \bar{r}_4}(4) \rangle = |z_{14} z_{23}|^{-2/N_C^2} \sum_{i,j=1}^{2} I_i^r \bar{I}_j^{\bar{r}} F_{ij}(z, \bar{z}) \tag{3.9}$$

where r denotes the ordered sequence of flavour (replica) indices r_1, r_2, r_3, r_4, and where the invariant tensors I_1 and I_2 are defined as $I_1^r = \delta^{r_1, r_2} \delta^{r_3, r_4}$ and $I_2^r = \delta^{r_1, r_3} \delta^{r_2, r_4}$, together with similar equations for $\bar{I}_1$ and $\bar{I}_2$. The anharmonic ratio z is defined as $z = z_{12} z_{34} / z_{14} z_{32}$ and similarly for $\bar{z}$. The functions $F_{ij}(z, \bar{z})$ are single-valued combinations of the solutions to the $\widehat{u}(N_F)_{N_C}$ Knizhnik–Zamolodchikov equations, which in the replica limit ($N_F \to 0$) are given by equations (A22) with $k = N_C$. For example, setting all replica indices equal to 1 yields

$$\langle Q_{1\bar{1}}(1) Q^\dagger_{1\bar{1}}(2) Q^\dagger_{1\bar{1}}(3) Q_{1\bar{1}}(4) \rangle \sim |\Upsilon|^2 \left[K_{N_C}(z) K_{N_C}(1 - \bar{z}) + K_{N_C}(1 - z) K_{N_C}(\bar{z}) \right] \tag{3.10}$$

where $\Upsilon = [z_{14} z_{23} z(1 - z)]^{-1/N_C^2}$ and $K_{N_C}(z)$ and $E_{N_C}(z)$ are natural generalizations of the complete elliptic integrals — see equation (A19). In addition one may consider 'mixed' correlation functions of the form:

$$\langle Q_{1\bar{1}}(1) Q^\dagger_{1\bar{1}}(2) Q^\dagger_{2\bar{2}}(3) Q_{2\bar{2}}(4) \rangle \sim |\Upsilon|^2 \left[E_{N_C}(z)(K_{N_C}(1 - \bar{z}) - E_{N_C}(1 - \bar{z})) + c.c. \right] \tag{3.11}$$

where *c.c* stands for complex conjugation — replacement of z by $\bar{z}$. Correlation functions calculated with respect to this effective (replicated) action correspond to correlation functions averaged with respect to the initial action with quenched disorder [39,54]:

$$\overline{\langle Q(1) Q^\dagger(2) Q^\dagger(3) Q(4) \rangle}_A = \langle Q_{r_i \bar{r}_i}(1) Q^\dagger_{r_i \bar{r}_i}(2) Q^\dagger_{r_i \bar{r}_i}(3) Q_{r_i \bar{r}_i}(4) \rangle_{\mathrm{rep}} \tag{3.12a}$$

$$\overline{\langle Q(1) Q^\dagger(2) \rangle}_A \overline{\langle Q^\dagger(3) Q(4) \rangle}_A = \langle Q_{r_i \bar{r}_i}(1) Q^\dagger_{r_i \bar{r}_i}(2) Q^\dagger_{r_j \bar{r}_j}(3) Q_{r_j \bar{r}_j}(4) \rangle_{\mathrm{rep}}, \quad r_i \neq r_j \tag{3.12b}$$

The left hand side of (3.12a) represents the four-point correlation function of the Q-field (calculated with respect to the original action in the presence of quenched disorder) averaged over disorder (denoted by an overline); the left hand side of (3.12b) represents the product of two two-point correlation function of the Q-field (again calculated with respect to the original action in the presence of quenched disorder) averaged over disorder.

We note that as a direct consequence of the rather simple replica index structure of (3.9) the results (3.12a) and (3.12b) are independent of which replicas are actually considered; this is a manifestation of replica symmetry — see for example §3.3 of reference [55]. The correlation functions of the LDOS may be obtained from these results by means of the decomposition (3.1) together with crossing symmetry. We emphasize that one does *not* perform a trace over replica indices in order to extract the LDOS; the traces over replica indices appearing in equation (60) of reference [31] and equation (5) of reference [33] are erroneous. We further draw attention to the simplicity and manifest crossing symmetry of the results (3.10) and (3.11) as compared with those obtained in reference [33].

B. Strong Disorder Approach

As was first discussed in [34, 39], and subsequently developed in [37], the random Dirac fermion problem is amenable to a direct treatment in the limit of infinite disorder strength ($g_A \to \infty$) without invoking either replicas or supersymmetry. We note that in this limit the probability measure (2.3) is absent. We summarize here only the important details. We separate the action (2.1) into its chiral components

$$S = \int d^2\xi \sum_{\alpha=1}^{N_C} \left[R_\alpha^\dagger (\bar{\partial} + i\bar{A}) R_\alpha + L_\alpha^\dagger (\partial + iA) L_\alpha \right] \tag{3.13}$$

and parameterize the gauge fields in terms of group elements $g(\xi) \in SU^{\mathbb{C}}(N) \sim SL(N;\mathbb{C})$ residing in the complex extension of $SU(N)$

$$A = i\partial g g^{-1}, \quad \bar{A} = i\bar{\partial}\bar{g}\bar{g}^{-1}. \tag{3.14}$$

This enables one to perform the chiral gauge transformations

$$L \to g\mathcal{L}, \quad R \to \bar{g}\mathcal{R}, \tag{3.15}$$

so as to render a theory of free fermions ($\mathcal{R}, \mathcal{L}$) decoupled from the gauge fields. An important feature of this procedure is that the Jacobian of the transformations (3.15) is proportional to the partition function of the original action (2.1) at fixed disorder, $Z[A_\mu]$ — this cancels the normalizing partition function in (fixed disorder) correlation functions and removes the need to invoke replicas or supersymmetry in order to perform disorder averaging. The Jacobian associated with the change of variables (3.14) is well known to involve the WZNW action (3.4) on the (non-compact) manifold $h = g^\dagger g \in SU^{\mathbb{C}}(N)/SU(N)$ at level $k = -2N_C$ [34, 39, 56, 57]:

$$\mathcal{D}A = \mathcal{D}g \, \exp\left(2N_C W_{-2N_C}[g^\dagger g]\right) \tag{3.16}$$

In reference [37] the Wakimoto free-field representation of the $SU^{\mathbb{C}}(N)/SU(N)$ WZNW model at $k = -2N$ was constructed, and it was shown to generate the $\widehat{su}(N)_{-2N}$ Kač–Moody algebra; in other words, the correlation functions of the h-fields may be obtained from the solution of the $\widehat{su}(N)_{-2N}$ Knizhnik–Zamolodchikov equations — see appendix A 2. In this approach the Q-fields of the random Dirac fermion problem with N_C colours are expressed as primary fields of the $\widehat{su}(N_C)_{-2N_C}$ WZNW model 'dressed' by free fermions:

$$Q = \sum_{\alpha,\bar{\alpha}=1}^{N_C} \mathcal{R}_\alpha^\dagger \, h_{\alpha\bar{\alpha}} \, \mathcal{L}_{\bar{\alpha}}, \qquad Q^\dagger = \sum_{\alpha,\bar{\alpha}=1}^{N_C} \mathcal{L}_{\bar{\alpha}}^\dagger \, h_{\bar{\alpha},\alpha}^\dagger \, \mathcal{R}_\alpha, \tag{3.17}$$

We shall study the conformal dimensions and correlation functions of these fields in the subsections below.

1. Conformal Dimensions

It is readily seen from the strong disorder decomposition (3.17) that the conformal dimension of the Q-field is that of free Dirac fermion ($h = 1/2$) and an $\widehat{su}(N_C)_{-2N_C}$ primary field — see equation (A4) and set $N = N_C$ and $k = -2N_C$:

$$h_Q = \frac{1}{2} + \frac{N_C^2 - 1}{2N_C(N_C - 2N_C)} = \frac{1}{2N_C^2} \tag{3.18}$$

This agrees with the replica result (3.8) and ensures the equality of the two-point and three-point correlation functions in both approaches. The coincidence of higher correlation functions will be discussed below.

2. Correlation functions of the Q-field

The correlation functions of the Q-field are thus obtained by a fermionic 'dressing' of the correlation functions of the $\widehat{su}(N_C)_{-2N_C}$ WZNW model. We note that whilst the chiral solutions to the $\widehat{su}(N_C)_{-2N_C}$ WZNW model given in [37] are correct, their normalization constants are erroneous — they do not satisfy the conformal bootstrap. We provide the correct conformal blocks in equations (A25) and their expression in terms of generalized elliptic integrals in equations (A26). Moreover, in the light of the new results obtained in the replica approach, we shall find it convenient to generalize the decomposition (3.17) slightly so as to accommodate disorder averages of *products* of quenched correlation functions such as those appearing in equation (3.12b). To this end we introduce as many *additional* fermionic species (denoted by an index p) as quenched correlation functions we wish to disorder average. We emphasize that these additional indices are *not* required to *perform* the disorder averaging (as would be true of replicas) but simply encode which combinations of quenched correlation functions are to be averaged over disorder — in this approach we do *not* perform an $N_p \to 0$ limit. The Q-matrix acquires a pseudo-replica index structure

(labelled by the index p) analogous to that appearing in equation (3.7):

$$Q_{p,\bar{p}} = \sum_{\alpha,\bar{\alpha}=1}^{N_C} \mathcal{R}_{\alpha,p}^{\dagger} h_{\alpha,\bar{\alpha}} \mathcal{L}_{\bar{\alpha},\bar{p}}, \quad Q_{p,\bar{p}}^{\dagger} = \sum_{\alpha,\bar{\alpha}=1}^{N_C} \mathcal{L}_{\bar{\alpha},\bar{p}}^{\dagger} h_{\bar{\alpha},\alpha}^{\dagger} \mathcal{R}_{\alpha,p}, \tag{3.19}$$

The correlation functions of the fields (3.19) may be written in the form

$$\langle Q_{p_1\bar{p}_1}(1)Q_{p_2\bar{p}_2}^{\dagger}(2)Q_{p_3\bar{p}_3}^{\dagger}(3)Q_{p_4\bar{p}_4}(4)\rangle = |z_{14}z_{23}|^{-4h} \sum_{a,b=1}^{2} C_{ab}\, \mathcal{G}_p^{(a)}(z_i)\, \mathcal{G}_{\bar{p}}^{(b)}(\bar{z}_i) \tag{3.20}$$

where $h = (1 - N_C^2)/2N_C^2$ is the conformal dimension of the $\widehat{su}(N_C)_{-2N_C}$ primary field $h_{\alpha,\bar{\alpha}}$ as given by equation (A4). $\mathcal{G}_p^{(a)}(z_i)$ is the (holomorphic) product of the free Dirac and $\widehat{su}(N_C)_{-2N_C}$ correlation functions traced over the colour indices:

$$\mathcal{G}_p^{(a)}(z_i) = \sum_{\alpha=1}^{N_C} \left(\frac{I_1^\alpha I_1^p}{z_{12}z_{34}} - \frac{I_2^\alpha I_2^p}{z_{13}z_{24}} \right) \times \left(I_1^\alpha F_1^{(a)}(z) + I_2^\alpha F_2^{(a)}(z) \right) \tag{3.21}$$

The coefficients C_{ab} appearing in equation (3.20) have the values $C_{12} = C_{21} = 1$, $C_{11} = C_{22} = 0$ to ensure single-valuedness and their off-diagonal form reflects the logarithmic nature of the underlying $\widehat{su}(N_C)_{-2N_C}$ LCFT. We use the symbol p here to denote the ordered sequence of pseudo-replica indices p_1, p_2, p_3, p_4 and the invariant tensors I_1 and I_2 are defined as $I_1^p = \delta^{p_1,p_2}\delta^{p_3,p_4}$, $I_2^p = \delta^{p_1,p_3}\delta^{p_2,p_4}$; analogous expressions hold for α. Performing the trace over colour indices appearing in (3.21) by means of the identities

$$\sum_{\alpha=1}^{N_C} I_1^\alpha I_1^\alpha = N_C^2 \quad \sum_{\alpha=1}^{N_C} I_1^\alpha I_2^\alpha = N_C \tag{3.22}$$

and utilizing the relations (A29) satisfied by the $\widehat{su}(N_C)_{-2N_C}$ chiral blocks one obtains:

$$\mathcal{G}_p^{(1)}(z_i) = \frac{c'}{z_{14}z_{32}} \Lambda \big\{ I_1^p E_{N_C}(z) + I_2^p \big[K_{N_C}(z) - E_{N_C}(z) \big] \big\} \tag{3.23a}$$

$$\mathcal{G}_p^{(2)}(z_i) = \frac{2c'N_C}{z_{14}z_{32}} \Lambda \left\{ I_1^p \left[\tilde{E}_{N_C}(z) - \tilde{K}_{N_C}(z) \right] - I_2^p \tilde{E}_{N_C}(z) \right\} \tag{3.23b}$$

where $c' = cN(N^2 - 1)$ is a normalization constant, $\Lambda = [z(1-z)]^{-1/N_C^2}$, and where we have adopted the notation that $\tilde{f}(z) \equiv f(1-z)$ for an arbitrary function $f(z)$. In particular by inserting the explicit results (3.23) into the decomposition (3.20) and collecting the coefficients of $I_i^p \bar{I}_j^{\bar{p}}$ one may recast (3.20) so as to read:

$$\langle Q_{p_1\bar{p}_1}(1)Q_{p_2\bar{p}_2}^{\dagger}(2)Q_{p_3\bar{p}_3}^{\dagger}(3)Q_{p_4\bar{p}_4}(4)\rangle = -\frac{2c'^2}{X_{12}}|z_{14}z_{23}|^{-2/N_C^2} \sum_{i,j=1}^{2} I_i^p \bar{I}_j^{\bar{p}} F_{ij}(z,\bar{z}) \tag{3.24}$$

where the $F_{ij}(z,\bar{z})$ are (both fortunately and remarkably) single-valued combinations of the solutions to the $\widehat{u}(0)_{N_C}$ Knizhnik–Zamolodchikov equations as given by equations (A22) with $k = N_C$. That is to say (upto an irrelevant normalization) we have recovered the replica result (3.9) in which our psuedo-replica indices play the rôle of replica indices. In particular the results (3.10) and (3.11) together with their interpretations (3.12) follow straightforwardly.

C. Supersymmetric Approach

In the supersymmetric approach to disordered fermionic systems one introduces bosonic copies of the original Grassmann fields [42, 43]. The partition function of the resulting supersymmetric theory is equal to unity due to the inverse relationship between ordinary c-number Gaussian functional integrals and their Grassmann counterparts. The absence of a disorder dependent partition function normalizing quenched correlation functions drastically simplifies their disorder averaging. The supersymmetric approach to the random Dirac fermion problem has been outlined by Bernard and LeClair [58] who, following the general principles of the supersymmetric approach, have introduced bosonic copies of the Grassmann fields coupled to the same gauge potential. The resulting action is given by (3.3) where the summation over replicas is replaced by a summation over fermionic (Grassmann) and bosonic (c-number) fields:

$$S^{\text{SUSY}} = \int d^2\xi \sum_{i=1}^{2} \sum_{\alpha,\beta=1}^{N_C} \bar\psi^{\alpha,i} \not{D}^{\alpha\beta} \psi^{\beta,i} \tag{3.25}$$

The symmetry of the free supersymmetrized Dirac action (in the absence of any gauge fields) is $OSp(2N_C|2N_C)$ [39] and it may be recast as the $\widehat{osp}(2N_C|2N_C)_1$ WZNW model. In particular, the random $su(N_C)$ gauge potential couples only to currents J^a (and $\bar J^a$) residing in the $\widehat{su}(N_C)_0$ subalgebra of the complete $\widehat{osp}(2N_C|2N_C)_1$ Kač–Moody algebra:

$$S^{\text{SUSY}} = \widehat{osp}(2N_C|2N_C)_1 + \int d^2\xi \, (J^a A^a + \bar J^a \bar A^a) \tag{3.26}$$

In the special case $N_C = 2$, Bernard and LeClair have demonstrated that the Suguwara energy momentum tensor for $\widehat{osp}(2N_C|2N_C)_1$ may be decomposed into the sum of two commuting pieces pertaining to different symmetries [58]:

$$T_{\text{osp}(4|4)_1} = T_{\text{osp}(2|2)_{-2}} + T_{\text{su}(2)_0} \tag{3.27}$$

Exploiting the decomposition (3.27) one may rewrite equation (3.26) in the *decoupled* form

$$S^{\text{SUSY}} = \widehat{osp}(2|2)_{-2} + \widehat{su}(2)_0 + \int d^2\xi \, (J^a A^a + \bar J^a \bar A^a) \tag{3.28}$$

We note that the rôle of the decomposition (3.27) in this approach closely mirrors that of the decomposition (3.6) employed in the replica approach — both allow us to decouple the effects of the gauge potential disorder. Indeed, the latter two terms appearing in equation (3.28) are precisely those which appear in the replica approach as $N_F \to 0$. As was rigorously proven in references [30, 31], and utilized to our advantage in section III A, this $su(2)$ sector becomes massive for the simple Gaussian distribution of the gauge fields given in equation (2.3); a perturbative renormalization group argument was given in reference [58] where the one loop beta function was calculated with the result that the coupling constant g_A

flows to strong coupling.[1] That is to say, the low-energy effective theory governing the Q-field is the $\widehat{osp}(2|2)_{-2}$ WZNW model. Fortunately, the $\widehat{osp}(2|2)_k$ WZNW model has been discussed quite extensively in the work of Maassarani and Serban [59]. In particular the model undergoes a dramatic simplification at $k = -2$ [60] and we provide a rather extensive discussion of this model in Appendix B. In this approach the Q-field may be represented as

$$Q = Q^{1,\bar{1}}, \quad Q^\dagger = Q^{4,\bar{4}} \tag{3.29}$$

where the $Q^{\alpha,\bar{\alpha}}$ are primary fields transforming in the $[0, 1/2]$ representaion of $osp(2|2)$ — see Appendix B.

1. Conformal Dimensions

In the supersymmetric approach to the random $su(2)$ Dirac fermion problem, the conformal dimension of the Q-field coincides with that of a primary field transforming in the fundamental $[0, 1/2]$ representation of $\widehat{osp}(2|2)_{-2}$ — see equation (B7) and set $k = -2$:

$$h_Q = \frac{1}{4 - 2(-2)} = \frac{1}{8} \tag{3.30}$$

This is in agreement with the replica result (3.8) and the strong disorder result (3.18) when $N_C = 2$.

2. Correlation functions of the Q-field

The four-point correlation function of the Q-matrix admits the $osp(2|2) \times osp(2|2)$ invariant decomposition — see equations (B4), (B5) and (B8):

$$\langle Q^{\alpha_1,\bar{\alpha}_1}(1)Q^{\alpha_2,\bar{\alpha}_2}(2)Q^{\alpha_3,\bar{\alpha}_3}(3)Q^{\alpha_4,\bar{\alpha}_4}(4)\rangle = |z_{14}z_{23}|^{-1/2} \sum_{i,j=1}^{3} I_i^\alpha \bar{I}_j^{\bar{\alpha}} F_{ij}(z,\bar{z}) \tag{3.31}$$

where α denotes the ordered sequence of indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, which label the basis states of the four-dimensional representation of $osp(2|2)$, and where the invariant tensors I_1, I_2 and I_3 are defined in equations (B9) together with similar equations for $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$. The anharmonic ratio z is defined as $z = z_{12}z_{34}/z_{14}z_{32}$ and similarly for $\bar{z}$. The functions $F_{ij}(z,\bar{z})$ are single-valued combinations of the solutions to the $\widehat{osp}(2|2)_{-2}$ Knizhnik–Zamolodchikov

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[1]Although a formal proof of the gap generation exists only for the case when the disorder distribution is Gaussian, it appears reasonable to assume that the $su(2)$ sector remains massive for a broader choice of disorder distributions. At scales smaller than the gap, the effective action is given by the critical $\widehat{osp}(2|2)_{-2}$ WZNW model. The critical point is stable with respect to variations of the disorder: all such variations affect only the massive modes and hence do not generate relevant perturbations of the critical action.

equations, which are summarized by equations (B29) — (B33). For example, focusing on the correlation function pertaining to the decomposition (3.29):

$$\langle Q^{1\bar{1}}(1)Q^{4\bar{4}}(2)Q^{4\bar{4}}(3)Q^{1\bar{1}}(4)\rangle \sim |\Upsilon|^2 \left[K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})\right]. \tag{3.32}$$

where here $\Upsilon = [z_{14}z_{23}\,z(1-z)]^{-1/4}$ and $K(z)$ and $E(z)$ are the complete elliptic integrals. This is in agreement with both the replica result (3.10) and the strong disorder result (3.24) when the number of colours $N_C = 2$.

IV. DENSE POLYMERS AND TWIST OPERATORS

It turns out that the supersymmetric description provides an extremely economical and straightforward approach to the disordered Dirac fermion problem. It follows from the work of Rasmussen [61], that the action of the $\widehat{osp}(2|2)_{-2}$ WZNW model may be represented as the direct sum of three simple theories:

$$\widehat{osp}(2|2)_{-2} = \widehat{su}(2)_1 + \frac{1}{4\pi}\int d^2\xi\,(\partial_\mu\varphi)^2 + \int d^2\xi\,\epsilon_{ab}\,\partial_\mu\chi^a\partial^\mu\chi^b, \tag{4.1}$$

where φ is a non-compact bosonic field, χ^a is a two-component symplectic fermion [62], and ϵ_{ab} is a two-component antisymmetric tensor. As expected in this supersymmetric theory, the bosonic sector ($c = 2$) and the symplectic fermions ($c = -2$) together yield a total central charge of zero. We note that the presence of the $\widehat{su}(2)_1$ WZNW model in the decomposition (4.1) of the $\widehat{osp}(2|2)_{-2}$ WZNW model reflects an underlying $\widehat{su}(2)_{-k/2}$ Kač–Moody subalgebra residing in the $\widehat{osp}(2|2)_k$ algebra [63]. The decomposition (4.1) may be simplified even further by noting that the $\widehat{su}(2)_1$ WZNW model admits the following free field representation:

$$\widehat{su}(2)_1 = \frac{1}{4\pi}\int d^2\xi\,(\partial_\mu\phi)^2, \tag{4.2}$$

where ϕ is a compact free boson; note that our choice of normalization is for latter convenience in equation (4.9). That is to say, the $\widehat{osp}(2|2)_{-2}$ WZNW model may be represented as the sum of a compact bosonic field ϕ ($c = 1$) a non-compact bosonic field φ ($c = 1$) and a two-component symplectic fermion ($c = -2$):

$$\widehat{osp}(2|2)_{-2} = \int d^2\xi\left[\frac{1}{4\pi}(\partial_\mu\phi)^2 + \frac{1}{4\pi}(\partial_\mu\varphi)^2 + \epsilon_{ab}\,\partial_\mu\chi^a\partial^\mu\chi^b\right]. \tag{4.3}$$

In view of the decomposition (4.3) one anticipates a representation of the $\widehat{osp}(2|2)_{-2}$ primary fields in terms of the primary fields of the models appearing on the right-hand side of equation (4.3) — namely vertex operators $e^{\alpha\phi}$ and $e^{\beta\varphi}$ and the primary fields of the $c = -2$ theory. As we shall discover in sections IV A – IV C this is indeed possible. The non-unitary $c = -2$ minimal model has received a great deal of attention in recent years as a theory of dense polymers [64,65], as a celebrated example of a logarithmic conformal field theory [62,66–70], and as a conformal ghost system [71]. The structure of this theory is rather rich and is known to consist of several sectors. As we shall see below, the so-called $\mathbb{Z}_2$ twisted sector

of the $c = -2$ theory will play a crucial rôle here; this sector consists of a scalar primary field μ of conformal dimension $-1/8$ and tensor primary fields $\nu^\pm$ of conformal dimension $3/8$. The correlation functions and operator product expansions of these fields have been studied in reference [64] and more fully in reference [69]. In section IV A we shall recast the four-point correlation functions of the $\widehat{osp}(2|2)_{-2}$ WZNW in a form which facilitates the elucidation of the desired operator correspondence. In section IV B we shall discuss the $\mathbb{Z}_2$ twist operator correlation functions of the $c = -2$ model [69]. In section IV C we shall compare the correlation functions (summarized in Table I and Table II of sections IV A and IV B respectively) and arrive at the aforementioned correspondence.

A. The $\widehat{osp}(2|2)_{-2}$ WZNW model

In view of the large number of components of the generic $\widehat{osp}(2|2)_{-2}$ non-chiral four-point correlation function and the rather cumbersome and opaque invariant tensors (B9) we shall refashion the correlator somewhat.[2] In particular the $c = -2$ operator correspondence will follow quite naturally. We study the four-point correlation function of the supersymmetric Q-matrix:

$$\mathcal{F}^{\alpha, \bar{\alpha}}(z_i, \bar{z}_i) = \langle Q^{\alpha_1, \bar{\alpha}_1}(z_1, \bar{z}_1) Q^{\alpha_2, \bar{\alpha}_2}(z_2, \bar{z}_2) Q^{\alpha_1, \bar{\alpha}_3}(z_3, \bar{z}_3) Q^{\alpha_4, \bar{\alpha}_4}(z_4, \bar{z}_4) \rangle, \qquad (4.4)$$

where on the left hand side we use the symbol α to denote the ordered sequence of indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. The indices α_i assume the values $1, 2, 3, 4$ and label the basis states of the four-dimensional $[0, 1/2]$ representation of $osp(2|2)$; in the notation of [59] states 1 and 4 are bosonic, whilst states 2 and 3 are fermionic. As may be seen from our more detailed studies in Appendix B, this correlation function may be written in the (off-diagonal) form

$$\mathcal{F}^{\alpha, \bar{\alpha}}(z_i, \bar{z}_i) = -\mathcal{F}^{\alpha, (1)}(z_i) \mathcal{F}^{\bar{\alpha}, (2)}(\bar{z}_i) - \mathcal{F}^{\alpha, (2)}(z_i) \mathcal{F}^{\bar{\alpha}, (1)}(\bar{z}_i), \qquad (4.5)$$

in which we have set the overall normalization of the four-point function to minus unity for latter convenience, and where

$$\mathcal{F}^{\alpha, (a)}(z_i) \equiv |z_{14} z_{23}|^{-1/4} \sum_{i=1}^{3} I_i^\alpha F_i^{(a)}(z). \qquad (4.6)$$

Using the explicit form of the chiral blocks $F_i^{(a)}(z)$ appearing in equations (B23), (B26) and (B27) together with the known transformation properties of the invariant tensors — see equations (B48) and (B50) — we deduce that

$$\mathcal{F}^{\alpha, (2)}(z_i) = -\tilde{\mathcal{P}} \mathcal{F}^{\bar{\alpha}, (1)}(\tilde{z}_i). \qquad (4.7)$$

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[2]Restricting our attention to four-point correlation functions pertaining to the four-dimensional $[0, 1/2]$ representation we have a total of 4^8 components. Since the vast majority of these vanish however, and many are related by symmetry, it is desirable to distill them further.

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Sector	α				$\mathcal{F}^{\alpha,(1)}(z_i)$
Bosonic	1144	4411			$(4\epsilon\gamma)^{-1}\,\Upsilon\,zK(z)$
	1414	4141			$(4\epsilon\gamma)^{-1}\,\Upsilon\,(1-z)K(z)$
	1441	4114			$-(4\epsilon\gamma)^{-1}\,\Upsilon\,K(z)$
Fermionic	2233	3322			$(4\epsilon\gamma)\,\Upsilon\,[2E(z)-(2-z)K(z)]$
	2323	3232			$(4\epsilon\gamma)\,\Upsilon\,[2E(z)-(1-z)K(z)]$
	2332	3223			$(4\epsilon\gamma)\,\Upsilon\,[2E(z)-K(z)]$
Mixed	1234	1324	2143	3142	$\pm\,\Upsilon\,[E(z)-(1-z)K(z)]$
	2413	3412	4231	4321	
	1243	1342	2134	3124	$\pm\,\Upsilon\,[K(z)-E(z)]$
	2431	3421	4213	4312	
	1423	1432	2314	3214	$\pm\,\Upsilon\,E(z)$
	2341	3241	4123	4132	

TABLE I. Chiral correlation functions of the $\widehat{osp}(2|2)_{-2}$ WZNW model displaying regular behaviour in the vicinity of $z = 0$ and logarithmic behaviour in the vicinity of $z = 1$; $\Upsilon = [z_{14}z_{23}\,z(1-z)]^{-1/4}$ and $z = z_{12}z_{34}/z_{14}z_{32}$ — see Appendix B for further details.

We use the tilde to denote the interchange of the coordinates or indices 2 and 3, and $\tilde{\mathcal{P}}$ to denote the fermionic parity of this permutation; note that this permutation induces the transformation $z \rightarrow 1 - z$ in these functions. Substituting equation (4.7) into equation (4.5) we obtain the result

$$\mathcal{F}^{\alpha,\bar{\alpha}}(z_i, \bar{z}_i) = \tilde{\mathcal{P}}\mathcal{F}^{\alpha,(1)}(z_i)\mathcal{F}^{\bar{\alpha},(1)}(\tilde{\bar{z}}_i) + \tilde{\mathcal{P}}\mathcal{F}^{\tilde{\alpha},(1)}(\tilde{z}_i)\mathcal{F}^{\bar{\alpha},(1)}(\bar{z}_i). \tag{4.8}$$

That is to say, one may build the full quota of single-valued and crossing symmetric nonchiral correlation functions of the $\widehat{osp}(2|2)_{-2}$ WZNW from our knowledge of the chiral functions $\mathcal{F}^{\alpha,(1)}(z_i)$ which display regular behaviour in the vicinity of $z = 0$ and logarithmic behaviour in the vicinity of $z = 1$. Using the explict results for the chiral blocks (B23), (B26) and (B27) together with the invariant tensors (B9) we are able to gather these nontrivial functions in Table I. We emphasize that the functions $\mathcal{F}^{\alpha,(1)}(z_i)$ are not simply the conformal blocks of the $\widehat{osp}(2|2)_{-2}$ WZNW model, but also embody the non-trivial tensorial structure of $osp(2|2)$ — see equation (4.6).

B. Twist Operator Correlation Functions

The correlation functions of the twist operators μ and ν in the non-unitary $c = -2$ minimal model have been extensively studied by Gaberdiel and Kausch [69]. Their nonchiral four-point functions are built from linear combinations of their chiral counterparts

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Sector	Correlator	Regular contribution
μ	$\langle\mu\mu\mu\mu\rangle$	$\Upsilon^{-1} K(z)$
ν	$\langle\nu^{\pm}\nu^{\pm}\nu^{\mp}\nu^{\mp}\rangle$	$\Upsilon^3 z\,[2E(z) - (2-z)K(z)]$
	$\langle\nu^{\pm}\nu^{\mp}\nu^{\pm}\nu^{\mp}\rangle$	$\Upsilon^3 (1-z)\,[2E(z) - (1-z)K(z)]$
	$\langle\nu^{\pm}\nu^{\mp}\nu^{\mp}\nu^{\pm}\rangle$	$\Upsilon^3\,[2E(z) - K(z)]$
Mixed	$\langle\mu\nu^{\pm}\nu^{\mp}\mu\rangle \times z_{14}^{-1/2} z_{23}^{+1/2}$ $\langle\nu^{\pm}\mu\mu\nu^{\mp}\rangle \times z_{14}^{+1/2} z_{23}^{-1/2}$	$\pm\,\Upsilon\,[E(z) - (1-z)K(z)]$
	$\langle\mu\nu^{\pm}\mu\nu^{\mp}\rangle \times z_{13}^{-1/2} z_{24}^{+1/2}$ $\langle\nu^{\pm}\mu\nu^{\mp}\mu\rangle \times z_{13}^{+1/2} z_{24}^{-1/2}$	$\pm\Upsilon\,[K(z) - E(z)]$
	$\langle\mu\mu\nu^{\pm}\nu^{\mp}\rangle \times z_{12}^{-1/2} z_{34}^{+1/2}$ $\langle\nu^{\pm}\nu^{\mp}\mu\mu\rangle \times z_{12}^{+1/2} z_{34}^{-1/2}$	$\pm\,\Upsilon\, E(z)$

TABLE II. Chiral correlation functions of the $\mathbb{Z}_2$ twist operators of $c = -2$ displaying regular behaviour in the vicinity of $z = 0$ and logarithmic behaviour in the vicinity of $z = 1$; $\Upsilon = [z_{14}z_{23}\,z(1-z)]^{-1/4}$ and $z = z_{12}z_{34}/z_{14}z_{32}$ — see Gaberdiel and Kausch for further details [69] and note that our definition of the anharmonic ratio differs from theirs.

so as to respect the stringent constraints of single-valuedness and crossing symmetry. As may be seen from §4 of their work, the contributions from either chiral sector are typically composed of two elliptic integral solutions posessing logarithmic branch cuts; one of these solutions displays regular behaviour in the vicinity of $z = 0$ and logarithmic behaviour in the vicinity of $z = 1$, whilst the other displays regular behaviour in the vicinity of $z = 1$ and logarithmic behaviour in the vicinity of $z = 0$. As in section IV A, in order to establish our correspondence (4.9) we find it convenient to focus on the chiral contributions to the four-point functions which exhibit regular behaviour in the vicinity of $z = 0$. The chiral four-point functions most relevant to our discussion and displaying such regular behaviour in the vicinity of the origin are summarized in table II.

C. Twist Operator Correspondence

Comparing our correlation functions for the $\widehat{osp}(2|2)_{-2}$ WZNW model (summarized in Table I) with those of Gaberdiel and Kausch for the $\mathbb{Z}_2$ twist fields of the $c = -2$ minimal model [69] (summarized in Table II) we see a clear correspondence at the level of their elliptic integrals. In particular we see that the bosonic and fermionic states of the $\widehat{osp}(2|2)_{-2}$ WZNW model are naturally associated with the $\mathbb{Z}_2$ twist fields μ and ν_α respectively. The remaining powers of z_{ij} which distinguish Table I from Table II are provided by bosonic vertex operators as suggested by the decomposition (4.3). Explicitly, we establish the following equivalence between the chiral operators of the $\widehat{osp}(2|2)_{-2}$ WZNW model and the 'dressed'

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chiral operators of the $c = -2$ minimal model:[3]

$$Q_1 \sim \mathcal{A}^{-1} e^{i\phi} \mu, \quad Q_2 \sim \mathcal{A} e^{-\varphi} \nu^-, \quad Q_3 \sim \mathcal{A} e^{\varphi} \nu^+, \quad Q_4 \sim \mathcal{A}^{-1} e^{-i\phi} \mu, \qquad (4.9)$$

where $\mathcal{A} = (4\epsilon\gamma)^{1/4}$ is a free parameter corresponding to the arbitrary relative normalizations of the $su(2)$ doublet (Q_1, Q_4) and the two singlets (Q_2, Q_3) in the four-dimensional representation of $osp(2|2)$ [59, 72]. With the normalization chosen in equation (4.3) the compact bosonic exponents $e^{\pm i\phi}$ have conformal dimension $h_\phi = 1/4$, and the non-compact bosonic exponents $e^{\pm\varphi}$ have $h_\varphi = -1/4$. It is thus straightforward to see that the dimensions add up to the correct value of $1/8$ — equation (B7) — in the decomposition (4.9).

We have now highlighted the important rôle played by the $c = -2$ model in the chiral structure of the $\widehat{osp}(2|2)_{-2}$ WZNW model, and have provided a prescription for constructing the non-chiral correlation functions (4.8). Although there are many interesting and notable exceptions, we emphasize that in general, the non-chiral correlation functions of the $\widehat{osp}(2|2)_{-2}$ WZNW are *not* simply related to *single* non-chiral correlation functions of the $\mathbb{Z}_2$ twist operators dressed by non-chiral bosons. For example, it follows from our solution of the $\widehat{osp}(2|2)_{-2}$ Knizhnik–Zamolodchikov equations that

$$\langle Q^{2,\bar{2}}(1)Q^{3,\bar{3}}(2)Q^{2,\bar{2}}(3)Q^{3,\bar{3}}(4)\rangle = \mathcal{C}\{[(1-z)K - 2E][(1+\bar{z})\tilde{K} - 2\tilde{E}] + c.c.\}, \qquad (4.10)$$

where $\mathcal{C} = (16\epsilon^2\gamma^2)^2 |z_{14}z_{23}|^{-1/2} \Lambda$, and Λ is given by equation (B33). We note that this is *not* simply related to a *single* correlation function of the non-chiral operators $\mu(z,\bar{z})$ and $\nu_{\alpha,\bar{\alpha}}(z,\bar{z})$. That is to say, no *single* non-chiral four-point function of the $\mathbb{Z}_2$ twist operators gives rise to this particular single-valued combination of the elliptic integrals. We invite the reader to verify this statement with the aid of §4 of Gaberdiel and Kausch [69]. It may be interesting to study the non-chiral aspects of this correspondence in greater detail.

V. CONVERGENCE OF APPROACHES

In section III we demonstrated that three different approaches to the random Dirac fermion problem yield identical results for the four-point correlation functions of the local density of states. In section IV we have further demonstrated that the supersymmetric approach inherits its non-trivial logarithmic structure from the $c = -2$ non-unitary minimal model. In this section we shall discuss the convergence of the these approaches — summarized in table III — and the emergence of the $c = -2$ theory in a little more detail. Indeed, we shall point to a remarkable proliferation of $c = -2$ theories and effects. In table III we have highlighted the separation of the active (critical WZNW) degrees of freedom from the passive (massive or decoupled Jacobian) degrees of freedom in each approach.

An interesting aspect of this marriage of approaches is the agreement between the weak coupling (replica and supersymmetry) limit and the strong coupling limit. Although such an

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[3]We use the (overused and rather abused) symbol $\sim$ to emphasize that we have derived the correspondence from the chiral regular solutions and that it is valid up to phase.

Approach	Fermions	Bosons	Active	Passive
Replicas	$N_C \times N_F$	0	$\widehat{u}(1) + \widehat{su}(N_F)_{N_C}$	$\widehat{su}(N_C)_{N_F}$
Strong Disorder	N_C	0	$\widehat{u}(1) + \widehat{su}(N_C)_1 + \widehat{su}(N_C)_{-2N_C}$	$(c = -2)^{N_C^2-1}$
Supersymmetry	2	2	$\widehat{osp}(2|2)_{-2}$	$\widehat{su}(2)_0$

TABLE III. Current algebra approaches used to investigate the critical behaviour of the random non-Abelian Dirac fermion problem and the separation of the active (critical WZNW) degrees of freedom from the passive (massive or decoupled Jacobian) degrees of freedom. The central charges and the conformal dimensions may be seen to add up correctly in each approach.

agreement might have been anticipated purely on physical grounds, its emergence is quite remarkable from a field-theoretic perspective. At the very least, it is quite surprising that such superficially different conformal field theories may nonetheless yield equivalent results, albeit for a specific subset of physically motivated correlation functions. In particular, the resultant central charge of the active degrees of freedom in the strong disorder approach differs from that in the replica and supersymmetric approaches, namely zero. As we have discussed in section III B — and indicate in table III — the active degrees of freedom in the strong disorder limit may be expressed as the sum of three different models:

$$S = \widehat{u}(1) + \widehat{su}(N_C)_1 + \widehat{su}(N_C)_{-2N_C}. \tag{5.1}$$

where the first two models describe N_C colours of free massless Dirac fermions, and the remaining model encodes the non-trivial logarithmic structure. We use the well known result for the central charge of the $\widehat{g}_k$ WZNW model [53]

$$c = \frac{k \dim g}{k + g^\vee}, \tag{5.2}$$

where $g^\vee$ is the dual Coxeter number of the algebra g (equal to N for $su(N)$) together with the fact that a free boson $\widehat{u}(1)$ has central charge 1, to find the resultant central charge of (5.1):

$$c = 1 + \frac{N_C^2 - 1}{1 + N_C} + \frac{-2N_C(N_C^2 - 1)}{-2N_C + N_C} = 2N_C^2 + N_C - 2 \neq 0. \tag{5.3}$$

That is to say, the active degrees of freedom in the strong disorder approach yield a positive central charge, in contrast to the replica and supersymmetric degrees of freedom which yield a net zero central charge.

As we have discused in section IV, the supersymmetric approach to the disordered Dirac fermion problem reveals a hidden substructure which is inherited from the $c = -2$ minimal model. In view of the convergence of approaches outlined in this paper it follows quite naturally that the active degrees of freedom in the replica and strong disorder treatments also inherit non-trivial traits from the $c = -2$ model. In addition to the rather natural

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proliferation of elliptic integrals in the conformal blocks presented in this paper, we note that a relation between the $\widehat{su}(2)_{-4}$ WZNW model (which arises in the strong disorder treatment with $N_C = 2$) and the $c = -2$ theory has already been discussed in the string theory literature [73]: the $\widehat{su}(2)_{-4}$ WZNW model is cohomologically equivalent to an $N = 4$ supersymmetric bosonic string with $c = -2$ matter.[4]

A curious by-product of these investigations is that the presence of the $c = -2$ theory may apparantly be seen in the 'passive' degrees of freedom outlined in table III. Indeed, a number of $c = -2$ models (arising as Jacobians) explicitly decouple in the strong disorder approach and play a passive (or spectator) rôle in regards the LDOS. A possible explanation for the closely related structure of the active and passive theories indicated in table III may arise from the requirement of the mutual cancellation of their logarithmic singularities [60]. In particular we anticipate a relationship between the $\widehat{su}(2)_0$ WZNW model and the $c = -2$ theory and a correspondence analogous to equation (4.9). We note that the four-point functions of the $\widehat{su}(2)_0$ model have been studied in references [76, 77] and assume a simple form in terms of the complete elliptic integrals [60]. The investigation of the $\widehat{su}(2)_0$ WZNW model will be continued in reference [78].

As we close this section we comment briefly on the stability of the critical point with respect to variations of the disorder distribution. In the replica and supersymmetric approaches the gauge potential disorder is coupled directly to the massive sector and it is natural to assume that the critical theory is protected from such variations. In the strong disorder approach, however, the disorder variations are coupled directly to the critical $\widehat{su}(N_C)_{-2N_C}$ subsector, and its stability is far from obvious. It is therefore interesting to study the stability properties of this critical theory. A simple perturbation of the critical theory that one might consider is the deformation of the WZNW action by its kinetic term. This operator has scaling dimension zero and is strongly relevant. However, at level $k = -2N$ (or more generally $-2g^\vee$) this operator commutes with all the Kač–Moody currents and therefore does not affect the correlation functions — see Appendix 6 of reference [12] and note the different sign conventions for the level k. In a more general framework, the WZNW models at level $k = -2g^\vee$ arise quite naturally in both two-dimensional ($c = 0$) topological field theories possessing a non-Abelian current algebra [79], and in the strong disorder treatment of arbitrary WZNW models coupled to random vector potentials [39]. The critical level arises from a BRST (Becchi–Rouet–Stora–Tyutin) [80, 81] symmetry nilpotency condition and is required for the coexistence of a Kač–Moody algebra symmetry and a topological algebra symmetry [79]. We emphasize that the level of the underlying Kač–Moody algebra, the existence of a topological algebra and the stability of our strongly disordered theory are therefore intimately related.

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[4]We note that this matter + string theory admits a description in terms of the $c = -2$ model dressed by $c = 28$ Liouville theory and $c = -26$ string ghosts. We remind the interested reader that this (and other) Liouville LCFTs [35, 74, 75] emerged in a very natural way in the closely related problem of prelocalization in disordered conductors [35, 38].

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VI. CONCLUSIONS

We briefly summarize here a number of our main results:

- We have solved the random non-Abelian Dirac fermion problem by means of three different non-perturbative procedures based on the replica approach, supersymmetry, and in the limit of strong disorder. We have demonstrated that this *ménage à trois* of approaches yields identical results (for the four-point correlation functions of the local density of states) in this relatively simple, but quite non-trivial model of disorder.

- We have emphasized the special rôle played by the level k of the Kač–Moody algebra. As we have seen both here and in reference [60] the level $k = -2$ is rather special for the $\widehat{osp}(2|2)_k$ WZNW model which arises in the supersymmetric treatment of the random Dirac fermion problem: an entire conformal block decouples from the spectrum and the resulting theory is drastically simplified.

- We have established that the $c = -2$ minimal model plays an important rôle in the random non-Abelian Dirac fermion problem. We have found a rather simple and suggestive form for the supersymmetric critical action (4.3):

$$\widehat{osp}(2|2)_{-2} = u(1) + gl(1) + [c = -2].$$

We have highlighted the relevance of the twist operators of the $c = -2$ theory in the disorder averaged correlation functions of the Dirac fermion problem. The $c = -2$ symplectic fermions are ubiquitous in the construction of supersymmetric sigma models [42, 43]; this example indicates that this sector may well be responsible for many remarkable features of disordered critical points including the presence of logarithmic operators [66]. We draw attention to the fact that the $c = -2$ theory emerges in the theoretical description of dense polymers [64, 65]. We hope that this connection may ultimately yield a more intuitive picture of the random non-Abelian Dirac fermion problem and other disordered critical points.

- We have argued for the stability of the critical point with respect to variations in the disorder, and noted that the level of the underlying Kač–Moody algebra, the existence of a topological algebra and the stability of our strongly disordered theory are rather intimately linked. It is an interesting open question whether such considerations may yield Kač–Moody level selection mechanisms in models displaying lines of critical points [11].

VII. ACKNOWLEDGEMENTS

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Note added in proof: whilst this paper was being completed we became aware of a preprint by A. W. W. Ludwig [82] in which one of our results, namely the equivalence (4.3), was established.

APPENDIX A: THE $\widehat{su}(N)_k$ WZNW MODEL

In this appendix we consider the appearance of logarithms in the correlation functions of the $\widehat{su}(N)_k$ WZNW model. Following §4 of Knizhnik and Zamolodchikov [53] we compute the four-point functions

$$\mathcal{F}^{\alpha,\bar{\alpha}}(z_i,\bar{z}_i) = \langle g(z_1,\bar{z}_1)g^\dagger(z_2,\bar{z}_2)g^\dagger(z_3,\bar{z}_3)g(z_4,\bar{z}_4)\rangle, \qquad (A1)$$

of the field $g(z_i,\bar{z}_i) = g^{\alpha_i,\bar{\alpha}_i}(z_i,\bar{z}_i)$ transforming in the fundamental representation of SU(N)×SU(N). We use the symbol $\boldsymbol{\alpha}$ to denote the ordered sequence of indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Global conformal invariance restricts this correlation function to have the form

$$\mathcal{F}^{\alpha,\bar{\alpha}}(z_i,\bar{z}_i) = (z_{14}z_{23}\bar{z}_{14}\bar{z}_{23})^{-2h}F^{\alpha,\bar{\alpha}}(z,\bar{z}) \qquad (A2)$$

where z and $\bar{z}$ are the anharmonic ratios

$$z = \frac{z_{12}z_{34}}{z_{14}z_{32}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{32}} \qquad (A3)$$

and the conformal dimension h of the field g is

$$h = \frac{N^2-1}{2N(N+k)} \qquad (A4)$$

The correlation function (A2) admits the su(N)×su(N) invariant decomposition

$$F^{\alpha,\bar{\alpha}}(z,\bar{z}) = \sum_{ij=1}^{2} I_i^\alpha \bar{I}_j^{\bar{\alpha}} F_{ij}(z,\bar{z}) \qquad (A5)$$

where the invariant tensors I_1 and I_2 are defined as

$$I_1^\alpha = \delta^{\alpha_1,\alpha_2}\delta^{\alpha_3,\alpha_4}, \quad I_2^\alpha = \delta^{\alpha_1,\alpha_3}\delta^{\alpha_2,\alpha_4} \qquad (A6)$$

with similar equations for $\bar{I}_1$ and $\bar{I}_2$. The four scalar functions F_{ij} satisfy the coupled first-order differential equations [53]

$$\frac{dF}{dz} = \left[\frac{1}{z}P + \frac{1}{z-1}Q\right]F, \quad \text{where} \quad F = \begin{pmatrix} F_{1j} \\ F_{2j} \end{pmatrix} \quad \forall j \qquad (A7)$$

where the matrices P and Q are given by

$$P = -\frac{1}{N(N+k)}\begin{pmatrix} N^2-1 & N \\ 0 & -1 \end{pmatrix}, \quad Q = -\frac{1}{N(N+k)}\begin{pmatrix} -1 & 0 \\ N & N^2-1 \end{pmatrix} \qquad (A8)$$

There are similar equations for the antiholomorphic dependence. Suppressing the antiholomorphic index j from the functions F_{1j} and F_{2j}, one may obtain the second-order differential equation satisfied by $F_1(z)$:

$$N^2(N+k)^2z^2(1-z)^2F_1''(z) -$$
$$N(N+k)z(1-z)\left[2 - N(2N+k) + (3N^2 + Nk - 4)z\right]F_1'(z) + \qquad (A9)$$
$$\left[1 - N^2 - (N^4 - 6N^2 - Nk + 4)z + (N^4 - 5N^2 + 4)z^2\right]F_1(z) = 0.$$

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The corresponding $F_2(z)$ may be obtained from the $F_1(z)$ solutions:

$$F_2(z) = -(N + k)z F_1'(z) + N^{-1}(1 - z)^{-1} \left[1 - N^2 + (N^2 - 2)z\right] F_1(z). \tag{A10}$$

Upon the change of variables

$$F_1(z) = z^{-N/(N+k)}[z(1 - z)]^{1/N(N+k)}G_1(z) \tag{A11}$$

one obtains an equation of the hypergeometric form:

$$z(1 - z)G_1'' + [\gamma - (\alpha + \beta + 1)z] G_1' - \alpha\beta G_1 = 0. \tag{A12}$$

where $\alpha = -1/(N + k)$, $\beta = 1/(N + k)$ and $\gamma = k/(N + k)$. For non-integer values of γ the solutions of these equations are given by Knizhnik and Zamolodchikov — see equations (4.10a) and (4.10b) of [53]. For integer values of γ, however, the solution involves *logarithms* — the roots of the indicial equation corresponding to (A12) (namely 0 and $1 - \gamma$) differ by an integer. Of particular importance in the study of the disordered Dirac fermion problem are the cases $N \to 0$ ($\gamma = 1$) appearing in the replica treatment (section III A) and $k = -2N$ ($\gamma = 2$) appearing in the strong disorder treatment (section III B). We consider both of these cases below.

1. The $\widehat{u}(N)_k$ WZNW Model as $N \to 0$

As may be seen from equation (A4) the conformal dimensions of the $\widehat{su}(N)_k$ WZNW model *diverge* as $N \to 0$, as do some of the prefactors in the chiral blocks (A11); the $N \to 0$ limit of the $\widehat{su}(N)_k$ WZNW model *alone* fails to yield a well defined CFT. However, the replica approach to the random Dirac fermion problem instructs us to consider the $\widehat{u}(N)_k$ WZNW model. As we shall demonstrate below, the $\widehat{u}(N)_k$ WZNW model *does* have a well defined $N \to 0$ limit.

An arbitrary element u, of the group U(N), may be expressed as an element of SU(N), g, multiplied by a phase: $u = e^{i\alpha\varphi}g$. Using the Polyakov–Wiegmann identity for the WZNW model [83]

$$W_k[ab] = W_k[a] + W_k[b] + \frac{k}{2\pi} \int d^2\xi \, \text{Tr}'(a^{-1}\bar{\partial}ab\partial b^{-1}) \tag{A13}$$

with $a = e^{i\alpha\varphi}$ and $b = g$ one obtains[5]

$$W_k[u] = W_k[g] + \frac{\alpha^2 Nk}{8\pi} \int d^2\xi \, \partial_\mu\varphi\partial^\mu\varphi \tag{A14}$$

---

[5]We have used the facts that the current $g\partial g^{-1}$ residing in the su(N) *algebra* is traceless, thereby eliminating the second term in (A13), that the WZNW term (3.4) vanishes for the U(1) element $e^{i\alpha\varphi}$, and that $\text{Tr}' \equiv 2\text{Tr}$.

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The $\widehat{u}(N)_k$ WZNW model is therefore seen to be the $\widehat{su}(N)_k$ WZNW model augmented by a free scalar field of *prescribed* normalization. The conformal dimension of the field $e^{i\alpha\varphi}$ is *fixed* by this normalization to be[6] $h_\alpha = 1/(2Nk)$. The conformal dimension of the composite field $u = e^{i\alpha\varphi}g$ is thus

$$h_{\widehat{u}(N)_k} = \frac{1}{2Nk} + \frac{N^2-1}{2N(N+k)} \xrightarrow{N\to 0} \frac{1}{2k^2} \tag{A15}$$

and is seen to have the finite replica limit $1/2k^2$. In addition, the (non-vanishing) holomorphic four-point correlation function of the U(1) fields $\mathcal{V}_\alpha = e^{i\alpha\varphi}$ is given by

$$\langle \mathcal{V}_\alpha(1)\mathcal{V}_{-\alpha}(2)\mathcal{V}_{-\alpha}(3)\mathcal{V}_\alpha(4)\rangle = \left(z_{14}z_{32}\right)^{-1/Nk}[z(1-z)]^{-1/Nk} \tag{A16}$$

The divergences occuring in the SU(N)$_k$ results (A2) and (A11) as $N \to 0$ are seen to be compensated by those of the U(1) phase (A16). In the replica ($N \to 0$) limit the hypergeometric equation (A12) reads

$$z(1-z)G_1'' + (1-z)G_1' + k^{-2}G_1 = 0 \tag{A17}$$

and one is able to find the $N \to 0$ limit of the $\widehat{u}(N)_k$ conformal blocks:

$$F_1^{(1)} = \gamma\ {}_2F_1[-\tfrac{1}{k}, \tfrac{1}{k}; 1; z] \tag{A18a}$$

$$F_1^{(2)} = \gamma\,(1-z)\ {}_2F_1[1 - \tfrac{1}{k}, 1 + \tfrac{1}{k}; 2; 1-z] \tag{A18b}$$

$$F_2^{(1)} = \gamma\,\frac{z}{k}\ {}_2F_1[1 - \tfrac{1}{k}, 1 + \tfrac{1}{k}; 2; z] \tag{A18c}$$

$$F_2^{(2)} = \gamma\, k\ {}_2F_1[-\tfrac{1}{k}, \tfrac{1}{k}; 1; 1-z] \tag{A18d}$$

where $\gamma = [z(1-z)]^{-1/k^2}$. We find it convenient to introduce generalizations of the complete elliptic integrals of the first and second kind [84]:

$$K_k(z) \equiv \frac{\pi}{k}\ {}_2F_1[1 - \tfrac{1}{k}, \tfrac{1}{k}; 1; z] \tag{A19a}$$

$$E_k(z) \equiv \frac{\pi}{k}\ {}_2F_1[-\tfrac{1}{k}, \tfrac{1}{k}; 1; z] \tag{A19b}$$

In terms of these functions (and rescaling) the $N \to 0$ $\widehat{u}(N)_k$ conformal blocks read:

$$F_1^{(1)} = \gamma\, E_k(z) \tag{A20a}$$

$$F_1^{(2)} = \gamma\, k\, [K_k(1-z) - E_k(1-z)] \tag{A20b}$$

$$F_2^{(1)} = \gamma\, [K_k(z) - E_k(z)] \tag{A20c}$$

$$F_2^{(2)} = \gamma\, k\, E_k(1-z) \tag{A20d}$$

---

[6]For a free scalar field governed by the action $S = \frac{1}{2}g\int d^2\xi\ \partial_\mu\varphi\partial^\mu\varphi$, the field $e^{i\alpha\varphi}$ has $h_\alpha = \alpha^2/8\pi g$ — see for example equations (5.73) and (6.60) of [17].

The full $U(N) \times U(N)$ invariant correlation function is built from single-valued combinations of these conformal blocks; it is straightforward to see that one can only have

$$F_{ij}(z, \bar{z}) = X_{12} \left[F_i^{(1)}(z) F_j^{(2)}(\bar{z}) + F_i^{(2)}(z) F_j^{(1)}(\bar{z}) \right] \tag{A21}$$

Substituting the explicit forms (A20) into (A21) one obtains

$$F_{11} = X_{12} \, k \, |\gamma|^2 \left[E_k (\tilde{\bar{K}}_k - \tilde{\bar{E}}_k) + \bar{E}_k (\tilde{K}_k - \tilde{E}_k) \right] \tag{A22a}$$

$$F_{12} = X_{12} \, k \, |\gamma|^2 \left[E_k \tilde{\bar{E}}_k + (\tilde{K}_k - \tilde{E}_k)(\bar{K}_k - \bar{E}_k) \right] \tag{A22b}$$

$$F_{21} = X_{12} \, k \, |\gamma|^2 \left[\bar{E}_k \tilde{E}_k + (\tilde{\bar{K}}_k - \tilde{\bar{E}}_k)(K_k - E_k) \right] \tag{A22c}$$

$$F_{22} = X_{12} \, k \, |\gamma|^2 \left[\tilde{\bar{E}}_k (\bar{K}_k - \bar{E}_k) + \tilde{E}_k (K_k - E_k) \right] \tag{A22d}$$

where we have adopted the notation that $\tilde{f}(z) \equiv f(1 - z)$, $\bar{f}(z) \equiv f(\bar{z})$ and $\tilde{\bar{f}}(z) \equiv f(1 - \bar{z})$ for an arbitrary function f.

2. The $\widehat{su}(N)_{-2N}$ WZNW Model

As may be seen from equations (A11) and (A12) with $k = -2N$, the $F_1(z)$ chiral blocks for the $\widehat{su}(N)_{-2N}$ WZNW Model are given by

$$F_1(z) = z \left[z(1 - z) \right]^{-1/N^2} G_1(z) \tag{A23}$$

where G_1 satisfies the hypergeometric equation with $\alpha = 1/N$, $\beta = -1/N$, $\gamma = 2$:

$$z(1 - z)G_1'' + (2 - z)G_1' + N^{-2}G_1 = 0 \tag{A24}$$

Solving this equation and applying the relation (A10) enables one to obtain the full set of $\widehat{su}(N)_{-2N}$ chiral blocks:[7]

$$F_1^{(1)} = \Lambda z \, {}_2F_1[-\tfrac{1}{N}, \tfrac{1}{N}; 2; z] \tag{A25a}$$

$$F_1^{(2)} = \Lambda (1 - z)^2 \, {}_2F_1[1 - \tfrac{1}{N}, 1 + \tfrac{1}{N}; 3; 1 - z] \tag{A25b}$$

$$F_2^{(1)} = -\frac{\Lambda z^2}{2N} \, {}_2F_1[1 - \tfrac{1}{N}, 1 + \tfrac{1}{N}; 3; z] \tag{A25c}$$

$$F_2^{(2)} = -2N\Lambda (1 - z) \, {}_2F_1[-\tfrac{1}{N}, \tfrac{1}{N}; 2; 1 - z] \tag{A25d}$$

---

[7]We note that (A25b) may be obtained by substituting $G_1(z) = z^{-1}(1 - z)^2 H_1(z)$ into (A24) and replacing z by $1 - z$ — this yields a hypergeometric equation. The result (A25c) may be obtained by straightforward application of (A10) to (A25a). In deriving (A25d) one may utilize the result $(N^2 - 1)z(1 - z) \, {}_2F_1[2 - \tfrac{1}{N}, 2 + \tfrac{1}{N}; 4; z] + 3N^2(2 - z) \, {}_2F_1[1 - \tfrac{1}{N}, 1 + \tfrac{1}{N}; 3; z] = 6N^2 \, {}_2F_1[-\tfrac{1}{N}, \tfrac{1}{N}; 2; z]$ which is easily verified by means of the Gauss recursion relations.

where $\Lambda = [z(1 - z)]^{-1/N^2}$. In terms of generalized elliptic integrals the $\widehat{su}(N)_{-2N}$ chiral blocks read:

$$F_1^{(1)} = c\Lambda \left[[1 + (N - 1)z] E_N - (1 - z)K_N\right] \tag{A26a}$$

$$F_1^{(2)} = 2Nc\Lambda \left[[1 + (N - 1)z] \tilde{E}_N - Nz\tilde{K}_N\right] \tag{A26b}$$

$$F_2^{(1)} = -c\Lambda \left[[N - (N - 1)z] E_N - N(1 - z)K_N\right] \tag{A26c}$$

$$F_2^{(2)} = -2Nc\Lambda \left[[N - (N - 1)z] \tilde{E}_N - z\tilde{K}_N\right] \tag{A26d}$$

where $c = N^2/[\pi(N^2 - 1)]$. We have used the Gauss recursion formula GR 9.137(13) [85]

$$\gamma[\alpha - (\gamma - \beta)z] \, _2F_1[\alpha, \beta; \gamma; z] - \alpha\gamma(1 - z) \, _2F_1[\alpha + 1, \beta; \gamma; z] \\ + (\gamma - \alpha)(\gamma - \beta)z \, _2F_1[\alpha, \beta; \gamma + 1; z] = 0 \tag{A27}$$

with $\alpha = -1/N$, $\beta = 1/N$, $\gamma = 1$ to obtain (A26a), and GR 9.137(6) [85]

$$\gamma(\gamma + 1) \, _2F_1[\alpha, \beta; \gamma; z] - \gamma(\gamma + 1) \, _2F_1[\alpha, \beta; \gamma + 1; z] \\ - \alpha\beta z \, _2F_1[\alpha + 1, \beta + 1; \gamma + 2; z] = 0 \tag{A28}$$

with $\alpha = -1/N$, $\beta = 1/N$, $\gamma = 1$ to obtain (A26b). One may obtain (A26c) and (A26d) by noting that $F_2^{(1)}(z) = -F_1^{(2)}(1 - z)/2N$ and $F_2^{(2)}(z) = -2NF_1^{(1)}(1 - z)$ as follows from (A25). In particular we note that

$$N^2 F_1^{(1)} + N F_2^{(1)} = c'\Lambda z E_N \tag{A29a}$$

$$N F_1^{(1)} + N^2 F_2^{(1)} = c'\Lambda(1 - z)[K_N - E_N] \tag{A29b}$$

$$N^2 F_1^{(2)} + N F_2^{(2)} = -2Nc'\Lambda z[\tilde{K}_N - \tilde{E}_N] \tag{A29c}$$

$$N F_1^{(2)} + N^2 F_2^{(2)} = -2Nc'\Lambda(1 - z)\tilde{E}_N \tag{A29d}$$

where $c' \equiv cN(N^2 - 1)$ is simply another constant.

APPENDIX B: THE $\widehat{osp}(2|2)_k$ WZNW MODEL

1. Representations of $osp(2|2)$

The Lie superalgebra $osp(2|2) \sim spl(2|1)$ consists of four bosonic generators Q_3, Q_+, Q_-, B, and four fermionic generators W_+, W_-, V_+, V_-. The bosonic subalgebra is $sl(2) \oplus u(1)$:

$$[Q_3, Q_\pm] = \pm Q_\pm, \quad [Q_+, Q_-] = 2Q_3, \quad [B, Q_\pm] = 0, \quad [B, Q_3] = 0 \tag{B1}$$

and the eigenvalues of Q_3 and B (called isospin and baryon number respectively) are used to classify the basis states of finite-dimensional representations [72, 86]. A representation $[b, q]$ contains at most four multiplets of states:

$$|b, q\rangle, \quad |b + \tfrac{1}{2}, q - \tfrac{1}{2}\rangle, \quad |b - \tfrac{1}{2}, q - \tfrac{1}{2}\rangle, \quad |b, q - 1\rangle, \tag{B2}$$

the states within a given multiplet $|b, q\rangle$ being labelled by their third component of isospin $|b, q, q_3\rangle$. In particular the $[0, \frac{1}{2}]$ representation is four-dimensional and contains a doublet and two singlets (the multiplet $|b, q - 1\rangle$ being absent):

$$|0, \tfrac{1}{2}, \tfrac{1}{2}\rangle, \quad |0, \tfrac{1}{2}, -\tfrac{1}{2}\rangle, \quad |\tfrac{1}{2}, 0, 0\rangle, \quad |-\tfrac{1}{2}, 0, 0\rangle. \tag{B3}$$

Following [59] we denote these states as $|1\rangle$, $|4\rangle$, $|3\rangle$ and $|2\rangle$ respectively. In addition to their isospin and baryon number, the basis states carry a grading denoted by $\varepsilon = \pm 1$. States $|1\rangle$ and $|4\rangle$ are even/bosonic and have $\varepsilon = 0$, whilst states $|2\rangle$ and $|3\rangle$ are odd/fermionic and have $\varepsilon = 1$. On this basis of states one may construct 4×4 matrix representation $[0, \frac{1}{2}]$ of the generators of osp(2|2).

2. Knizhhnik–Zamolodchikov Equations

Let us study the four-point function of the supersymmetric Q-matrix:

$$\mathcal{F}^{\alpha,\bar{\alpha}}(z_i, \bar{z}_i) = \langle Q^{\alpha_1,\bar{\alpha}_1}(z_1, \bar{z}_1) Q^{\alpha_2,\bar{\alpha}_2}(z_2, \bar{z}_2) Q^{\alpha_1,\bar{\alpha}_3}(z_3, \bar{z}_3) Q^{\alpha_4,\bar{\alpha}_4}(z_4, \bar{z}_4) \rangle, \tag{B4}$$

where on the left hand side we use the symbol α to denote the ordered sequence of indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Global conformal invariance restricts this correlation function to have the form

$$\mathcal{F}^{\alpha,\bar{\alpha}}(z_i, \bar{z}_i) = (z_{14} z_{23} \bar{z}_{14} \bar{z}_{23})^{-2h} F^{\alpha,\bar{\alpha}}(z, \bar{z}) \tag{B5}$$

where z and $\bar{z}$ are the anharmonic ratios

$$z = \frac{z_{12} z_{34}}{z_{14} z_{32}}, \quad \bar{z} = \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{14} \bar{z}_{32}} \tag{B6}$$

and the conformal dimension h of the field ϕ is

$$h = \frac{1}{4 - 2k} \tag{B7}$$

The correlation function (B5) has the osp(2|2)×osp(2|2) invariant decomposition

$$F^{\alpha,\bar{\alpha}}(z, \bar{z}) = \sum_{ij=1}^{3} I_i^{\alpha} \bar{I}_j^{\bar{\alpha}} F_{ij}(z, \bar{z}) \tag{B8}$$

The tensors I and $\bar{I}$ are given in appendix A of [59]:

$$I_1 = (1144) + (1234)4\epsilon\gamma + (1324)4\epsilon\gamma - (1414) + (2143)4\epsilon\gamma$$
$$+ (2233)16\epsilon^2\gamma^2 + (2323)16\epsilon^2\gamma^2 - (2413)4\epsilon\gamma + (3142)4\epsilon\gamma + (3232)16\epsilon^2\gamma^2$$
$$+ (3322)16\epsilon^2\gamma^2 - (3412)4\epsilon\gamma - (4141) - (4231)4\epsilon\gamma - (4321)4\epsilon\gamma + (4411) \qquad \text{(B9a)}$$

$$I_2 = (1234)4\epsilon\gamma - (1243)4\epsilon\gamma + (1324)4\epsilon\gamma - (1342)4\epsilon\gamma$$
$$- (1414) + (1441) - (2134)4\epsilon\gamma + (2143)4\epsilon\gamma + (2233)32\epsilon^2\gamma^2 + (2323)16\epsilon^2\gamma^2$$
$$+ (2332)16\epsilon^2\gamma^2 - (2413)4\epsilon\gamma + (2431)4\epsilon\gamma - (3124)4\epsilon\gamma + (3142)4\epsilon\gamma$$
$$+ (3223)16\epsilon^2\gamma^2 + (3232)16\epsilon^2\gamma^2 + (3322)32\epsilon^2\gamma^2 - (3412)4\epsilon\gamma + (3421)4\epsilon\gamma$$
$$+ (4114) - (4141) + (4213)4\epsilon\gamma - (4231)4\epsilon\gamma + (4312)4\epsilon\gamma - (4321)4\epsilon\gamma \qquad \text{(B9b)}$$

$$I_3 = (1234) - (1243) + (1324) - (1342) + (1423) + (1432) - (2134) + (2143)$$
$$+ (2233)8\epsilon\gamma + (2314) + (2323)8\epsilon\gamma + (2332)8\epsilon\gamma - (2341) - (2413) + (2431)$$
$$- (3124) + (3142) + (3214) + (3223)8\epsilon\gamma + (3232)8\epsilon\gamma - (3241) + (3322)8\epsilon\gamma$$
$$- (3412) + (3421) - (4123) - (4132) + (4213) - (4231) + (4312) - (4321) \qquad \text{(B9c)}$$

and the nine scalar functions F_{ij} satisfy the coupled first-order differential equations [59]

$$x\frac{dF}{dz} = \left[\frac{1}{z}P + \frac{1}{z-1}Q\right]F, \quad \text{where} \quad F = \begin{pmatrix} F_{1j} \\ F_{2j} \\ F_{3j} \end{pmatrix} \quad \forall j \qquad \text{(B10)}$$

There are similar equations for the antiholomorphic dependence. Suppressing the antiholomorphic index j from the functions $F_{1j} \cdots F_{3j}$, one may reduce this first-order matrix differential equation to the following set of equations [59]

$$x^3z^3(1-z)^3\frac{d^3F_3(z)}{dz^3} + x^2(1+2x)z^2(1-z)^2(1-2z)\frac{d^2F_3(z)}{dz^2} +$$
$$xz(1-z)[-1-x+2xz-2x(2+x)z(1-z)]\frac{dF_3(z)}{dz} + \qquad \text{(B11a)}$$
$$[-1-x+2z+2xz(1-z)]F_3(z) = 0$$

$$F_2(z) = -\frac{1}{4\epsilon\gamma xz(1-z)}\left[x^2D^2F_3(z) + 2x(1-z)DF_3(z) + (1-2z)F_3(z)\right] \qquad \text{(B11b)}$$

$$F_1(z) = \frac{1}{4\epsilon\gamma}\left[xDF_3(z) - F_3(z)\right] + (z-2)F_2(z) \qquad \text{(B11c)}$$

where $D = z(1-z)d/dz$. Equation (B11a) has three independent solutions which we denote $F_3^{(a)}$ where $a = 1,2,3$. Equations (B11b) and (B11c) yield the corresponding solutions $F_2^{(a)}$ and $F_1^{(a)}$. The nine scalar functions F_{ij} appearing in equation (B8) may be expressed as a linear combination of these nine functions (the so-called chiral/current blocks.)

$$F_{ij}(z,\bar{z}) = \sum_{a,b=1}^{3} X_{ab}F_i^{(a)}(z)F_j^{(b)}(\bar{z}) \qquad \text{(B12)}$$

The values of the coefficients X_{ab} are determined by single valuedness (monodromy invariance) and crossing symmetry to be discussed later.

a. Factorization at Level $k = 1$

At level $k = 1$ ($x = 1$) the Knizhnik–Zamolodchikov equation (B11a) takes the form

$$z^3(1 - z)F_3''' + 3z^2(1 - 2z)F_3'' - 2z(1 + 3z)F_3' - 2F_3 = 0 \qquad \text{(B13)}$$

This may be factorized to read

$$\mathcal{D}^{(2)}\mathcal{D}^{(1)}F_3 = 0 \qquad \text{(B14)}$$

where

$$\mathcal{D}^{(1)} = z\frac{d}{dz} + 1 \qquad \text{(B15)}$$

$$\mathcal{D}^{(2)} = z^2(1 - z)\frac{d^2}{dz^2} - 3z^2\frac{d}{dz} - 2 \qquad \text{(B16)}$$

It is straightforward to see that the equation $\mathcal{D}^{(1)}F_3 = 0$, and thus (B14) has the solution

$$F_3^{(1)} = \frac{1}{z} \qquad \text{(B17)}$$

Applying the formulae (B11b) and (B11c) to this single solution of the Knizhnik–Zamolodchikov equation (B14) one obtains:

$$F_2^{(1)} = \frac{2z - 1}{4\epsilon\gamma z(1 - z)}, \qquad F_1^{(1)} = \frac{z - 2}{4\epsilon\gamma(1 - z)} \qquad \text{(B18)}$$

Analysis of the full set of solutions of (B14) reveals that it is only these blocks which enter the physical correlator - a fact which is intimately related to the factorization of the Knizhhnik–Zamolodchikov equation (B14). In the $su(2)$ WZNW model the factorization properties of the differential operators are related to a finite closure of the underlying operator algebra [87].

b. Factorization at Level $k = -2$

At level $k = -2$ ($x = 4$) the differential equation (B11a) takes the form

$$[4z(1 - z)]^3 F_3''' + 9[4z(1 - z)]^2(1 - 2z)F_3'' +$$
$$4z(1 - z)[-5 + 8z - 48z(1 - z)]F_3' + \qquad \text{(B19)}$$
$$[-5 + 2z + 8z(1 - z)]F_3 = 0$$

This equation may be factorized to read

$$\mathcal{D}^{(1)}\mathcal{D}^{(2)}F_3 = 0 \qquad \text{(B20)}$$

where this time the differential operators $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ take the form

$$\mathcal{D}^{(1)} = 4z(1-z)\frac{d}{dz} - (5-6z) \tag{B21}$$

$$\mathcal{D}^{(2)} = [4z(1-z)]^2\frac{d^2}{dz^2} + 8z(1-z)(3-4z)\frac{d}{dz} + (1-4z) \tag{B22}$$

In the light of the factorization which occured at $k = 1$ one might expect that the subset of solutions obtained from $\mathcal{D}^{(2)}F_3 = 0$, which are closed under analytic continuation on the Riemann sphere, might serve as a reduced basis on which to perform the conformal bootstrap. This indeed turns out to be the case. It is easily seen that the equation $\mathcal{D}^{(2)}F_3 = 0$, and therefore the full Knizhnik–Zamolodchikov equation (B20) admits the two solutions[8]

$$F_3^{(1)}(z) = \frac{E(z)}{[z(1-z)]^{1/4}} \qquad F_3^{(2)}(z) = \frac{E(1-z) - K(1-z)}{[z(1-z)]^{1/4}} \tag{B23}$$

where $K(z)$ is the complete elliptic integral of the first kind and $E(z)$ is the complete elliptic integral of the second kind:[9]

$$E(z) = \int_0^1 \frac{\sqrt{1 - zx^2}}{\sqrt{1 - x^2}}dx \qquad K(z) = \int_0^1 \frac{1}{\sqrt{(1 - x^2)(1 - zx^2)}}dx \tag{B24}$$

The representation (B23) is particularly useful owing to the very simple manner in which the elliptic integrals behave under differentiation with respect to the parameter z:

$$\frac{dE(z)}{dz} = \frac{E(z) - K(z)}{2z} \qquad \frac{dK(z)}{dz} = \frac{E(z) - (1-z)K(z)}{2z(1-z)} \tag{B25}$$

Applying (B11b) and (B11c) to these solutions yields:

$$F_2^{(1)} = -\frac{K(z)}{4\epsilon\gamma[z(1-z)]^{1/4}} \qquad F_1^{(1)} = \frac{zK(z)}{4\epsilon\gamma[z(1-z)]^{1/4}} \tag{B26}$$

$$F_2^{(2)} = \frac{K(1-z)}{4\epsilon\gamma[z(1-z)]^{1/4}} \qquad F_1^{(2)} = -\frac{zK(1-z)}{4\epsilon\gamma[z(1-z)]^{1/4}} \tag{B27}$$

As we shall subsequently demonstrate, one may satisfy the demands of single-valuedness and crossing symmetry on the subspace of functions (B23), (B26) and (B27). Once again this is intimately connected with the factorization of the Knizhnik–Zamolodchikov equation (B20). Having found closed form expressions for the chiral blocks, one must now construct

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[8]Upon the change of variables $F_3 = [z(1-z)]^{-1/4}H(z)$, the equation $\mathcal{D}^{(2)}F_3 = 0$ reduces to the canonical form of the hypergeometric equation $z(1-z)H'' + [c - (a+b+1)z]H' - abH = 0$ with $a = -1/2$, $b = 1/2$, $c = 1$. This has solutions $H^{(1)} = {}_2F_1\left[-\frac{1}{2}, \frac{1}{2}; 1; z\right]$ and $H^{(2)} = (1-z)\,{}_2F_1\left[\frac{1}{2}, \frac{3}{2}; 2; 1-z\right]$. Using the well known fact that ${}_2F_1\left[-\frac{1}{2}, \frac{1}{2}; 1; z\right] = \frac{2}{\pi}E(z)$ together with it's derivative, the result follows.

[9]Note that many texts on the theory of elliptic integrals denote the parameter z by k^2 - the so-called modulus. This is purely a mater of convention.

the physical correlation functions to be single-valued on the whole Riemann sphere. It is enough to ensure this property at the two singular points $z = 0$ and $z = 1$. The blocks $F_i^{(1)}(z)$ are regular at $z = 0$ and logarithmic at $z = 1$, whilst the blocks $F_i^{(2)}(z)$ are logarithmic at $z = 0$ and regular at $z = 1$. It is straightforward to see that in this subspace of functions one can only have[10]

$$F_{ij}(z, \bar{z}) = X_{12} \left[F_i^{(1)}(z) F_j^{(2)}(\bar{z}) + F_i^{(2)}(z) F_j^{(1)}(\bar{z}) \right] \tag{B28}$$

One may gather the explicit expressions for the F_{ij} at level $k = -2$ in the Hermitian matrix:

$$F_{ij} = \begin{pmatrix} |z|^2 F_{22} & -z F_{22} & -z F_{23} \\ -\bar{z} F_{22} & F_{22} & F_{23} \\ -\bar{z} \bar{F}_{23} & \bar{F}_{23} & F_{33} \end{pmatrix} \tag{B29}$$

in which we have singled out the elements

$$F_{22} = -\Lambda \left[K \tilde{\bar{K}} + \bar{K} \tilde{K} \right] \tag{B30}$$

$$F_{23} = 4\epsilon\gamma\Lambda \left[K \tilde{\bar{K}} + \tilde{K} \bar{E} - K \tilde{\bar{E}} \right] \tag{B31}$$

$$F_{33} = (4\epsilon\gamma)^2 \Lambda \left[E \tilde{\bar{E}} - E \tilde{\bar{K}} + \bar{E} \tilde{E} - \bar{E} \tilde{K} \right] \tag{B32}$$

where

$$\Lambda = \frac{X_{12}}{(4\epsilon\gamma)^2 |z(1 - z)|^{1/2}} \tag{B33}$$

and where we have adopted the notation that $\tilde{f}(z) \equiv f(1 - z)$, $\bar{f}(z) \equiv f(\bar{z})$ and $\tilde{\bar{f}}(z) \equiv f(1 - \bar{z})$ for an arbitrary function f. One may also demonstrate that this combination is consistent with the crossing symmetry constraints on the correlator:

$$F^{\alpha, \bar{\alpha}}(z, \bar{z}) = \tilde{\mathcal{P}} \tilde{\bar{\mathcal{P}}} F^{\tilde{\alpha}, \tilde{\bar{\alpha}}}(1 - z, 1 - \bar{z}), \qquad F^{\alpha, \bar{\alpha}}(z, \bar{z}) = z^{-2\Delta} \bar{z}^{-2\Delta} \hat{\mathcal{P}} \hat{\bar{\mathcal{P}}} F^{\hat{\alpha}, \hat{\bar{\alpha}}}(1/z, 1/\bar{z}) \tag{B34}$$

where $\tilde{\alpha}$ denotes the permuted sequence of indices $\alpha_1, \alpha_3, \alpha_2, \alpha_4$, $\hat{\alpha}$ denotes the sequence $\alpha_1, \alpha_4, \alpha_3, \alpha_2$, and $\mathcal{P}$ denotes the parity of the permutation:

$$\tilde{\mathcal{P}} = (-1)^{\varepsilon_{\alpha_2} \varepsilon_{\alpha_3}}, \qquad \hat{\mathcal{P}} = (-1)^{\varepsilon_{\alpha_2}(\varepsilon_{\alpha_3} + \varepsilon_{\alpha_4}) + \varepsilon_{\alpha_3} \varepsilon_{\alpha_4}} \tag{B35}$$

The proof of this requires the use of the analytic continuation formulae of the elliptic integrals and is presented in appendix B 4.

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[10]A detailed proof of this statement in terms of monodromy matrices is given in appendix B 3.

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3. Monodromy Invariance

A monodromy transformation of a function of z consists in letting z circulate around some other point (typically a singular point). We define

$$\mathcal{C}_0\, F(z,\bar{z}) = \lim_{t\to 1^-} F(ze^{2i\pi t}, \bar{z}e^{-2i\pi t}) \tag{B36}$$

$$\mathcal{C}_1\, F(z,\bar{z}) = \lim_{t\to 1^-} F(1+(z-1)e^{2i\pi t}, 1+(\bar{z}-1)e^{-2i\pi t}) \tag{B37}$$

Using the standard analytic continuation formulae for the hypergeometric series, it is easily seen that the elliptic integrals have the following nontrivial monodromy properties:

$$\mathcal{C}_0 K(1-z) = K(1-z) - 2iK(z) \tag{B38}$$

$$\mathcal{C}_0 E(1-z) = E(1-z) + 2i[E(z) - K(z)] \tag{B39}$$

$$\mathcal{C}_1 K(z) = K(z) - 2iK(1-z) \tag{B40}$$

$$\mathcal{C}_1 E(z) = E(z) + 2i[E(1-z) - K(1-z)] \tag{B41}$$

together with the trivial transformations $\mathcal{C}_0 K(z) = K(z)$, $\mathcal{C}_0 E(z) = E(z)$, $\mathcal{C}_1 K(1-z) = K(1-z)$, $\mathcal{C}_1 E(1-z) = E(1-z)$. Using these results it is straightforward to see that

$$\mathcal{C}_0\, F_i^{(a)}(z) = (g_0)_{ab}\, F_i^{(b)}(z), \quad i = 1,2,3 \tag{B42}$$

$$\mathcal{C}_1\, F_i^{(a)}(z) = (g_1)_{ab}\, F_i^{(b)}(z), \quad i = 1,2,3 \tag{B43}$$

where (on this reduced subspace) the matrices g_0 and g_1 are given by

$$g_0 = \begin{pmatrix} -i & 0 \\ 2 & -i \end{pmatrix}, \quad g_1 = \begin{pmatrix} -i & 2 \\ 0 & -i \end{pmatrix}. \tag{B44}$$

Under the monodromy transformation $\mathcal{C}_0$, the combination

$$F_{ij}(z,\bar{z}) = \sum_{a,b=1}^{2} X_{ab} F_i^{(a)}(z) F_j^{(b)}(\bar{z}) \tag{B45}$$

transforms in the following manner

$$\mathcal{C}_0 F_{ij}(z,\bar{z}) = F_i^{(1)}(z)F_j^{(1)}(\bar{z})\left[X_{11} - 2i(X_{12} - X_{21}) + 4X_{22}\right] + F_i^{(2)}(z)F_j^{(2)}(\bar{z})\left[X_{22}\right]$$
$$+ F_i^{(1)}(z)F_j^{(2)}(\bar{z})\left[X_{12} + 2iX_{22}\right] + F_i^{(2)}(z)F_j^{(1)}(\bar{z})\left[X_{21} - 2iX_{22}\right]$$

Invariance under the monodromy transformation $\mathcal{C}_0$ thus requires $X_{12} = X_{21}$, and $X_{22} = 0$. That is to say

$$F_{ij}(z,\bar{z}) = X_{11} F_i^{(1)}(z)F_j^{(1)}(\bar{z}) + X_{12}\left[F_i^{(1)}(z)F_j^{(2)}(\bar{z}) + F_i^{(2)}(z)F_j^{(1)}(\bar{z})\right]$$

Under the monodromy transformation $\mathcal{C}_1$ this simplified function transforms as

$$\mathcal{C}_1 F_{ij}(z,\bar{z}) = F_i^{(1)}(z)F_j^{(1)}(\bar{z})\left[X_{11}\right] + F_i^{(1)}(z)F_j^{(2)}(\bar{z})\left[X_{12} - 2iX_{11}\right]$$
$$+ F_i^{(2)}(z)F_j^{(1)}(\bar{z})\left[X_{12} + 2iX_{11}\right] + F_i^{(2)}(z)F_j^{(2)}(\bar{z})\left[4X_{11}\right]$$

Invariance under the monodromy transformation $\mathcal{C}_1$ therefore imposes the additional constraint $X_{11} = 0$. Hence monodromy invariance restricts $F_{ij}(z, \bar{z})$ to have the form

$$F_{ij}(z, \bar{z}) = X_{12} \left[F_i^{(1)}(z) F_j^{(2)}(\bar{z}) + F_i^{(2)}(z) F_j^{(1)}(\bar{z}) \right].$$ (B46)

as stated in the text.

4. Crossing Symmetry

a. Invariance under $z \to 1 - z$

Crossing symmetry requires that

$$F^{\alpha, \bar{\alpha}}(z, \bar{z}) = \tilde{\mathcal{P}} \tilde{\bar{\mathcal{P}}} F^{\tilde{\alpha}, \tilde{\bar{\alpha}}}(1 - z, 1 - \bar{z})$$ (B47)

where $\boldsymbol{\alpha}$ denotes the sequence of indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, $\tilde{\alpha}$ denotes the permuted sequence of indices $\alpha_1, \alpha_3, \alpha_2, \alpha_4$, $\tilde{\mathcal{P}} = (-1)^{\varepsilon_{\alpha_2} \varepsilon_{\alpha_3}}$ denotes the parity of the interchange in the holomorphic sector (ε_α is 0 for bosons and 1 for fermions.) Introducing the following tensor [59]

$$J_i^\alpha = \tilde{\mathcal{P}} I_i^{\tilde{\alpha}}$$ (B48)

the crossing symmetry constraint may be written

$$\sum_{i,j=1}^{3} I_i^\alpha \bar{I}_j^{\bar{\alpha}} F_{ij}(z, \bar{z}) = \sum_{i,j=1}^{3} J_i^\alpha \bar{J}_j^{\bar{\alpha}} F_{ij}(1 - z, 1 - \bar{z})$$ (B49)

The tensor J admits the following decomposition [59]

$$J_i^\alpha = C_1^{ij} I_j^\alpha \quad C_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -4\epsilon\gamma \\ 0 & 0 & -1 \end{pmatrix}$$ (B50)

Substituting this decomposition into equation (B49) and equating the coefficients of $I_i \bar{I}_j$ on both sides, one finds the following nine identities which must be satisfied by the $F_{ij}(z, \bar{z})$ if this crossing symmetry is to be satisfied. Denoting $F_{ij}(z, \bar{z})$ by F_{ij}, and $F_{ij}(1 - z, 1 - \bar{z})$ by $\tilde{F}_{ij}$ these are as follows:

$$F_{11} = \tilde{F}_{11} + \tilde{F}_{12} + \tilde{F}_{21} + \tilde{F}_{22}$$ (B51a)

$$F_{12} = -\tilde{F}_{12} - \tilde{F}_{22}$$ (B51b)

$$F_{13} = 4\epsilon\gamma(\tilde{F}_{12} + \tilde{F}_{22}) + \tilde{F}_{13} + \tilde{F}_{23}$$ (B51c)

$$F_{21} = -\tilde{F}_{21} - \tilde{F}_{22}$$ (B51d)

$$F_{22} = \tilde{F}_{22}$$ (B51e)

$$F_{23} = -4\epsilon\gamma \tilde{F}_{22} - \tilde{F}_{23}$$ (B51f)

$$F_{31} = 4\epsilon\gamma(\tilde{F}_{21} + \tilde{F}_{22}) + \tilde{F}_{31} + \tilde{F}_{32}$$ (B51g)

$$F_{32} = -4\epsilon\gamma \tilde{F}_{22} - \tilde{F}_{32}$$ (B51h)

$$F_{33} = 4\epsilon\gamma(4\epsilon\gamma \tilde{F}_{22} + \tilde{F}_{23} + \tilde{F}_{32}) + \tilde{F}_{33}$$ (B51i)

It is straightforward to show that these relations are indeed satisfied.

Crossing symmetry requires that

$$F^{\alpha,\bar{\alpha}}(z,\bar{z}) = z^{-2\Delta}\bar{z}^{-2\Delta}\hat{\mathcal{P}}\hat{\bar{\mathcal{P}}}F^{\alpha,\hat{\bar{\alpha}}}(1/z,1/\bar{z}) \tag{B52}$$

where α denotes the sequence of indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, $\hat{\alpha}$ denotes the permuted sequence of indices $\alpha_1, \alpha_4, \alpha_3, \alpha_2$, $\hat{\mathcal{P}} = (-1)^{\varepsilon_{\alpha_2}(\varepsilon_{\alpha_3}+\varepsilon_{\alpha_4})+\varepsilon_{\alpha_3}\varepsilon_{\alpha_4}}$ denotes the parity of the interchange in the holomorphic sector (ε_α is 0 for bosons and 1 for fermions.) Introducing the following tensor [59]

$$K_i^\alpha = \hat{\mathcal{P}}I_i^{\hat{\alpha}} \tag{B53}$$

the crossing symmetry constraint may be written

$$\sum_{i,j=1}^{3} I_i^\alpha \bar{I}_j^\alpha F_{ij}(z,\bar{z}) = z^{-1/x}\bar{z}^{-1/x}\sum_{i,j=1}^{3} K_i^\alpha \bar{K}_j^\alpha F_{ij}(1/z,1/\bar{z}) \tag{B54}$$

The tensor K admits the following decomposition [59]

$$K_i^\alpha = C_2^{ij}I_j^\alpha \quad C_2 = \begin{pmatrix} 0 & 1 & -4\epsilon\gamma \\ 1 & 0 & -4\epsilon\gamma \\ 0 & 0 & -1 \end{pmatrix} \tag{B55}$$

Substituting this decomposition into equation (B54) and equating the coefficients of $I_i\bar{I}_j$ on both sides, one finds the following nine identities which must be satisfied by the $F_{ij}(z,\bar{z})$ if this crossing symmetry is to be satisfied. Denoting $F_{ij}(z,\bar{z})$ by F_{ij}, and $F_{ij}(1/z,1/\bar{z})$ by $\hat{F}_{ij}$ these are as follows:

$$F_{11} = |z|^{-2/x}\hat{F}_{22} \tag{B56a}$$

$$F_{12} = |z|^{-2/x}\hat{F}_{21} \tag{B56b}$$

$$F_{13} = |z|^{-2/x}[-4\epsilon\gamma(\hat{F}_{21} + \hat{F}_{22}) - \hat{F}_{23}] \tag{B56c}$$

$$F_{21} = |z|^{-2/x}\hat{F}_{12} \tag{B56d}$$

$$F_{22} = |z|^{-2/x}\hat{F}_{11} \tag{B56e}$$

$$F_{23} = |z|^{-2/x}[-4\epsilon\gamma(\hat{F}_{11} + \hat{F}_{12}) - \hat{F}_{13}] \tag{B56f}$$

$$F_{31} = |z|^{-2/x}[-4\epsilon\gamma(\hat{F}_{12} + \hat{F}_{22}) - \hat{F}_{32}] \tag{B56g}$$

$$F_{32} = |z|^{-2/x}[-4\epsilon\gamma(\hat{F}_{11} + \hat{F}_{21}) - \hat{F}_{31}] \tag{B56h}$$

$$F_{33} = |z|^{-2/x}[16\epsilon^2\gamma^2(\hat{F}_{11} + \hat{F}_{12} + \hat{F}_{21} + \hat{F}_{22}) + $$
$$4\epsilon\gamma(\hat{F}_{13} + \hat{F}_{23} + \hat{F}_{31} + \hat{F}_{32}) + \hat{F}_{33}] \tag{B56i}$$

In order to demonstrate that these identities are satisfied we shall make use of the following rather simple transformation laws of the elliptic integrals under $z \to 1/z$:[11]

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[11] We note that replacing i by $-i$ in equations (B57a) and (B57b) changes the domain of validity of the transformation from $\Im mz < 0$ to $\Im mz > 0$. Since E and K appear together in the F_{ij} it is essential that their transformations be defined in the same region of the complex plane.

$$K(1/z) = z^{+1/2}\left[K(z) + iK(1-z)\right], \quad \Im m z < 0 \tag{B57a}$$

$$E(1/z) = z^{-1/2}\left[D(z) - iD(1-z)\right], \quad \Im m z < 0 \tag{B57b}$$

$$K(1 - 1/z) = z^{+1/2}K(1-z) \tag{B57c}$$

$$E(1 - 1/z) = z^{-1/2}E(1-z) \tag{B57d}$$

where

$$D(z) = E(z) - (1-z)K(z). \tag{B58}$$

These are easily obtained using the standard analytic continuation formulae for ordinary hypergeometric functions. Recalling the form of F_{22} appearing in equation (B30) namely

$$F_{22} = -\Lambda\left[K(1-z)K(\bar{z}) + K(z)K(1-\bar{z})\right] \tag{B59}$$

it is straightforward to see how it transforms under $z \to 1/z$:

$$\hat{F}_{22} = |z|^{5/2}F_{22} \tag{B60}$$

Since $F_{11} = |z|^2 F_{22}$ from equation (B29) it follows that that constraint (B56a) is satisfied. Replacing now z by $1/z$ in (B56a) one may infer the validity of (B56e). Further, since $F_{21} = -\bar{z}F_{22}$ from equation (B29) it follows by reciprocity that

$$\hat{F}_{21} = -\bar{z}^{-1}\hat{F}_{22}. \tag{B61}$$

Substituting for $\hat{F}_{22}$ using (B60) and using the fact that $F_{12} = -zF_{22}$ one obtains

$$\hat{F}_{21} = |z|^{1/2}F_{12} \tag{B62}$$

That is to say, constraint (B56b) is satisfied. By reciprocity we see that constraint (B56d) is also satisfied. Recalling now the form of F_{23} appearing in equation (B31) namely

$$F_{23} = 4\epsilon\gamma\Lambda\left[K\tilde{\bar{K}} + \tilde{K}\bar{E} - K\tilde{\bar{E}}\right] \tag{B63}$$

it is straightforward to see how it transforms under $z \to 1/z$:

$$\hat{F}_{23} = 4\epsilon\gamma|z|^{3/2}\Lambda\left[z^{1/2}\left(K + i\tilde{K}\right)\bar{z}^{1/2}\tilde{\bar{K}} + \right.$$
$$\left. z^{1/2}\tilde{K}\bar{z}^{-1/2}\left(\bar{D} + i\tilde{\bar{D}}\right) - z^{1/2}\left(K + i\tilde{K}\right)\bar{z}^{-1/2}\tilde{\bar{E}}\right] \tag{B64}$$

$$= 4\epsilon\gamma|z|^{1/2}\Lambda\left[|z|^2\left(K + i\tilde{K}\right)\tilde{\bar{K}} + z\tilde{K}\left(\bar{E} - (1-\bar{z})\bar{K}\right)\right.$$
$$\left. + iz\tilde{K}\left(\tilde{\bar{E}} - \bar{z}\tilde{\bar{K}}\right) - z\left(K + i\tilde{K}\right)\tilde{\bar{E}}\right] \tag{B65}$$

$$= 4\epsilon\gamma|z|^{1/2}\Lambda\left[|z|^2\left(K\tilde{\bar{K}} + \tilde{K}\bar{K}\right) + z\left(\tilde{K}\bar{E} - \tilde{K}\bar{K} - K\tilde{\bar{E}}\right)\right] \tag{B66}$$

$$= 4\epsilon\gamma|z|^{1/2}\Lambda\left[|z|^2\left(K\tilde{\bar{K}} + \tilde{K}\bar{K}\right) - z\left(K\tilde{\bar{K}} + \tilde{K}\bar{K}\right) + z\left(K\tilde{\bar{K}} + \tilde{K}\bar{E} - K\tilde{\bar{E}}\right)\right] \tag{B67}$$

$$= 4\epsilon\gamma|z|^{1/2}\left[-|z|^2 F_{22} + zF_{22} + \frac{z}{4\epsilon\gamma}F_{23}\right] \tag{B68}$$

$$= |z|^{1/2}\left[-4\epsilon\gamma(F_{11} + F_{12}) - F_{13}\right] \tag{B69}$$

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Replacing z by $1/z$ on both sides of this equation one obtains the relation (B56f). Taking now the complex conjugate of (B56f) and using the Hermiticity property $\bar{F}_{ij} = F_{ji}$, one obtains (B56h). Further, rearranging (B69) for F_{13} and replacing F_{11} and F_{12} using (B56a) and (B56b) respectively one obtains (B56c). Taking now the complex conjugate of (B56c) and using Hermiticity, one obtains (B56g). We consider finally how F_{33}, namely

$$F_{33} = (4\epsilon\gamma)^2\Lambda \left[E(\tilde{\bar{E}} - \tilde{K}) + \text{c.c.}\right] \tag{B70}$$

behaves under the transformation $z \to 1/z$:

$$\hat{F}_{33} = (4\epsilon\gamma)^2|z|^{3/2}\Lambda \left[z^{-1/2}(D - i\tilde{D})(\bar{z}^{-1/2}\tilde{\bar{E}} - \bar{z}^{1/2}\tilde{K}) + c.c.\right] \tag{B71}$$

$$= (4\epsilon\gamma)^2|z|^{1/2}\Lambda \left[(D - i\tilde{D})(\tilde{\bar{E}} - z\tilde{K}) + c.c.\right] \tag{B72}$$

$$= (4\epsilon\gamma)^2|z|^{1/2}\Lambda \left[E\tilde{\bar{E}} - (1-z)K\tilde{\bar{E}} - z\bar{E}\tilde{K} + \bar{z}(1-z)K\tilde{K} + c.c.\right] \tag{B73}$$

$$= (4\epsilon\gamma)^2|z|^{1/2}\Lambda \left[E\tilde{\bar{E}} - \bar{E}\tilde{K} + (1-z)(\bar{E}\tilde{K} - K\tilde{\bar{E}} + K\tilde{K}) - |1-z|^2K\tilde{K} + c.c.\right] \tag{B74}$$

$$= |z|^{1/2} \left[(4\epsilon\gamma)^2|1-z|^2F_{22} + 4\epsilon\gamma\left[(1-z)F_{23} + c.c.\right] + F_{33}\right] \tag{B75}$$

$$= |z|^{1/2} \left[(4\epsilon\gamma)^2(F_{11} + F_{12} + F_{21} + F_{22}) + 4\epsilon\gamma(F_{13} + F_{23} + F_{31} + F_{32}) + F_{33}\right] \tag{B76}$$

which may be seen to be the relation (B56i) with z replaced by $1/z$. This completes our proof of crossing symmetry in the reduced subspace.

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Applications to disorder.

General points.

1. Non-unitarity.

Follows from the fact that the corresponding field theories have trivial partition function

$$Z = 1$$

In SUSY formalism it is particularly clear that there are states with negative norms.

Each energy level (or rather transfer matrix eigenvalue) is degenerate —

at: the # of bosonic states = # of fermionic states in the multiplet

$$Z = 1 + \sum [<b|b> - <f|f>] e^{-\beta \mathcal{E}_n} = 1$$

$$= Str \, e^{-\beta H}$$

negative norms.

2. Non-compactness

   – this property is related to multifractality of wave functions – the property ubiquitous in models of disorder.

Applications to Disorder

General distinctive properties

1. Non-unitarity $\longrightarrow$ $Z = 1$

2. Non-compactness $\longrightarrow$ multy fractality

Distinct properties of disorder critical points.

1. Presence of logarithms in correlation functions

   (Gurarie, Caux et al., Cardy)

Non - compactness

- related to the fact that

$$\sum_n |\psi_n|^{2j} \delta(E-E_n) = \rho^j(x)$$

and their correlation functions
are finite

In the $\mathbb{Z}$-model approach

$$\rho^{(j)}(x) = \lim_{\delta \to 0} \gamma^{(j-1)} (G_R(x) - G_A(x))^j$$

$$= \left[\sum_n |\psi_n(x)|^2 \delta_\gamma (E - E_n) \right]^{(j)} \gamma^{(j-1)}$$

$$= \lim_{\gamma \to 0} \left\{ \sum_n |\psi_n(x)|^2 \frac{\gamma}{\gamma^2 + (E-E_n)^2} \right\}^j \gamma^{(j-1)}$$

$$= \lim_{\gamma \to 0} Q^j \gamma^{(j-1)}$$

$$Q - matrix$$

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Logarithms at disordered critical points.

Replicas

$$E_a(r) \quad (a = 1, \ldots n) \qquad - \text{some physical quantities}$$

At the critical point there is a group of permutations $S(n)$ and operators transform according to this group

$$E = \sum_a E_a$$

$$\tilde{E}_a = E_a - \frac{1}{n} \sum_b E_b$$

belong to different reps. and therefore have different dimensions at $n \neq 0$.

$$\langle E(r) E(0) \rangle = n \langle E_1(r) E_1(0) \rangle + n(n-1) \langle E_1(r) E_2(0) \rangle$$

$$= n \, A(n) \Big/ r^{2 d_E(n)}$$

$$\langle \tilde{E}_a(r) \tilde{E}_a(0) \rangle = \left(1 - \frac{1}{n}\right) \left(\langle E_1 E_1 \rangle - \langle E_1 E_2 \rangle \right)$$

$$= \left(1 - \frac{1}{n}\right) \tilde{A}(n) \Big/ r^{2 d_{\tilde{E}}(n)}$$

$A, \tilde{A}$ are finite

$$\langle E_1 E_1 \rangle - \langle E_1 E_2 \rangle = \frac{\widetilde{A}(n)}{r^{2d_{\widetilde{E}}(n)}}$$

$$\left(\langle E_1 E_1 \rangle - \langle E_1 E_2 \rangle \right) + n \langle E_1 E_2 \rangle = \frac{A(n)}{r^{2d_E(n)}}$$

Taking the limit $n \to 0$ we see that

$$\widetilde{A}(0) = A(0)$$

$$d_{\widetilde{E}}(0) = d_E(0)$$

In the next order in n we get

$$\langle E_1(r) E_2(0) \rangle = \frac{\left(-\ln r \; 2d'_E(0) + c \right)}{r^{2d_E(0)}}$$

$$d'_E(0) = \left. \frac{\partial d_E(n)}{\partial n} \right|_0 .$$