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SUMMER SCHOOL
on
LOW-DIMENSIONAL QUANTUM SYSTEMS:
Theory and Experiment
(16 - 27 JULY 2001)

PLUS

PRE-TUTORIAL SESSIONS
(11 - 13 JULY 2001)

THE SCALING TWO-DIMENSIONAL ISING MODEL IN A MAGNETIC FIELD

G. DELFINO
S.I.S.S.A.
International School for Advanced Studies
Via Beirut 2-4
34123 Trieste
Italy

The scaling two-dimensional Ising model in a magnetic field

Gesualdo Delfino

SISSA–Trieste

Goals :

– Review some recent progress on a fundamental model

– Illustrate a more general framework

Contents :

– Introduction

– Ising field theory

– From scattering theory to correlation functions

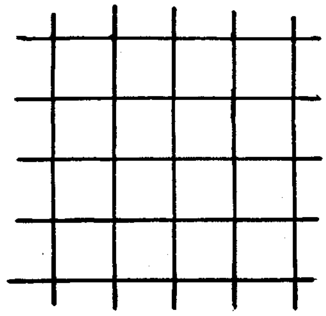
– Universality

– Breaking integrability

The two-dimensional Ising model

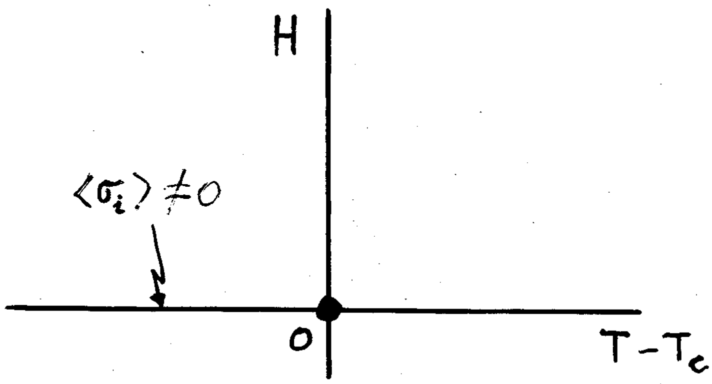
$$E = -\frac{1}{T}\sum_{\langle i,j\rangle} \sigma_i\sigma_j + H\sum_i \sigma_i\,, \qquad \sigma_i = \pm 1$$

$$Z = \sum_{conf} e^{-E}$$



Spin-reversal symmetry for $H = 0$

The simplest model of statistical mechanics exhibiting a phase transition. The critical point is located at $H = 0$, $T = T_c$



- Thermal case : $H = 0$

– L. Onsager, 1944 : free energy

– C.N. Yang, 1952 : spontaneous magnetisation

– B. Mccoy and T.T. Wu, 1967 : correlation functions

- $H \neq 0$: No lattice solution

Scaling limit : $\xi \to \infty$ as $T \to T_c$, $H \to 0$

$\Longrightarrow$ continous (field theoretic) description

Universality : the scaling limit does not depend on the microscopic realisation (e.g. choice of the lattice)

Ising field theory

1. Critical point: conformal field theory (CFT)

$$z = x_1 + ix_2\,, \qquad \bar{z} = x_1 - ix_2$$

conformal scaling operators: $\varphi(x) = \varphi_L(z)\varphi_R(\bar{z})$

$$\varphi(x) \quad \longrightarrow \quad (\Delta_\varphi, \bar{\Delta}_\varphi) \qquad \text{conformal dimensions}$$

$$\langle \varphi(x)\varphi(0)\rangle \sim z^{-2\Delta_\varphi}\,\bar{z}^{-2\bar{\Delta}_\varphi}$$

$$X_\varphi \equiv \Delta_\varphi + \bar{\Delta}_\varphi \qquad \text{scaling dimension}$$

$$S_\varphi \equiv \Delta_\varphi - \bar{\Delta}_\varphi \qquad \text{spin}$$

$\varphi(x)$ local if $S_\varphi \in Z/2 \quad \longrightarrow \quad \langle \varphi(x)\varphi(0)\rangle$ single valued

$\varphi(x)$ scalar if $S_\varphi = 0 \quad \longrightarrow \quad \langle \varphi(x)\varphi(0)\rangle \sim |x|^{-2X_\varphi}$

$\varphi(x)$ relevant if $X_\varphi < 2$

Ising: the simplest (unitary) CFT

central charge: $C = 1/2$

conformal dimensions: 0, 1/16, 1/2 (mod $n > 0$)

relevant local operators:

$$\begin{cases} I = (0,0) & \text{identity} \\ \psi = (1/2, 0)\,, \quad \bar{\psi} = (0, 1/2) & \text{free neutral fermions} \\ \sigma = (1/16, 1/16)\,, \quad \varepsilon = (1/2, 1/2) & \end{cases}$$

Symmetry:

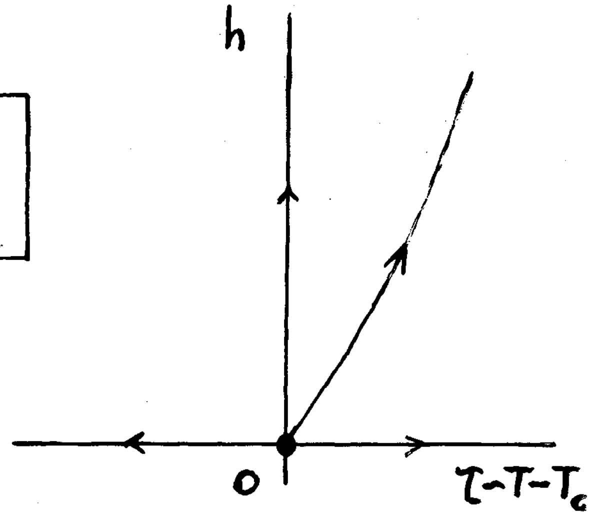
$$\begin{cases} \sigma \times \sigma \sim I + \varepsilon \\ \sigma \times \varepsilon \sim \sigma \\ \varepsilon \times \varepsilon \sim I \end{cases} \Longrightarrow \quad \begin{matrix} \sigma \;\; Z_2\text{-odd} & \text{(spin)} \\ \varepsilon \;\; Z_2\text{-even} & \text{(energy)} \end{matrix}$$

[redacted]

2. **Scaling limit:** deformed CFT

The operator space is isomorphic to the conformal case, but $\varphi(x) \neq \varphi_L(z)\varphi_R(\bar{z})$

$$\boxed{\mathcal{A} = \mathcal{A}_{CFT} - \tau \int d^2x\, \varepsilon(x) - h \int d^2x\, \sigma(x)}$$



$\tau \sim m^{2-X_\varepsilon} = m\,, \quad h \sim m^{2-X_\sigma} = m^{15/8}$

One-parameter family of renormalisation group trajectories labeled by $\eta \equiv \tau/|h|^{8/15}$

- Thermal case: $h = 0$

  $\varepsilon \sim \psi\bar{\psi} \quad \Longrightarrow \quad$ free massive fermion theory

  but the spin sector is non-trivial

- The fermionic language is not helpful at $h \neq 0$

A. Zamolodchikov, 1988: **integrable** deformations of CFTs

In particular: the Ising field theory is integrable when $h = 0$ **or** $\tau = 0$

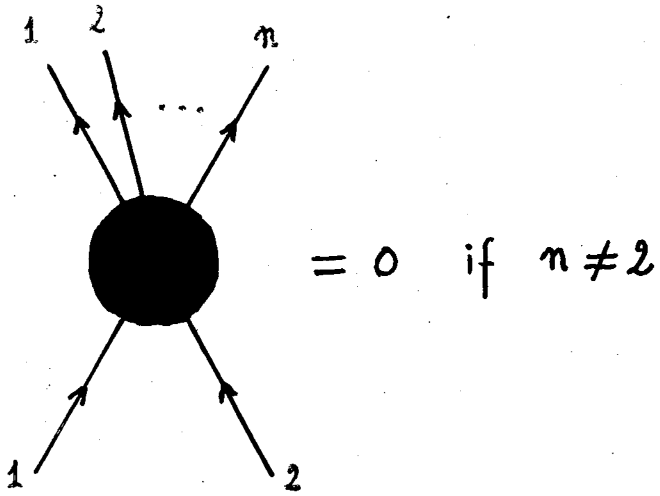
Waarnar, Nienhuis, Seaton, 1992: an integrable lattice model in the same universality class as Ising at $T = T_c$

Integrable Quantum Field Theories

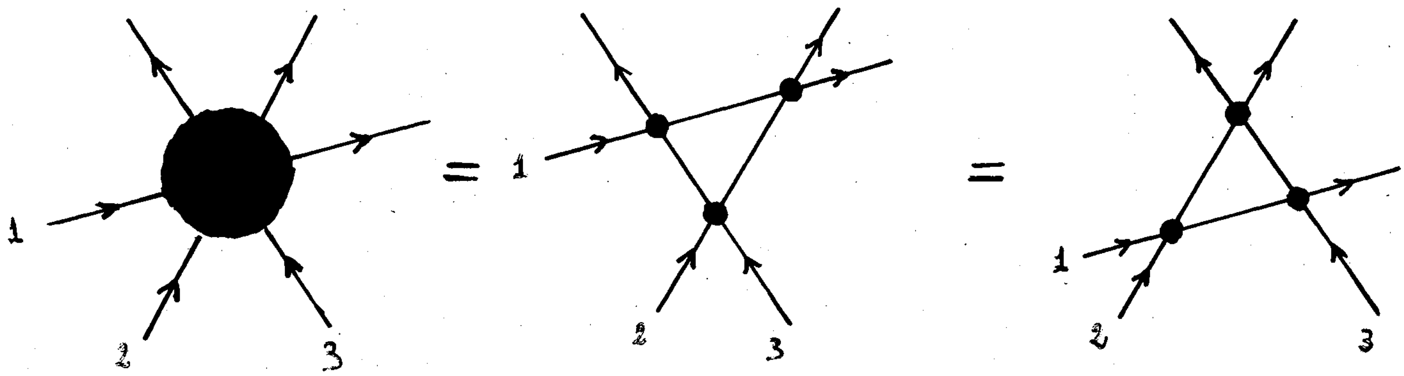
Possess an infinite number of quantum integrals of motion

Consequences for the **scattering theory**:

- Elasticity:



- Factorisation:



The scattering amplitudes are determined exactly by requiring analyticity, unitarity, crossing symmetry, ...

Scattering theory for self-conjugated particles

$|0\rangle$ vacuum state

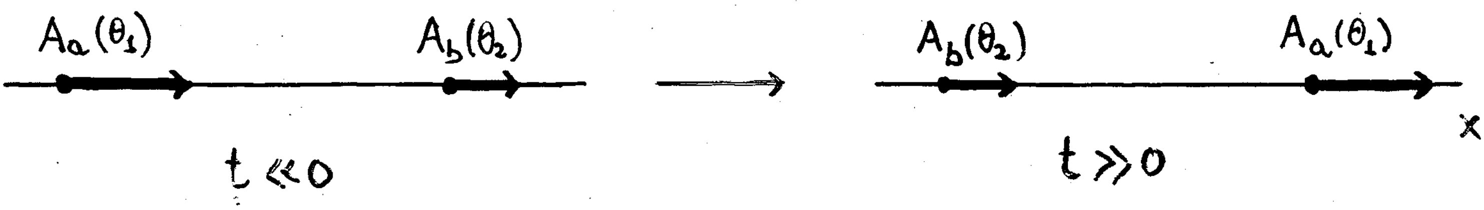
$A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n)|0\rangle = |A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n)\rangle$ particle states

$A_a(\theta)$ creates a particle of species a (mass m_a) with energy-momentum

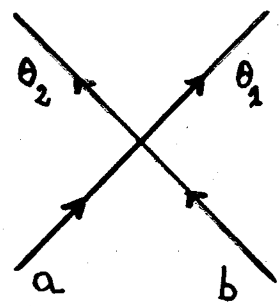
$$(p_a^0, p_a^1) = (m_a \cosh\theta, m_a \sinh\theta)$$

$$p_a^\mu (p_a)_\mu = (p_a^0)^2 - (p_a^1)^2 = m_a^2$$

Two-particle scattering : $(\theta_1 > \theta_2)$



$$A_a(\theta_1)A_b(\theta_2) = S_{ab}(\theta_1-\theta_2)A_b(\theta_2)A_a(\theta_1)$$



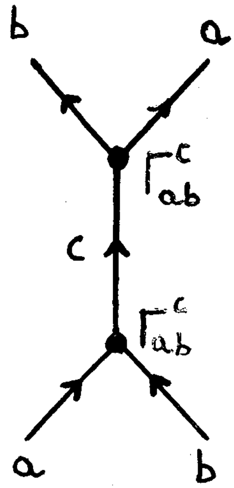
$s_{ab} = (p_1+p_2)^2 = m_a^2+m_b^2+2m_am_b\cosh(\theta_1-\theta_2)$ (centre of mass energy)^2

- Unitarity : $S_{ab}(\theta)S_{ab}(-\theta) = 1$
- Crossing symmetry : $S_{ab}(\theta) = S_{ab}(i\pi-\theta)$

Solution : $S_{ab}(\theta) = \pm \prod_{\alpha \in \mathcal{A}_{ab}} t_\alpha(\theta)$

$$t_\alpha(\theta) = \frac{\tanh \frac{1}{2}(\theta + i\pi\alpha)}{\tanh \frac{1}{2}(\theta - i\pi\alpha)}$$

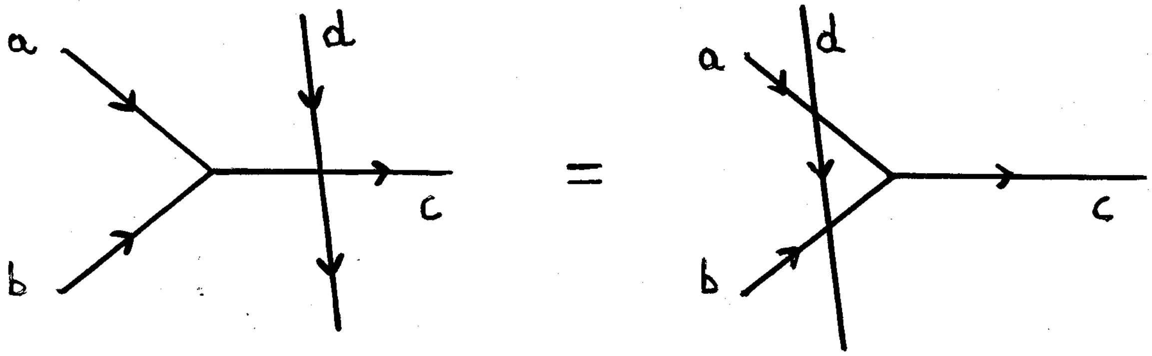
Bound states :



$$S_{ab}(\theta \simeq iu^c_{ab}) \simeq \frac{i(\Gamma^c_{ab})^2}{\theta - iu^c_{ab}}$$

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u^c_{ab}$$

Bootstrap : $\bar{u}^c_{ab} \equiv \pi - u^c_{ab}$



$$S_{dc}(\theta) = S_{da}(\theta - i\bar{u}^b_{ca}) S_{db}(\theta + i\bar{u}^a_{bc})$$

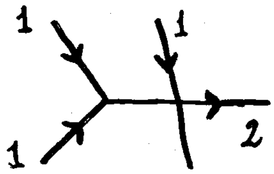
Strategy :

start with the lightest particle A_1 ;

if S_{11} has a pole corresponding to A_2 ,

use the bootstrap eq. to compute S_{12} ;

check the poles of S_{12} ;



⋮

continue until no new particle is generated .

Scattering theory for the Ising integrable directions

1. $h = 0$

A neutral **free** fermion $\Longrightarrow \quad S(\theta) = -1$

$$m \sim |\tau|$$

2. $\tau = 0\,,\ h \neq 0$ (A. Zamolodchikov, 1988)

The conserved currents give information on the poles of S_{11}. Then the bootstrap closes on 8 particles A_a ($a = 1, \ldots, 8$) with masses

$$\begin{aligned}
m_1 &\sim |h|^{8/15} \\
m_2 &= 2m_1 \cos\frac{\pi}{5} = (1.6180339887..)\, m_1 \\
m_3 &= 2m_1 \cos\frac{\pi}{30} = (1.9890437907..)\, m_1 \\
m_4 &= 2m_2 \cos\frac{7\pi}{30} = (2.4048671724..)\, m_1 \\
m_5 &= 2m_2 \cos\frac{2\pi}{15} = (2.9562952015..)\, m_1 \\
m_6 &= 2m_2 \cos\frac{\pi}{30} = (3.2183404585..)\, m_1 \\
m_7 &= 4m_2 \cos\frac{\pi}{5} \cos\frac{7\pi}{30} = (3.8911568233..)\, m_1 \\
m_8 &= 4m_2 \cos\frac{\pi}{5} \cos\frac{2\pi}{15} = (4.7833861168..)\, m_1
\end{aligned}$$

Scattering amplitudes:

$$S_{11}(\theta) = t_{2/3}(\theta)t_{2/5}(\theta)t_{1/15}(\theta) \quad \Longrightarrow \quad A_1A_1 \to A_1, A_2, A_3$$

$$S_{12}(\theta) = t_{4/5}(\theta)t_{3/5}(\theta)t_{7/15}(\theta)t_{4/15}(\theta) \quad \Longrightarrow \quad A_1A_2 \to A_1, A_2, A_3, A_4$$

and so on ... (36 amplitudes)

Form factors for self-conjugated particles

$\Phi(x)$ scalar operator

$$F^{\Phi}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n) = \langle 0|\Phi(0)|A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n)\rangle$$

General properties : (Karowski et al, 1978; Smirnov)

i) $F^{\Phi}_{\ldots,a_i,a_{i+1},\ldots}(\ldots,\theta_i,\theta_{i+1},\ldots) =$

$$S_{a_i,a_{i+1}}(\theta_i-\theta_{i+1})F^{\Phi}_{\ldots,a_{i+1},a_i,\ldots}(\ldots,\theta_{i+1},\theta_i,\ldots)$$

ii) $F^{\Phi}_{a_1,\ldots,a_n}(\theta_1+2i\pi,\theta_2,\ldots,\theta_n) = (-1)^{l_\Phi}F^{\Phi}_{a_2,\ldots,a_n,a_1}(\theta_2,\ldots,\theta_n,\theta_1)$

iii) $-i\,\mathrm{Res}_{\theta_a-\theta_b=iu^c_{ab}}\,F^{\Phi}_{a,b,a_1,\ldots,a_n}(\theta_a,\theta_b,\theta_1,\ldots,\theta_n) =$

$$\Gamma^c_{ab}\,F^{\Phi}_{c,a_1,\ldots,a_n}(\theta_c,\theta_1,\ldots,\theta_n)$$

iv) $-i\,\mathrm{Res}_{\theta'=\theta+i\pi}\,F^{\Phi}_{a,a,a_1,\ldots,a_n}(\theta',\theta,\theta_1,\ldots,\theta_n) =$

$$\left[1-(-1)^{l_\Phi}\prod_{j=1}^{n}S_{a_j,a}(\theta_j-\theta)\right]F^{\Phi}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$$

l_Φ is 0 if Φ and the particles are mutually local, 1 otherwise

These equations are linear in the operator

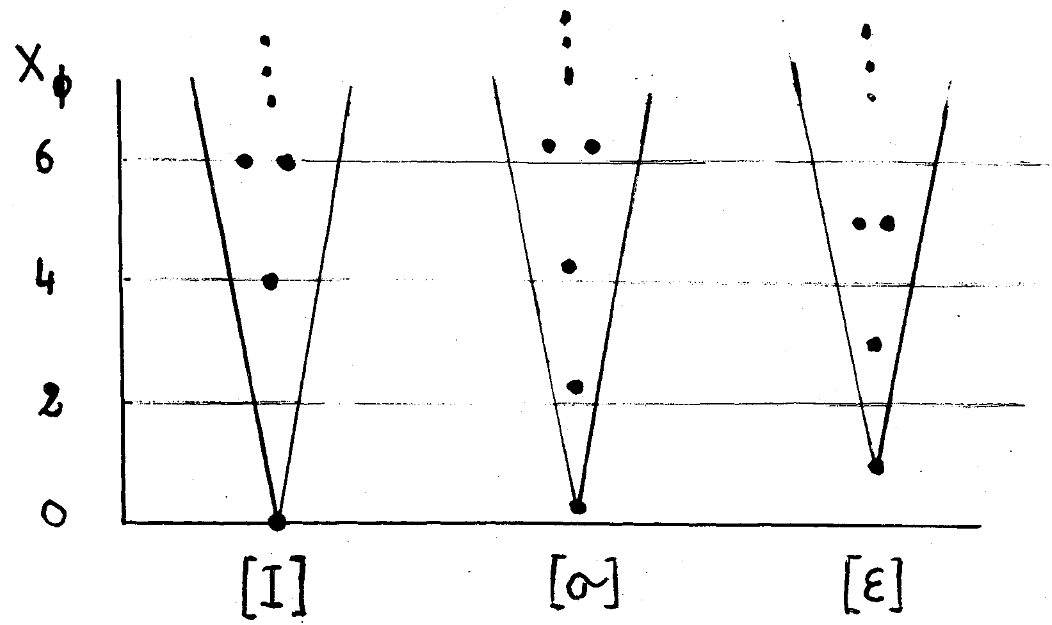
Correlation functions :

$$\langle\Phi_1(r)\Phi_2(0)\rangle = \sum_n(2\pi)^{-n}\int_{\theta_1>\cdots>\theta_n}d\theta_1\cdots d\theta_n\,F^{\Phi_1}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$$

$$\times\left[F^{\Phi_2}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)\right]^* e^{-r\sum_k m_k\cosh\theta_k}$$

The identification problem I

Operator space:



How do we select the different operators in the ff approach?

Asymptotic bound (G.D. and G. Mussardo, 1995)

$$\langle \Phi(r)\Phi(0)\rangle \to r^{-2X_\Phi}, \qquad r \to 0$$

$$\Longrightarrow \quad M_p \equiv \int d^2r\, r^p \langle \Phi(r)\Phi(0)\rangle \; < +\infty \qquad \text{if } p > 2X_\Phi - 2$$

$$M_p \propto \sum_n \int (\prod_k d\theta_k)\, |F^\Phi_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)|^2 \int d^2r\, r^p\, e^{-r\sum_k m_k \cosh\theta_k}$$

$$\Longrightarrow \quad \boxed{\lim_{|\theta_i|\to\infty} F^\Phi_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n) \le \text{constant} \exp(X_\Phi|\theta_i|/2)}$$

Form factors in the thermal Ising model (h=0)

(Berg, Karowski, Weisz, 1979; Yurov, Al. Zamolodchikov, 1991)

$$S(\theta) = -1, \quad \forall\, \tau \qquad \text{(Duality)}$$

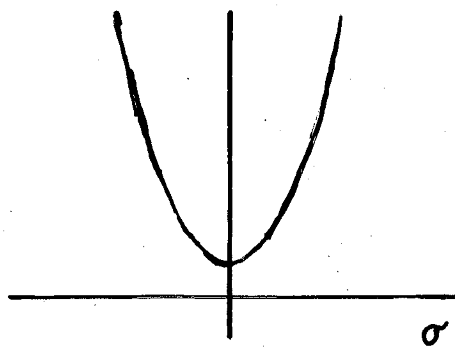
$$\text{Res}_{\theta'=\theta+i\pi}\langle 0|\Phi(0)|\theta',\theta,\theta_1,..,\theta_n\rangle = i\left[1-(-1)^{n+l_\Phi}\right]\langle 0|\Phi(0)|\theta_1,..,\theta_n\rangle$$

$$\langle 0|\varepsilon(0)|\theta_1,..,\theta_n\rangle \propto \delta_{n,2}\sinh\frac{\theta_1-\theta_2}{2}, \quad \forall\,\tau \qquad (\varepsilon\sim\bar\psi\psi,\ l_\varepsilon=0)$$

$$\langle 0|\sigma(0)|\theta_1,..,\theta_n\rangle \propto \prod_{i<j}\tanh\frac{\theta_i-\theta_j}{2}, \qquad \begin{cases} n \text{ odd}, & \tau>0 \\ n \text{ even}, & \tau<0 \end{cases}$$

$\tau > 0$: $\qquad l_\sigma = 0$

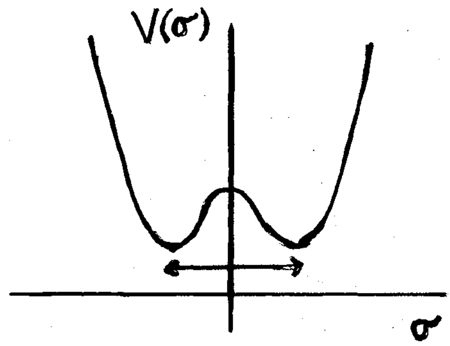
$$\langle 0|\sigma|\theta\rangle \neq 0$$



$\tau < 0$: $\qquad l_\sigma = 1$

$$\text{Res}_{\theta_1-\theta_2=i\pi}\langle 0|\sigma|\theta_1,\theta_2\rangle = 2i\langle\sigma\rangle \neq 0$$

$$\langle 0|\sigma|\theta_1,\theta_2\rangle = i\langle\sigma\rangle\tanh\tfrac{\theta_1-\theta_2}{2} = \langle\sigma\rangle\sqrt{\tfrac{4m^2-s}{s}}$$



N.B.: The asymptotic bound uniquely selects the solutions. For example $\sinh^k(\theta_1-\theta_2)/2$, k odd positive, satisfies the ff eqs. but growes faster that $\exp(X_\varepsilon\theta_i/2) = \exp(\theta_i/2)$ if $k \neq 1$

Form factors in the magnetic Ising model (τ=0)

(G.D., G. Mussardo, 1995; G.D., P. Simonetti, 1996)

$S_{ab}(\theta) = \prod_{\alpha\in\mathcal{A}_{ab}} t_\alpha(\theta), \qquad S_{ab}(0) = (-1)^{\delta_{ab}}, \qquad a,b = 1,\ldots,8$

Two-particle ff: $F^{\Phi}_{ab}(\theta_1,\theta_2) \equiv F^{\Phi}_{ab}(\theta_1-\theta_2)$

i) $F^{\Phi}_{ab}(\theta) = S_{ab}(\theta) F^{\Phi}_{ba}(-\theta)$

ii) $F^{\Phi}_{ab}(\theta+2i\pi) = F^{\Phi}_{ba}(-\theta)$

iii) $\text{Res}_{\theta=iu^c_{ab}} F^{\Phi}_{ab}(\theta) = i\,\Gamma^c_{ab} F^{\Phi}_c$

iv) $\text{Res}_{\theta=i\pi} F^{\Phi}_{ab}(\theta) = 0$

$$\boxed{F^{\Phi}_{ab}(\theta) = \frac{Q^{\Phi}_{ab}(\theta)}{D_{ab}(\theta)}\, F^{min}_{ab}(\theta)}$$

- $F^{min}_{ab}(\theta) = \left(-i\sinh\frac{\theta}{2}\right)^{\delta_{ab}} \prod_{\alpha\in\mathcal{A}_{ab}} T_\alpha(\theta)$ solves (i) and (ii)

$$T_\alpha(\theta) = \exp\left\{2\int_0^\infty \frac{dt}{t}\frac{\cosh\left(\alpha-\frac{1}{2}\right)t}{\cosh\frac{t}{2}\sinh t}\sin^2\frac{(i\pi-\theta)t}{2\pi}\right\} \sim e^{|\theta|/2} \text{ as } |\theta|\to\infty$$

- $D_{ab}(\theta) = \prod_{\alpha\in\mathcal{A}_{ab}} \frac{\cos\pi\alpha-\cosh\theta}{2\cos^2(\pi\alpha/2)}$ accounts for the poles

- $Q^{\Phi}_{ab}(\theta) = \sum_{k=0}^{N_{ab}} c^{(k)}_{ab}\cosh^k\theta$ operator-dependent

$$[F^{\Phi}_{ab}(\theta)]^* = F^{\Phi}_{ab}(-\theta) \quad\Longrightarrow\quad c^{(k)}_{ab} \text{ real}$$

The asymptotic bound imposes the same constraints for any **relevant operator** $\varphi(x)$ $(X_\varphi < 2)$:

$N_{11} \leq 1 \quad \longrightarrow \quad$ 1 coefficient (+ an overall normalisation)

$N_{12} \leq 2 \quad \longrightarrow \quad$ 3 coefficients

S_{11} and S_{12} have 3 common poles $\quad \Longrightarrow$

$$F_c^\Phi = \frac{1}{i\Gamma_{11}^c} \text{Res}_{\theta=iu_{11}^c} F_{11}^\Phi(\theta) = \frac{1}{i\Gamma_{12}^c} \text{Res}_{\theta=iu_{12}^c} F_{12}^\Phi(\theta), \quad c = 1,2,3$$

$\Longrightarrow \quad$ There is a single free parameter :

$$\boxed{\varphi(x) \sim \sigma(x) + a\,\varepsilon(x)}$$

$$z_\varphi \equiv \frac{c_{11}^0}{c_{11}^1} \qquad Q_{11}^\varphi(\theta) = c_{11}^1(\cosh\theta + z_\varphi)$$

No solution if $c_{11}^1 = 0$

Problem : determine z_σ and z_ε

On the energy-momentum tensor form factors

$$\mathcal{A} = \mathcal{A}_{CFT} + g \int d^2x \, \phi(x)$$

$\Theta(x) \sim g\,\phi(x)$ Trace of the energy-momentum tensor

Conservation : $\begin{cases} \partial_- T^{++} = \partial_+ \Theta \\ \partial_+ T^{--} = \partial_- \Theta \end{cases}$ $\qquad x_\pm = x_0 \pm x_1$

•

$$P^\pm \equiv \sum_{k=1}^{n} p^\pm_{a_k}$$

$$\langle 0|\Phi(x)|A_{a_1}(\theta_1)\dots A_{a_n}(\theta_n)\rangle = e^{i[P^+x_+ + P^-x_-]} F^\Phi_{a_1\dots a_n}(\theta_1,\dots,\theta_n)$$

$$\Longrightarrow \quad \begin{cases} F^{T^{++}}_{a_1\dots a_n}(\theta_1,\dots,\theta_n) = \frac{P^+}{P^-} F^\Theta_{a_1\dots a_n}(\theta_1,\dots,\theta_n) \\ F^{T^{--}}_{a_1\dots a_n}(\theta_1,\dots,\theta_n) = \frac{P^-}{P^+} F^\Theta_{a_1\dots a_n}(\theta_1,\dots,\theta_n) \end{cases}$$

The ff of Θ, T^{++}, T^{--} have the same singularities

$\Longrightarrow \quad F^\Theta_{a_1\dots a_n} \propto P^+P^-$ unless P^+, P^- have the same zeros

$$n = 2 : \quad \frac{P^+}{P^-} = \frac{m_1 e^{\theta_1} + m_2 e^{\theta_2}}{m_1 e^{-\theta_1} + m_2 e^{-\theta_2}} = \frac{m_1 e^{\theta_1-\theta_2} + m_2}{m_1 + m_2 e^{\theta_1-\theta_2}} \, e^{\theta_1+\theta_2}$$

$$\Longrightarrow \quad F^\Theta_{ab}(\theta) \propto (P^+P^-)^{1-\delta_{ab}} = \left(\cosh\theta + \frac{m_a^2 + m_b^2}{2m_a m_b}\right)^{1-\delta_{ab}}$$

Back to magnetic Ising:

$$\Theta(x) \sim h\,\sigma(x)$$

$$\Longrightarrow \quad Q^{\sigma}_{12}(\theta) = \left(\cosh\theta + \frac{m_1^2 + m_2^2}{2m_1m_2}\right)(a_1\cosh\theta + a_0)$$

3 residue equations for 3 unknowns (a_1, a_0, z_σ)

$$\Longrightarrow \qquad z_\sigma = \frac{2m_1^2 + m_3m_7}{2m_1^2} = 4.869840..$$

How can we determine z_ε ?

The identification problem II.
Asymptotic factorisation

(G.D., P. Simonetti and J. Cardy, 1996)

Consider a massive integrable theory without internal symmetries:

$$S(\theta) = \prod_{\alpha} t_{\alpha}(\theta) \to 1\,, \qquad \theta \to \pm\infty$$

$\varphi(x)$ relevant, scalar, scaling operator

$$\langle \varphi \rangle \sim m^{X_\varphi} \neq 0\,, \qquad \widehat{\varphi}(x) \equiv \frac{\varphi(x)}{\langle \varphi \rangle}$$

$$F_n^{\Phi}(\theta_1, \ldots, \theta_n) \equiv \langle 0|\Phi(0)|A(\theta_1)\ldots A(\theta_n)\rangle$$

Massless (conformal) limit:

$$\lim_{m\to 0}(p^0, p^1) = \lim_{m\to 0,\, \alpha\to +\infty} \left(m \cosh\left(\theta \pm \frac{\alpha}{2}\right), m \sinh\left(\theta \pm \frac{\alpha}{2}\right)\right) =$$

$$= \left(\frac{M}{2}e^{\pm\theta}, \pm\frac{M}{2}e^{\pm\theta}\right) \qquad \text{R/L movers}$$

$M \equiv m\, e^{\alpha/2}$ finite parameter

$$\begin{cases} S_{RL}(\theta) = \lim_{\alpha\to+\infty} S(\theta+\alpha) = 1 \\ S_{RR}(\theta) = S_{LL}(\theta) = S(\theta) \end{cases}$$

Massless ff:

$$\mathcal{F}_{r,l}^{\Phi}(\theta_1, \ldots, \theta_r|\theta'_1, \ldots, \theta'_l) \equiv \langle 0|\Phi(0)|A_R(\theta_1)\ldots A_R(\theta_r)A_L(\theta'_1)\ldots A_L(\theta'_l)\rangle$$

$$\varphi(x) = \varphi_R(z)\varphi_L(\bar{z}) \quad \text{at criticality} \quad \Longrightarrow$$

$$\mathcal{F}^{\widehat{\varphi}}_{r,l}(\theta_1, \ldots, \theta_r | \theta'_1, \ldots, \theta'_l) = \mathcal{F}^{\widehat{\varphi}_R}_{r,0}(\theta_1, \ldots, \theta_r |)\, \mathcal{F}^{\widehat{\varphi}_L}_{0,l}(|\theta'_1, \ldots, \theta'_l)$$

Lorentz invariance $\Longrightarrow$

$$\begin{aligned} F^{\widehat{\varphi}}_n(\theta_1, \ldots, \theta_n) &= \lim_{\alpha \to \pm\infty} F^{\widehat{\varphi}}_n(\theta_1 \pm \alpha/2, \ldots, \theta_n \pm \alpha/2) \\ &= \begin{cases} \mathcal{F}^{\widehat{\varphi}_R}_{n,0}(\theta_1, \ldots, \theta_n |)\, \mathcal{F}^{\widehat{\varphi}_L}_{0,0} \\ \\ \mathcal{F}^{\widehat{\varphi}_R}_{0,0}\, \mathcal{F}^{\widehat{\varphi}_L}_{0,n}(|\theta_1, \ldots, \theta_n) \end{cases} \end{aligned}$$

$$\begin{aligned} &\lim_{\alpha \to +\infty} F^{\widehat{\varphi}}_{r+l}\left(\theta_1 + \frac{\alpha}{2}, \ldots, \theta_r + \frac{\alpha}{2}, \theta'_1 - \frac{\alpha}{2}, \ldots, \theta'_l - \frac{\alpha}{2}\right) \\ &= \mathcal{F}^{\widehat{\varphi}}_{r,l}(\theta_1, \ldots, \theta_r | \theta'_1, \ldots, \theta'_l) \\ &= F^{\widehat{\varphi}}_r(\theta_1, \ldots, \theta_r)\, F^{\widehat{\varphi}}_l(\theta'_1, \ldots, \theta'_l) / (\mathcal{F}^{\widehat{\varphi}_R}_{0,0}\, \mathcal{F}^{\widehat{\varphi}_L}_{0,0}) \end{aligned}$$

$$\mathcal{F}^{\widehat{\varphi}_R}_{0,0}\, \mathcal{F}^{\widehat{\varphi}_L}_{0,0} = \mathcal{F}^{\widehat{\varphi}}_{0,0} = \langle \widehat{\varphi} \rangle = 1$$

$$\boxed{\begin{aligned} &\lim_{\alpha \to \infty} F^{\varphi}_{r+l}(\theta_1 + \alpha, \ldots, \theta_r + \alpha, \theta'_1, \ldots, \theta'_l) \\ &= \frac{1}{\langle \varphi \rangle} F^{\varphi}_r(\theta_1, \ldots, \theta_r)\, F^{\varphi}_l(\theta'_1, \ldots, \theta'_l) \end{aligned}}$$

Non-linear equation for **scaling** operators

Magnetic Ising :

$$\lim_{\theta\to\infty} F^{\varphi}_{ab}(\theta) = \frac{F^{\varphi}_a F^{\varphi}_b}{\langle\varphi\rangle}, \qquad a,b = 1,\ldots,8$$

In particular :

$$\langle\varphi\rangle = \frac{(F^{\varphi}_1)^2}{\lim_{\theta\to\infty} F^{\varphi}_{11}(\theta)} = \frac{F^{\varphi}_1 F^{\varphi}_2}{\lim_{\theta\to\infty} F^{\varphi}_{12}(\theta)}$$

Two real solutions for z_φ :

$$\boxed{z_\varphi = \begin{cases} 4.869840.. = z_\sigma \\ 1.255585.. = z_\varepsilon \end{cases}}$$

Notice that we have access to $\langle\varphi\rangle$

Having fixed z_σ and z_ε the ff bootstrap can be carried through using the residue eqs (this requires a discussion of higher order poles)

We get rid of the non-universal overall normalisation considering $\hat{\varphi} = \varphi/\langle\varphi\rangle$

$$Q^{\widehat{\varphi}}_{ab}(\theta) = \sum_{k=0}^{N_{ab}} c^{(k)}_{ab} \cosh^k \theta$$

$\widehat{\varphi}$	$\widehat{\sigma}$	$\widehat{\varepsilon}$
c^1_{11}	−2.093102832	−70.00917205
c^0_{11}	−10.19307727	−87.90247670
c^2_{12}	−7.979022182	−466.3008246
c^1_{12}	−71.79206351	−1307.331521
c^0_{12}	−70.29218939	−853.2803886
c^3_{13}	−582.2557366	−43021.45153
c^2_{13}	−6944.416956	−182413.2733
c^1_{13}	−13406.48877	−241929.7678
c^0_{13}	−7049.622303	−102574.1349
c^3_{22}	−21.48559881	−2193.896354
c^2_{22}	−333.8125724	−10870.05277
c^1_{22}	−791.3745549	−16161.44508
c^0_{22}	−500.2535896	−7510.235388
c^3_{14}	22.57778351	2074.636471
c^2_{14}	318.7122159	9881.413381
c^1_{14}	672.2210098	14357.04570
c^0_{14}	377.4586311	6568.762583
c^4_{15}	−260.7643072	−30333.56619
c^3_{15}	−4719.877128	−198757.2340
c^2_{15}	−15172.07643	−447504.5720
c^1_{15}	−17428.22924	−422808.9295
c^0_{15}	−6716.787925	−143743.2050
c^4_{23}	−92.73452350	−11971.94909
c^3_{23}	−1846.579035	−81253.72269
c^2_{23}	−6618.297073	−186593.8661
c^1_{23}	−8436.850082	−178494.3378
c^0_{23}	−3579.556465	−61194.62416

c_{33}^5	−1197.056497	−195385.7662
c_{33}^4	−30166.99117	−1743171.802
c_{33}^3	−150512.4122	−5603957.324
c_{33}^2	−301093.9432	−8422606.859
c_{33}^1	−267341.1276	−6035102.896
c_{33}^0	−87821.70785	−1668721.004
c_{25}^6	1425.995027	289831.4882
c_{25}^5	44219.03877	3275586.983
c_{25}^4	286184.1535	13872077.63
c_{25}^3	788413.2178	29236961.96
c_{25}^2	1078996.488	32979257.31
c_{25}^1	725356.4417	19100224.04
c_{25}^0	191383.5734	4471623.121
c_{17}^5	190.8548023	30394.23374
c_{17}^4	4633.706068	274294.8033
c_{17}^3	21406.72691	897781.3229
c_{17}^2	39514.82959	1375919.456
c_{17}^1	32456.91939	1004969.466
c_{17}^0	9906.265607	282938.1974
c_{44}^7	−7249.785565	−1830120.693
c_{44}^6	−276406.7236	−25699492.93
c_{44}^5	−2299573.212	−138411873.8
c_{44}^4	−849276.3526	−384776478.8
c_{44}^3	−16615618.39	−608371427.1
c_{44}^2	−17950817.11	−553818699.0
c_{44}^1	−10139089.36	−270964337.7
c_{44}^0	−2341590.241	−55283137.91

One-particle form factors:

$$F_a^{\Phi} = \langle 0|\Phi(0)|A_a(\theta)\rangle\ , \qquad a = 1,\ldots,8$$

$$\langle\Phi_1(r)\Phi_2(0)\rangle = \langle\Phi_1\rangle\langle\Phi_2\rangle + \pi^{-1}\sum_{a=1}^{3} F_a^{\Phi_1}F_a^{\Phi_2}K_0(m_a r) + O(e^{-2m_1 r})$$

$\hat{\varphi}$	$\hat{\sigma}$	$\hat{\varepsilon}$
F_1	−0.64090211..	−3.70658437..
F_2	0.33867436..	3.42228876..
F_3	−0.18662854..	−2.38433446..
F_4	0.14277176..	2.26840624..
F_5	0.06032607..	1.21338371..
F_6	−0.04338937..	−0.96176431..
F_7	0.01642569..	0.45230320..
F_8	−0.00303607..	−0.10584899..

- Numerical estimates:

M. Caselle, M. Hasenbusch, 2000 (transfer matrix)

$\hat{\varphi}$	$\hat{\sigma}$	$\hat{\varepsilon}$
$\lvert F_1\rvert$	0.6408(3)	3.707(7)
$\lvert F_2\rvert$	0.325(25)	3.38(7)

- Short distance expansion of the correlators (conformal perturbation theory) : Guida, Magnoli, 1996; Caselle, Grinza, Magnoli, 2000

Sum rules

$T_{\mu,\nu}$ energy-momentum tensor

$$T \equiv (T_{11} - T_{22} - 2iT_{12})/4\,, \qquad \Theta \equiv T_{11} + T_{22}$$

Nearby a fixed point:

$$\langle T(x)T(0)\rangle \simeq C/(2z^4)\,, \qquad \langle T(x)\hat{\varphi}(0)\rangle \simeq X_\varphi/(2z^2)$$

$$S_T = 2\,, \quad S_\Theta = S_\varphi = 0 \quad \Longrightarrow$$

$$\begin{cases} \langle T(x)T(0)\rangle = F(z\bar{z})/z^4 \\ \langle T(x)\Theta(0)\rangle = G(z\bar{z})/z^3\bar{z} \\ \langle \Theta(x)\Theta(0)\rangle_{conn} = H(z\bar{z})/(z\bar{z})^2 \end{cases}$$

$$\begin{cases} \langle T(x)\hat{\varphi}(0)\rangle = L(z\bar{z})/z^2 \\ \langle \Theta(x)\hat{\varphi}(0)\rangle_{conn} = M(z\bar{z})/z\bar{z} \end{cases}$$

$$\Gamma \equiv 2F - G - 3H/8\,, \qquad D \equiv L + M/4$$

$$\partial T/\partial\bar{z} + (1/4)\,\partial\Theta/\partial z = 0 \quad \Longrightarrow$$

$$\dot{\Gamma} = -3H/4\,, \qquad \dot{D} = M/4 \qquad \left(\dot{f} \equiv z\bar{z}\frac{d}{dz\bar{z}}\right)$$

At a fixed point: $\Theta = 0 \Longrightarrow \Gamma = C\,, \quad D = X_\varphi/2$

Then, integration over $z\bar{z}$ gives:

- $$\boxed{C^{UV} - C^{IR} = 3/(4\pi)\int d^2x\,|x|^2\langle\Theta(x)\Theta(0)\rangle_{conn}}$$

  (A. Zamolodchikov, 1986; J. Cardy, 1988)

- $$\boxed{X_\varphi^{UV} - X_\varphi^{IR} = -1/(2\pi)\int d^2x\,\langle\Theta(x)\hat{\varphi}(0)\rangle_{conn}}$$

  (G.D., P. Simonetti, J. Cardy, 1996)

$C^{IR} = X_\varphi^{IR} = 0$ in a massive theory

Sum rules in the Ising model

$F^{\Theta}_{aa}(\theta = i\pi) = 2\pi m_a^2$

- $h = 0$:

$$\begin{cases} \langle 0|\Theta(0)|\theta_1,\ldots,\theta_n\rangle = -2\pi i m^2\,\delta_{n,2}\,\sinh\frac{\theta_1-\theta_2}{2} \\ \langle 0|\hat{\sigma}(0)|\theta_1,\theta_2\rangle_{\tau<0} = i\tanh\frac{\theta_1-\theta_2}{2} \end{cases}$$

$$\Longrightarrow \quad \begin{cases} C = \frac{3}{2}\int \frac{\sinh^2 x}{\cosh^4 x}\,dx = \frac{1}{2} \\ X_\sigma = \frac{1}{2\pi}\int \frac{\sinh^2 x}{\cosh^3 x}\,dx = \frac{1}{8} \end{cases}$$

- $\tau = 0$:

$C_{ab..} \equiv$ contribution of the state $A_a A_b..$ to C

C_1	0.472038282
C_2	0.019231268
C_3	0.002557246
C_{11}	0.003919717
C_4	0.000700348
C_{12}	0.000974265
C_5	0.000054754
C_{13}	0.000154186
C_{partial}	0.499630066

$\Delta_\varphi = \frac{X_\varphi}{2}$

	σ	ε
Δ_1	0.0507107	0.2932796
Δ_2	0.0054088	0.0546562
Δ_3	0.0010868	0.0138858
Δ_{11}	0.0025274	0.0425125
Δ_4	0.0004351	0.0069134
Δ_{12}	0.0010446	0.0245129
Δ_5	0.0000514	0.0010340
Δ_{13}	0.0002283	0.0065067
Δ_{partial}	0.0614934	0.4433015
Δ_{exact}	0.0625	0.5

Ising universality class

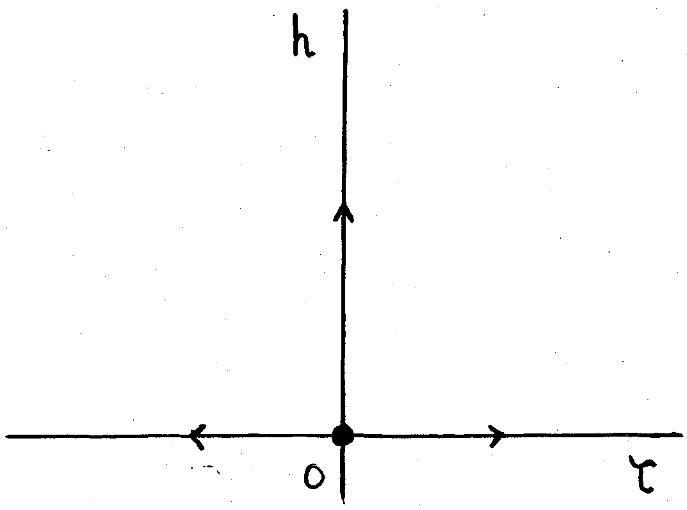
Critical exponents and critical amplitudes:

Correlation length: $$\xi = \begin{cases} f_\pm\,|\tau|^{-\nu}, & \tau \to 0^\pm,\ h=0 \\ f_c\,|h|^{-\nu_c}, & \tau = 0,\ h \to 0 \end{cases}$$

Specific heat: $$C = \begin{cases} (A_\pm/\alpha)\,|\tau|^{-\alpha}, & \tau \to 0^\pm,\ h=0 \\ (A_c/\alpha_c)\,|h|^{-\alpha_c}, & \tau = 0,\ h \to 0 \end{cases}$$

Magnetisation: $$M = \begin{cases} B\,(-\tau)^{\beta}, & \tau \to 0^-,\ h=0 \\ (|h|/D)^{1/\delta}, & \tau = 0,\ h \to 0 \end{cases}$$

Susceptibility: $$\chi = \begin{cases} \Gamma_\pm\,|\tau|^{-\gamma}, & \tau \to 0^\pm,\ h=0 \\ \Gamma_c\,|h|^{-\gamma_c}, & \tau = 0,\ h \to 0 \end{cases}$$



$$\mathcal{A} = \mathcal{A}_{fixed\ point} - \tau \int d^d x\, \varepsilon(x) - h \int d^d x\, \sigma(x)$$

$$Z = \mathrm{Tr}\, e^{-\mathcal{A}}\,, \qquad f = -\frac{1}{V} \ln Z \sim m^d$$

$$\begin{cases} \xi & \sim & m^{-1} \\ C & = & -\dfrac{\partial^2 f}{\partial \tau^2} = \displaystyle\int d^d x\, \langle \varepsilon(x) \varepsilon(0) \rangle_{conn} \sim m^{2X_\varepsilon - d} \\ M & = & -\dfrac{\partial f}{\partial h} = \langle \sigma \rangle \sim m^{X_\sigma} \\ \chi & = & -\dfrac{\partial^2 f}{\partial h^2} = \displaystyle\int d^d x\, \langle \sigma(x) \sigma(0) \rangle_{conn} \sim m^{2X_\sigma - d} \end{cases}$$

$$\tau \sim m^{d - X_\varepsilon} \qquad h \sim m^{d - X_\sigma} \qquad \Longrightarrow$$

$$\boxed{\begin{array}{ll} \nu = 1/(d - X_\varepsilon)\,, & \nu_c = 1/(d - X_\sigma) \\ \alpha = (d - 2X_\varepsilon)\, \nu\,, & \alpha_c = (d - 2X_\varepsilon)\, \nu_c \\ \beta = X_\sigma\, \nu\,, & 1/\delta = X_\sigma\, \nu_c \\ \gamma = (d - 2X_\sigma)\, \nu\,, & \gamma_c = (d - 2X_\sigma)\, \nu_c \end{array}}$$

$d = 2$ Ising model: $\quad X_\sigma = 1/8,\ X_\varepsilon = 1 \quad \Longrightarrow$

$\nu = 1$, $\nu_c = 8/15$, $\alpha = \alpha_c = 0$, $\beta = 1/8$, $\delta = 15$, $\gamma = 7/4$, $\gamma_c = 14/15$

The exponents are universal, the amplitudes are not

Universal amplitude ratios :

8 exponents $\longrightarrow$ 6 scaling relations

		$f_+/f_-, \quad A_+/A_-, \quad \Gamma_+/\Gamma_-$
$\alpha + 2\beta + \gamma = 2$		$R_c \equiv A_+\Gamma_+/B^2$
$\alpha + d\nu = 2$		$R_\xi^+ \equiv A_+^{1/d} f_+$
$\gamma_c = 1 - 1/\delta$		$\delta \Gamma_c D^{1/\delta} = 1$
	$\longrightarrow$	
$\gamma = \beta(\delta - 1)$		$R_\chi \equiv \Gamma_+ D B^{\delta-1}$
$\alpha_c = \alpha/\beta\delta$		$R_A \equiv A_c D^{-(1+\alpha_c)} B^{-2/\beta}$
$\nu_c = \nu/\beta\delta$		$Q_2 \equiv (\Gamma_+/\Gamma_c)(f_c/f_+)^{\gamma/\nu}$

Example:

$$\lim_{\tau \to 0^+,\, h=0} \frac{f\chi}{M^2} = -\frac{A_+\,\tau^{2-\alpha}}{\alpha(1-\alpha)(2-\alpha)} \frac{\Gamma_+\,\tau^{-\gamma}}{(B\,\tau^{-\beta})^2}$$

$$= -\frac{R_c}{\alpha(1-\alpha)(2-\alpha)}$$

Ising critical amplitudes:

$$\langle\sigma(x)\sigma(0)\rangle = C_\sigma |x|^{-1/4}, \quad x \to 0$$

$$\langle\varepsilon(x)\varepsilon(0)\rangle = C_\varepsilon |x|^{-2}, \quad x \to 0$$

"conformal" normalisation: $C_\sigma = C_\varepsilon = 1$

1) Exponential correlation length:

$$\langle\sigma(x)\sigma(0)\rangle_{conn} \sim \frac{e^{-|x|/\xi}}{|x|^{(d-1)/2}}, \quad |x| \to \infty$$

$$h = 0 : \qquad \xi = \begin{cases} 1/m, & \tau > 0 \\ 1/2m, & \tau < 0 \end{cases}, \qquad m = \mathcal{C}_\tau |\tau|$$

$$\tau = 0 : \qquad \xi = 1/m_1, \qquad m_1 = \mathcal{C}_h |h|^{8/15}$$

Thermodynamic Bethe ansatz (Al. Zamolodchikov, 1995; Fateev, 1994):

$$\mathcal{C}_\tau = 2\pi, \qquad \mathcal{C}_h = 4.40490857..$$

2) $\alpha = \alpha_c = 0 \quad \Longrightarrow \quad C = \begin{cases} -A_\pm \ln|\tau| \\ -A_c \ln|h| \end{cases}$

$$C = 2\pi \int r dr \, \langle\varepsilon(r)\varepsilon(0)\rangle \sim 2\pi \int_{r_0} dr/r \sim -2\pi \ln m r_0$$

$$\Longrightarrow \quad A_\pm = 2\pi, \qquad A_c = \frac{8}{15} 2\pi$$

3) $$M = \langle \sigma \rangle$$

$$\langle \sigma \rangle_{h=0,\, \tau<0} = B(-\tau)^{1/8}\,, \qquad \langle \sigma \rangle_{\tau=0} = (|h|/D)^{1/15}$$

Fateev, Lukyanov, A. and Al. Zamolodchikov, 1997:

$$B = 1.70852190..\,, \qquad D = 0.0253610264..$$

4) $$\chi = \int d^2x\, \langle \sigma(x)\sigma(0) \rangle_{conn}$$

$h = 0$ (Wu, Mccoy, Tracy, Barouch, 1976):

$$\Gamma_+ = 0.148001214..\,, \qquad \Gamma_- = 0.00392642280..$$

$$\tau = 0: \qquad \chi = \begin{cases} \Gamma_c\, h^{-\gamma_c} \\ (\partial/\partial h)\langle \sigma \rangle = (\partial/\partial h)(h/D)^{1/\delta} \end{cases}$$

$$\Longrightarrow \qquad \gamma_c = 1 - 1/\delta\,, \qquad \Gamma_c = \frac{1}{\delta D^{1/\delta}} = 0.0851721517..$$

Thermal ratios (known since Wu et al, 1976, lattice solution) :

$$A_+/A_- = 1$$

$$\Gamma_+/\Gamma_- = 37.6936520..$$

$$\xi_0^+/\xi_0^- = 2$$

$$R_C = 0.318569391..$$

$$R_\xi^+ = 1/\sqrt{2\pi}$$

Magnetic ratios (G.D., 1998):

$$R_\chi = 6.77828502..$$

$$R_A = 0.0250658794..$$

$$Q_2 = 3.23513834..$$

Numerical estimates:

Tarko, Fisher, 1975, series expansions :

$$R_\chi \sim 6.78$$

Caselle, Hasenbusch, 2000, transfer matrix :

$$R_\chi = 6.7782(8), \qquad Q_2 = 3.233(4)$$

Perturbing integrable theories

(G.D., G. Mussardo and P. Simonetti, 1996)

$$\mathcal{A} = \mathcal{A}_{integrable} + \lambda \int d^2x\, \Psi(x)$$

Perturbation theory in λ involving the ff of $\Psi(x)$ computed in $\mathcal{A}_{integrable}$

First order corrections to the energy spectrum:

$$\delta\mathcal{E}_{vac} \simeq \lambda \langle \Psi \rangle$$

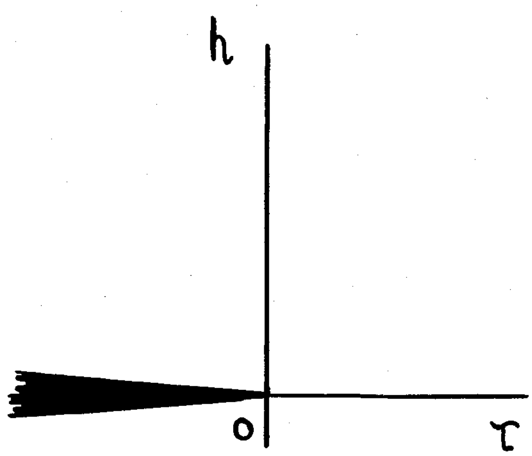
$$\delta m_a^2 \simeq \lambda \langle A_a(\theta)|\Psi(0)|A_a(\theta)\rangle = \lambda\, F_{aa}^{\Psi}(\theta_1 - \theta_2 = i\pi)$$

Ising model

- $h \to 0\,, \quad \tau < 0:$

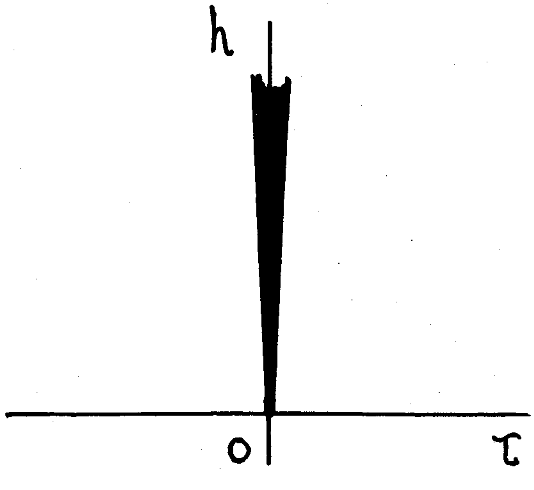
$$\langle 0|\sigma(0)|\theta_1, \theta_2\rangle = i\langle\sigma\rangle \tanh\frac{\theta_1-\theta_2}{2} \quad \Longrightarrow$$

$\delta m = \infty$! kink confinement



- $\tau \to 0:$

	form factors	Numerical
$\delta\mathcal{E}_{vac}/\delta m_1$	$-0.0558..\, m_1$	$-0.05\, m_1$
$\delta m_2/\delta m_1$	0.8616..	0.87
$\delta m_3/\delta m_1$	1.5082..	1.50



One-point functions on the cylinder

$$\langle\varphi(x)\rangle_R = \frac{\mathrm{Tr}\,\varphi(x)\,e^{-RH}}{\mathrm{Tr}\,e^{-RH}}$$

Leading corrections as $R \to \infty$:

$$\frac{\langle\varphi\rangle_R}{\langle\varphi\rangle_{R=\infty}} = 1 + \frac{1}{\pi}\sum_a A_a^\varphi\, K_0(m_a R) + O(e^{-2m_1 R})$$

$$A_a^\varphi \equiv \left.\frac{F_{aa}^\varphi(i\pi)}{\langle\varphi\rangle}\right|_{R=\infty}$$

Ising field theory:

$h = 0$:

$$\langle 0|\sigma(0)|\theta_1,\theta_2\rangle = \begin{cases} 0\,, & \tau > 0 \\ i\langle\sigma\rangle \tanh\frac{\theta_1-\theta_2}{2}\,, & \tau < 0 \end{cases}$$

$$\langle\sigma\rangle_R = \begin{cases} 0\,, & \tau > 0 \\ \text{to be defined}\,, & \tau < 0 \end{cases}$$

$\tau = 0$ (G.D., 2001):

φ	σ	ε
A_1^φ	−8.0999744..	−17.893304..
A_2^φ	−21.206008..	−24.946727..
A_3^φ	−32.045891..	−53.679951..

M. Caselle and M. Hasenbusch (to appear):

$$A_1^\sigma = -8.11(2)\,, \qquad A_1^\varepsilon = -17.5(5)$$

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