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# A quasilinear approximation for the three-dimensional Navier-Stokes system

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# A quasilinear approximation for the three–dimensional Navier–Stokes system

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A commonly accepted view is that, for typical initial data, the three-dimensional Navier–Stokes system (NSS) has a unique strong solution. In this article we consider the NSS in whole space and without an external force. The energy inequality shows that this solution must decrease in time, which does not preclude growth of enstrophy. (On the NSS see, e.g., [1]).

Below we suggest a modification of the NSS that, in our opinion, preserves the basic character of the nonlinearity but is quasilinear in the Fourier space. Because of this fact, it possesses characteristics along which the nonlinearity propagates. There is some evidence that in a certain sense the asymptotic behavior of the NSS and our system is identical.

Consider the NSS describing the motion of a viscous incompressible fluid in the three-dimensional space $R^3$:

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \Delta u \\ \qquad \nabla \cdot u = 0, \end{cases} \tag{0.1}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity vector, $p = p(x,t)$ is the pressure, $x = (x_1, x_2, x_3)$, and $\nu > 0$ is the viscosity.

Applying the operator $\nabla$ to both sides of the first equation and using the second equation, we obtain the following expression for $p$:

$$p = (-\Delta)^{-1}(\nabla \cdot (u \cdot \nabla)u),$$

or in coordinates of the vector $u$

$$
\begin{aligned}
p(x,t) &= (-\Delta)^{-1}\left(\sum_{n=1}^{3}\frac{\partial}{\partial x_n}\left(\sum_{m=1}^{3} u_m\frac{\partial u_n}{\partial x_m}\right)\right) \\
&= (-\Delta)^{-1}\left(\sum_{n,m=1}^{3}\frac{\partial u_m}{\partial x_n}\frac{\partial u_n}{\partial x_m}+\sum_{m=1}^{3} u_m\frac{\partial}{\partial x_m}\left(\sum_{n=1}^{3}\frac{\partial u_n}{\partial x_n}\right)\right) \\
&= (-\Delta)^{-1}\left(\sum_{m,n=1}^{3}\frac{\partial u_m}{\partial x_n}\frac{\partial u_n}{\partial x_m}\right).
\end{aligned}
$$

Let $\hat{u}(k,t)$ be the Fourier transform of the vector $u(x,t)=\int\limits_{R^3}\hat{u}(k,t)e^{ikx}dk$. For $\hat{u}(k,t)$ we obtain the following system of equations:

$$
\begin{cases}
\frac{\partial \hat{u}_l(k,t)}{\partial t}+i\sum_{j=1}^{3}\int \hat{u}_j(k',t)(k_j-k'_j)\hat{u}_l(k-k',t)dk'- \\
-\frac{ik_l}{|k|^2}\sum_{n,m=1}^{3}\int k'_n\hat{u}_m(k',t)(k_m-k'_m)\hat{u}_n(k-k',t)dk'=-\nu|k|^2\hat{u}_l(k,t), \quad l=1,2,3 \\
\sum_{j=1}^{3} ik_j\hat{u}_j(k,t)=0.
\end{cases}
$$

Since $u(x,t)$ is a real vector, we see that $\hat{u}(k,t)=\overline{\hat{u}(-k,t)}$.

*We will consider (2) in the invariant space of pure imaginary odd functions.*

Setting $\hat{u}(k,t)=iv(k,t)$, we get

$$
\begin{cases}
\frac{\partial v_l(k,t)}{\partial t}-\sum_{j=1}^{3}\int v_j(k',t)(k_j-k'_j)v_l(k-k',t)dk'+ \\
+\frac{k_l}{|k|^2}\sum_{m,n=1}^{3}\int k'_n v_m(k',t)(k_m-k'_m)v_n(k-k',t)dk'=-\nu|k|^2 v_l(k,t), \quad l=1,2,3 \\
\sum_{j=1}^{3} k_j v_j(k,t)=0.
\end{cases}
$$

and $v(k,t)=-v(-k,t)$. Using the last equation of (3), we obtain a system equivalent to (3):

$$
\begin{cases}
\frac{\partial v_l(k,t)}{\partial t}-\sum_{j=1}^{3}k_j\int v_j(k',t)v_l(k-k',t)dk'+ \\
\frac{k_l}{|k|^2}\sum_{m,n=1}^{3}k_m\int k'_n v_m(k',t)v_n(k-k',t)dk'=-\nu|k|^2 v_l(k,t), \quad l=1,2,3 \quad (0.4) \\
\sum_{j} k_j v_j(k,t)=0.
\end{cases}
$$

Our main hypothesis, which constitutes the basis for the suggested approximation, is that solutions $u$ decay rather slowly at infinity and therefore the main contribution to the Fourier transform comes from the two domains $|k'| \ll |k|$ and $|k-k'| \ll |k|$. In other words, the integral in (4) may be represented as a sum of two terms, one being the integral over these domains and the other over their complement; this last integral is discarded since we regard it as a small quantity of higher order.

In what follows, we drop the argument $t$ in the unknown functions.

1) First consider the domain $|k'| \ll |k|$. Taylor expanding in $k'$, we obtain

$$\int v_j(k')v_l(k-k')dk' = v_l(k)\int v_j(k')dk' - \sum_{s=1}^{3}\int v_j(k')\frac{\partial v_l}{\partial k_s}k'_s dk' + \cdots =$$
$$= -\sum_{s=1}^{3}\frac{\partial v_l}{\partial k_s}\int v_j(k')k'_s dk' + \cdots, \tag{0.5}$$

since the first integral is equal to zero because $v(k)$ is odd, and

$$\int k'_n v_m(k')v_n(k-k')dk' = v_n(k)\int k'_n v_m(k')dk' - \sum_{s=1}^{3}\frac{\partial v_n}{\partial k_s}\int k'_n k'_s v_m(k')dk' + \cdots =$$
$$= v_n(k)\int k'_n v_m(k')dk' + \cdots \tag{0.6}$$

(similarly, the second integral equals zero since $v(k)$ is odd).

2) To obtain the expansion in the domain $|k-k'| \ll |k|$, we set $k-k'=k''$. Then $k'=k-k''$ and we get

$$\int v_j(k')v_l(k-k')dk' = \int v_j(k-k'')v_l(k'')dk'' = -\sum_{s=1}^{3}\frac{\partial v_j}{\partial k_s}\int v_l(k')k'_s dk' + \cdots, \tag{0.7}$$

which follows from (5) after the interchange of the indices $j$ and $l$. Further,

$$\int k'_n v_m(k')v_n(k-k')dk' = \int (k_n - k''_n)v_m(k-k'')v_n(k'')dk'' =$$

$$= v_m(k)\int(k_n - k''_n)v_n(k'')dk'' - \sum_{s=1}^{3}\frac{\partial v_m}{\partial k_s}\int(k_n - k''_n)k''_s v_n(k'')dk'' + \cdots$$

$$= -v_m(k)\int k'_n v_n(k')dk' - k_n\sum_{s=1}^{3}\frac{\partial v_m}{\partial k_s}\int k''_s v_n(k'')dk'' + \cdots. \qquad (0.8)$$

In the derivation of (8) we use the identities $\int v_n(k'')dk'' = 0$ and $\int k''_n k''_s v_n(k'')dk'' = 0$ following from the fact that $v(k)$ is odd.

In expansions (5)–(8) dots denote the higher order terms that we discard. Substituting (5)–(8) in (4) and using the identities

$$\sum_{j=1}^{3} k_j \int v_j(k')v_l(k-k')dk' = -\sum_{j=1}^{3} k_j \sum_{s=1}^{3}\frac{\partial v_l}{\partial k_s}\int v_j(k')k'_s dk' - \sum_{j=1}^{3} k_j \sum_{s=1}^{3}\frac{\partial v_j}{\partial k_s}\int v_l(k$$

$$\sum_{j=1}^{3} k_j \frac{\partial v_j}{\partial k_s} = \frac{\partial \sum_{j=1}^{3} k_j v_j}{\partial k_s} - \delta_{js}v_j,$$

$$\sum_{j=1}^{3} k_j \int v_j(k')v_l(k-k')dk' = -\sum_{s=1}^{3}\frac{\partial v_l}{\partial k_s}\left(\sum_{j=1}^{3} k_j \int v_j(k')k'_s dk'\right) + \sum_{s=1}^{3} v_s \int v_l(k')k'_s d$$

we obtain that up to the terms of higher order

$$\sum_{m,n} k_m \int k'_n v_m(k')v_n(k-k')dk' = \sum_{m,n} k_m v_n(k)\int k'_n v_m(k')dk' - \sum_{m,n} k_m v_m(k)\int k'_n v$$

$$-\sum_{m,n} k_m k_n \sum_{s=1}^{3}\frac{\partial v_m}{\partial k_s}\int k'_s v_n(k')dk' =$$

$$= \sum_{m,n} k_m v_n \int k'_n v_m(k')dk' - \sum_{n} k_n \sum_{s=1}^{3}\left(\frac{\partial \sum_m k_m v_m}{\partial k_s} - \sum_m \delta_{ms}v_m\right)\cdot\int k'_s v_n(k')dk' =$$

$$= \sum_{m,n} k_m v_n(k)\int k'_n v_m(k')dk' + \sum_{m,n} k_n v_m \sum_{s=1}^{3}\delta_{sm}\int k'_s v_n(k')dk'$$

$$= 2\sum_{m,n} k_m v_n(k)\int k'_n v_m(k')dk'.$$

(Here we interchange $m$ and $n$ in the second sum.)

Finally we get the following system of equations:

$$\begin{cases} \frac{\partial v_l}{\partial t} + \sum_{s=1}^{3} \frac{\partial v_l}{\partial k_s} B_s = -\nu|k|^2 v_l + \sum_{s=1}^{3} v_s A_{sl} - \frac{2k_l}{|k|^2} \sum_{m,n} k_m v_n(k) A_{nm}, \quad l = 1,2,3 \\ \sum k_s v_s = 0, \end{cases} \tag{0.11}$$

where $A_{ij} = \int k_i v_j(k,t) dk$, $B_i = \sum_{j=1}^{3} k_j A_{ij}$. System (11) is our suggested approximation of the NSS. The conjecture is that, for a wide class of initial data, solutions of system (11) decay at infinity slowly whenever system (3) has a solution with the same property.

System (11) may be rewritten in the form

$$\begin{cases} \frac{\partial v_l}{\partial t} + \sum_{s=1}^{3} \frac{\partial v_l}{\partial k_s} B_s = -\nu|k|^2 v_l + \sum_{s=1}^{3} v_s A_{sl} - \frac{2k_l}{|k|^2} \sum_{s=1}^{3} v_s B_s, \quad l = 1,2,3 \\ \sum_{s=1}^{3} k_s v_s = 0, \end{cases} \tag{0.12}$$

System (12) has the following important property.

*Suppose that the initial data $v(k,0) = v^{(0)}(k)$ of system (12) satisfy the incompressibility condition $\sum_{i=1}^{3} k_i v_i^{(0)}(k) = 0$. Then for any $t > 0$ solutions of system (12) satisfy the same condition $\sum_{i=1}^{3} k_i v_i(k,t) = 0$ provided a solution exists for this $t$.*

**Proof.** We have

$$\frac{\partial(\sum_{i=1}^{3} k_i v_i(t))}{\partial t} = -\sum_{i=1}^{3} k_i \sum_{j=1}^{3} \frac{\partial v_i(k,t)}{\partial k_j} B_j - \nu|k|^2 \sum_{i=1}^{3} k_i v_i(k,t) +$$
$$+\sum_{i=1}^{3} k_i \sum_{j=1}^{3} v_j(k,t) A_{ji} - 2 \sum_{i,j=1}^{3} \frac{|k_i|^2}{|k|^2} \sum_{j=1}^{3} v_j(k,t) B_j.$$

If the incompressibility condition is satisfied at the moment $t$, then:

1) the second term equals 0;

2) the first sum equals

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\left(\frac{\partial(k_i v_i(k,t))}{\partial k_j}-\delta_{ji}v_i(k,t)\right)B_j=-\sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ji}v_i(k,t)B_j=-\sum_{i=1}^{3}v_i(k,t)B_i.$$

Since $\sum_{i=1}^{3} k_i A_{ji} = B_j$ and $\sum_{i=1}^{3} \frac{|k_i|^2}{|k|^2} = 1$, we finally have

$$\frac{\partial \sum_{i=1}^{3} k_i v_i(k,t)}{\partial t}=\sum_{i=1}^{3} v_i(k,t)B_i+\sum_{j=1}^{3} v_j(k,t)B_j-2\sum_{j=1}^{3} v_j(k,t)B_j=0.$$

Thus if $\sum k_i v_i(k,t)=0$ for $t \geq 0$, then $\frac{\partial}{\partial t}(\sum_{i=1}^{3} k_i v_i(k,t))=0$ for this $t$. Therefore $\sum_{i=1}^{3} k_i v_i(k,t)=0$ for all $t$ such that the solution exists.

System (12) has characteristics satisfying the following system of equations:

$$\begin{cases} \frac{dv_l}{dt}=-\nu|k|^2 v_l+\sum_{s=1}^{3} v_s A_{sl}-\frac{2k_l}{|k|^2}\sum_{s=1}^{3} v_s B_s, \\ \frac{dk_l}{dt}=B_l, \quad l=1,2,3 \\ \sum k_s v_s = 0, \end{cases} \tag{0.13}$$

$A_{jl}$ and $B_l$ are nonlocal functionals of $v$ in this system.

Finite-dimensional approximations

Let the initial data $v_i(0)$ be nonzero only at a finite number of points $k^{(s)}=(k_1^{(s)},k_2^{(s)},k_3^{(s)})$, $1 \leq s \leq N$; denote $v_i(k^{(s)},0)=v_i^{(s)}$. Instead of a partial differential equation, for initial data of this kind we obtain a system of $6N$ ordinary differential equations

$$\begin{cases} \frac{dk_i^{(s)}}{dt}=\sum_{j=1}^{3} k_j^{(s)}\left(\sum_{n=1}^{N} k_i^{(n)} v_j^{(n)}\right), \quad i=1,2,3, \quad s=1,2,\cdots,N \\ \frac{dv_i^{(s)}}{dt}=-\nu|k^{(s)}|^2 v_i^{(s)}-\sum_{j=1}^{3} v_j^{(s)}\left(\sum_{n=1}^{N} k_j^{(n)} v_i^{(n)}\right)-\frac{2k_i^{(s)}}{|k^{(s)}|^2}\sum_{j=1}^{3} v_j^{(s)}\Big(\sum_{l=1}^{3} k_l^{(s)}\Big(\sum_{n=1}^{N} k_j^{(n)} v_l^{(n} \end{cases}$$

Also, the incompressibility condition $\sum_{i=1}^{3} k_i^{(s)} v_i^{(s)}(t)=0$, $s=1,2,\cdots,N$, holds if it is satisfied at the initial moment: $\sum_{i=1}^{3} k_i^{(s)}(0)v_i^{(s)}(0)$ for all $s=1,2,\cdots,N$.

For $N = 1$ system (14) is trivial. We study some properties of this system in the case $N = 2$. For the two points $k^{(1)} = \left(k_1^{(1)}, k_2^{(1)}, k_3^{(1)}\right)$ and $k^{(2)} = \left(k_1^{(2)}, k_2^{(2)}, k_3^{(2)}\right)$, the finite-dimensional system consists of 12 equations

$$\begin{cases} \frac{dk_i^{(1)}}{dt} = k_i^{(2)} \sum\limits_{j=1}^{3} k_j^{(1)} v_j^{(2)}, \\ \frac{dk_i^{(2)}}{dt} = k_i^{(1)} \sum\limits_{j=1}^{3} k_j^{(2)} v_j^{(1)}, \\ \frac{dv_i^{(1)}}{dt} = -\nu |k^{(1)}|^2 v_i^{(1)} + v_i^{(2)} \sum\limits_{j=1}^{3} k_j^{(2)} v_j^{(1)} - \frac{2k_i^{(1)}}{|k^{(1)}|^2} \sum\limits_{m,n=1}^{3} k_m^{(1)} v_n^{(1)} k_n^{(2)} v_m^{(2)}, \\ \frac{dv_i^{(2)}}{dt} = -\nu |k^{(2)}|^2 v_i^{(2)} + v_i^{(1)} \sum k_j^{(1)} v_j^{(2)} - \frac{2k_i^{(2)}}{|k^{(2)}|^2} \sum\limits_{m,n} k_m^{(1)} v_n^{(1)} k_n^{(2)} v_m^{(2)}, \quad i = 1, 2, 3 \end{cases} \tag{0.15}$$

In addition, $\sum\limits_{j=1}^{3} k_j^{(i)} v_j^{(i)} = 0$, $i = 1, 2$.

*System (15) admits three first integrals*

$$k_i^{(1)} k_j^{(2)} - k_i^{(2)} k_j^{(1)} = C_{i,j}, \quad i = 1, 2, 3, \quad i < j.$$

In other words, the vector product $[k^{(1)}, k^{(2)}]$ is invariant under the flow.

**Proof.** From the first six equations we have

$$\frac{dk_i^{(1)}}{dk_j^{(1)}} = \frac{k_i^{(2)}}{k_j^{(2)}}, \quad \frac{dk_i^{(2)}}{dk_j^{(2)}} = \frac{k_i^{(1)}}{k_j^{(1)}}, \quad i \neq j$$

(to obtain this, divide the equations by each other). Hence

$$\begin{cases} k_j^{(2)} dk_i^{(1)} - k_i^{(2)} dk_j^{(1)} = 0 \\ k_i^{(1)} dk_j^{(2)} - k_j^{(1)} dk_i^{(2)} = 0, \quad i \neq j. \end{cases}$$

Adding these two equations, we obtain

$$d(k_i^{(1)} k_j^{(2)} - k_j^{(1)} k_i^{(2)}) = 0, \qquad i \neq j.$$

4. Let

$$V^{(1)} = \sum_{j=1}^{3} k_j^{(2)} v_j^{(1)}$$

and

$$V^{(2)} = \sum_{j=1}^{3} k_j^{(1)} v_j^{(2)}.$$

In this notation system (15) assumes a simpler form

$$\begin{cases} \frac{dk_i^{(1)}}{dt} = k_i^{(2)} V^{(2)}, \\ \frac{dk_i^{(2)}}{dt} = k_i^{(1)} V^{(1)}, \\ \frac{dv_i^{(1)}}{dt} = -\nu|k_i^{(1)}|^2 v_i^{(1)} + v_i^{(2)} V^{(1)} - \frac{2k^{(1)}}{|k^{(1)}|^2} V^{(1)} V^{(2)}, \\ \frac{dv_i^{(2)}}{dt} = -\nu|k^{(2)}|^2 v_i^{(1)} + v_i^{(1)} V^{(2)} - \frac{2k_i^{(2)}}{|k^{(2)}|^2} V^{(2)} V^{(1)}, \\ i = 1, 2, 3. \end{cases} \tag{0.16}$$

We compute $dV^{(1)}/dt$:

$$\frac{dV^{(1)}}{dt} = \sum_{j=1}^{3} k_i^{(2)} \frac{dV^{(1)}}{dt} + \sum_{j=1}^{3} v_i^{(1)} \frac{dk^{(2)}}{dt} = -\nu|k^{(1)}|^2 V^{(1)} + \left(\sum_{j=1}^{3} k_i^{(2)} v_i^{(2)}\right) V^{(1)}$$
$$- \sum_{i=1}^{3} \frac{2k_i^{(2)} k_i^{(1)}}{|k^{(1)}|^2} V^{(1)} V^{(2)} - \sum_{i=1}^{3} v_i^{(1)} k_i^{(1)} V^{(1)} = -\nu|k^{(1)}|^2 V^{(1)} - \frac{d\ln|k^{(1)}|^2}{dt} V^{(1)}.$$

In the last step of this derivation we use the equalities $\sum_{j=1}^{3} k_j^{(i)} v_j^{(i)} = 0$ for $i = 1, 2$ and the equations $dk_i^{(1)}/dt = k_i^{(2)} V^{(2)}$, $i = 1, 2, 3$, of system (16). We finally obtain

$$\frac{d(\ln|V^{(1)}||k^{(1)}|^2)}{dt} = -\nu|k^{(1)}|^2. \tag{0.17}$$

Similarly, we obtain

$$\frac{d(\ln |V^{(2)}||k^{(2)}|^2)}{dt} = -\nu |k^{(2)}|^2. \qquad (0.18)$$

Thus the following lemma is proved.

**Lemma 1.** *System (15) admits two Lyapunov functions* $F_1 = \ln\left(|V^{(1)}||k^{(1)}|^2\right)$ *and* $F_2 = \ln\left(|V^{(2)}||k^{(2)}|^2\right)$.

From (17) and (18) we get that for all $t > t_0 \geq 0$

$$V^{(s)}(t)|k^{(s)}(t)|^2 = V^{(s)}(t_0)|k^{(s)}(t_0)|^2 \exp\left(-\nu \int_{t_0}^{t} |k^{(s)}(\tau)|^2 d\tau\right), \quad s = 1, 2.$$

Recently one of us (E.D.) proved for the system (16) the following theorem.

**Theorem.**

*For any initial conditions* $k^{(1)}(0) \neq 0,\ k^{(2)}(0) \neq 0,\ v^{(1)}(0),\ v^{(2)}(0)$ *satisfying incompressibility conditions the solution* $k^{(1)}(t),\ k^{(2)}(t),\ v^{(1)}(t),\ v^{(2)}(t)$ *exist for all* $t > 0$ *and*

$$1) \lim_{t\to\infty} k^{(i)}(t) = k^{(i)}(\infty) \neq 0 \text{ exists, } i = 1, 2;$$

$$2) \lim_{t\to\infty} v^{(i)}(t) = 0, \quad i = 1, 2.$$

This result can be considered as a manifestation of the absence of blow-up in solutions of (12).

The proof of the theorem will be published elsewhere.

In E. Ein's diploma equations for vorticity are obtained that are similar in form to our equations (11).

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## REFERENCES

[1] C. R. Doering, J. D. Gibbon. Applied analysis of the Navier–Stokes equations. Cambrifge University Press, Cambridge, 1995.