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FORM FACTORS AND CORRELATION FUNCTIONS IN LOW-DIMENSIONAL QFTs

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Form Factors and Correlation Functions in low-dimensional QFT's

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Introduction

Over the past few years, considerable progress has been made in the use of conformal invariance methods and scattering theory for the understanding of the critical points and the nearby scaling region for a large class of two-dimensional models.

It has been possible to find exact solutions of an incredible large number of problems, each of them of extreme interest for the community of statistical mechanics, condensed matter and high-energy physics.

To name just few:

1) The exact solution of the Ising Model, both at $T \neq T_c$ or at $T = T_c$ but in the presence of a magnetic field

2) The exact solution of quantum spin chains with the associated Non-linear Sigma Models

3) Genuine QFT model, like the Sine-Gordon, of great interest for problems of strongly correlated systems, via the bosonization procedure

4) problems of geometrical origin, like polymers and percolation

5) Theories with boundaries (surface critical behaviour, Kondo effects, etc.) and defects

6) Transport properties ~~in~~ strongly correlated systems and quantum impurity problems

7) exact solutions of some random system, like the random bond Ising model

8) Finite temperature correlators

As we will see, all these problems can be elegantly formulated in terms of Conformal Field Theories and deformations thereof. This provides an unifying formalism of analyse. Moreover, the integrability of the resulting deformation of CFT comes out to be the magic technical key point which allows us to arrive to their complete solution.

A crucial method of analyse is the so-called <u>Bootstrap Approach</u>, on which it is probably worth spending some words on its hystorical genesis.

Historical considerations

First of all, the dynamics of any of the theories which will consider is encoded into the spectrum of their quasi-particle excitations (genuine particle, kinks, solitons, etc.) and their interactions.
The bootstrap approach is somehow summarized in the statement

All particles are equal but one
is more equal than the others

Out of joke, the bootstrap approach provides a coherent method to solve exactly models of interacting particles in 1+1 dimensions.
The bootstrap approach received a considerable attention during the '50 and the '60, in connection with the strong interaction felt by the hadrons.
The first, seminal paper on the subject is due to Fermi and Yang. The bootstrap idea that all particles are merely bound or resonance states of each others, produced by the exchange of the particle themselves, has been a natural outgrowth of the development of dispersion theory.

Early attempts to cope theoretically with strong interacting particles by following the methods that had proved -4- successfully in QED simply failed. This because the naive Lagrangian QFT approach presents large coupling constants and the convergence of the perturbative series was highly questionable. Therefore a modified approach was needed. At that time this was the method purely based on the S-matrix theory, especially pushed by Chew, Mandelstam, etc.

It should be emphasized that since the S-matrix represents a common point between theory and experiments, most of the outcoming results are independent whether or not one believes in some underlying microscopic structure, as a Lagrangian QFT!

The first studies of the S-matrix as a function of the energy, momentum transfer, angular momentum etc. produced a very suggestive fact, namely that

<u>The analytic structure of the S-matrix is as simple as possible</u>
(a fact which goes under the name of principle of maximal analiticity)

If this hypothesis is correct, there are <u>no</u> arbitrary constants in strongly correlated systems, except $\underline{c}$, $\underline{\hbar}$ and one mass scale. Said in another way, all strongly interacting particles are composite and may be regarded on the same footing.
This is the so-called bootstrap principle.

As you probably know, it was however extremely difficult to implement in practice all general constraints of the S-matrix (unitarity, crossing symmetry, etc.) in 4-dimensions, and the subject became then extremely baroque and abstract.

The S-matrix approach was in serious competition with the QFT formalism and it was not even clear if the two points of view can be conciliated.
The status of art can be summarized by the following story.

Once there was a student who wanted to know
'Can one prove the Mandelstam representation for the scattering amplitude starting from QFT?'
He went to Weisskopf, who replies: "Field Theory? What is Field Theory?" Then the student saw Wigner, who said 'Mandelstam, who is Mandelstam?'
Finally, the persistent student found his way to Chew, who heard the question, replied:
'Proof, what is a proof?'

As I said, the S-matrix approach became very abstract and complicate and it was finally abandoned not because it was proved wrong rather because physicists gave up trying to make any progress with it. With the discovery of asymptotical freedom in QCD, the field theory point of view finally took over.
The main revenge of the bootstrap ideas and of the S-matrix approach came just in these recent years, in the exact solution of low dimensional systems, where there is a drastic simplification of the kinematic which allows us to fully implement all the constraints.

In what follows, I will expose the main motivations, the main results and, ~~[illegible]~~, I will feel free to use also a language of QFT to motivate many arguments.

Topics of the seminar

1) Conformal Field Theories
2) Deformation of CFT
3) Off-critical quantities, universal ratios
4) S-matrix formulation
5) Form Factors and their relevance
6) Functional Eqs of the FF.
7) Examples
   i) Ising Model (Thermal deformation)
   ii) Sinh-Gordon Model

General Aspects of 2-D QFT

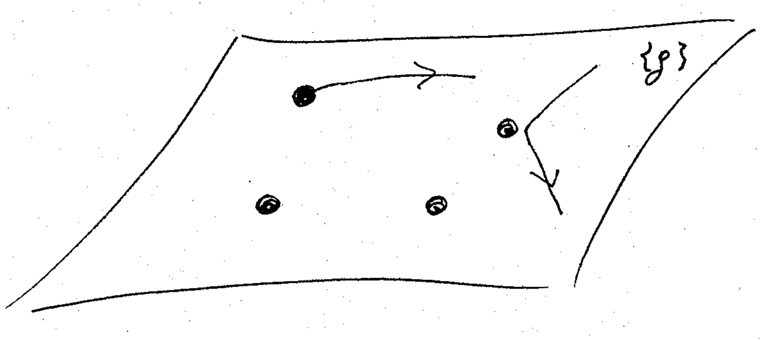
We have to think the space of QFT's as an infinite dimensional space, the one of all possible coupling constants. To be more concrete, one selects initially an order parameter field, say φ ($\varphi \to -\varphi$ under parity, for instance) and then, at fixed cutoff Λ, write the most general lagrangian involving all local interactions, for instance

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \sum_{k=1}^{\infty} g_k :\varphi^k:$$

In this ∞-dimensional space, there are some special points, which are those fixed under RG transformation $\Lambda \to \Lambda + d\Lambda$.

They are characterized by the condition

$$\beta(g_i) = 0 \qquad \Longrightarrow \qquad \xi = \infty$$



In 2-dimensions there is a clear and net description of the QFT relative to the fixed points, thanks to the work of Belavin, Polyakov and Zamolodchikov on Conformal Field Theories.

Let us summarize the main results.

The fields are characterized by their scaling properties

$$\varphi_i(\lambda x) = \lambda^{-2\Delta_i} \varphi_i(x)$$

They close an OPE algebra

$$\varphi_i(x)\,\varphi_j(0) = \sum_k \frac{d_{ij}^{\ k}}{|x|^{2(\Delta_i+\Delta_j-\Delta_k)}}\,\varphi_k(0)$$

Any CFT is identified by the value of the central charge anomaly $\underline{c}$, entering the OPE of the holomorphic part of the mom.-energy tensor

$$T(z)\,T(0) = \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{1}{z}\partial T$$

To find a consistent values of $\underline{c}$, $d_{ij}^{\ k}$ and Δ_j is a dynamical problem.

Examples of consistent CFT's, with a finite number of primary fields, are given by the so-called Minimal Models

$$c = 1 - \frac{6\,(p-q)^2}{pq} \qquad\qquad (p,q)=1$$

$$\Delta_{n,m} = \frac{(np-mq)^2-(p-q)^2}{4pq} \qquad \begin{array}{l} n=1\ldots\ p-1 \\ m=1\ldots\ q \end{array}$$

Easiest examples of unitary Theories

1) Ising Model $\qquad c = 1/2$

$$\begin{array}{|ccc} 1/2 & 1/16 & 0 \\ 0 & \frac{1}{16} & 1/2 \\ \hline \end{array}$$

$$\sigma\cdot\sigma = 1+\varepsilon$$

$$\varepsilon\cdot\varepsilon = 1$$

$$\varepsilon\cdot\sigma = \sigma$$

2) TIM $\qquad c = 7/10$

$$\begin{array}{|cccc} 3/2 & 6/10 & 1/10 & 0 \\ 7/16 & 3/80 & 3/80 & 7/16 \\ 0 & \frac{1}{10} & \frac{6}{10} & 3/2 \\ \hline \end{array}$$

$$\varphi = [3/80,\ 3/80]$$

$$\varphi^2 = [\tfrac{1}{10},\ \tfrac{1}{10}]$$

$$\varphi^3 = [7/16,\ 7/16]$$

$$\varphi^4 = [6/10,\ 6/10]$$

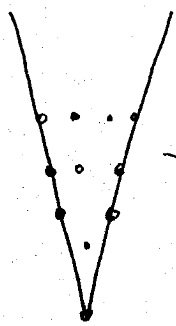
$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + :\varphi^6:$$

Easiest example non-unitary Theory

Yang-Lee Model $\quad c = -22/5 \quad$ $\underline{|\, 0 \;\; -1/5 \;\; -1/5 \, .} \; 0$

$$\varphi \cdot \varphi = 1 + \varphi$$

At the critical point we have a precise characterization of the operator content of the theory, since all conformal fields are organized into conformal families (Verma module)



φ

$$L_0 \varphi = \Delta \varphi \qquad\qquad L_n \varphi = 0 \quad n > 1$$

$$L_{-n_1} \cdots L_{-n_k} \varphi \qquad\qquad \sum_i n_i = N$$

$$T(z) = \sum_{-\infty}^{+\infty} \frac{L_n}{z^{n+2}}$$

The correlators satisfy linear differential equations and these equations can be explicitly solved

$$\mathcal{L} \langle \varphi_1 \cdots \varphi_n \rangle = 0$$

Given a CFT, we can easily identify the set of the relevant operators, i.e. those with

$$\Delta_i < 1$$

The number R of the relevant operators is an indication of the instability of the fixed point

(For the unitary M.M. $(p, p+1) \Rightarrow R = 2(p-2)$)

We can use one of these to perturb the system and go away from criticality

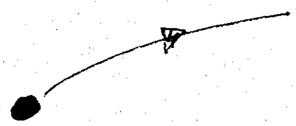
$$A = A_{CFT} + \lambda \int \varphi(x)\, d^2x$$

The lowest-order of the β-function is

$$\beta(\lambda) \simeq 2(1-\Delta)\lambda \quad ; \quad \boxed{\Theta = 2\pi\lambda(2-2\Delta)\varphi}$$

and therefore for $\Delta < 1$, λ tends to grow at large distances.

The above action is certainly a faithful representation for the QFT associated to the RG flow which departures from the fixed point.



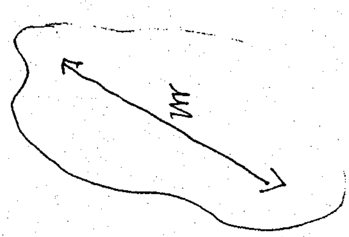
There are several consequences coming from the coupling the system to one of the relevant field.

a) First of all, there is now an explicit mass scale

$$M \sim \lambda^{\frac{1}{2(1-\Delta)}}$$

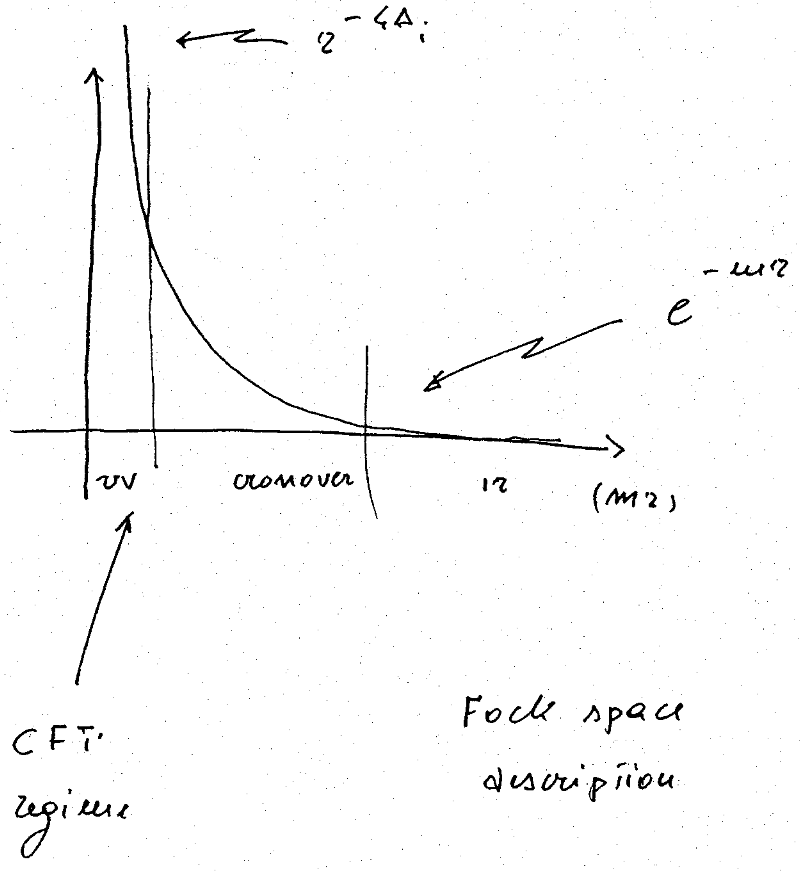
which breaks the conformal invariance of the problem

b) The most common scenario is that in which the mass M corresponds indeed to a finite correlation length present in the system.
The standard example is the critical Ising Model coupled to a magnetic field



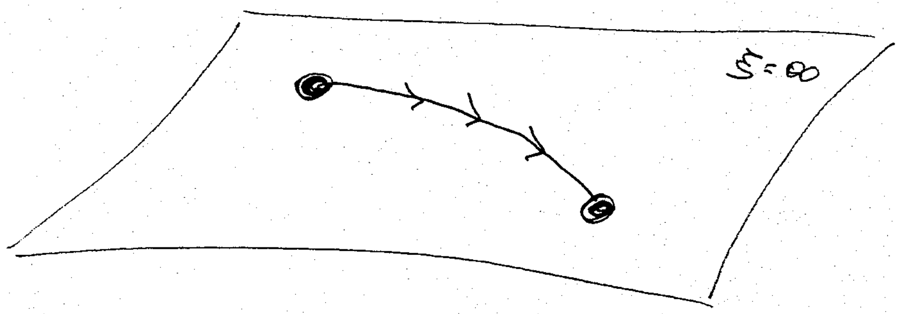
In this case the system flows to the trivial fixed point (i.e. $c=0$) where there are no massless degrees of freedom left

The 2-point correlation function will have in this case the following behaviour. $G(r) = \langle \varphi_i(r) \varphi_i(0) \rangle$

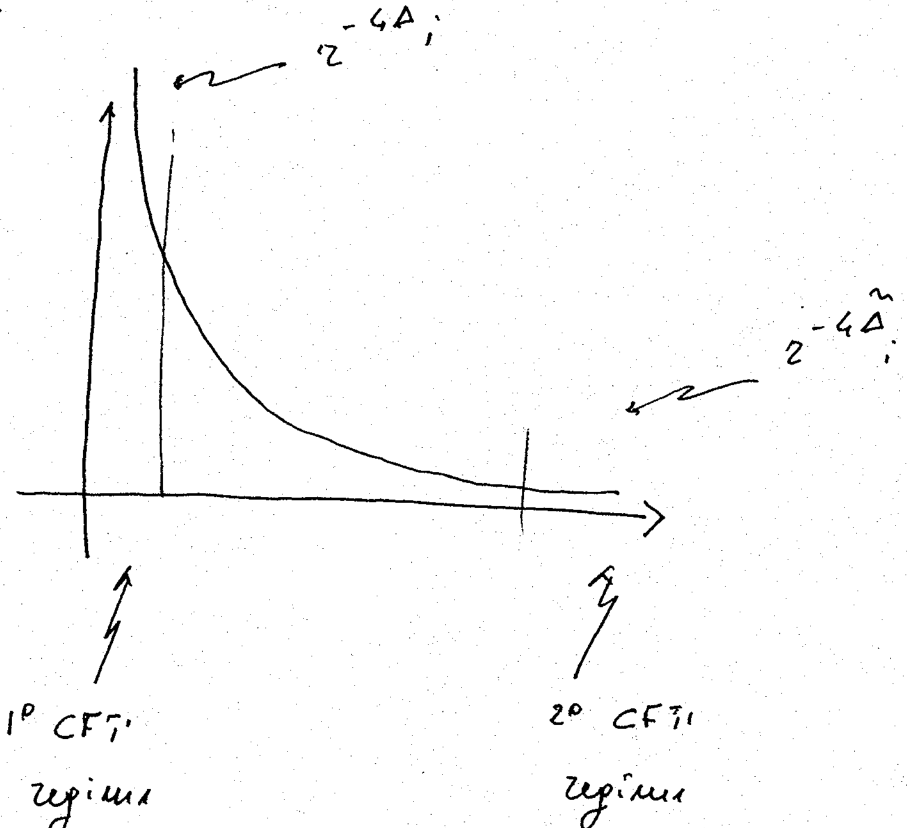


The large-distance can be efficiently described in terms of massive QFT, with its Hilbert space spanned by the multi-particle states.

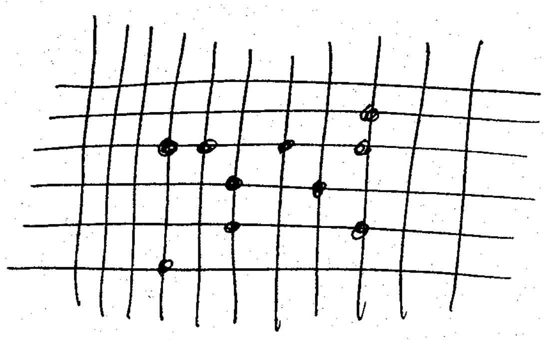
c) It may happen another interesting situation, i.e. that the RG flow remains on the critical surface (i.e. $\xi = \infty$), and therefore gives rise to a massless flow between two critical points



In this case the correlation functions will present a <u>power-law behaviour</u>, both at short and large distances although with different power laws.



Let me give a simple example of this kind of field theory associated to massless flow. This is provided by the RG flow which links the Tricritical Ising Model (i.e. the Ising Model with vacancies) to the standard, critical Ising Model.



A microscopic lattice Hamiltonian (in its $\mathbb{Z}_2$ invariant subspace) can be written as

$$H = -\beta \sum_{\langle ij \rangle} t_i t_j \sigma_i \sigma_j + \mu \sum_i (t_i - 1)$$

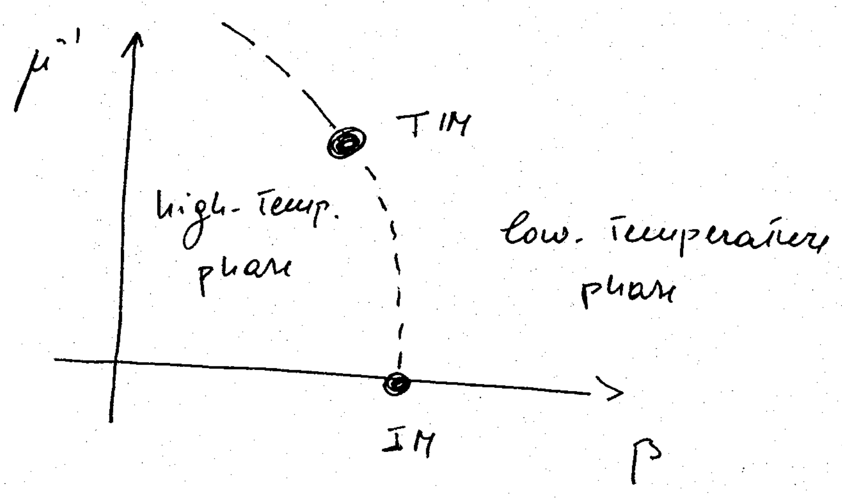
where

$$\{\sigma_i\} = \pm 1 \qquad , \quad \{t_i\} = \{0, 1\}$$

β - is the temperature like coupling constant whereas μ plays the role of the chemical potential for the vacancies.

The TIM has a tricritical point, i.e. a point where a first order line meets a second order line. At the tricritical point there are fluctuations both in

The spin and vacancy variables. The phase diagram
looks like



It is obvious that if we make a fine-tuning of the two variables μ and β, we end up to the Ising fixed point.

In fact, increasing μ we will suppress the vacancies (i.e. we fill more and more lattice sites) but varying β we can let the system to remain critical. Hence

$$\mu \to \infty$$
$$\beta \to \beta_{critical}$$

is just the Ising Model. Notice that the operator associated to the vacancy density is $\varphi_{1,3} = \left(\frac{6}{10}, \frac{6}{10}\right)$.

This is a general features in the unitary Minimal Models, i.e. the perturbation of the p-th Minimal Model by the operator $\varphi_{1,3}$ brings the system along a massless flow to the (p-1)th Min. Model.

$$p \xrightarrow{\varphi_{1,3}} p-1$$

The massless flows are very interesting in Condensed Matter and Statistical Mechanics. For instance, it is believed that coupling the system to a disorder, move the system from a given class of universality to another.

In summary, CFT's provide a very efficient characterization of the fixed points but, in order to characterize the class of universality, the RG flows and the whole space of QFT; we have to go away from criticality.

Away from criticality, we can study the approach to the critical point by means of different susceptibilities and the so-called universal ratios, which provide a very efficient fingerprints of the universality classes.

Universal Ratios

$$Z[g_i] = \int \mathcal{D}\varphi \exp\left[-\left(A_{CFT} + \sum_i g_i \int \varphi_i(x)\, d^2x\right)\right]$$

$$\equiv \exp\left[-V f(g)\right]$$

The free-energy takes the scaling form

$$f[g] = (K_i g_i)^{\frac{1}{1-\Delta_i}} \; F\left(\frac{K_j g_j}{(K_i g_i)^{\phi_{ji}}}\right)$$

$$\phi_{ji} \equiv \frac{1-\Delta_j}{1-\Delta_i} \qquad ; \qquad K_i \sim \left(\frac{1}{g_i}\right)_0$$

$$\langle \varphi_j \rangle_i = -\left.\frac{\partial f}{\partial g_j}\right|_{\substack{g_\ell = 0 \\ g_i \neq 0}} = B_{ji}\, g_i^{\Delta_j / 1-\Delta_i}$$

$$\hat{\Gamma}^i_{jk} = \frac{\partial}{\partial g_k} \langle \varphi_j \rangle_i = -\left.\frac{\partial^2 f}{\partial g_j \partial g_k}\right|_{\substack{g_\ell = 0 \\ g_i \neq 0}} =$$

$$= \Gamma^i_{jk}\, g_i^{(\Delta_j + \Delta_k - 1)/1-\Delta_i} = -\int d^2x \, \langle \varphi_j(x)\, \varphi_k(0) \rangle_i$$

The correlation length satisfies an analogous scaling law

$$\xi = a\,(K_i g_i)^{\frac{1}{2(1-\Delta_i)}} \; L\left(\frac{K_j g_j}{(K_i g_i)^{\phi_{ji}}}\right)$$

All the above quantities, obtained by taking the derivative of the free energy, contain metric factors (the quantities K_i) which make their values not universal.

However, it is always possible to consider special combinations thereof in such a way that all metric factors cancel out and we have the universal ratios

$$(R_c)^i_{jk} = \frac{\Gamma^i_{ii}\,\Gamma^i_{jk}}{B_{ji}\,B_{ki}}$$

$$R^i_\xi = (\Gamma^i_{ii})(\xi_i^2)_0$$

$$Q = \frac{\Gamma^i_{jj}}{\Gamma^k_{jj}}\left(\frac{\xi_k}{\xi_j}\right)_0^{2-4\Delta_i}$$

They present an obvious advantage from an experimental point of view. In fact, the universal ratios are numbers which present significant variations between different classes of universality, whereas the critical exponents usually assume small values that vary by only a small percent on changing the universality class.

Main questions

Once clarified the importance of going away from criticality, let us list some important questions we would like to answer

① Is there a way to determine the mass spectrum?

② Can we identify the operator content?

③ Once we have identified the spectrum of the operators, can we calculate their correlation functions

④ Do these correlator satisfy some differential equation? Of what kind?

⑤ Can we work out the thermodynamics of these QFT?

⑦ Is it possible to relate off-critical data (as the mass spectrum) to the conformal data (as, for instance, central charge and anomalous dimensions)?

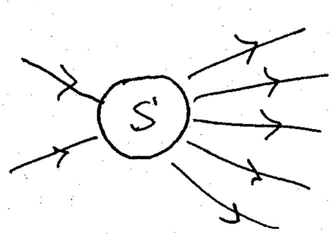
Important progress on all these questions has been made by means of the integrable models and the associated exact S-matrix.

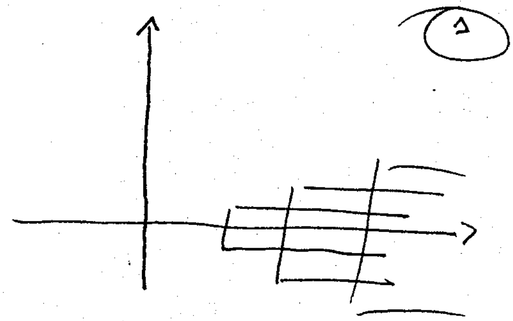
Let us discuss then the theory of analytic S-matrix.

Analytic S-matrix Theory

Let us assume that the deformation of CFT leads the system to the pure massive case (the massless case can be dealt by an appropriate analytic continuation).
In the Minkowski space we may think to characterize the system (on-shell) by means of the S-matrix.
It is well known that the knowledge of the S-matrix is enough to permit the full reconstruction of the theory, i.e. the calculation of correlation functions, finite-size energies, etc.

However, in ordinary situations the determination of the S-matrix is a difficult problem, due to the complexity of the analytic structure of the amplitudes induced by the production and annihilation processes





There are however situations in which the calculation of the S-matrix is drastically simplified. This occurs when in the QFT under consideration there is an infinite number of conserved charges, which we express as

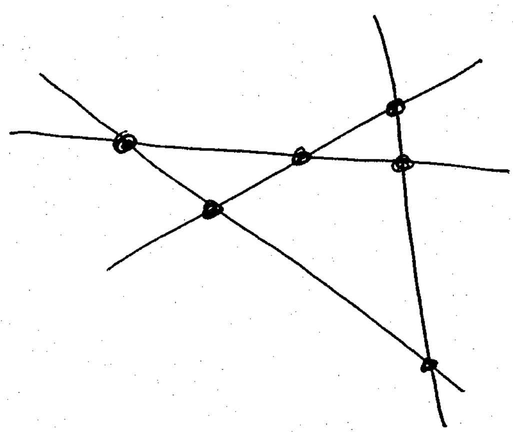
$$Q_s = \int \left[T_{s+1}\, dz + \Theta_{s-1}\, d\bar{z} \right]$$

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}$$

These charges implies not only the conservation of the total momentum, but also of its higher powers

$$Q_s \sim \sum p^{s'}$$

Under these circumstances, the scattering processes are completely elastic and factorizable and



$$S = \prod_{\langle ij \rangle} S^{(2)}_{ij}$$

Hence the scattering problem is solved once we determine the two-body scattering amplitudes only.

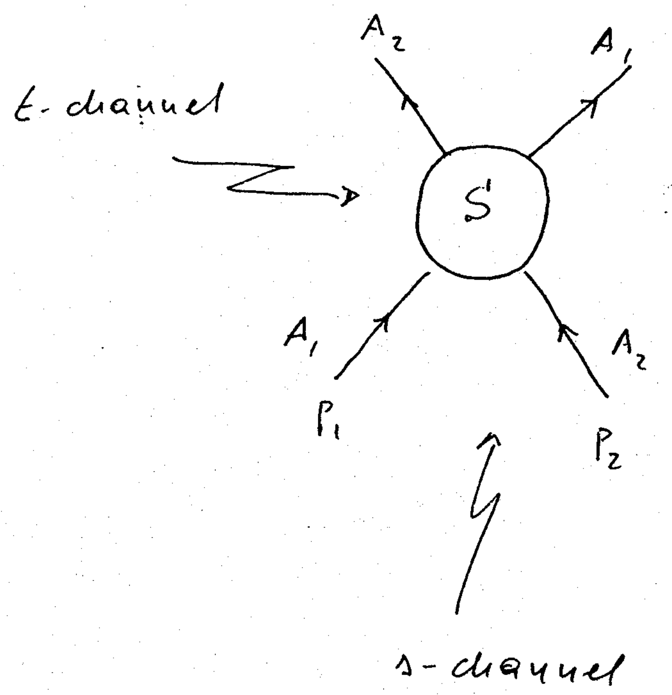
Two-body S-matrix

We will develop the theory in the simplest case in which there is no degeneracy in the mass spectrum and all particles are self-conjugated (i.e. neutral).

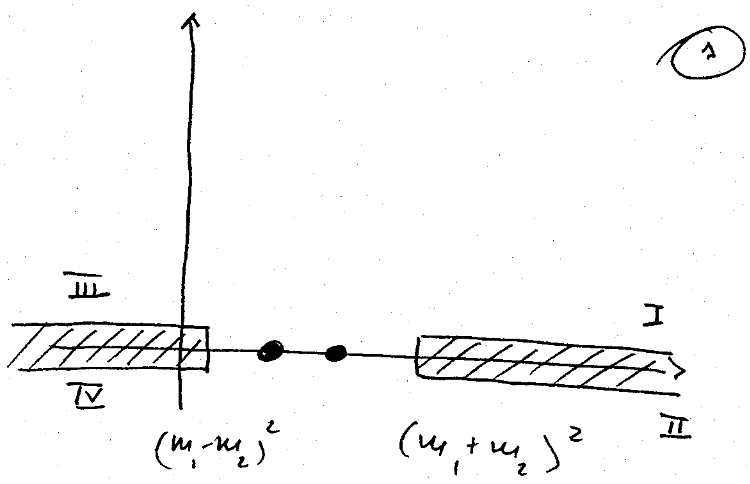
The two-body S-matrix is a function of the relativistic invariant quantities, which are the Mandelstam variables

$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_3)^2 = (p_1 - p_2)^2$$



In the Mandelstam plane s, the analytic structure is the simplest possible

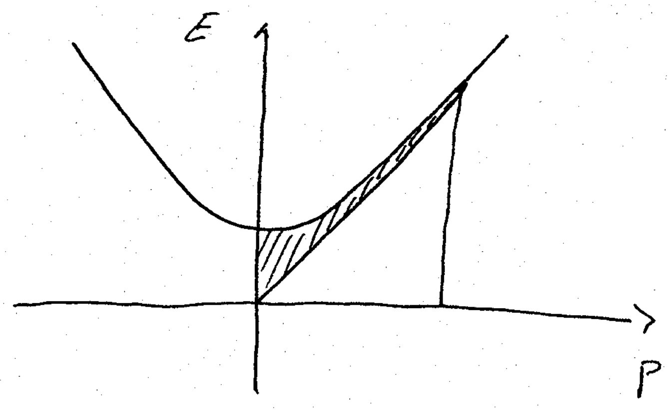


It is convenient to introduce the parameterization

$$E_a = m_a \operatorname{ch} \beta \qquad\qquad E^2 - p^2 = m_a^2$$

$$P_a = m_a \operatorname{sh} \beta$$

which has the graphical interpretation

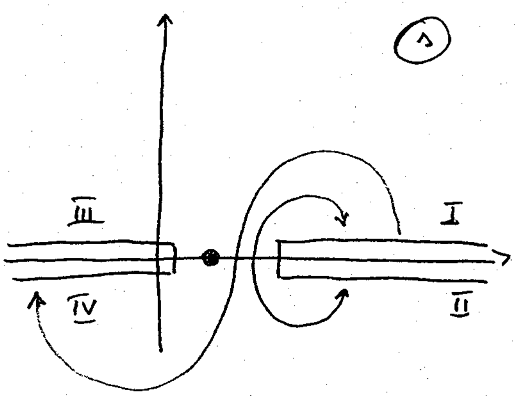


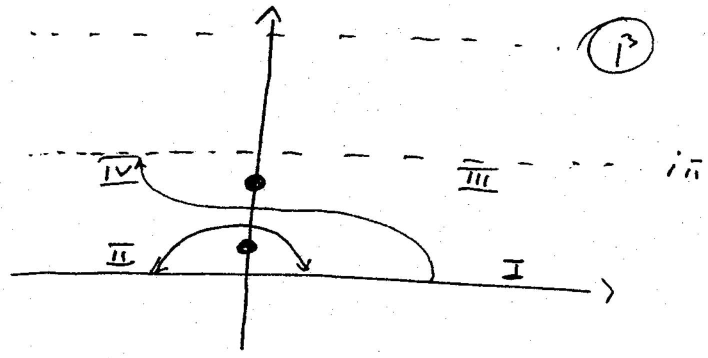
$$s = (P_1 + P_2)^2 = m_1^2 + m_2^2 + 2 m_1 m_2 \operatorname{ch}(\beta_1 - \beta_2)$$

$$t = (P_1 - P_2)^2 = m_1^2 + m_2^2 - 2 m_1 m_2 \operatorname{ch}(\beta_1 - \beta_2) =$$

$$= m_1^2 + m_2^2 + 2 m_1 m_2 \operatorname{ch}(i\pi - (\beta_1 - \beta_2))$$

Let us write the general equations of unitarity and crossing symmetry





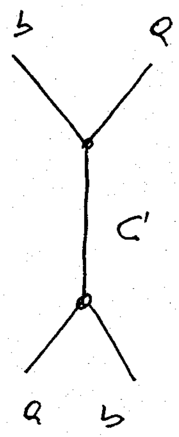
$$\begin{cases} S_{ab}(\beta)\, S_{ab}(-\beta) = 1 \\ S'_{ab}(\beta) = S_{ab}(i\pi - \beta) \end{cases}$$

Interesting enough, there exists a general solution of these equations, in terms of meromorphic functions in the strip

$$S'_{ab} = \prod_{x} \frac{\operatorname{th} \frac{1}{2}(\beta + i\pi x)}{\operatorname{th} \frac{1}{2}(\beta - i\pi x)} \qquad 0 < x < 1$$

The meaning of $\underline{x}$ is clear. It is the location of the pole singularity of the S-matrix, related to the bound states in the channel (a, b)

In the vicinity of the pole we have the factorization

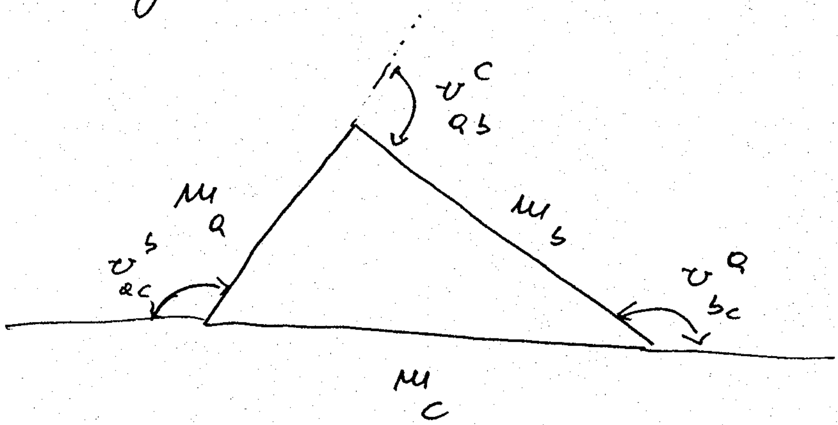


$$S(\beta) \simeq \frac{i\,(\Gamma_{ab}^{c})^{2}}{\beta - i u_{ab}^{c}}$$

The angle u_{ab}^{c} identifies the mass of the bound state. In fact, substituting into the Mandelstam variable s, we have

$$m_c^2 = m_a^2 + m_b^2 + 2 m_a m_b \cos u_{ab}^{c}$$

which is nothing else but the Carnot formula for a triangle with sides m_a, m_b and m_c



An obvious result is that if in the channel (a, b) there is a pole due to the bound state $\underline{c}$ hence in the channel (a, c) there is the bound state $\underline{b}$ and in (b, c) the bound state $\underline{a}$, with the location of the poles related by

$$u_{ab}^{c} + u_{ac}^{b} + u_{bc}^{a} = 2\pi$$

A useful notation is

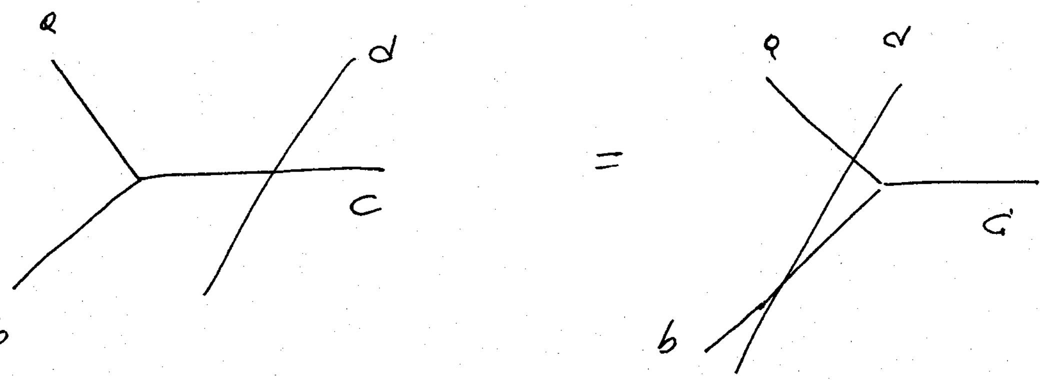
$$\bar{u} \equiv \pi - u$$

In order to determine the location of the poles we have to impose the following dynamical requirement

<u>Bootstrap hypothesis</u>

All bound states are on the same footing of the asymptotic states. There is a complete democracy among all particles.

Mathematically, this translates into the equation



$$S_{dc}(\beta) = S_{da}(\beta + i\bar{u}^{b}_{ac})\, S_{db}(\beta - i\bar{u}^{a}_{bc})$$

Simplest Theory

Consider a QFT with only one-particle, bound state of itself

$$A \times A \to A \to A \cdot A$$

The above equation is

$$S(\beta) = S\left(\beta - \frac{i\pi}{3}\right) S\left(\beta + \frac{i\pi}{3}\right)$$

whose simplest solution is

$$S(\beta) = \frac{\tanh \frac{1}{2}\left(\beta + \frac{2\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\beta - \frac{2\pi i}{3}\right)}$$

Mathematical digression

An understanding of the form of the S-matrix is obtained by means of the infinite product representation of the hyperbolic functions due to Euler.

$$(a) \equiv \frac{\operatorname{th}\frac{1}{2}(\beta+i\pi a)}{\operatorname{th}\frac{1}{2}(\beta-i\pi a)} = \frac{\operatorname{sh}\frac{1}{2}(\beta+i\pi a)}{\operatorname{sh}\frac{1}{2}(\beta-i\pi a)} \; \frac{\operatorname{ch}\frac{1}{2}(\beta-i\pi a)}{\operatorname{ch}\frac{1}{2}(\beta+i\pi a)}$$

$$\equiv [a]\,[1-a]$$

Let us consider

$$\operatorname{sh} x = x \prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2\pi^2}\right) = x \prod_{k=1}^{\infty}\left(1+\frac{x}{ik\pi}\right)\left(1-\frac{x}{ik\pi}\right)$$

$$\operatorname{ch} x = \prod_{k=0}^{\infty}\left(1+\frac{2x}{i(2k+1)\pi}\right)\left(1-\frac{2x}{i(2k+1)\pi}\right)$$

Hence

$$[a] = \frac{\operatorname{sh}\frac{1}{2}(\beta+i\pi a)}{\operatorname{sh}\frac{1}{2}(\beta-i\pi a)} = \frac{\beta+i\pi a}{\beta-i\pi a}\prod_{k=1}^{\infty}\frac{\beta+i\pi a+2k\pi i}{\beta-i\pi a+2k\pi i}\;\frac{\beta+i\pi a-2k\pi i}{\beta-i\pi a-2k\pi i}$$

$$= \prod_{k=-\infty}^{+\infty}\frac{\beta+i\pi a+2k\pi i}{\beta-i\pi a+2k\pi i}$$

Hence the scattering is nothing else but the relativistic generalization of the quantum-mechanics scattering on a δ-function potential of strength Q.

In fact, in the quantum mechanics problem the amplitude is given by

$$S_0 = \frac{\beta + i\pi Q}{\beta - i\pi Q}$$

But, if we want to make this amplitude relativistic invariant, we have to remember that β is defined up to multiples of $2\pi i$. and therefore we have to make the above function invariant under

$$\beta \to \beta + 2\pi K i$$

Exponential Integral Representation

By using the formula

$$\int_0^\infty e^{-px} \sin qx \frac{dx}{x} = \frac{1}{2i} \ln \frac{1+i\,q/p}{1-i\,q/p}$$

The building block (a) can be written as

$$(a) = -\exp\left[2\,\mathrm{sign}(a) \int_0^\infty \frac{dx}{x} \frac{\mathrm{ch}\,\frac{x}{2}(1-2|a|)}{\mathrm{ch}\,\frac{x}{2}} \,\mathrm{sh}\,\frac{\beta x}{i\pi} \right]$$

This will become useful when we compute

the Form Factors.

Our main examples

In my lectures I will concentrate the attention on some simple bootstrap models, those with only one particle in the spectrum.

① S-matrix of the Thermal Ising Model

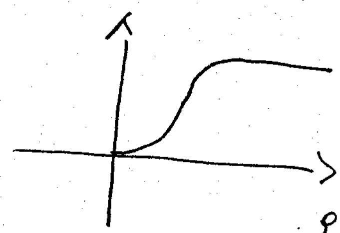
$$S = -1$$

② S-matrix of the Sh-Gordon Model

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{g^2}(\cosh g\varphi - 1)$$

$$S = (-B)$$

$$B = \frac{g^2}{8\pi}\,\frac{1}{1+g^2/8\pi}$$



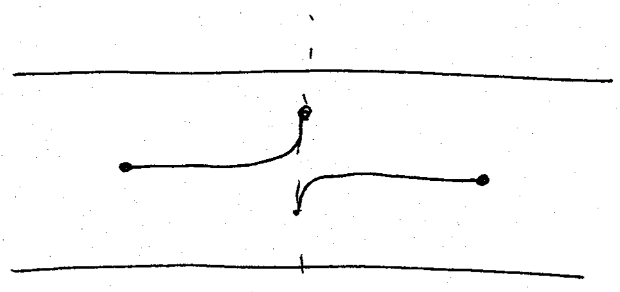
S-matrix invariant under

$$B \to 1 - B \qquad \Longleftrightarrow \qquad g = \frac{8\pi}{g}$$

self-dual model

Making the analytic continuation

$$\beta = \frac{1}{2} + \frac{i\beta_0}{\pi}$$



$$S_{sh} = \frac{sh\,\beta - i\,ch\,\beta_0}{sh\,\beta + i\,ch\,\beta_0} \xrightarrow[\beta_0 \to \infty]{} -1$$

Hence there should be a mapping between the two models

(3) S-matrix of Yang-Lee Model

$$S = \left(\tfrac{2}{3}\right)$$

(4) S-matrix of the BD Model

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{6\lambda^2}\left(2e^{\lambda\varphi} + e^{-2\lambda\varphi}\right)$$

$$S = \left(\tfrac{2}{3}\right)\left(\tfrac{B-2}{3}\right)\left(-\tfrac{B}{3}\right)$$

$$B = \frac{\lambda^2}{2\pi}\,\frac{1}{1+\lambda^2/4\pi}$$

The last is invariant under

$$\beta \to 2-\beta \qquad \Longleftrightarrow \qquad \lambda \to \frac{4\pi}{\lambda}$$

Making the analytic continuation

$$\beta = 1 + \frac{3}{i\pi}\,\beta_0$$

and taking $\beta_0 \to \infty$

$$S_{BD} \longrightarrow S'_{YL}$$

Also in this case there should be a mapping between the two models

Form Factors

Once the S-matrix and the exact spectrum are known, we can proceed further. In particular we will show that this enough to determine the matrix elements of the various operators, on the multiparticle states

$$F^{\varphi}(\beta_1 \cdots \beta_n) \equiv \langle 0 | \varphi | A(\beta_1) \ldots A(\beta_n) \rangle$$

The relevant equations which determine these quantities will be discussed below. In the meantime let us discuss some important issues.

a) Operator content

b) correlation functions

c) Thermal effects at equilibrium

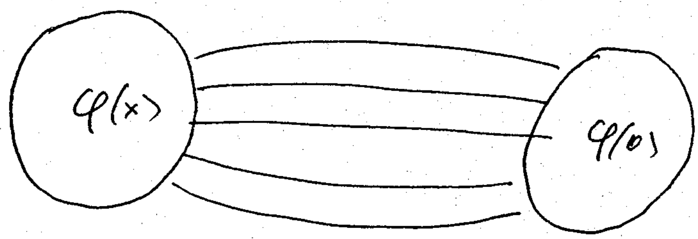
d) non-integrable theories

a) Since an operator φ is known once all its matrix elements are given, a classification of the operator content is reached by finding all possible solutions to the Form Factor equations

b) Knowing all Form Factors of an operator, we can write down its correlation functions by using the spectral representations. For the two point function, for instance, we have

$$\langle 0|\varphi(x)\varphi(0)|0\rangle = \sum_n \langle 0|\varphi(x)|n\rangle\langle n|\varphi(0)|0\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} \left|\langle 0|\varphi(0)|\beta_1 \ldots \beta_n\rangle\right|^2 e^{-mr \sum_i^N ch \beta_i}$$



This can be considered as large distance expansion of the correlators. This is a very effective formula due to the following fact

① There is no divergencies (compare with Feynman diagram, for instance), because from the very begining all physical quantities are involved

② If the theory depends on a coupling constant g (as Sh. Gordon or BD model), each matrix element is the full resummation of the perturbative series.

(3) For operators which do not present a violent singularity at the origin (as, for instance, the relevant operators), the corresponding spectral series present a very fast rate of convergence, even for small values of the scaling variable mr.

This is due to a series of reasons. To see them, let us consider the Fourier transform

$$G(x) = \langle 0 | \varphi(x) \varphi(0) | 0 \rangle =$$

$$= \int \frac{d^2p}{(2\pi)^2} e^{ip\cdot x} G(p)$$

$$G(p) = \int_0^\infty dm^2 \rho(m^2) \frac{1}{p^2+m^2}$$

$$\rho(p^2) = 2\pi \sum_M \int d\Omega_1 \dots d\Omega_n \, \delta^2(p-\hat{p}_n) \, |\langle 0|\varphi(0)|n\rangle|^2$$

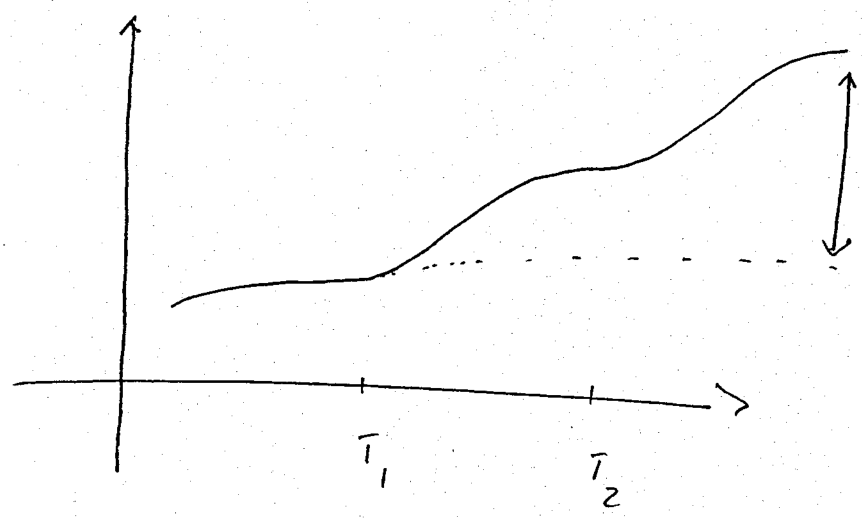
$$d\Omega = \frac{dp}{2\pi(2E)} = \frac{d\beta}{2\pi} \qquad \hat{P}_M = p_1 + \dots p_n$$

The quantity

$$R_M \equiv \int \prod_{k=1}^M d\Omega_k \, \delta^2(p-\hat{P}_n)$$

is called the phase-space of n-particles.

Intuitively one would expect for $\rho(m^2)$



However, notice that in 2-dimension at high-energy, the phase-space shrinks (rather than enlarging)

$$R_n(s) \underset{s \to \infty}{\simeq} \frac{n(n-1)}{2} \frac{1}{s} \left(\frac{1}{4\pi} \ln \frac{s}{m^2} \right)^{n-2}$$

(In 4-dimensions, instead

$$R_n(s) \simeq s^{n-2} \quad)$$

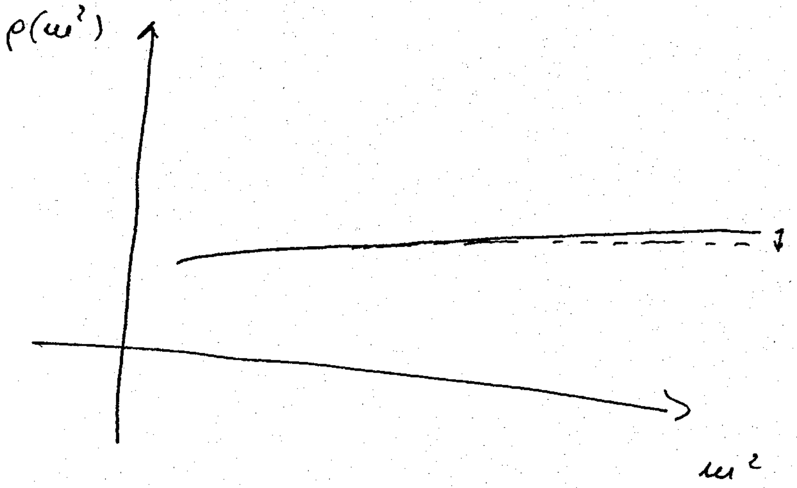
Moreover, at thresholds

$$R_n \simeq A_n \left(\sqrt{s} - n M \right)^{\frac{n-3}{2}}$$

but in interacting theories, the form factors have zero at threshold of higher orders, so that the spectral function goes like

$$\rho(s) \simeq \left(\sqrt{s} - nM\right)^{\frac{n-3}{2}}$$

This means that in the vicinity of the threshold is extremely flat



Feynman Gas

There is an easy way to see that the above expression in the limit $m \to 0$ will become a power ~~law~~ law

$$G(\tau) = \langle 0 | \varphi(\tau) \varphi(0) | 0 \rangle \underset{\tau \to 0}{\simeq} \left(\frac{1}{\tau}\right)^{4\Delta}$$

In fact, notice that the spectral representation expression looks like the grand-canonical partition function of a fictitious one-dimensional gas

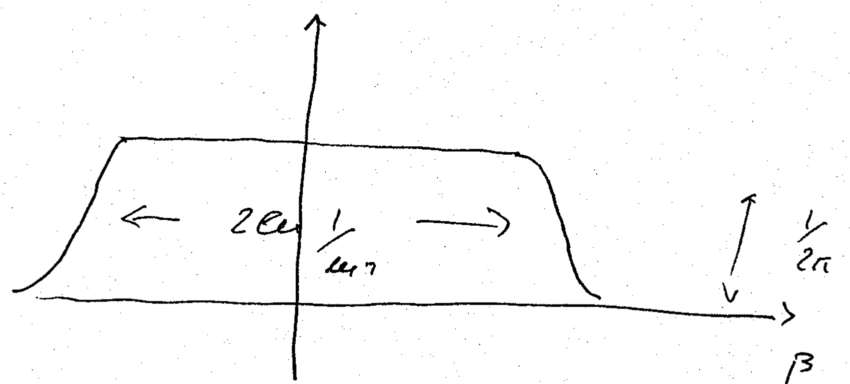
$$G(z) = \sum_N z^N Q_N$$

with fugacity which depends on the 'coordinate' β

$$z(\beta) = \frac{1}{2\pi} e^{-mr\,ch\beta}$$

It is easy to see that in limit $mr \to 0$, this gas <u>effectively</u> occupies a finite volume of length

$$V \simeq 2 \ln \frac{1}{mr}$$



Hence, by applying the equation of state

$$G \simeq e^{p(z)V} = e^{2p(z)\ln\frac{1}{mr}} = \left(\frac{1}{mr}\right)^{2p(z)}$$

Hence the anomalous dimension is nothing else but the pressure of this fictitious one-dimensional gas at the value of the fugacity $z = 1/2\pi$

Another conformal datum, the central charge, can be obtained by using the c-theorem sum rule

$$c = \frac{3}{4\pi} \int d^2x \, |x|^2 \langle \Theta(x)\, \Theta(0) \rangle =$$

$$= \frac{3}{2} \int_0^\infty dr \, r^3 \langle \Theta(x)\, \Theta(0) \rangle$$

By using the spectral representation of the correlator, we have

$$c = \sum_{n=1}^{\infty} c_n$$

where the n-particle contribution is given by

$$c_n = \frac{12}{n!} \int_0^\infty \frac{d\mu}{\mu^3} \int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi}$$

$$\delta\left(\sum_{i=1}^{n} \sinh\beta_i\right) \delta\left(\sum_{i=1}^{n} \cosh\beta_i - \mu\right) \left|F^{\Theta}(\beta_1 \ldots \beta_n)\right|^2$$

Usually this series converges extremely fast (because the term r^3 kills the ultraviolet singularity and the term weights more the large-distance behaviour of the function where the spectral representation is more accurate)

often, the one and the two-particle contribution saturate the actual value of c within few percents. The one and the two-particle contribution are given respectively by

$$c^1 = \frac{6}{m^4 \pi} \left| \langle 0 | \Theta | A(0) \rangle \right|^2$$

$$c^2 = \frac{3}{2} \int_0^\infty \frac{d\beta}{\cosh^4 \beta} \left| \frac{F^\Theta(2\beta)}{F^\Theta(i\pi)} \right|^2$$

c) Thermal Effects at Equilibrium

Knowing the Form Factors of the Theory at $T=0$, we can in principle address the computation of the correlation functions at finite temperature.
By using the imaginary time formalism, we have in fact

$$\langle \varphi \cdots \varphi \rangle_T = \frac{1}{Z} \sum_n \langle n | \varphi \cdots | n \rangle \, e^{-E_n/T}$$

and all quantities can be computed, although there are certain subtleties which we cannot discuss here.

d) Non-Integrable Quantum Field Theories

Not all Field Theories are integrable. However, important examples of Non-Integrable QFT can be often written as

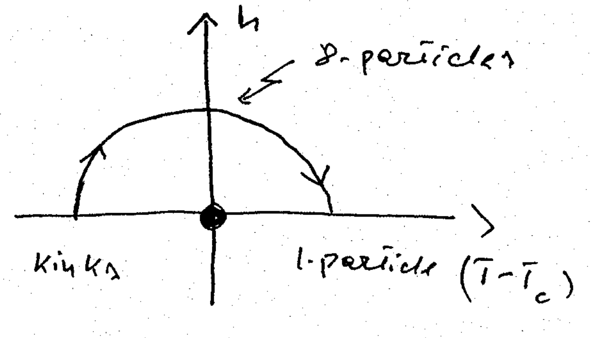
$$A = A_{CFT} + \lambda \int \varphi(x)\, d^2x + g \int \psi(x)\, d^2x$$

where each individual deformation defines an integrable QFT, characterized by its mass spectrum, exact S-matrix and set of Form Factors.

Examples

i) Ising Model

$$A = A_{CFT} + (T - T_c) \int \varepsilon(x)\, d^2x + h \int \sigma(x)\, d^2x$$



ii) Multi-frequency Sine-Gordon

$$\mathcal{L} = \frac{1}{2} (\partial\varphi)^2 + \lambda \cos \beta\varphi + g \cos \omega \varphi$$

Under these fortunate circumstances, a big deal of information can be obtained by employing a Form-Factor Perturbation Theory (Nucl. Phys. B473 (1996) 469
Nucl. Phys. B516 (1998) 6

We can answer questions like:

i) how does the spectrum change by switching on the new interaction?

ii) what is the inelastic correction to the S-matrix?

iii) what is the decay rate of the highest mass particle

iv) how does it change the ground state energy?

At the first order in g, we have

i) Mass correction given by

$$\delta M^2_{ab} \simeq 2g \langle A_a(0) | \psi | A_b(0) \rangle$$

ii) Inelastic correction S-matrix

$$\delta S_{ab} \simeq -ig \frac{\langle A_a(\beta_1) A_b(\beta_2) | \psi | A_a(\beta_1) A_b(\beta_2) \rangle}{m_a m_b \, sh(\beta_1 - \beta_2)}$$

iii) Decay rate

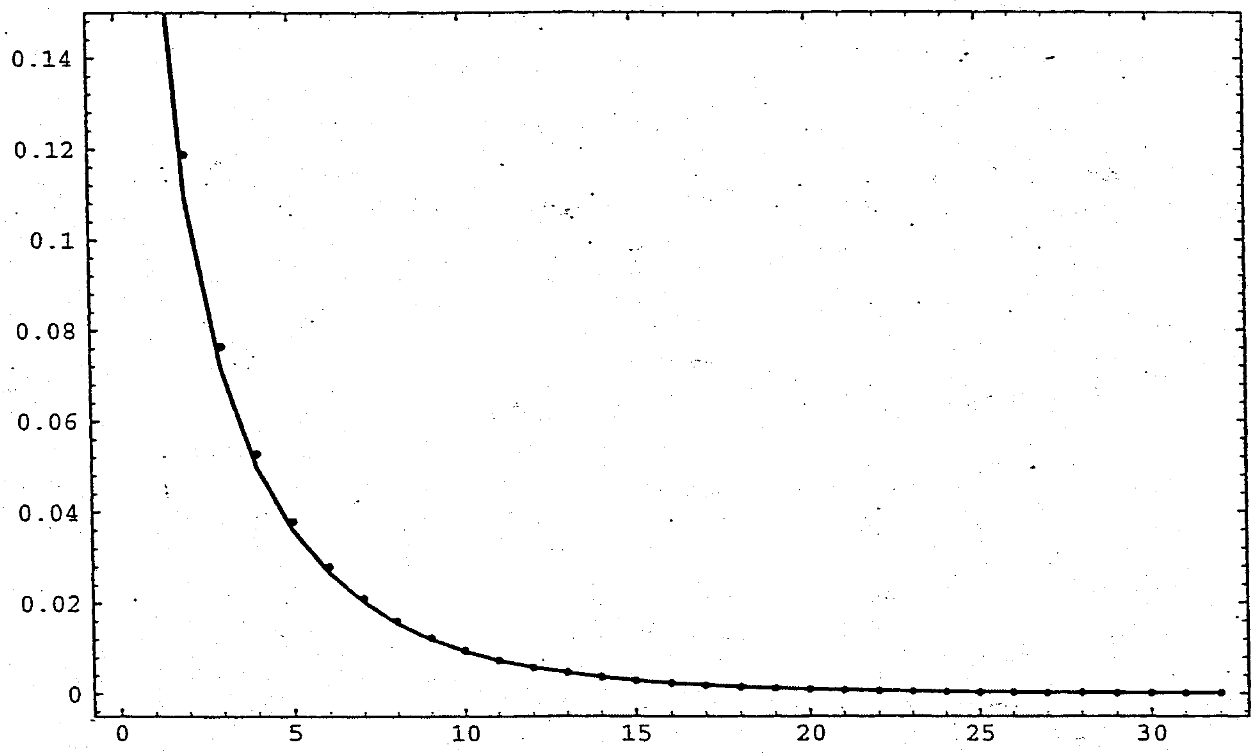
$$\Gamma = \frac{g^2}{2M} \int d\Omega_M \, | \langle M | \psi | m \rangle |^2$$

2-D Ising Model in a Magnetic Field at $T = T_c$

$$\langle \sigma(x)\, \sigma(0) \rangle$$

(G. Delfino & G.M.) Nucl. Phys. B 455

$\langle \sigma(x)\, \sigma(0) \rangle$



$|x|$

$$\xi \sim 10$$

$$\frac{x}{\xi} \sim \frac{1}{10}$$

Figure 14

Form Factor Calculation

Given the exact spectrum of the theory, an operator φ is defined in the Hilbert space once all its matrix elements are assigned

$$_{out}\langle A(\beta_1)\ldots A(\beta_m) | \varphi(0) | A(\beta_{m+1})\ldots A(\beta_n)\rangle_{in}$$

Without loosing in generality, we can always restrict ourselves to the matrix elements of the form

$$F^{\varphi}(\beta_1 \ldots \beta_n) \equiv \langle 0 | \varphi(0) | \beta_1 \ldots \beta_n \rangle$$

In fact, the most general ones, are obtained by the analytic continuations

$$\langle \beta_1 \ldots \beta_m | \varphi | \beta_{m+1} \ldots \beta_n \rangle = F(i\pi + \beta_1, \ldots i\pi + \beta_m, \beta_{m+1}, \ldots \beta_n)$$

This equation holds as far as the set $(\beta_1 \ldots \beta_m)$ is disconnected from the set $(\beta_{m+1} \ldots \beta_n)$, otherwise we can the usual δ- function singularities.

Notice, in fact the under the shift

$$\beta \to i\pi + \beta$$

$$\hat{E} = m \, ch\,(i\pi + \beta) = -E$$

$$\hat{P} = m \, sh\,(i\pi + \beta) = -P$$

Faddeev-Zamolodchikov algebra and physical states

The S-matrix can be seen as a quantity entering the FZ algebra

$$A_a^+(\beta_1) A_b^+(\beta_2) = S_{ab}^{cd}(\beta_1-\beta_2)\, A_c^+(\beta_2) A_d^+(\beta_1)$$

$$A_a^+(\beta_1) A_b(\beta_2) = S_{bc}^{da}(\beta_1-\beta_2)\, A_d^+(\beta_2) A_c(\beta_1) +$$

$$+ 2\pi\, \delta_{ab}\, \delta(\beta_1-\beta_2)$$

The physical in-states are identified by the condition

$$\beta_1 > \beta_2 > \beta_3 \cdots > \beta_n \qquad (\text{in})$$

$$\beta_1 < \beta_2 < \beta_3 \cdots < \beta_n \qquad (\text{out})$$

From now on, we will adopt these orderings to identify the in- and -out states.

FF of operator of spin s

FF. of an operator φ of spin $\underline{s}$ satisfy the equation

$$F(\beta_1+\Lambda, \beta_2+\Lambda, \dots \beta_n+\Lambda) = e^{s\Lambda} F(\beta_1 \dots \beta_n)$$

This equation implies that FF of scalar operators (the only ones we are interested in these lectures) can only depend on the differences of rapidities.

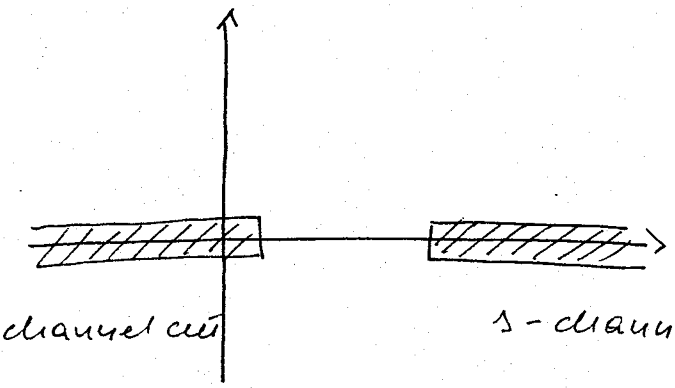
Watson Equations

The FF of local operators satisfy a set of functional equations which are quite general, since they are only based on unitarity and crossing symmetry conditions. These eqs. are known in the literature as Watson Eqs.

In order to deduce them, let us first consider the case of the two-particle FF of a scalar operator $\varphi(x)$.

Two-particle FF

The analytic structure of the S-matrix is shown in fig.



We want to determine the discontinuity of the matrix element between the two cuts.

Let us first consider the discontinuity across the s-channel cut

$$F_{ab}(s+i\varepsilon) = \langle 0 | \varphi(0) | A_a(\beta_1) A_b(\beta_2) \rangle_{in}$$

At the lower edge of the unitarity cut we have

$$F_{ab}(s-i\varepsilon) = \langle 0 | \varphi(0) | A_a(\beta_1) A_b(\beta_2) \rangle_{out}$$

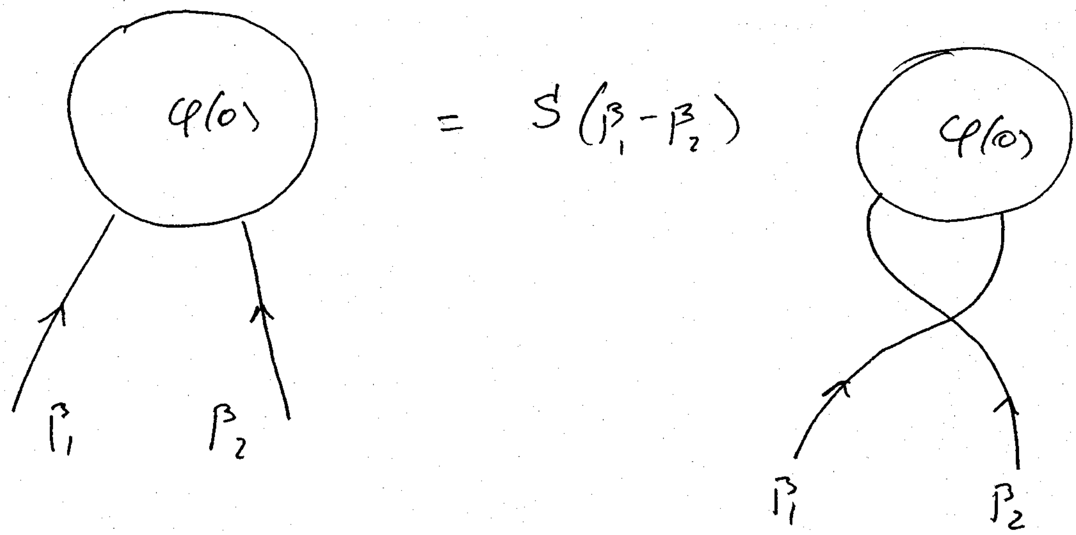
By inserting a complete set of out-states in the first eq., and by using the definition of the S-matrix we have

$$F_{ab}(s+i\varepsilon) = S_{ab}^{cd}(s+i\varepsilon)\, F_{cd}(s-i\varepsilon)$$

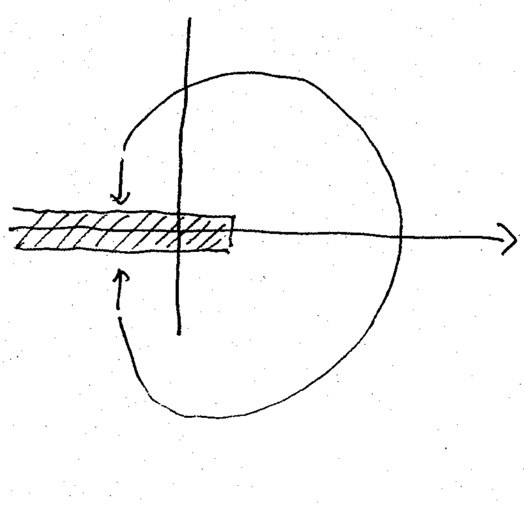
which in terms of the rapidities can be written as

$$F_{ab}(\beta_1-\beta_2) = S_{ab}^{cd}(\beta_1-\beta_2)\, F_{cd}(\beta_2-\beta_1)$$

graphically represented by



This is the only branch-cut discontinuity which this function can have. Let us show, in fact, that it is regular across the t-channel (this result holds for local operators)



Consider

$$\langle A_{\bar b}(p_2)\,|\,\varphi(0)\,|\,A_a(p_1)\rangle_{in}^{out} = \langle 0\,|\,\varphi(0)\,|\,A_a(p_1)\,A_b(-p_2)\rangle_{in}$$

$$= F_{ab}\left(2m_a^2 + 2m_b^2 - s - i\varepsilon\right)$$

$$\langle A_{\bar b}(p_2)\,|\,\varphi(0)\,|\,A_a(p_1)\rangle_{out}^{in} = \langle 0\,|\,\varphi(0)\,|\,A_a(p_1)\,A_b(-p_2)\rangle_{out}$$

$$= F_{ab}\left(2m_a^2 + 2m_b^2 - s + i\varepsilon\right)$$

But the two above matrix elements should coincide, as derived from the stability of the one-particle states.

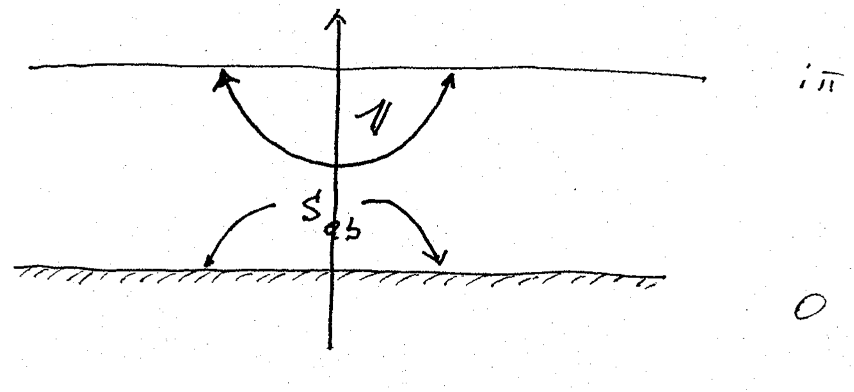
Hence :

$$F_{ab}\left(2m_a^2 + 2m_b^2 - s - i\varepsilon\right) = F_{ab}\left(2m_a^2 + 2m_b^2 - s + i\varepsilon\right)$$

or, in terms of the rapidities

$$F_{ab}(i\pi - \beta) = F_{ab}(i\pi + \beta)$$

Hence, in the physical strip we have



From the above analysis we have the equations

$$\begin{cases} F_{ab}(\beta) = S_{ab}(\beta)\, F_{ab}(-\beta) \\ F_{ab}(i\pi-\beta) = F_{ab}(i\pi+\beta) \end{cases}$$

Notice that the second equations can be written as

$$F_{ab}(\beta_1 + 2\pi i, \beta_2) = F_{ab}(\beta_2, \beta_1)$$

The generalization of the above equations to the case of n-particle FF is given by

$$\begin{cases} F(\beta_1 \cdots \beta_i\, \beta_{i+1} \cdots \beta_n) = S(\beta_i - \beta_{i+1})\, F(\beta_1 \cdots \beta_{i+1}\, \beta_i \cdots \beta_n) \\ F(\beta_1 + 2\pi i, \beta_2 \cdots \beta_n) = F(\beta_2 \cdots \beta_n, \beta_1) \end{cases}$$

The above equations can be solved in a factorized form, in terms of a function, called $F_{min}(\beta)$ solution of the above two-particle FF equation, with the further condition that it does not have poles and zeros in the physical strip.

$$0 < \mathrm{Im}\,\beta \leqslant \pi$$

It is quite interesting to analyse the solution of these equation in the case of the building blocks of the S-matrix of neutral particle.

The recipe is

$$S(\beta) = -\exp\left[\int_0^\infty \frac{dx}{x} f(x)\, sh\, \frac{x\beta}{i\pi}\right]$$

Then

$$F_{min}(\beta) = N\, sh\frac{\beta}{2} \exp\left[\int_0^\infty \frac{dx}{x} \frac{f(x)}{shx} \sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)\right]$$

$$\hat{\beta} = i\pi - \beta$$

For :

$$S(\beta) = \frac{th\frac{1}{2}(\beta + i\pi a)}{th\frac{1}{2}(\beta - i\pi a)} = -\exp\left[2\int_0^\infty \frac{dx}{x} \frac{ch\frac{x}{2}(1-2a)}{ch\frac{x}{2}} sh\frac{\beta x}{i\pi}\right]$$

we have

$$\hat{g}_a(\beta) = N\, sh\,\beta/2 \exp\left[2\int_0^\infty \frac{dx}{x} \frac{ch\frac{x}{2}(1-2a)}{ch\frac{x}{2}\, shx} \sin^2\left(\frac{i\pi\hat{\beta}}{2\pi}\right)\right]$$

$$\equiv N\, sh\,\beta/2\; g_a(\beta)$$

By using the formulas

$$\frac{1}{\operatorname{ch} x} = 2 \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)x}$$

$$\frac{1}{\operatorname{sh} x} = 2 \sum_{k=0}^{\infty} e^{-(2k+1)x}$$

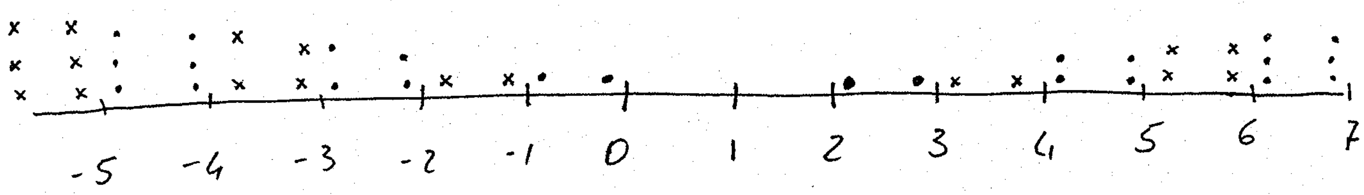
$$\int_0^\infty \frac{dx}{x} e^{-\beta x} \sin^2 ax = \frac{1}{4} \ln \left[1 + \left(\frac{2a}{\beta} \right)^2 \right]$$

The corresponding function can be written as

(a part the prefactor $\operatorname{sh} \frac{\beta}{2}$)

$$g_a(\beta) = \prod_{k=0}^{\infty} \left(\frac{\left[1 + \left(\frac{\hat{\beta}/2\pi}{k+1-a/2} \right)^2 \right] \left[1 + \left(\frac{\hat{\beta}/2\pi}{k+\frac{1}{2}+\frac{a}{2}} \right)^2 \right]}{\left[1 + \left(\frac{\hat{\beta}/2\pi}{k+1+a/2} \right)^2 \right] \left[1 + \left(\frac{\hat{\beta}/2\pi}{k+\frac{3}{2}-\frac{a}{2}} \right)^2 \right]} \right)^{k+1}$$

and has the following analytic structure along the imaginary axis



× poles

∘ zeros

The function $g_a(\beta)$ satisfies the functional equations

$$g_a(\beta+i\pi)\, g_a(\beta) = -\frac{i\, g_a(0)}{\sin \pi a}\left(\sh \beta + i \sin \pi a\right)$$

$$g_a(\beta)\, g_{-a}(\beta) = \frac{\cos \pi a - \ch \beta}{2\cos^2 \frac{\pi a}{2}}$$

which can be obtained graphically.

Moreover

$$g_a(\beta) \simeq e^{-|\beta|/2} \qquad \beta \to \infty$$

so that

$$\hat{g}_a(\beta) \longrightarrow \text{const} \qquad \beta \to \infty$$

Notice that if $S(0) = -1$, then

$$F_{min}(\beta) \simeq \beta \qquad \beta \to 0$$

i.e. the FF should vanish at threshold

We can write the most general solution of the Watson equations as

$$\rightarrow \beta^{\frac{n(n-1)}{2}} \quad \beta \to 0$$

$$F(\beta_1 \dots \beta_n) = Q(\beta_1 \dots \beta_n) \prod_{i<j} F_{min}(\beta_{ij})$$

where $Q(\beta_1 \dots \beta_n)$ is a function which is blind under the two operations

$$\beta_{ij} \to -\beta_{ij}$$

$$\beta_i \to \beta_i + 2\pi i$$

Therefore it has to be a function of

$$Q(\beta_1 \dots \beta_n) = Q(\mathrm{ch}\, \beta_{ij})$$

This function must contain all poles expected for the matrix elements under considerations.

~~By~~ E. The poles in the FF occur when a cluster of m-particles goes on one-particle mass shell. By factorization we have to consider only two type of poles, those related to two-body cluster (bound state poles) and those relative to three-body cluster (kinematical poles)

Bound state Pole

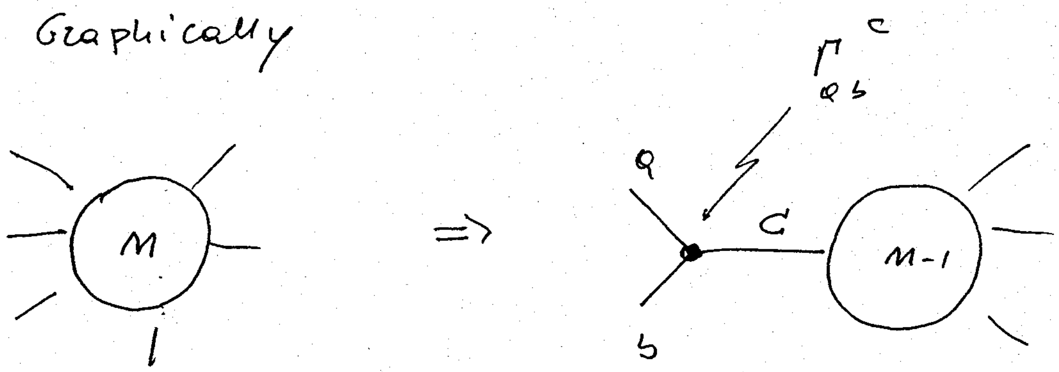
Suppose $A_a(\beta_a)$ and $A_b(\beta_b)$ have a bound state $A_c(\beta_c)$ at a particular (complex) value of their rapidity difference.

Then

$$-i \lim_{\beta_{ab} \to i v_{ab}^c} (\beta_{ab} - i v_{ab}^c)\, F(\beta_a\, \beta_b\, \beta_3 \cdots \beta_n) =$$

$$= \Gamma_{ab}^C\; F(\beta_c, \beta_3 \cdots \beta_n)$$

Graphically



This provides a recursive equation

$$F_M \longrightarrow F_{n-1}$$

Kinematical Poles

These are poles which appear when there are particles A_1 and A_2 which are conjugate particles each other, i.e. they can be annihilated to the vacuum, if their rapidity difference is $\underline{i\pi}$.

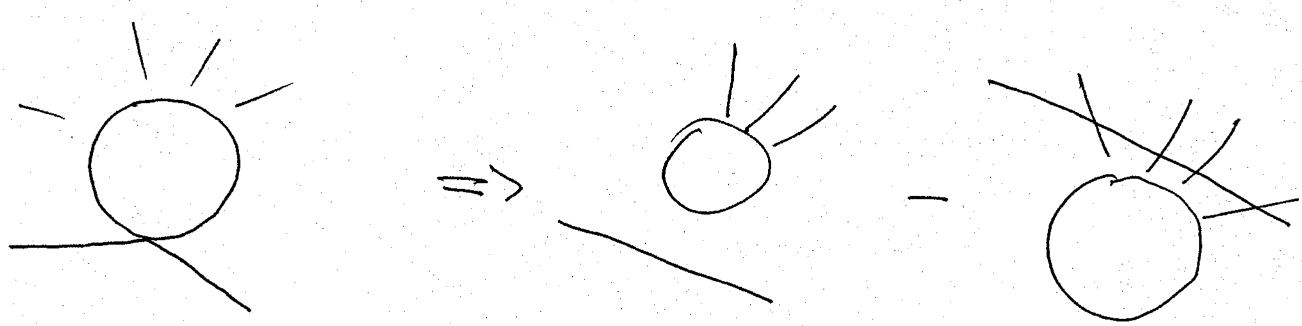
The technical derivation of the corresponding residue can be found below. Here we just report the final equation

$$-i \lim_{\beta' \to \beta} (\beta' - \beta) \, F_{\bar{a} a a_1 \dots a_n} (\beta' + i\pi, \beta, \beta_1 \dots \beta_n) =$$

$$= \left(1 - e^{2\pi i w} \prod_{i=1}^{n} S(\beta - \beta_i)\right) F_{a_1 \dots a_n} (\beta_1 \dots \beta_n)$$

where w is the semi-locality index of the operator with respect the asymptotical particles

Graphically



This pole can be seen as due to an interference phenomenon.

This equation provides a recursive equation

$$F_n \to F_{n-2}$$

Finally, there is an <u>asymptotical bound</u> ruled by the anomalous dimension of the field (valid, at least in unitary theories). Namely, if

$$\langle \varphi(x)\,\varphi(0)\rangle \underset{x\to 0}{\simeq} \frac{1}{|x|^{4\Delta}}$$

hence

$$y \leq \Delta$$

where y is defined by

$$\lim_{|\beta_i|\to\infty} F(\beta_1 \cdots \beta_n) \simeq e^{y|\beta_i|}$$

Kinematical Poles

There are poles which can appear only when there are two particles A_1 and A_2 among the $A_1 - A_n$ which form a conjugate pair of particle - antiparticle. This pair has the quantum number of the vacuum and therefore it can produce "annihilation processes". They occur when their rapidity difference is equal to $i\pi$. These kinematical annihilation poles can be seen as result of the contact terms present in the matrix elements. We want now to fix their residue. To this aim consider initially

$$F^{a'}_{a a_1 - a_n}(\theta' | \theta, \theta_1 - \theta_n) = F_{\bar{a}' a a_1 - a_n}(\theta' + i\pi, \theta, \theta_1 - \theta_n) \qquad (31)$$
$$+ 2\pi \delta_{aa'} \delta(\theta - \theta') F_{a_1 - a_n}(\theta_1 - \theta_n)$$

Let us consider now

$$F^{a'}_{a_1 - a_n a}(\theta' | \theta_1 - \theta_n \theta) = F_{a_1 - a_n a a'}(\theta_1 - \theta_n, \theta, \theta' - i\pi) \qquad (32)$$
$$+ 2\pi \delta_{aa'} \delta(\theta - \theta') F_{a_1 - a_n}(\theta_1 - \theta_n)$$

To move the rapidity θ at the beginning of the string of θ_i', we use the S-matrix

$$F^{a'}_{a_1\ldots a_M,a}(\theta'|\theta_1\ldots\theta_M,\theta) = \prod_{i=1}^{M} S(\theta_i-\theta)\, F^{a'}_{a,a_1\ldots a_M}(\theta'|\theta,\theta_1\ldots\theta_M) =$$

$$= \prod_{i=1}^{M} S(\theta_i-\theta)\, F_{aa_1\ldots a_M a'}(\theta,\theta_1\ldots\theta_M,\theta'-i\pi) \tag{33}$$

$$+\, 2\pi\,\delta_{aa'}\,\delta(\theta-\theta')\, F_{a_1\ldots a_M}(\theta_1\ldots\theta_M)$$

Multiplying both terms in the LHS and in the RHS by

$$\prod_{i=1}^{M} S^{+}(\theta_i-\theta) = \prod_{i=1}^{M} S(\theta-\theta_i) \tag{34}$$

we arrive to

$$F^{a'}_{aa_1\ldots a_M}(\theta'|\theta\,\theta_1\ldots\theta_M) = F_{aa_1\ldots a_M a'}(\theta,\theta_1\ldots\theta_M,\theta'-i\pi) \tag{35}$$

$$+\, 2\pi\,\delta_{aa'}\,\delta(\theta-\theta') \prod_{i=1}^{M} S(\theta-\theta_i)\, F_{a_1\ldots a_M}(\theta_1\ldots\theta_M)$$

Comparing with eq. (31), we have

$$\boxed{\begin{aligned} F_{\bar{a}aa_1\ldots a_M}(\theta'+i\pi,\theta,\theta_1\ldots\theta_M) &= F_{aa_1\ldots a_M\bar{a}}(\theta,\theta_1\ldots\theta'-i\pi) \\ &+ 2\pi\,\delta_{aa'}\,\delta(\theta-\theta')\left[\prod_{i=1}^{M} S(\theta-\theta_i) - 1\right] F_{a_1\ldots a_M}(\theta_1\ldots\theta_M) \end{aligned}}$$

(36)

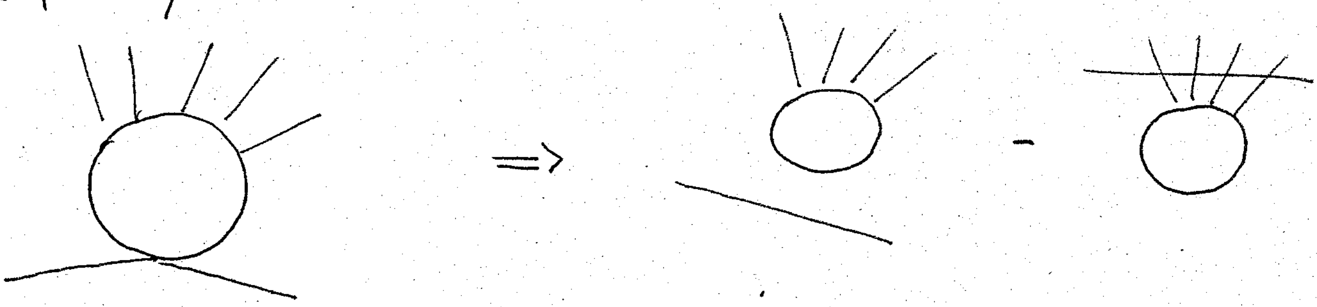
using now the formula

$$\boxed{\frac{1}{x-a-i\varepsilon} = P\left(\frac{1}{x-a}\right) + i\pi\,\delta(x-a)} \tag{37}$$

we see that eq. (36) predicts an "annihilation pole" at $\theta' = \theta + i\pi$ with residue equal

$$\boxed{\begin{aligned} & i \lim_{\beta' \to \beta} (\beta' - \beta)\, F_{\bar{a} a a_1 \ldots a_n}(\beta' + i\pi, \beta, \beta_1 \ldots \beta_n) = \\ & \qquad = \left(1 - \prod_{i=1}^{n} S(\beta - \beta_i)\right) F_{a_1 \ldots a_n}(\beta_1 \ldots \beta_n) \end{aligned}} \tag{38}$$

Graphically



i.e. one understand this as an effect of interference.

This provides a recursive equation

$$\boxed{F_n \longrightarrow F_{n-2}}$$

Quick Summary

(1) FF of scalar operators depend on β_{ij}

(2) The Watson equations, ruling the monodromy of the FF, can be easily solved in terms of the function $F_{min}(\beta_{ij})$

(3) The pole structure of the FF is entirely determined by the singularities entering the two-body channels. They are completely fixed by the poles of the S-matrix.

(4) Notice that, so far, no reference was made whatsoever to any specific operator! This arbitrariness of physical nature agrees with the extra degrees of freedom left in the mathematical problem, namely the parameters in the polynomial $N(\beta_{ij})$

$$Q = \frac{N}{D}$$

entering the general expression of the FF

(5) The anomalous dimension of the operator of which we want to compute the FF puts an upper bound on the partial degree of the polynomial

(6) There are more stringent equations, the so-called cluster Eqs. which give non-linear constraints

$$\lim_{\Lambda \to \infty} F_n^{\varphi}(\beta_1+\Lambda, \beta_2+\Lambda \ldots \beta_m+\Lambda, \beta_{m+1} \ldots \beta_n) = F_m^{\psi}(\beta_1 \ldots \beta_m)\, F_{n-m+1}^{\phi}(\beta_{m+1} \ldots \beta_n)$$

Thermal Perturbation of The Ising Model

Here we show how to solve and compute correlation function for the thermal deformation of the Ising Model.

The Ising Model is the class of universality of a 2^{nd} order phase transition with $\mathbb{Z}_2$ invariance. At the conformal point we have the Kac-Table

$$\begin{array}{|ccc} \frac{1}{2} & \frac{1}{16} & 0 \\ 0 & \frac{1}{16} & \frac{1}{2} \\ \hline \end{array} \qquad c = \frac{1}{2}$$

A local set of fields are given by the conformal families of the relevant operators

$$\{\mathbb{1}\} \qquad \{\sigma\} \qquad \{\varepsilon\}$$

$$\phi_{0,0} \qquad \phi_{\frac{1}{16}\frac{1}{16}} \qquad \phi_{\frac{1}{2}\frac{1}{2}}$$

Other fields of the model, which are non-local operators w.r.t. the previous ones are

$$\psi(z) \;, \quad \bar{\psi}(\bar{z}) \;, \quad \mu$$

The model has a self-dual invariance

$$\sigma \longleftrightarrow \mu$$

$$\varepsilon \longrightarrow -\varepsilon$$

which becomes manifest if we perturb the model

as

$$A = A_{CFT} + \tau \int \varepsilon(x)\, d^2x$$

$$\tau \equiv T - T_c$$

$\tau > 0$ high-temperature phase

$\tau < 0$ low-temperature phase

This deformation creates massive excitations. We have two equivalent ways to look at it

① Fermionic description

$$A = \int d^2x\; \bar{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x)$$

$$m = 2\pi\tau$$

$$\varepsilon = \bar{\psi}\psi$$

In this description the massive excitations are just the massive Majorana fermions, which are free particles with $S = 1$

However, these particles have a non-zero semi-local index in the magnetization sector, i.e. their FF are non-trivial

(2) Bosonic description

The particle is a scalar one, associated to the magnetization operator

$$|1\rangle = \sigma(0)\,|0\rangle$$

and its exact two-body S-matrix is

$$S = -1$$

i.e. the theory is interacting!

In the following we will adopt this second point of view.

Since there is no poles in the S-matrix, no bound states appear. We want to compute the FF of the operators

$$\sigma \longrightarrow (2n+1)\text{- particle states}$$

$$\mu \longrightarrow 2n\text{- particle states}$$

$$\varepsilon = \Theta \longrightarrow 2n\text{- particle states}$$

and the correlators

$$G_1(r) = \langle \sigma(r)\,\sigma(0)\rangle$$

$$G_2(r) = \langle \mu(r)\,\mu(0)\rangle$$

$$G_3(r) = \langle \Theta(r)\,\Theta(0)\rangle$$

F_{min} is simply

$$F_{min} = \sinh \beta/2$$

The easiest FF are those of θ

$$\theta = 2\pi\, \Sigma\, \mathcal{E}(x)$$

since

$$-i \operatorname*{Res}_{\varepsilon \to 0} F_{2n+2}(\beta+i\pi+\varepsilon, \beta, \beta_1 \ldots \beta_{2n}) = (1-(-1)^{2n})\, F_{2n}(\beta_1 - \beta_{2n})$$

$$= 0$$

This means that there are no kinematical poles and we can put consistently

$$F^{\theta} = \begin{cases} -2\pi i m^2 \sinh \beta/2 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

with the normalization condition for the trace of the mom.-energy tensor given by

$$\langle \beta | \theta | \beta \rangle = F^{\theta}(i\pi) = 2\pi m^2$$

The correlator is given by

$$G_3(r) = \langle \Theta(r)\,\Theta(0)\rangle = \frac{1}{2}\int \frac{d\beta_1}{2\pi}\frac{d\beta_2}{2\pi}\,|F(\beta_1-\beta_2)|^2\, e^{-mr(\mathrm{ch}\beta_1+\mathrm{ch}\beta_2)}$$

$$= \frac{(2\pi)^2 m^4}{2(2\pi)^2}\int d\beta_1 d\beta_2\, \mathrm{sh}^2\frac{\beta_1-\beta_2}{2}\, e^{-mr(\mathrm{ch}\beta_1+\mathrm{ch}\beta_2)} =$$

$$= \frac{m^4}{4}\int d\beta_1 d\beta_2\,\left[\mathrm{ch}(\beta_1-\beta_2)-1\right] e^{-mr(\mathrm{ch}\beta_1+\mathrm{ch}\beta_2)} =$$

$$= m^4\left(\left[\int d\beta\, \mathrm{ch}\beta\, e^{-mr\,\mathrm{ch}\beta}\right]^2 - \left[\int d\beta\, e^{-mr\,\mathrm{ch}\beta}\right]^2\right)$$

$$= m^4\left(K_1^2(mr) - K_0^2(mr)\right)$$

$$K_\nu(z) \equiv \int_0^\infty dt\; e^{-z\,\mathrm{ch}\,t}\,\mathrm{ch}\,\nu t$$

$$G_3(mr) = m^4\left[K_1^2(mr) - K_0^2(mr)\right]$$

Notice

$$G_3(mr) \longrightarrow \frac{m^2}{|x|^2}$$

$$c = \frac{3}{4\pi}\int d^2x\, |x|^2 \langle\Theta(x)\Theta(0)\rangle = \frac{3}{4\pi}\cdot\frac{2\pi}{3} = \frac{1}{2} \quad !$$

Different is the story in the magnetization sector.

In fact

$$-i \operatorname*{Res}_{\varepsilon \to 0} F_{2n+1}(\beta + i\pi + \varepsilon, \beta, \beta_1 \cdots \beta_{2n-1}) = [1 - (-1)^{2n-1}] F_{2n-1}(\beta_1 - \beta_{2n-1})$$

and the solution (for all n) is given by

$$\sigma \to F_{2n+1} = \prod_{i<j}^{2n+1} th \frac{\beta_{ij}}{2}$$

$$\mu \to F_{2n} = \prod_{i<j}^{2n} th \frac{\beta_{ij}}{2}$$

and we have, for instance

$$\langle \sigma(r)\sigma(0) \rangle = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int \prod_{k=1}^{2n+1} \frac{d\beta_k}{2\pi} e^{-mr \operatorname{ch} \beta_k} \prod_{i<j} th^2 \frac{\beta_{ij}}{2}$$

Now, notice that

$$th^2 \frac{\beta_{ij}}{2} = \left(\frac{v_i - v_j}{v_i + v_j} \right)^2 \qquad v_i \equiv e^{\beta_i}$$

and

$$\prod_{i<j} th^2 \frac{\beta_{ij}}{2} = \prod_{i<j} \left(\frac{v_i - v_j}{v_i + v_j} \right)^2 = \det \left(\frac{2\sqrt{v_i v_j}}{v_i + v_j} \right)$$

$$= \det W$$

Hence

$$\langle \mu(r)\mu(0)\rangle \pm \langle \sigma(r)\sigma(0)\rangle =$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \int \prod_{k=1}^{n} \frac{d\beta_k}{2\pi}\, e^{-mr\operatorname{ch}\beta_k} \det W$$

$$\equiv \operatorname{Det}\left(1 + zV\right) \qquad z = \pm 1$$

i.e. we have the Fredholm determinant of the integral operator V, whose kernel is given by

$$V(\beta_i \beta_j\, r) = \frac{E(\beta_i\, r)\, E(\beta_j, r)}{v_i + v_j}$$

$$E(\beta_i, r) = \left(2 v_i\, e^{-mr\operatorname{ch}\beta_i}\right)^{\frac{1}{2}}$$

If we define a function $\chi(s)$ – where $s = \frac{mr}{2}$ – by the formula

$$\begin{pmatrix} \langle \sigma(r)\,\sigma(0)\rangle \\ \langle \mu(r)\,\mu(0)\rangle \end{pmatrix} = \begin{pmatrix} \mathrm{sh}\,\chi/2 \\ \mathrm{ch}\,\chi/2 \end{pmatrix} \exp\left[-\frac{1}{4}\int_s^\infty dv\, v\left[\left(\frac{d\chi}{dv}\right)^2 - \mathrm{sh}^2\chi\right]\right]$$

Then the function $\chi(s)$ is a solution of the radial sh. Gordon equation

$$\frac{d^2\chi}{ds^2} + \frac{1}{s}\frac{d\chi}{ds} = 2\,\mathrm{sh}\,(2\chi)$$

That, with the substitution

$$\eta = e^{-\chi}$$

becomes the Painlevé III transcendental differential equation

$$\frac{\eta''}{\eta} = \left(\frac{\eta'}{\eta}\right)^2 - \frac{1}{s}\left(\frac{\eta'}{\eta}\right) + \eta^2 - \frac{1}{\eta^2}$$

A very interesting calculation is to recover the anomalous dimension of the fields σ and μ.

This can be done exactly by studying the solution of the Painlevé equation.

By using the trick of the Feynman gas, in the nearest neighborood approximation of the grand canonical partition function, we have to solve the integral equation

$$2\pi = \int_0^{\infty} dx \; th^2 \frac{x}{2} \; e^{-px}$$

$$p_{num} = 0.12529 \qquad \Leftrightarrow \qquad p_{ex} = 0.125$$

10^{-4} precision !

Form Factors of The Sinh-Gordon Model

A. Fring, G. Mussardo and P. Simonetti, Nucl. Phys B 393 (1992), 413

A. Koubek and G. Mussardo, Phys. Lett B 311 (1993), 193

The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{g^2}(\mathrm{ch}\, g\varphi - 1)$$

and the exact S-matrix

$$S = \frac{\mathrm{th}\,\frac{1}{2}(\beta - i\pi B)}{\mathrm{th}\,\frac{1}{2}(\beta + i\pi B)}$$

$$B = {g^2}/{8\pi}\ \frac{1}{1+{g^2}/{8\pi}}$$

The F_{min} is easily determined

$$F(\beta) = \prod_{k=0}^{\infty} \left| \frac{\Gamma\left(k+\frac{3}{2}+i\hat{\beta}/2\pi\right)\Gamma\left(k+\frac{1}{2}+\frac{B}{2}+\frac{i\hat{\beta}}{2\pi}\right)\Gamma\left(k+1-\frac{B}{2}+\frac{i\hat{\beta}}{2\pi}\right)}{\Gamma\left(k+\frac{1}{2}+i\hat{\beta}/2\pi\right)\Gamma\left(k+\frac{3}{2}-\frac{B}{2}+i\hat{\beta}/2\pi\right)\Gamma\left(k+1+\frac{B}{2}+i\hat{\beta}/2\pi\right)} \right|^2$$

and the general FF can be parameterized as

$$F_n(\beta_1 \cdots \beta_n) = H_M\, Q_M(x_1 \cdots x_n) \prod_{i<j} \frac{F_{min}(\beta_i - \beta_j)}{x_i + x_j}$$

where

$$x_i \equiv e^{\beta_i}$$

and $Q(x_1 \dots x_n)$ are symmetric polynomials in the x_i variables.

They can be expressed in terms of the <u>elementary symmetrical polynomials</u>, defined by the generating function

$$\prod_{i=1}^{n} (x + x_i) = \sum_{k=0}^{n} x^{n-k} \sigma_k(x_1 \dots x_n)$$

Explicitly

$$\sigma_0 = 1$$

$$\sigma_1 = x_1 + x_2 + \dots x_n$$

$$\sigma_2 = x_1 x_2 + \text{perm.}$$

$$\vdots$$

$$\sigma_n = x_1 x_2 \dots x_n$$

The total order of σ_k is $\underline{k}$ and their partial degree in each variable is 1. They satisfy the recursive equation

$$\sigma_k^{(n+2)}(-x, x, x_1 \dots x_n) = \sigma_k(x_1 \dots x_n) - x^2 \sigma_{k-2}^{n}(x_1 \dots x_n)$$

By enforcing the kinematical recursive equation, we have the following recursive eqs. for the polynomials Q_M

$$(-1)^M Q_{M+2}(-x, x, x_1 \ldots x_n) = x\, \mathcal{D}(x \mid x_1 \ldots x_n)\, Q_M(x_1 \ldots x_n)$$

where

$$\mathcal{D}(x \mid x_1 \ldots x_n) = \sum_{k=1}^{M} \sum_{m=1,\,odd}^{k} [m]\, x^{2(n-k)+m}\, \sigma_k\, \sigma_{k-m}\, (-1)^{k+}$$

$$[m] \equiv \frac{\sin \pi m B}{\sin \pi B}$$

Therefore to find the operator content of the model one has to look at the independent solutions of the above recursive equations. At each of this solution is associated an operator.

General properties of the space of the FF

① The kinematical eqs. imply that the FF on the even number of particles are decoupled from those with odd number of particles

$$\to F_{2n} \to F_{2n-2} \to F_{2n-4} \to \dots$$

$$\to F_{2n+1} \to F_{2n-1} \to F_{2n-3} \to \dots$$

This is related to the $\mathbb{Z}_2$ parity of the model.

② Since all FF equations are linear, the FF at level n form a linear space. What is its dimension?

Of course it depends on which space we are looking for the solutions.

For scalar operators, the total order of the polynomial Q is $n(n-1)/2$.

Now, the dimension of the space at level n satisfies

$$\dim \mathbb{F}_n = \dim \mathbb{F}_{n-2} + \dim (\text{kernel})$$

$$Q_n(-x, x_1 \dots x_{n-2}) = 0 \Rightarrow \Sigma(x_1 \dots x_n)$$

But in the space of polynomials of total degree $\frac{n(n-1)}{2}$ there is only one polynomial which satisfies the above condition

$$\Sigma(x_1 \cdots x_n) = \prod_{i<j}^{n} (x_i + x_j) = \det \sigma_{2i-j}$$

Therefore the dimension of the space increases by one unit at each level of iteration!

We have then the following structure

$$\begin{tikzcd}[row sep=small]
n=1 & \phi_1 \arrow[d] & & & \\
n=3 & \phi_1 \arrow[d] & \phi_2 \arrow[d] & & \\
n=5 & \phi_1 \arrow[d] & \phi_2 \arrow[d] & \phi_3 \arrow[d] & \\
n=7 & \phi_1 & \phi_2 & \phi_3 & \phi_4
\end{tikzcd}$$

To any tower is then associated a quantum operator of the theory, which is natural to put in correspondence with the composite operator

$$:\varphi^k(x):$$

Notice that the recursive equations are solved by the following polynomials

$$Q_n(k) = \det M_{ij}(k)$$

$$M_{ij}(k) = \det\left(\sigma_{2i-j}\,[i-j+k]\right)$$

$(n-1)\times(n-1)$ matrix

$$M_{ij} = \begin{vmatrix} [k]\sigma_1 & [k+1]\sigma_3 & [k+2]\sigma_5 & \cdots \\ [k-1] & [k]\sigma_2 & [k+1]\sigma_4 & \\ 0 & [k-1]\sigma_1 & [k]\sigma_3 & \cdots \\ 0 & \vdots & \vdots & \\ \vdots & & & \end{vmatrix}$$

In terms of these polynomials we can express the FF of the elementary field φ and Θ as

$$\langle 0|\varphi(0)|\beta_1\ldots\beta_n\rangle = \left(\frac{4\sin\pi B}{N(B)}\right)^{\frac{n-1}{2}} Q_n(0) \prod_{i<j}\frac{F_{min}}{x_i+x_j}$$

$$\langle 0|\Theta(0)|\beta_1\ldots\beta_n\rangle = \frac{2\pi m^2}{N(B)}\left(\frac{4\sin\pi B}{N(B)}\right)^{n-1} Q_n(1) \prod_{i<j}\frac{F_{min}}{x_i+x_j}$$

It is easy to check that all FF with even number of particles are zero for φ, whereas those with odd number of particles are zero for Θ.

As a check of the conformal limit, the two-particle contribution of the c-theorem give

$$c = \sum_{n=0}^{\infty} c_{2n}$$

$$c_2 = \frac{3}{2} \int_0^{\infty} \frac{d\beta}{\cosh^4 \beta} \, |F(2\beta)|^2$$

B	c_2
1/100	0.99998
1/10	0.9989
1/2	0.9897
4/5	0.9789
1	0.9774