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LEADING ASYMPTOTICS OF FERMION CORRELATION FUNCTIONS IN INTEGRABLE QFTs

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LEADING ASYMPTOTICS OF FERMION CORRELATION FUNCTIONS IN INTEGRABLE QFTs

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Based on a joint work with A.B. Zamolodchikov

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MASSIVE THIRRING MODEL

$$\mathcal{A}_{MTM} = \int d^2x \left\{ \bar{\Psi}\gamma^\mu \partial_\mu \Psi + \frac{g}{2} J_\mu J^\mu + \mathcal{M} \bar{\Psi}\Psi \right\}$$

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \qquad \longleftarrow \quad \textbf{Dirac fermion}$$

$J_\mu = \bar{\Psi}\gamma_\mu \Psi$ **is a non-anomalous vector current**

$$2\pi i \, \langle \, \Psi(x) \, \bar{\Psi}(0) \, \rangle = \frac{\gamma_\mu x^\mu}{|x|} \, G_1(|x|) + \hat{\mathbb{I}} \, G_2(|x|)$$

$$\langle \, \psi_R(x) \, \psi_R^\dagger(0) \, \rangle = i \, \frac{\mathrm{x} - i\mathrm{y}}{|x|} \, G_1(|x|) \, , \qquad \langle \, \psi_R(x) \, \psi_L^\dagger(0) \, \rangle = G_2(|x|)$$

- **Multiplicative unambiguity**

$$\boxed{\Psi \to const \ \Psi}$$

- **CFT normalization condition**

$$\mathcal{A}_{MTM} = \mathcal{A}_{TM} + \mathcal{M} \int d^2x \ \bar{\Psi}\Psi$$

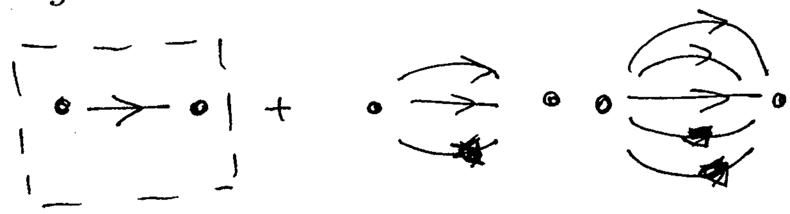
$$\langle \, \psi_R(x) \, \psi_R^\dagger(0) \, \rangle|_{|x| \to 0} \to \langle \, \psi_R(x) \, \psi_R^\dagger(0) \, \rangle_{CFT} = i \, \frac{\mathrm{x} - i\mathrm{y}}{|x|} \, \frac{const}{|x|^{2d_\Psi}}$$

CFT normalization : $G_1(|x|) \to \dfrac{1}{|x|^{2d_\Psi}}$ **as** $|x| \to 0$

Anomaly dimension: $d_\Psi = \frac{1}{2} + \frac{g^2}{4\pi(\pi+g)}$

Large distance: For $g < 0$ there are no bound states:

$\langle\, \psi_R(x)\, \psi_R^\dagger(0)\, \rangle =$



$$\int_{-\infty}^{\infty} \frac{dp}{2\pi E_p}\; |\,\langle\, vac\,|\, \psi_R(0)\,|\, p\,\rangle\;|^2\; e^{ip_\mu x^\mu} + \cdots$$

One particle form-factor

$$|\,\langle\, vac\,|\, \psi_R(0)\,|\, p\,\rangle\;|^2 = \mathbf{Z}_\Psi\, \sqrt{\frac{E_p + p}{E_p - p}} \implies$$

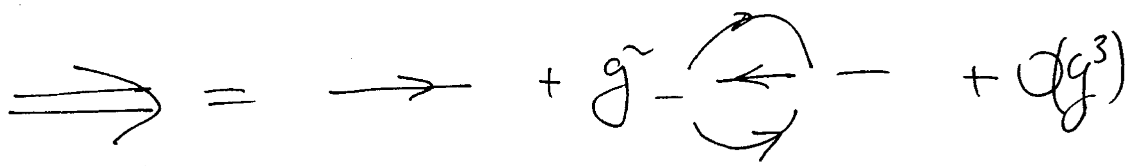
"physical" mass

$$G_1(|x|) \to \mathbf{Z}_\Psi\, \frac{K_1(M|x|)}{\pi} \simeq \mathbf{Z}_\Psi\, \frac{e^{-M|x|}}{\sqrt{2\pi M|x|}} \quad \text{as} \quad |x| \to \infty\ .$$

"Field-strength renormalization constant"

- $\mathbf{Z}_\Psi \sim M^{2d_\Psi}$
- $\mathbf{Z}_\Psi / M^{2d_\Psi}$ **- function of g**

Perturbation Expansion



$$\mathbf{Z}_\Psi = \pi M \left\{ 1 + \left(\frac{g}{2\pi}\right)^2 \left(\log(M^2) + 2\gamma_E + 6 - 2\log 2 - \frac{\pi^2}{3} \right) + O(g^3) \right\},$$

γ_E **is Euler's constant.**

THE PROBLEM

• **Form-factors of local field operator $\mathcal{O}(x)$**

$$\langle\, vac\,|\,\mathcal{O}(x)\,|\; n\;-\;\text{particle states}\,\rangle$$

allow one to generate exact large-distance expansions for the correlation functions by inserting complete set of states of asymptotic particles.

• **It is usually convenient to fix normalizations of the field operators in terms of the short-distance behavior of their correlation functions. If the short-distance behaviour is controlled by associated CFT, the two-point correlation function of a spin-S field $\mathcal{O}(x)$ has the asymptotic form**

$$\langle\, \mathcal{O}(x)\,\mathcal{O}^{\dagger}(0)\,\rangle \to \frac{1}{|x|^{2d_{\mathcal{O}}}}\;\frac{(i\,\mathrm{x}+\mathrm{y})^{2S}}{|x|^{2S}}$$

$$\boxed{\begin{array}{c}\textbf{Find the specific normalization of form}-\textbf{factors}\\ \textbf{which corresponds to the ``CFT normalization''.}\end{array}}$$

TOPOLOGICALLY CHARGED FIELDS IN SG

MTM ≡ sine-Gordon QFT

$$\frac{g}{\pi} = \frac{1}{\beta^2} - 1$$

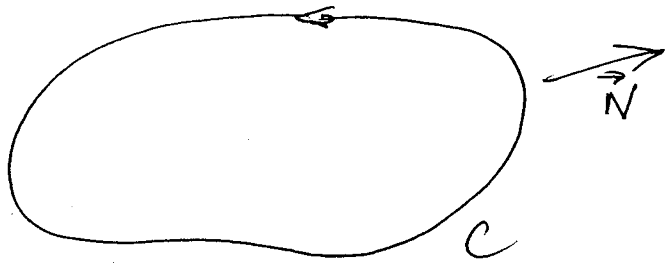
$$\mathcal{A}_{sG} = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\nu \varphi)^2 - 2\mu \cos(\beta\varphi) \right)$$

Global $\mathbb{Z}$ symmetry: $\varphi(x) \to \varphi(x) + 2\pi n/\beta \implies$ Disorder fields

$$\int \mathcal{D}\varphi \; e^{-\mathcal{A}_{sG}} \; \{local\ insertions\}$$

Change of the variable:

$$\varphi \to \varphi'(x) = \begin{cases} \varphi(x) \ \textbf{if } x \textbf{ outside the loop C} \\ \varphi(x) + 2\pi n/\beta \ \textbf{if } x \textbf{ inside the loop C} \end{cases}$$



$$\mathcal{D}\varphi = \mathcal{D}\varphi' \,, \qquad \cos(\beta\varphi) = \cos(\beta\varphi') \ .$$

$$\partial_\nu \varphi' = \partial_\nu \varphi - 2\pi n/\beta \; N_\nu \; \delta^{(2)}(x - C)$$

where N_ν is an external normal to the loop.

$$\frac{1}{16\pi} \int d^2x (\partial_\nu \varphi)^2 =$$

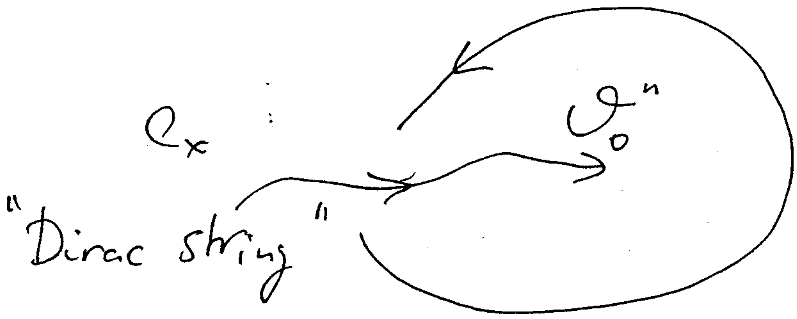
$$\frac{1}{16\pi} \int d^2x (\partial_\nu \varphi')^2 + \frac{n}{4\beta} \oint_C dl \, \partial_\nu \varphi' N^\nu + \frac{\pi n^2}{4\beta^2} \int d^2x \, \left(\delta^{(2)}(x - C)\right)^2$$

Notice that

$$\oint_C dl\,\partial_\nu\varphi'\,N^\nu = \oint_C dx^\mu\,\epsilon_{\mu\nu}\,\partial^\nu\varphi' \Longrightarrow$$

$$\langle\, e^{-\frac{n}{4\beta}\,\oint_C \epsilon_{\mu\nu}\partial^\nu\varphi\,dx^\mu}\,\cdots\rangle\; e^{-const} = \langle\cdots\rangle\;.$$

$$\boxed{\mathcal{O}_0^n(x) = \exp\left\{\frac{n}{4\beta}\int_{C_x}\epsilon_{\mu\nu}\partial^\mu\varphi\,dx^\nu\right\}}$$



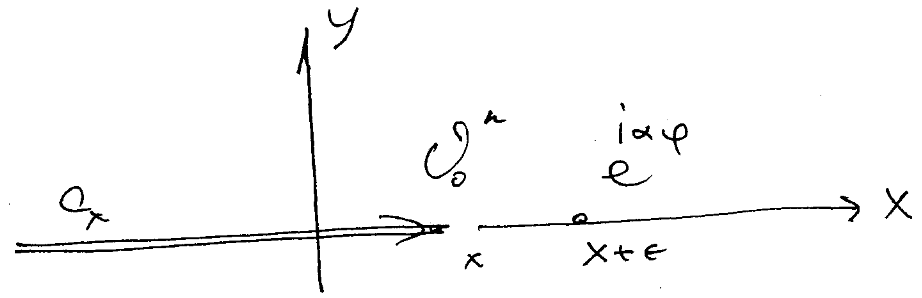
$$\Delta\varphi = \frac{2\pi n}{\beta}$$

$\tilde\varphi = \int_{C_x}\epsilon_{\mu\nu}\partial^\mu\varphi\,dx^\nu$ **is an ill-defined field in sG!**

$\mathcal{O}_0^n$ **are mutually local and they have zero Lorentz spin and the topological charge** n**. They are not local w.r.t.** $\varphi(x)$**.**

More general topologically charged, "semi-local" fields

$$\boxed{\begin{array}{c} \mathcal{O}_a^n(x) = \lim_{\epsilon \to +0} \exp\left\{ -\frac{n}{4\beta} \int_{-\infty}^{\mathrm{x}} \partial_{\mathrm{y}} \varphi(\mathrm{x}, \mathrm{y}) d\mathrm{x} \right\} \exp\left\{ ia\varphi(\mathrm{x}+\epsilon, \mathrm{y}) \right\} \\ \textbf{Lorentz spin}: \ \frac{an}{\beta} \\ \textbf{Topological charge}: \ n \end{array}}$$

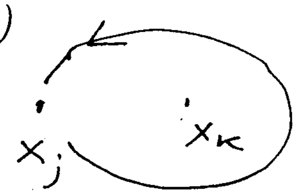


Correlation functions

$$\langle\, \mathcal{O}_{a_1}^{n_1}(\mathrm{x}_1, \mathrm{y}_1) \cdots \mathcal{O}_{a_N}^{n_N}(\mathrm{x}_N, \mathrm{y}_N) \,\rangle$$

- **multivalued function of the coordinates** $x_1, \cdots, x_N$
- **It acquires the phase factor** *(mutual locality index)*

$$\exp\big(-i\pi(a_j\, n_k + a_k\, n_j)/\beta \big)$$



when the point x_j **is brought around** x_k **counterclockwise**

Examples

- *(S.Mandelstam, 1975)* **MTM ($\frac{g}{\pi} = \frac{1}{2\beta^2} - 1$) fermions**

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \mathcal{O}^{\,1}_{-\beta/2} \\ \mathcal{O}^{\,1}_{\beta/2} \end{pmatrix}$$

- **"Spin-charge separation"**

$$\mathcal{A} = \mathcal{A}_{sG} + \int d^2x \, \frac{(\partial_\mu \omega)^2}{16\pi}$$

ω **is a free bozon field:** $\omega = \omega_R(\mathrm{x} + \mathrm{iy}) - \omega_L(\mathrm{x} - \mathrm{iy})$

$$\tilde{\mathcal{O}}^n_a(x) = e^{ia(\omega_R - \omega_L) - \frac{in}{4\gamma}\ (\omega_R + \omega_L)}$$

$$\boxed{\begin{array}{ll} \psi_{\downarrow L} = \eta_\downarrow \ \mathcal{O}^{-1}_{-\beta/4}\, \tilde{\mathcal{O}}^{\,1}_{\gamma/4}\ , & \psi_{\downarrow R} = \eta_\downarrow \ \mathcal{O}^{-1}_{\beta/4}\, \tilde{\mathcal{O}}^{\,1}_{-\gamma/4} \\ \psi_{\uparrow L} = \eta_\uparrow \ \mathcal{O}^{1}_{\beta/4}\, \tilde{\mathcal{O}}^{\,1}_{\gamma/4}\ , & \psi_{\uparrow R} = \eta_\uparrow \ \mathcal{O}^{\,1}_{-\beta/4}\, \tilde{\mathcal{O}}^{\,1}_{-\gamma/4} \end{array}}$$

where $\eta_\sigma = \eta^\dagger_\sigma$ **are Klein factors** ($\eta^2_\uparrow = \eta^2_\downarrow = 1,\ \eta_\uparrow \eta_\downarrow = -\eta_\uparrow \eta_\downarrow$)**.**

Each of the factors $\mathcal{O}$ **and** $\tilde{\mathcal{O}}$ **is nonlocal (they each have spin** $\frac{1}{4}$**), while**

$$\Psi_\sigma(x) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \psi_{\sigma R}(x) \\ \psi_{\sigma L}(x) \end{pmatrix}$$

are local fermi fields of spin $\frac{1}{2}$

DEFORMED $SU(2)$-THIRRING MODEL

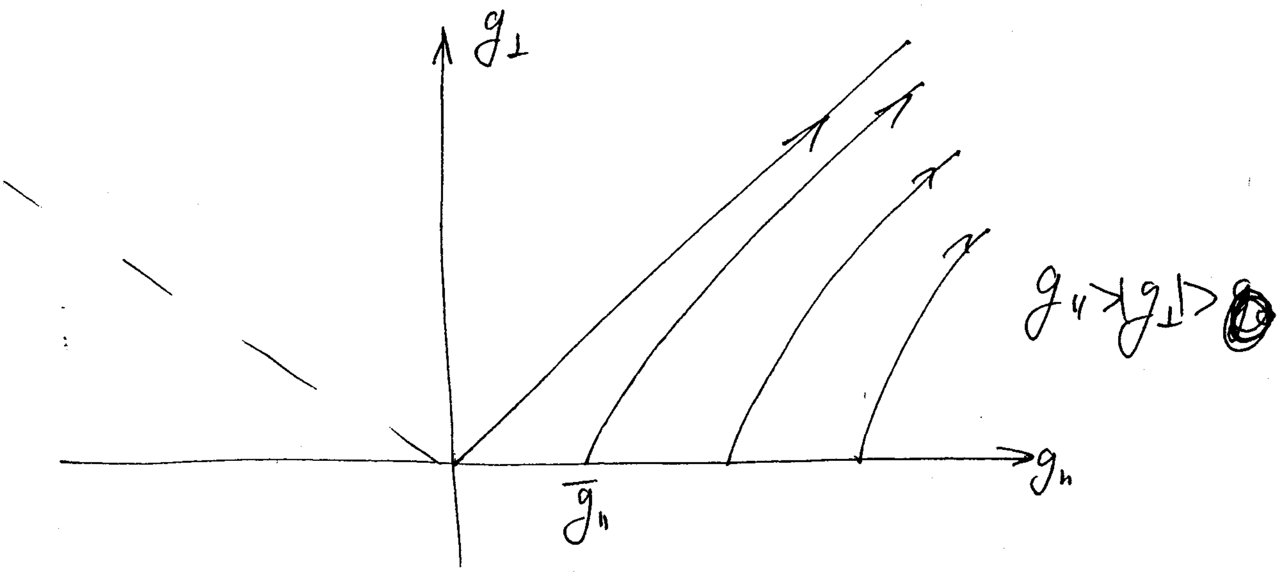
$$\mathcal{A}_{sG} + \int d^2x \, \frac{(\partial_\mu \omega)^2}{16\pi} \equiv$$

$$\int d^2x \Big\{ \sum_{\sigma=\uparrow,\downarrow} \bar{\Psi}^\sigma \gamma_\mu \partial_\mu \Psi_\sigma + \frac{g_0}{2} J_\mu J_\mu + \frac{g_\parallel}{2} J^3_\mu J^3_\mu + 2\, g_\perp J^+_\mu J^-_\mu \Big\}$$

$$J_\mu = \bar{\Psi} \gamma_\mu \Psi \,, \qquad J^A_\mu = \bar{\Psi} \gamma_\mu \tau^A \Psi$$

$g_\parallel > 0,\ g_\perp$ - **"running" coupling constants, g_0 does not flow**

RG (Kosterlitz-Thouless) flow



$$\bar{g}_\parallel = \lim_{L \to -\infty} g_\parallel(L) \qquad\qquad L = \log(\text{scale})$$

The theory depends on $\bar{g}_\parallel$ and g_0 besides the mass scale appearing through dimensional transmutation.

$$\boxed{\frac{1}{\beta^2} = 1 + \frac{2\,\bar{g}_\parallel}{\pi}\,, \qquad\qquad \frac{1}{\gamma^2} = 1 + \frac{2\,g_0}{\pi}}$$

FORM FACTORS

- **Conservation of the topological charge:**

$$\langle\, vac \mid \mathcal{O}^n_a(0) \mid p_1 \cdots p_{n+N}\,,\, p'_1 \cdots p'_N \,\rangle_{\underbrace{-\cdots-}_{N+n}\overbrace{+\cdots+}^{N}} \neq 0$$

- **Lorentz Transformation:** $E \pm p \to e^{\pm\Lambda}\ (E \pm p)$

$$\langle vac \mid \mathcal{O}^n_a(0) \mid p_1 \cdots\rangle_{-\cdots+} \to e^{\frac{na}{\beta}\Lambda}\ \langle vac \mid \mathcal{O}^n_a(0) \mid p_1 \cdots\rangle_{-\cdots+}$$

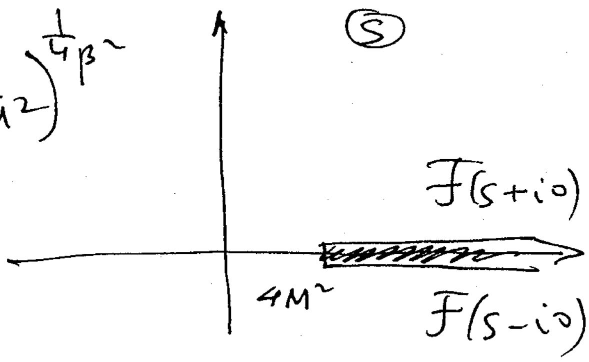
- **Factorizable Scattering**

Up to overall normalization, all form-factors can be written down in closed form, as certain N-fold integrals. For $N = 0$:

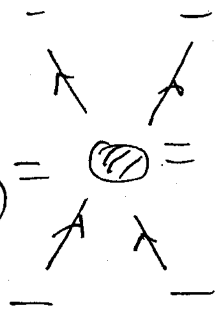
$$\boxed{\begin{aligned}&\langle\, vac\, |\mathcal{O}^n_a(0) \mid p_1 \cdots p_n \,\rangle^{(in)}_{-\cdots-} = \\ &\qquad \sqrt{\mathbf{Z}_n(a)}\ e^{\frac{i\pi n a}{2\beta}} \prod_{m=1}^{n} \left(\frac{E_m + p_m}{E_m - p_m}\right)^{\frac{a}{2\beta}} \prod_{m<j}^{n} \mathcal{F}(s_{mj} + i0)\end{aligned}}$$

$\mathcal{F}(s)$ - *"minimal form factor"* $(s_{12} = (p^\mu_1 + p^\mu_2)^2)$

$$\mathcal{F}(s) \to \left(\frac{s}{M^2}\right)^{\frac{1}{4\beta^2}}$$



$$\frac{\mathcal{F}(s+i0)}{\mathcal{F}(s-i0)} =$$



$$= S^{--}_{--}(s)$$

- **MTM fermions:** $\mathbf{Z}_\Psi = \mathbf{Z}_1(\beta/2)$

- **Deformed** $SU(2)$ **TM fermions:** $\mathbf{Z}_\psi = \mathbf{Z}_1(\beta/4)$

$\left(\xi = \frac{\beta^2}{1-\beta^2}\right)$

$U_q(\widehat{sl(2)})$ WARD IDENTITIES

$U_q(\widehat{sl(2)})$-symmetry in sG *(D.Bernard, A.LeClair, 1991):*

$$\mathcal{J}_\pm(x) = \mathcal{O}^{\pm 2}_{\pm\frac{1}{2\beta}}(x)\,, \quad \mathcal{H}_\pm(x) = \pi\,\xi\,\mu\;\mathcal{O}^{\pm 2}_{\pm(\frac{1}{2\beta}-\beta)}(x)$$
$$\bar{\mathcal{J}}_\pm(x) = \mathcal{O}^{\pm 2}_{\mp\frac{1}{2\beta}}(x)\,, \quad \bar{\mathcal{H}}_\pm(x) = \pi\,\xi\,\mu\;\mathcal{O}^{\pm 2}_{\mp(\frac{1}{2\beta}-\beta)}(x)$$

$$\boxed{\bar\partial\,\mathcal{J}_\pm(x) = \partial\,\mathcal{H}_\pm\,, \qquad \partial\,\bar{\mathcal{J}}_\pm(x) = \bar\partial\,\bar{\mathcal{H}}_\pm}$$

The conserved charges

$$Q_\pm = \frac{1}{\mathbf{Z}_Q}\int_{-\infty}^{\infty}\big(\mathcal{J}_\pm(\mathrm{x},\mathrm{y}) + \mathcal{H}_\pm(\mathrm{x},\mathrm{y})\big)\,d\mathrm{x}$$
$$\bar{Q}_\pm = \frac{1}{\mathbf{Z}_Q}\int_{-\infty}^{\infty}\big(\bar{\mathcal{J}}_\pm(\mathrm{x},\mathrm{y}) + \bar{\mathcal{H}}_\pm(\mathrm{x},\mathrm{y})\big)\,d\mathrm{x}$$
$$H = \frac{\beta}{2\pi}\int_{-\infty}^{\infty}\partial_{\mathrm{x}}\varphi(\mathrm{x},\mathrm{y})\,d\mathrm{x}$$

generate affine quantum group $U_q(\widehat{sl(2)})$ of level zero, with

$$q = e^{i\pi/\beta^2}$$

If $\mathbf{Z}_Q^2 = \mu\,\xi(1+\xi)\,\sin(\pi/\xi)\;\langle\, vac \mid e^{i(\beta-1/\beta)\varphi} \mid vac\,\rangle \quad \Longrightarrow$

$$\boxed{Q_-\bar{Q}_+ - q^2\bar{Q}_+Q_- = \frac{1-q^{2H}}{1-q^{-2}}, \quad Q_+\bar{Q}_- - q^2\bar{Q}_-Q_+ = \frac{1-q^{-2H}}{1-q^{-2}}}$$

$U_q(\widehat{sl(2)})$ action on asymptotic states

- **One-particle states**

$$Q_{\pm}\,|\,p\,\rangle_{\pm} = \bar{Q}_{\pm}\,|\,p\,\rangle_{\pm} = 0\,,$$

$$Q_{\mp}\,|\,p\,\rangle_{\pm} = \left(\frac{E+p}{E-p}\right)^{\frac{1}{4\xi}}\,|\,p\,\rangle_{\mp}\,, \qquad \bar{Q}_{\mp}\,|\,p\,\rangle_{\pm} = \left(\frac{E+p}{E-p}\right)^{-\frac{1}{4\xi}}\,|\,p\,\rangle_{\mp}$$

- **Multy-particle states: Coproduct**

$$\Delta(Q_{\pm}) = Q_{\pm}\otimes 1 + q^{\mp H}\otimes Q_{\pm}\,, \qquad \Delta(\bar{Q}_{\pm}) = \bar{Q}_{\pm}\otimes 1 + q^{\pm H}\otimes \bar{Q}_{\pm}$$

$U_q(\widehat{sl(2)})$ action on fields

For all integer n and m the fields

$$\mathcal{O}^{n}_{a_{n,m}} \qquad \textbf{with} \quad a_{n,m} = \frac{n}{4\beta} + \frac{m\beta}{2}$$

are local w.r.t. the currents $\bar{\mathcal{J}}_{\pm}$ and $\bar{\mathcal{H}}_{\pm}$, and

$$q^{-\frac{n}{2}}\,\bar{Q}_{+}\,\mathcal{O}^{n}_{a_{n,1}} - q^{\frac{n}{2}}\,\mathcal{O}^{n}_{a_{n,1}}\,\bar{Q}_{+} = \frac{2\pi i}{\mathbf{Z}_Q}\,\mathcal{O}^{n+2}_{a_{n-2,1}} \implies$$

$$\boxed{\frac{\mathbf{Z}_{n+2}(a_{n-2,1})}{\mathbf{Z}_{n}(a_{n,1})} = \left(\frac{2\,\mathbf{Z}_Q}{\pi\xi\,\mathcal{F}(-4M^2)}\right)^2}$$

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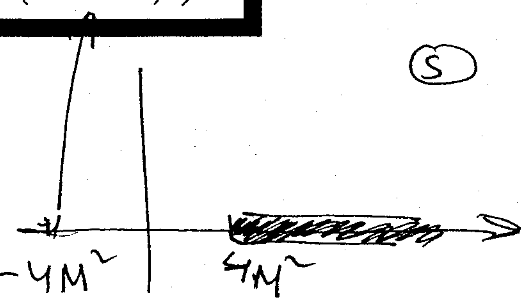

RESULT

$Z_n(a) =$

$$\left(\frac{\sqrt{8}}{\sqrt{\xi}\,\mathcal{F}(-4M^2)}\right)^n \left[\frac{\sqrt{\pi}M\Gamma(\frac{3}{2}+\frac{\xi}{2})}{\Gamma(\frac{\xi}{2})}\right]^{2d(a,n)} \exp\left[\int_0^\infty \frac{dt}{2t}\times\right.$$

$$\left.\left\{\frac{\cosh(4\xi at/\beta)e^{-(1+\xi)nt}-1}{\sinh(\xi t)\sinh\big((1+\xi)t\big)\cosh(t)} + \frac{n}{\sinh(t\xi)} - 4d(a,n)\ e^{-2t}\right\}\right]$$

Here

$$\mathcal{F}(-4M^2) = 2\,\xi^{-\frac{1}{4}}\ \exp\left(-\int_0^\infty \frac{dt}{t}\ \frac{\sinh(t)\ \sinh(t(\xi-1))}{\sinh(t\xi)\ \cosh^2(t)}\right)$$

$$d(a,n) = 2a^2 + \frac{n^2}{8\beta^2}$$

Sale dimension of $\mathcal{O}_a(0)$

$$\xi = \frac{\beta^2}{1-\beta^2}$$

CONCLUSION

- **Example:**

$$\mathcal{A}_{SU(2)} = \int d^2x \left\{ \sum_{\sigma=\uparrow,\downarrow} \bar{\Psi}^{\sigma}\gamma^{\mu}\partial_{\mu}\Psi_{\sigma} + \frac{g}{2}\, \vec{J}_{\mu}\vec{J}^{\mu} \right\} .$$

$$2\pi\, i\, \langle\, \Psi_{\sigma'}(x)\, \bar{\Psi}^{\sigma}(0)\, \rangle = \frac{\gamma_{\mu}x^{\mu}}{|x|^2}\, \delta^{\sigma}_{\sigma'} \times \begin{cases} \mathbf{1} & \text{as} \quad |x| \to 0 \\ \mathbf{C}\ e^{-M|x|} & \text{as} \quad |x| \to \infty \end{cases} ,$$

$$\mathbf{C} = 2^{-\frac{5}{6}}\ e^{-\frac{1}{4}}\ A_G^3 = 0.921862\ldots\ .$$

- **Subleading asymptotics**

  Short distances: Conformal Perturbation Theory

  Large distances: Next form factors

- **Exact asymptotics in lattice systems (XXZ, Hubbard,... chains).**

Short distances $|x| \to 0$

$$2\pi\, i\, \langle\, \Psi_{\sigma'}(x)\, \bar{\Psi}^{\sigma}(0)\,\rangle = \frac{\gamma_\mu x^\mu}{|x|^2}\delta^{\sigma}_{\sigma'}\Big(1-\frac{3}{16}g-\frac{75}{512}g^2-\frac{261}{8192}g^3+O(g^4)\Big)$$

Here

$$-g^{-1}+\frac{1}{2}\,\log(g) = \log\Big(\sqrt{\frac{\pi}{2}}\ e^{\gamma_E-\frac{5}{8}}\ M\,|x|\Big) \qquad (\gamma_E = 0.577216\ldots)$$

Large distances $|x| \to +\infty$

$$2\pi\, i\, \langle\, \Psi_{\sigma'}(x)\, \bar{\Psi}^{\sigma}(0)\,\rangle = \frac{\gamma_\mu x^\mu}{|x|^2}\ \delta^{\sigma}_{\sigma'}\ \Big(\mathbf{C}\ e^{-M|x|}+O(e^{-3M|x|})\Big)$$

$$\mathbf{C} = 2^{-\frac{5}{6}}\ e^{-\frac{1}{4}}\ A_G^3 = 0.921862\ldots\ .$$