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Interactions between homogeneous dynamics and Number Theory

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These are preliminary lecture notes, intended only for distribution to participants

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INTERACTIONS BETWEEN HOMOGENEOUS DYNAMICS AND NUMBER THEORY

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Plan of the lectures

1. $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$, basic facts
   Quadratic forms, Oppenheim Conjecture

2. Linear forms, badly approximable systems
   One-parameter partially hyperbolic actions

3. Diophantine approximation on manifolds
   Non-divergence of quasi-polynomial flows

4. Higher rank actions
   Multiplicative approximation, Littlewood's Conj.

Problem 1. Given a non-degenerate irrational indefinite quadratic form B of signature (m, n), study the set of its values at integer points.

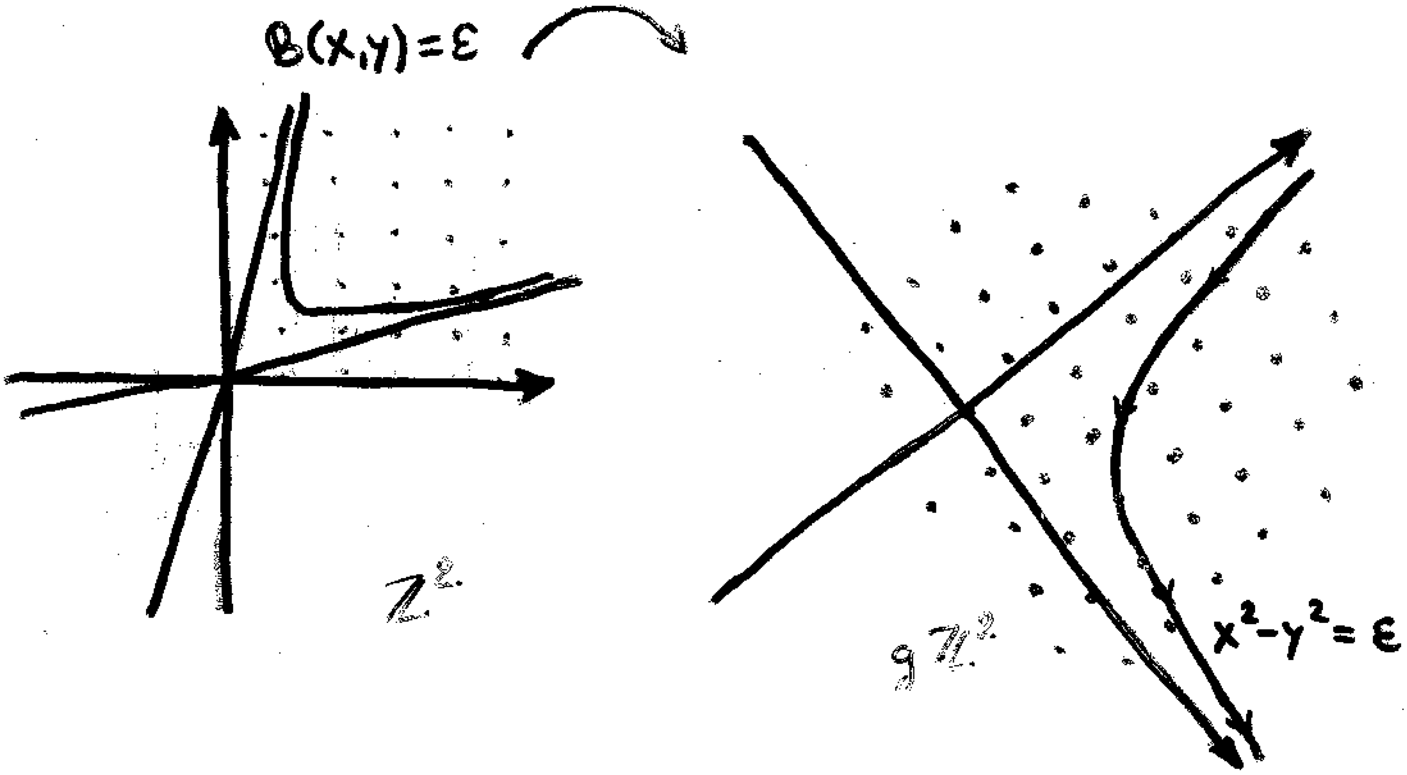
Approach: using a linear unimodular change of variables $g : \mathbb{R}^{m+n} \mapsto \mathbb{R}^{m+n}$ one can write

$$\boxed{B(\mathbf{x}) = \lambda S_{m,n}(g\mathbf{x})}$$

where $\lambda \in \mathbb{R}$, $g \in SL_{m+n}(\mathbb{R})$ and

$$S_{m,n}(x_1, \ldots, x_{m+n}) = x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2 ,$$

and then work with $\{g\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^{m+n}\}$.



Example The Oppenheim Conjecture:

for every indefinite irrational quadratic form B on $\mathbb{R}^k$, $k \geq 3$,

$$\inf_{x \in \mathbb{Z}^k \setminus \{0\}} |B(x)| = 0$$

Proved by Margulis in 1986

Not true for $k=2$: if $B(p,q) = (\alpha q - p)q$,

then $\inf_{x \in \mathbb{Z}^2 \setminus \{0\}} |B(x)| > 0 \iff \left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}$ for some $c > 0$

Problem 2. Given a system of m linear forms $A_1, \dots, A_m$ on $\mathbb{R}^n$, how small (simultaneously) can be the values of

$$|A_i(\mathbf{q}) + p_i|, \; p_i \in \mathbb{Z},$$

when $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}^n$ is far from 0?

Approach: Put together

$$\underline{A_1(\mathbf{q}) + p_1}, \dots, \underline{A_m(\mathbf{q}) + p_m} \quad \text{and} \quad \underline{q_1}, \dots, \underline{q_n},$$

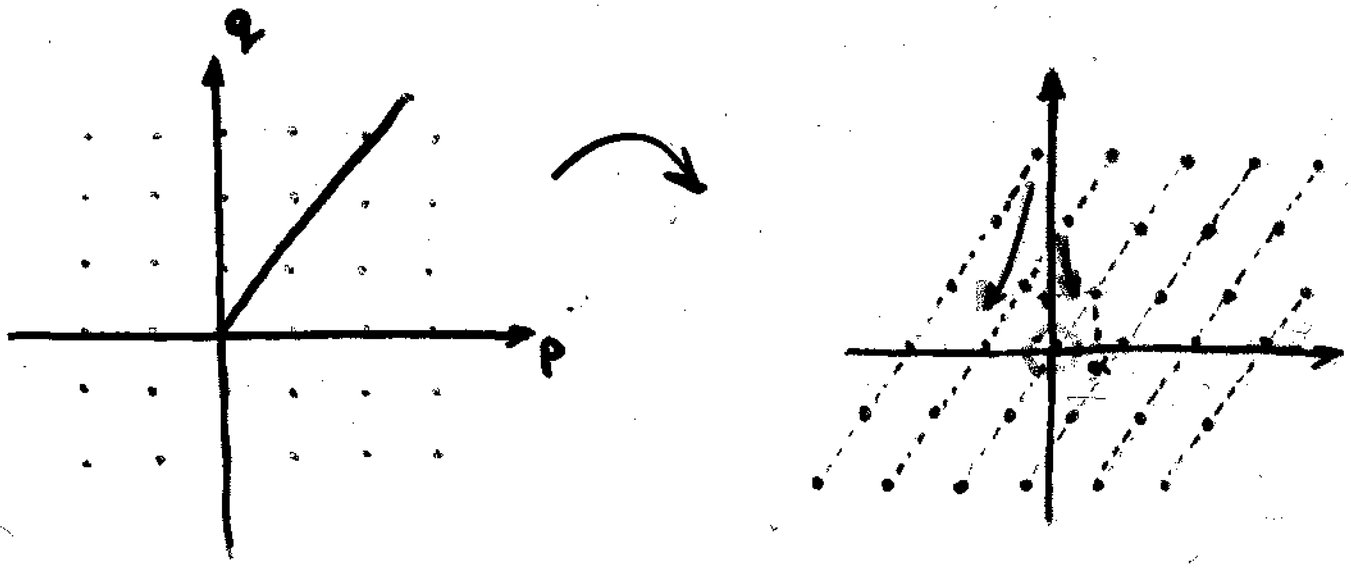
and consider the collection of vectors

$$\boxed{\left\{ \begin{pmatrix} A\mathbf{q} + \mathbf{p} \\ \mathbf{q} \end{pmatrix} \middle| \, \mathbf{p} \in \mathbb{Z}^m, \, \mathbf{q} \in \mathbb{Z}^n \right\} = L_A \mathbb{Z}^{m+n}}$$

where $L_A \overset{\text{def}}{=} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}$ and A is the matrix with rows $A_1, \dots, A_m$.

$m = n = 1$: $\quad |\alpha q + p|$ vs. $|q|$

$(\alpha \in \mathbb{R},\ p, q \in \mathbb{Z})$



<u>Example</u> Littlewood's Conjecture:

for any $y_1, \dots, y_n \in \mathbb{R}$

$$\inf_{\substack{p_1, \dots, p_n \in \mathbb{Z} \\ q \in \mathbb{Z} \setminus \{0\}}} |y_1 q + p_1| \cdot \ldots \cdot |y_n q + p_n| \cdot |q| = 0$$

$$\Updownarrow$$

$$\inf_{p \in \mathbb{Z},\ q_1, \dots, q_n \in \mathbb{Z} \setminus \{0\}} |y_1 q_1 + \ldots + y_n q_n + p| \cdot |q_1| \cdot \ldots \cdot |q_n| = 0$$

(not true if $n = 1$)

This motivates the use of the following dynamical system:

Phase space. Fix $k \in \mathbb{N}$ and consider

$$\boxed{\Omega \stackrel{\text{def}}{=} \text{the set of unimodular lattices in } \mathbb{R}^k}$$

(discrete subgroups with covolume 1).

That is, any $\Lambda \in \Omega$ is equal to $\mathbb{Z}\mathbf{x}_1 \oplus \cdots \oplus \mathbb{Z}\mathbf{x}_k$ where the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ (called a *generating set* of the lattice) is linearly independent, and $\|\mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_k\| = 1$.

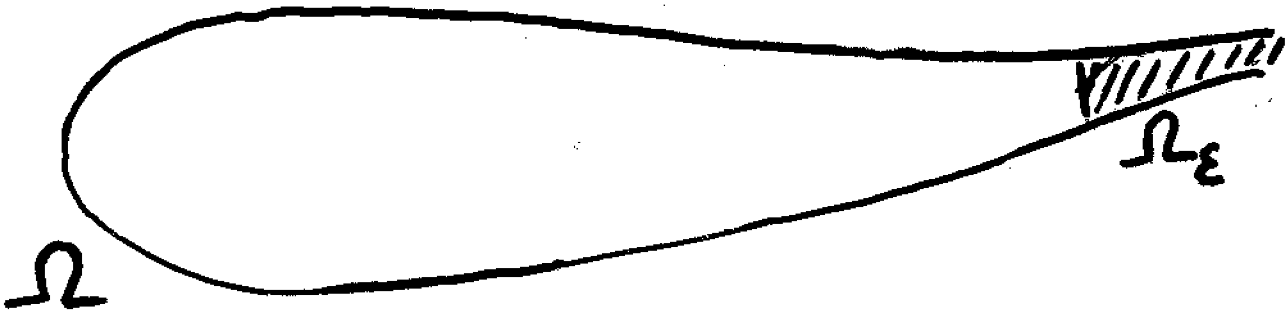
An element of Ω which is easy to distinguish is $\mathbb{Z}^k$ (the standard lattice). In fact, any $\Lambda \in \Omega$ is equal to $g\mathbb{Z}^k$ for some $g \in G \stackrel{\text{def}}{=} SL_k(\mathbb{R})$. That is, G acts transitively on Ω, and, further, $\Gamma \stackrel{\text{def}}{=} SL_k(\mathbb{Z})$ is the stabilizer of $\mathbb{Z}^k$. In other words, Ω is isomorphic to the homogeneous space G/Γ.

Topology. Two lattices are close if their generating sets are close. This defines a topology on Ω which coincides with the quotient topology on G/Γ.

Fact: Ω is not compact. More precisely, a subset K of Ω is bounded iff there exists $\varepsilon > 0$ such that for any $\Lambda \in K$ one has $\inf_{\mathbf{x} \in \Lambda \smallsetminus \{0\}} \|\mathbf{x}\| \geq \varepsilon$ (Mahler's Compactness Criterion). In other words, define

$$\boxed{\Omega_\varepsilon \stackrel{\text{def}}{=} \big\{\Lambda \in \Omega \;\big|\; \|\mathbf{x}\| < \varepsilon \text{ for some } \mathbf{x} \in \Lambda \smallsetminus \{0\}\big\}\,;}$$

then $\Omega \smallsetminus \Omega_\varepsilon$ is compact.



Measure. One can consider a Haar measure on G (both left and right invariant) and the corresponding left-invariant measure on Ω. Fact: the resulting measure is finite (Borel-Harish Chandra). We denote by μ the normalized Haar measure on Ω.

Action. Ω is a topological G-space, with the (continuous) left action defined by

$$\boxed{g\Lambda = \{g\mathbf{x} \mid \mathbf{x} \in \Lambda\} \quad \text{or} \quad g(h\Gamma) = (gh)\Gamma\,.}$$

One can consider the action of various subgroups (one- or multi-parameter) or subsets of G.

Features:

- uniformity of the geometry of the homogeneous space G/Γ

(a nbhd of every $\Lambda \in \Omega \cong$ a nbhd of $e \in G$)

- the representation theory of G

(the G-action on $\Omega \Leftarrow$ the regular repr-n of G on $L^2(\Omega)$)

- combinatorial structure of the space of lattices
- intuition coming from number theory

More general situations:

- G = (connected) Lie group

  $\Gamma \subset G$ a lattice

- G = connected semisimple Lie group, center-free, no compact factors

  $\Gamma \subset G$ an irreducible lattice

- $S = \{p_1, \dots, p_k\}$ a finite set of primes (possibly including ∞)

  $G = \prod_{i=1}^{k} G_i$, G_i = Lie group over $\mathbb{Q}_{p_i}$

  ($\mathbb{Q}_\infty = \mathbb{R}$)

($\Downarrow$ S-arithmetic number theory)

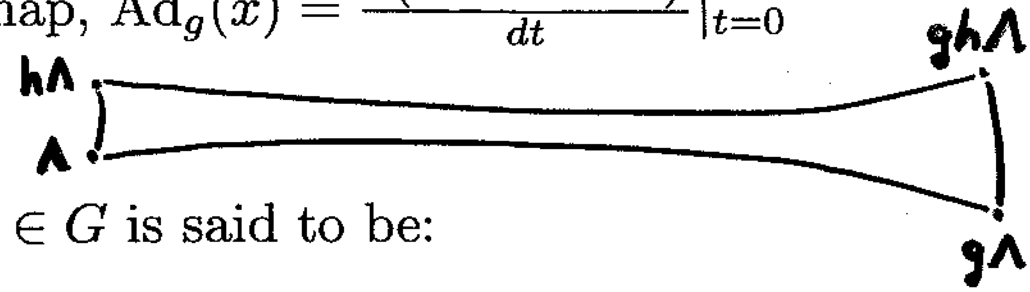
Roughly speaking, the problems arising from numbes theory are of the following type:

$$\boxed{\begin{array}{l}\text{given } H \subset G, \text{ describe} \\ \text{the set of } x \in G/\Gamma \\ \text{with prescribed behavior of } Hx\end{array}}$$

while <u>metric number theory</u> deals with problems like:

$$\boxed{\begin{array}{l}\text{describe the set of } \left[\begin{array}{l}\text{numbers}\\ \text{vectors}\\ \text{matrices}\end{array}\right. \\ \\ \text{with prescribed "approximation property"}\end{array}}$$

Since $g(h\Lambda) = (ghg^{-1})g\Lambda$, local properties of the g-action are determined by the differential of the conjugation map, $\mathrm{Ad}_g(x) = \frac{d\left(g\exp(tx)g^{-1}\right)}{dt}\big|_{t=0}$



An element $g \in G$ is said to be:

unipotent if $(\mathrm{Ad}_g - \mathrm{Id})^j = 0$ for some $j \in \mathbb{N}$ ($\Leftrightarrow$ all eigenvalues of Ad_g are equal to 1);

quasi-unipotent if ____ are of absolute value 1;

partially hyperbolic if it is not quasi-unipotent.

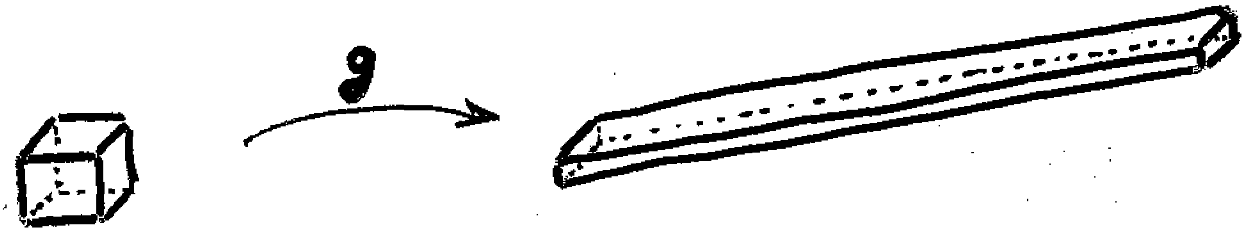
Equivalently: given $g \in G$, define

$$H_\pm(g) = \{h \in G \mid g^{-l}hg^l \to e \text{ as } l \to \pm\infty\}$$

(expanding and contracting horospherical subgroups)

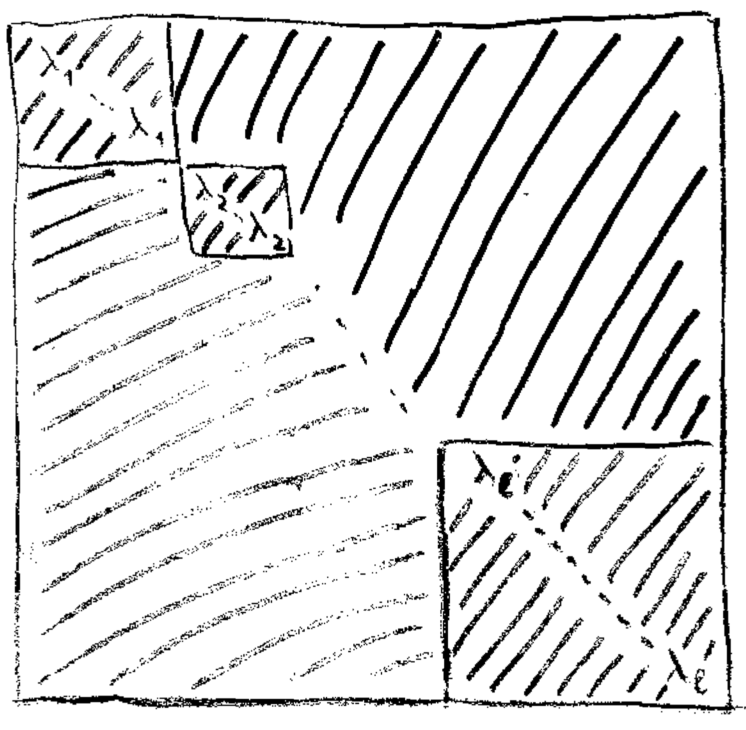
Then G is locally a direct product of $H_-(g)$, $H_+(g)$ and another subgroup $H_0(g)$, and g is quasiunipotent iff $H_0(g) = G$ (that is, $H_-(g)$ and $H_+(g)$ are trivial).

Furthermore, for any $\Lambda \in \Omega$ the orbits $H_-(g)\Lambda$, $H_+(g)\Lambda$ and $H_0(g)\Lambda$ are leaves of *stable*, *unstable* and *neutral* foliations on Ω.



Example. Suppose that $g \in G$ is diagonalizable over $\mathbb{R}$, and choose a basis of $\mathbb{R}^k$ in which $g = \operatorname{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{i_1 \text{ times}}, \ldots\ldots, \underbrace{\lambda_l, \ldots, \lambda_l}_{i_l \text{ times}}), \quad \lambda_1 > \cdots > \lambda_l.$

Then $\underline{H_-(g)}$ and $\underline{H_+(g)}$ are subgroups of lower- and upper-triangular groups:

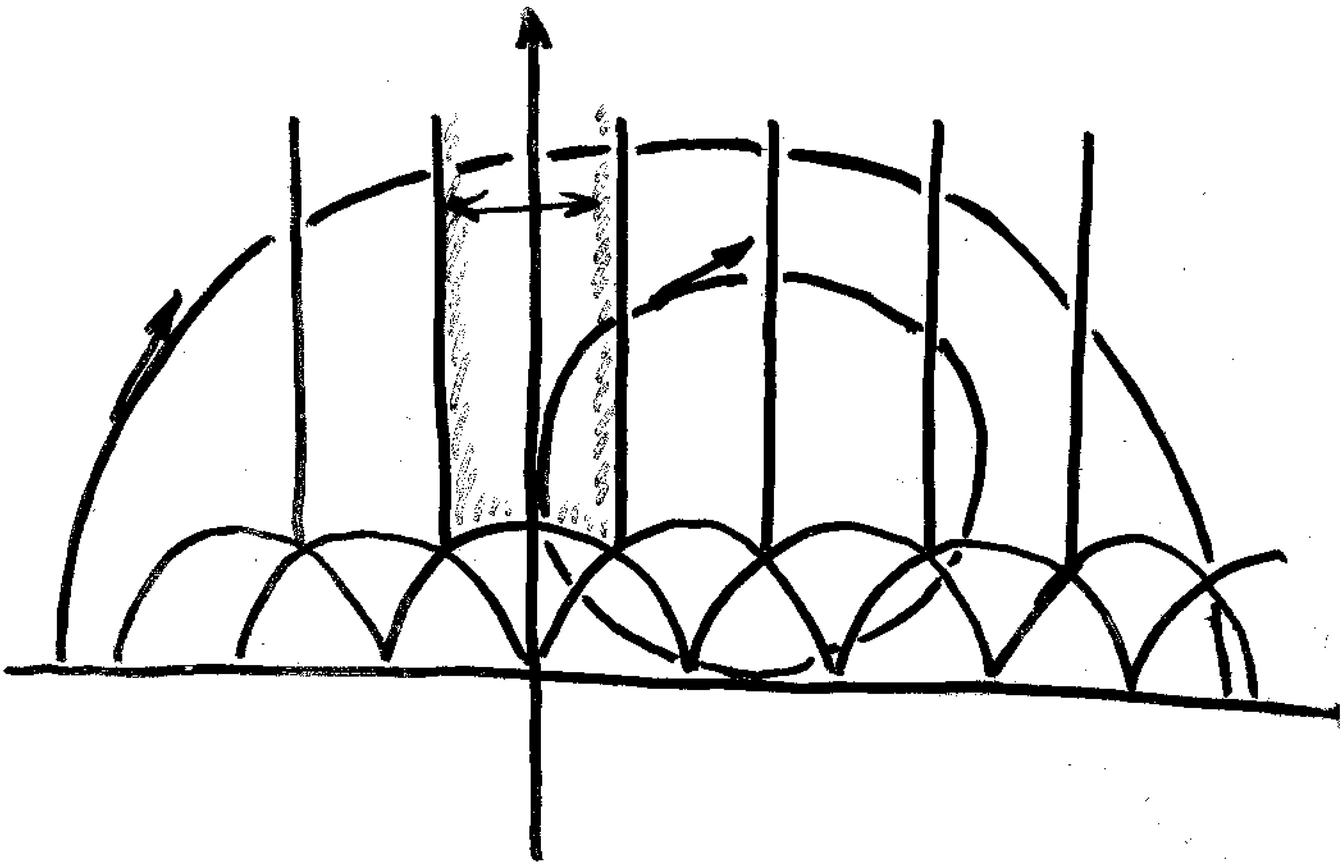


More examples. The simplest case $k = 2$: then $\Omega \cong$ the unit tangent bundle to $\mathbb{H}^2/SL_2(\mathbb{Z})$

The geodesic flow — the action of $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

The horocycle flow — the action of $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

(an example of a **unipotent flow**).



Ergodic properties.

Moore's Theorem: the action of any noncompact closed subgroup of G on Ω is ergodic and mixing.

Decay of correlations: there exists $\beta > 0$ such that for any two functions $\varphi, \psi \in C^{\infty}_{comp}(\Omega)$ with $\int \varphi = \int \psi = 0$ and any $g \in G$ one has

$$\boxed{\left|\int (g\varphi \cdot \psi)\right| \leq \mathrm{const}(\varphi, \psi) e^{-\beta\|g\|}\,.}$$

In particular, if g_t is partially hyperbolic, then

$$\boxed{\left|\int (g_t\varphi \cdot \psi)\right| \leq \mathrm{const}(\varphi, \psi, g_t) e^{-\gamma t}\,.}$$

(Moore, Ratner for $k = 2$,

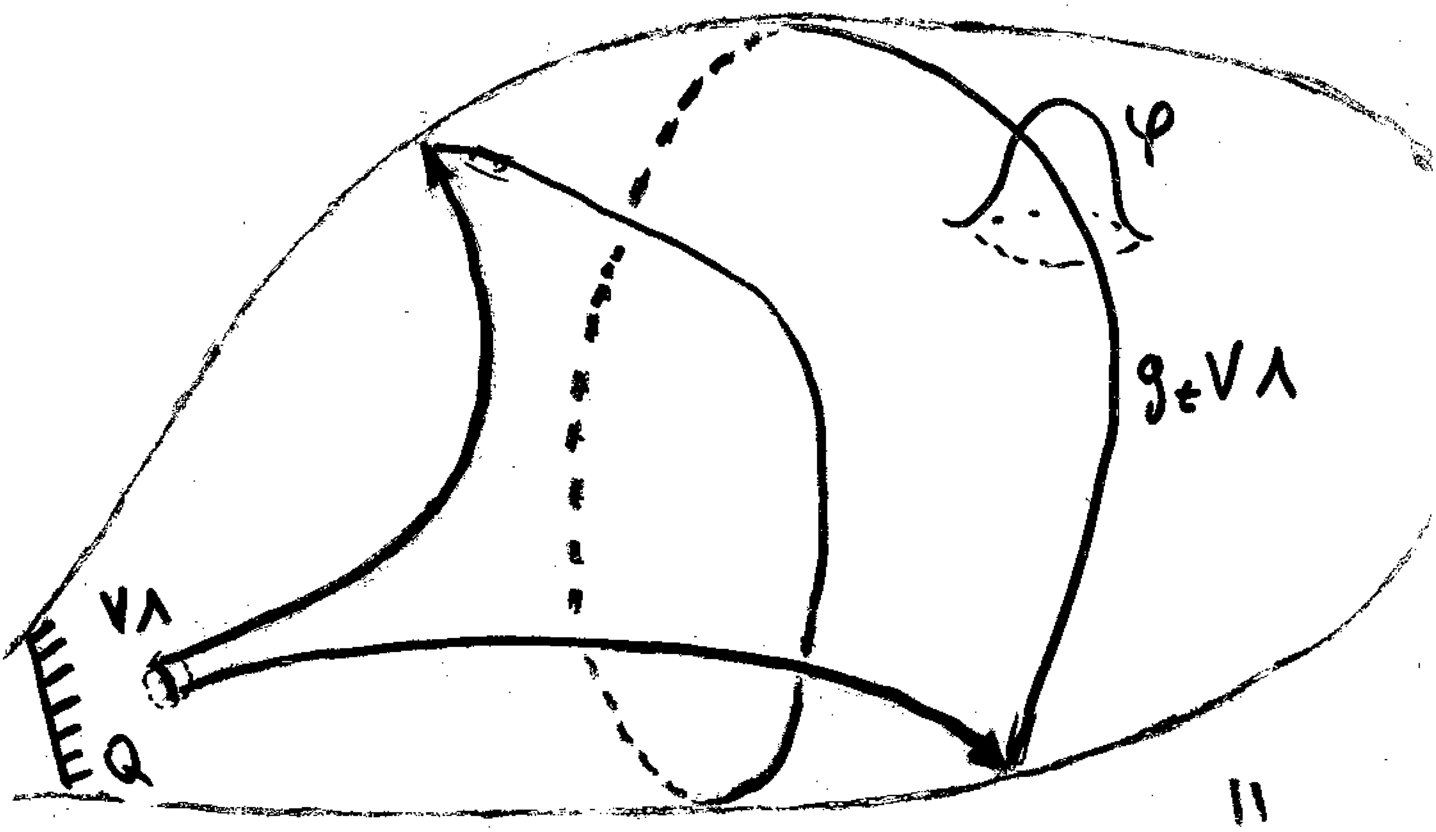
Howe, Cowling, Katok-Spatzier for $k > 2$).

Uniform distribution of unstable leaves: let g_t be a partially hyperbolic one-parameter subgroup of G, $H = H_+(g)$, ν a Haar measure on H. Then there exists $\lambda > 0$ with the following property:

for any open subset V of H, any $\varphi \in C^\infty_{comp}(\Omega)$ and any compact subset Q of Ω there exists $C > 0$ such that

$$\boxed{\left| \frac{1}{\nu(g_t V g_{-t})} \int_{g_t V g_{-t}} \varphi(h g_t \Lambda)\, d\nu(h) - \int_\Omega \varphi\, d\mu \right| \le C e^{-\lambda t}}$$

for all $\Lambda \in Q$ and $t \ge 0$. (K-Margulis 1996)



Orbit closures of unipotent flows.

Fact: any orbit of the horocycle flow on $SL_2(\mathbb{R})/\Gamma$ is either periodic or dense (Hedlund 1930s).

Theorem. *Let U be a unipotent subgroup of G. Then for any $\Lambda \in \Omega$ there exists a closed connected subgroup L of G containing U such that the closure of the orbit $U\Lambda$ coincides with $L\Lambda$ and there is an L-invariant probability measure supported on $L\Lambda$.*

(conjectured by Raghunathan, proved by Ratner)

Furthermore, $L = G$ for Λ not lying in a countable union of proper submanifolds of G/Γ.

(unipotent flows are "not very chaotic")

Corollary. *Let* $S(x_1, x_2, x_3) = 2x_1x_3 - x_2^2$, *and*

$$\boxed{H_S = \{h \in SL_3(\mathbb{R}) \mid S(h\mathbf{x}) = S(\mathbf{x})\ \forall \mathbf{x} \in \mathbb{R}^3\}} \cong SO(2,1)$$

(the stabilizer of S*). Then any relatively compact orbit* $H_S\Lambda$, Λ *a lattice in* $\mathbb{R}^3$, *is compact.*

Explanation: H_S is generated by its unipotent one-parameter subgroups,

$$V(t) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V^T(t) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{pmatrix},$$

and there are no intermediate subgroups between H_S and $SL_3(\mathbb{R})$.

Corollary. *Let B be a real nondegenerate indefinite quadratic form in 3 variables.*

If $\lvert B(\mathbf{x})\rvert \geq \varepsilon$ *for some* $\varepsilon > 0$ *and all* $\mathbf{x} \in \mathbb{Z}^3 \smallsetminus \{0\}$,	*then B is proportional to a rational form.*
$\Updownarrow$	$\Updownarrow$
$\lVert h\mathbf{x}\rVert \geq \varepsilon$ for some $\varepsilon > 0$ and all $h \in H_B$, $\mathbf{x} \in \mathbb{Z}^3 \smallsetminus \{0\}$	H_B is defined over $\mathbb{Q}$
$\Updownarrow$	$\Updownarrow$
$H_B \mathbb{Z}^3$ is relatively compact in Ω	$H_B \cap SL_3(\mathbb{Z})$ is Zariski dense in H_B
$\Updownarrow$	$\Updownarrow$
$H_S \Lambda$ is relatively compact	$\mathrm{vol}\,(\underbrace{H_B / H_B \cap SL_3(\mathbb{Z})}_{\cong H_B \mathbb{Z}^3}) < \infty$
	$\Updownarrow$
	$\mathrm{vol}\,(H_S \Lambda) < \infty$
	$\uparrow$
(here $B(\mathbf{x}) = \lambda S(g\mathbf{x})$ and $\Lambda = g\mathbb{Z}^3$)	$H_S \Lambda$ is compact

Basics of metric number theory

Let $\psi(x)$ be a non-increasing function $\mathbb{R}_+ \mapsto \mathbb{R}_+$.

Definition. Say that $A \in M_{m \times n}(\mathbb{R})$ (viewed as a system of linear forms $A_1, \ldots, A_m$ on $\mathbb{R}^n$) is *ψ-approximable* if there are infinitely many $q \in \mathbb{Z}^n$ such that

$$\|Aq + p\|^m \leq \psi(\|q\|^n) \quad \text{for some } p \in \mathbb{Z}^m.$$

Theorem 1. *Every* $A \in M_{m \times n}(\mathbb{R})$ *is* $\frac{1}{x}$*-approximable.* (Dirichlet 1842)

Theorem 2. *Almost every (resp. almost no)* A *is ψ-approximable, provided the integral* $\boxed{\int_1^\infty \psi(x)\, dx}$ *diverges (resp. converges).* (Groshev 1938)

(the Khintchine-Groshev Theorem)

Definition. $A \in M_{m\times n}(\mathbb{R})$ is *badly approximable* if it is not $\frac{c}{x}$-approximable for some $c > 0$; that is, if there exists $c > 0$ such that $\boxed{\|A\mathbf{q} + \mathbf{p}\|^m \|\mathbf{q}\|^n \geq c}$ $\forall\, \mathbf{p} \in \mathbb{Z}^m$ and all but finitely many $\mathbf{q} \in \mathbb{Z}^n$.
(If $m = n = 1$: $\alpha \in \mathbb{R}$ is badly approximable $\Leftrightarrow$ coefficients in the continued fraction expansion of α are bounded)

Facts. The set of badly approximable $A \in M_{m\times n}(\mathbb{R})$ is

- nonempty (Perron 1921)
- of measure zero (Khintchine 1926)
- of full Hausdorff dimension (Jarnik 1929 for $m = n = 1$, Schmidt 1969 for the general case)

Theorem 3. (Dani 1985) *$A \in M_{m\times n}(\mathbb{R})$ is* *badly approximable* *iff the trajectory $\{g_t L_A \mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\}$, with* $L_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}$ and

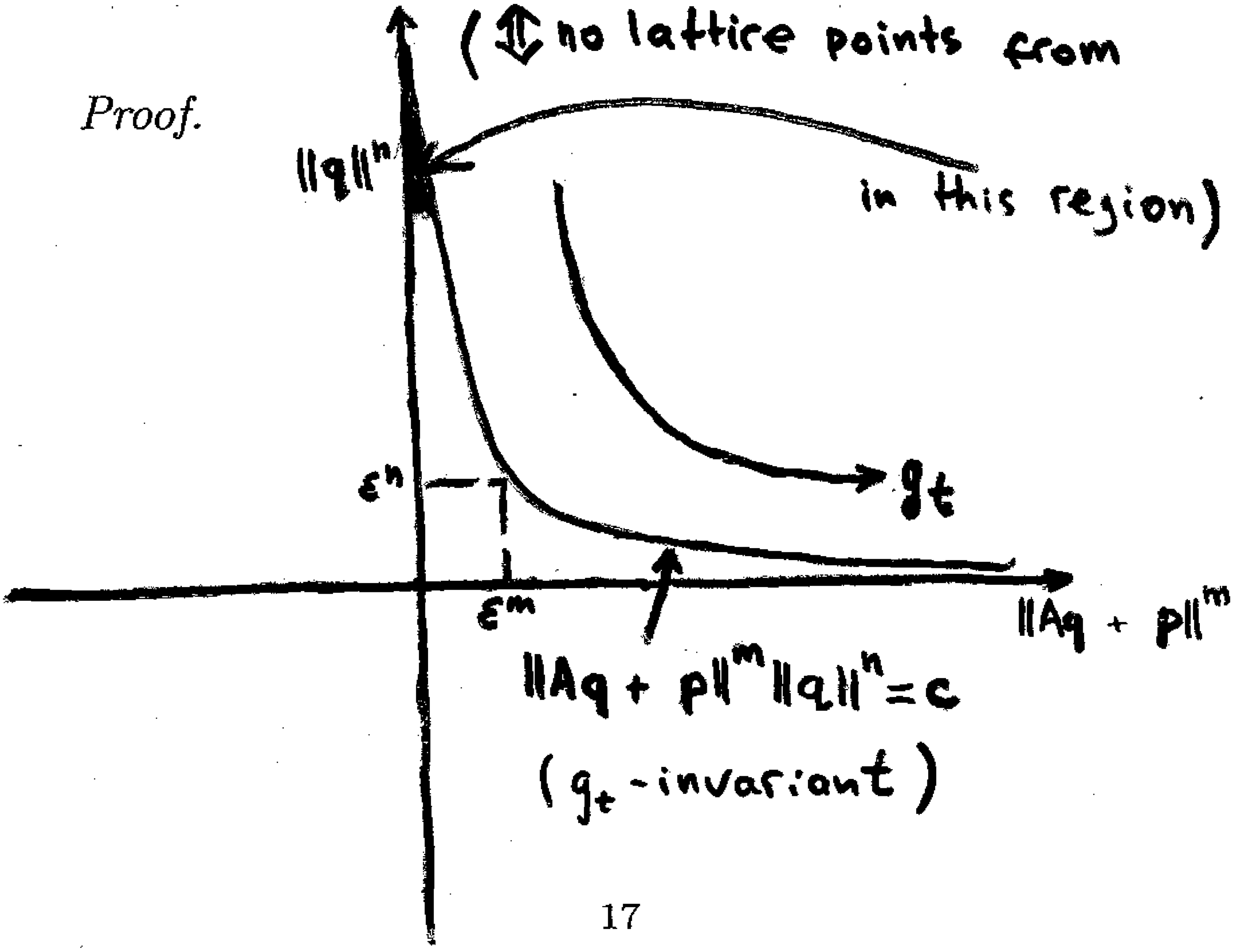
$$(*) \qquad g_t = \operatorname{diag}(e^{t/m}, \ldots, e^{t/m}, e^{-t/n}, \ldots, e^{-t/n}),$$

is bounded in the space $\overbrace{\Omega}^{\text{of unimodular lattices in } \mathbb{R}^{m+n}}$*. (Here $k = m+n$.)*

(Recall: $\Omega_\varepsilon = \{\Lambda \in \Omega \mid \Lambda \cap B(0,\varepsilon) \neq \{0\}\}$, $\Omega \setminus \Omega_\varepsilon$ compact)

($\Leftrightarrow$ for some $\varepsilon > 0$, $\{g_t L_A \mathbb{Z}^{m+n} \mid t \in \mathbb{R}_+\} \cap \Omega_\varepsilon = \varnothing$)

Proof.



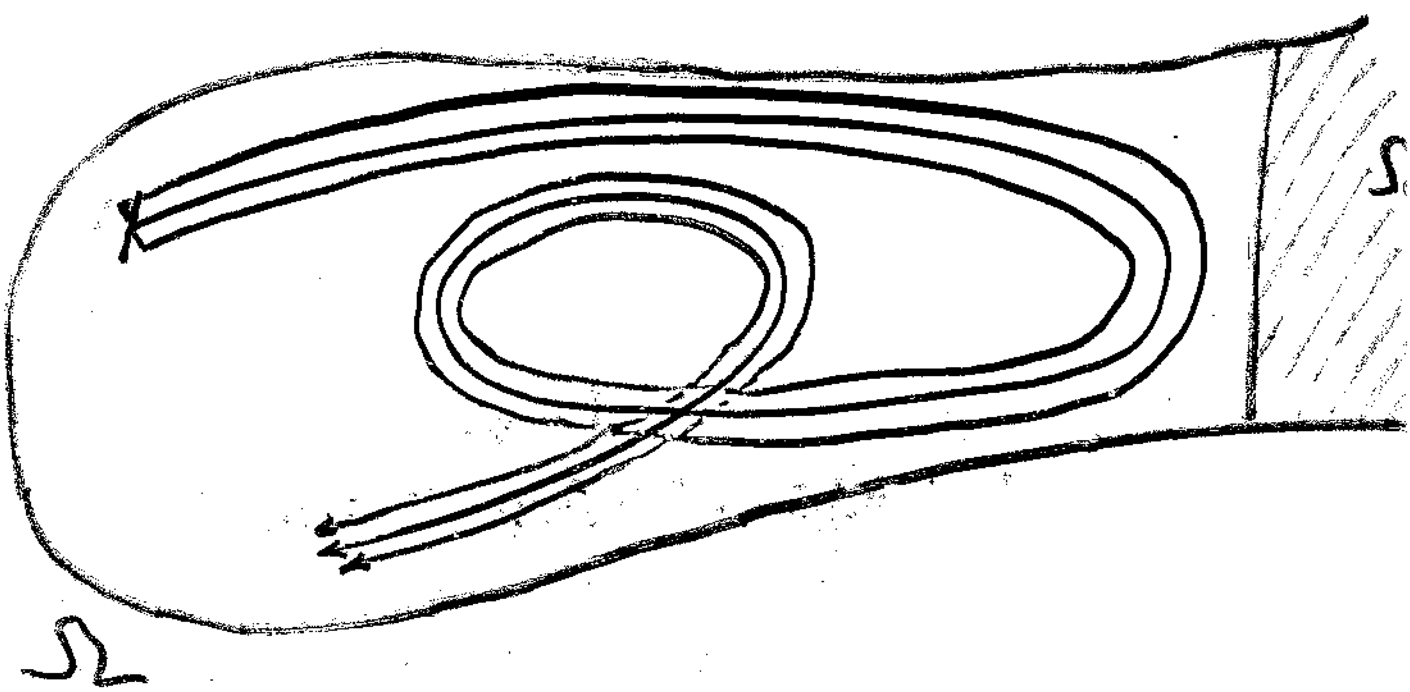
Corollary. (Dani 1985) *The set*

$$\{\Lambda \in \Omega \mid \{g_t\Lambda \mid t \geq 0\} \text{ is bounded}\},$$

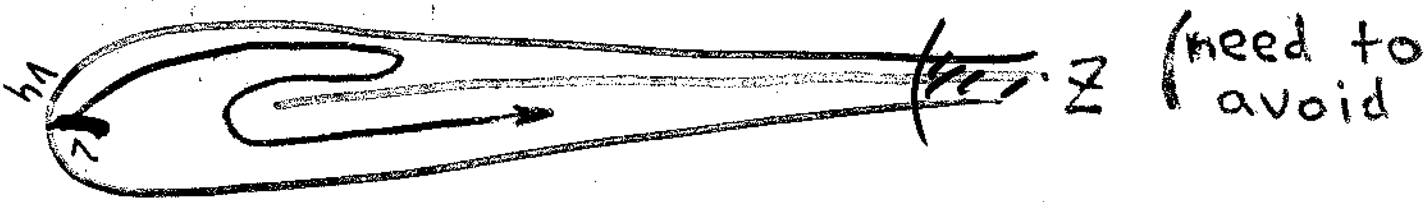
with $\{g_t\}$ *as in* $(*)$, *has full Hausdorff dimension.*

Proof. $\Lambda = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} L_A \mathbb{Z}^{m+n} \quad \Rightarrow$

$$g_t\Lambda = \underbrace{g_t \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} g_{-t}}_{\text{bounded}} \cdot g_t L_A \mathbb{Z}^{m+n}. \qquad \square$$



Theorem 4. (K-Margulis 1996) *Let $F = \{g_t \mid t \geq 0\}$ be a one-parameter subsemigroup of G, $H = H_+(g_1)$ the expanding horospherical subgroup. Then for any closed F-invariant null subset Z of Ω and any $\Lambda \in \Omega$, the set $\{h \in H \mid Fh\Lambda$ is bounded and $\overline{Fh\Lambda} \cap Z = \varnothing\}$ has full Hausdorff dimension.*

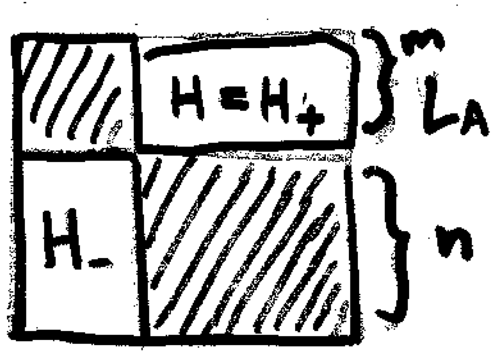


Corollary. *If $\{g_t\}$ is partially hyperbolic, then the set $\{\Lambda \in \Omega \mid F\Lambda$ is bounded and $\overline{F\Lambda} \cap Z = \varnothing\}$ has full Hausdorff dimension.*

Another Corollary: Schmidt's result on badly approximable systems of linear forms

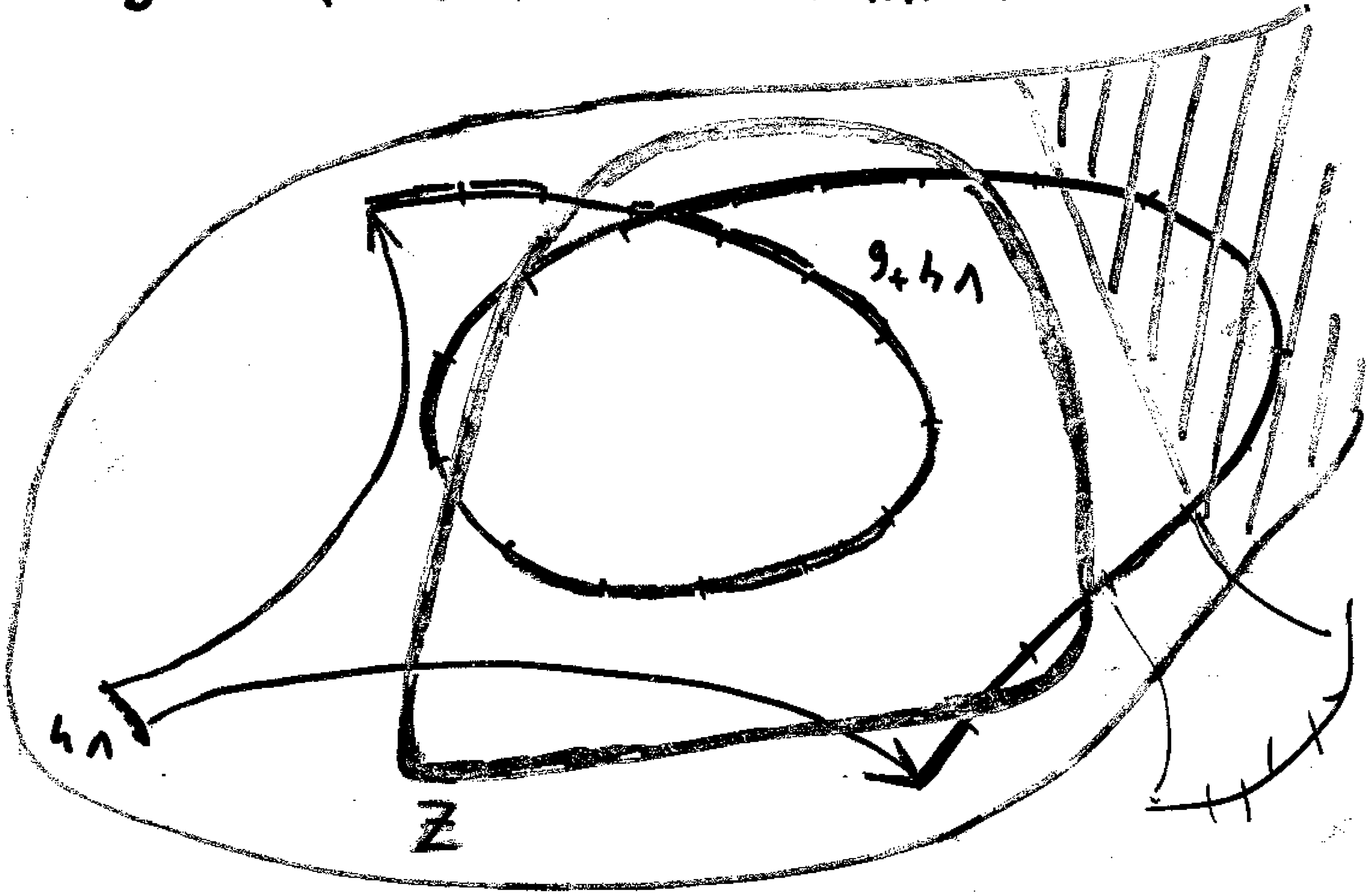
(since for g_t as in $(*)$, $H = \{L_A \mid A \in M_{m\times n}(\mathbb{R})\}$)

$$g_t = \begin{pmatrix} e^{t/m} & & & & & \\ & \ddots & & & \boxed{} & \\ & & e^{t/m} & & & \\ & & & e^{-t/n} & & \\ & & & & \ddots & \\ & & & & & e^{-t/n} \end{pmatrix} \Rightarrow$$



Proof of Theorem 4. Use uniform distribution of unstable leaves to create a Cantor-like set of big Hausdorff dimension.

Stage 1 of the Cantor set construction:



(Holds for any Lie group G and any lattice $\Gamma \subset G$ under an additional technical assumption on $\{g_t\}$.)

Inhomogeneous approximation

An *affine form* = a linear form plus a real number.

A system of m affine forms in n variables is given by a pair $\langle A, \mathbf{b}\rangle$, where $A \in M_{m\times n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$.

$(\mathbf{x} \to A\mathbf{x} + \mathbf{b} : \mathbb{R}^n \to \mathbb{R}^m)$

Definition. A system of affine forms given by $\langle A, \mathbf{b}\rangle \in M_{m\times n}(\mathbb{R}) \times \mathbb{R}^m$ is *ψ-approximable* if there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\boxed{\|A\mathbf{q} + \mathbf{b} + \mathbf{p}\|^m \le \psi(\|\mathbf{q}\|^n) \quad \text{for some } \mathbf{p} \in \mathbb{Z}^m,}$$

and it is *badly approximable* if it is not $\frac{c}{x}$-approximable for some $c > 0$; that is, if there exists $c > 0$ such that

$$\boxed{\|A\mathbf{q} + \mathbf{b} + \mathbf{p}\|^m \|\mathbf{q}\|^n \ge c}$$

$\forall\, \mathbf{p} \in \mathbb{Z}^m$ and all but finitely many $\mathbf{q} \in \mathbb{Z}^n$.

Fact: The set of badly approximable $\langle A, \mathbf{b}\rangle \in M_{m\times n}(\mathbb{R}) \times \mathbb{R}^m$ is of measure zero (an inhomogeneous version of Khintchine-Groshev)

<u>Examples</u> $(m=n=1)$ $(|\alpha q+\beta+p|$ vs. $|q|)$

1. $\alpha q_0+\beta+p_0=0$ for some $p_0, q_0 \in \mathbb{Z}$

$$\Downarrow$$

$$|\alpha q+\beta+p|\cdot|q| = |\alpha(q-q_0)+p-p_0|\cdot|q-q_0|\cdot\frac{|q|}{|q-q_0|}$$

$$\Downarrow$$

$\langle\alpha,\beta\rangle$ is badly approximable $\Longleftrightarrow$ α is badly approximable

2. $\alpha\in\mathbb{Q}, \beta\notin\mathbb{Q} \Rightarrow \{\alpha q+\beta+p \mid p,q\in\mathbb{Z}\}\not\ni 0$ is discrete

$$\Downarrow$$

$|\alpha q+\beta+p|\cdot|q| \geq \text{const}\cdot|q| \Rightarrow$ badly approximable

3. ???

All known examples of badly approximable $\langle A, \mathbf{b}\rangle \in M_{m\times n}(\mathbb{R})\times\mathbb{R}^m$ belong to a countable union of proper submanifolds of $M_{m\times n}(\mathbb{R}) \times \mathbb{R}^m \Rightarrow$ form a set of positive Hausdorff codimension.

A dynamical approach:

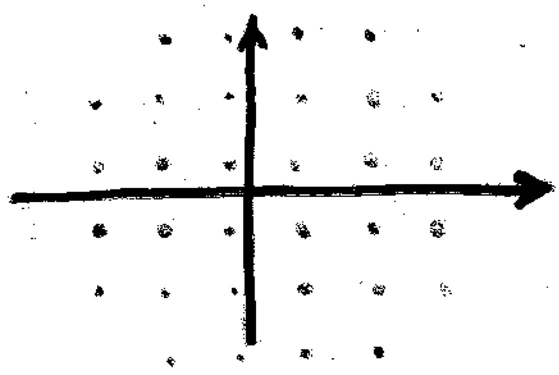
consider the collection of vectors

$$\left\{ \begin{pmatrix} A\mathbf{q}+\mathbf{b}+\mathbf{p} \\ \mathbf{q} \end{pmatrix} \middle| \, \mathbf{p}\in\mathbb{Z}^m, \, \mathbf{q}\in\mathbb{Z}^n \right\} = L_A\mathbb{Z}^{m+n} + \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}$$

This would be an element of the space $\hat{\Omega} = \hat{G}/\hat{\Gamma}$ of *affine lattices* in $\mathbb{R}^{m+n}$, where

$$\hat{G} \stackrel{\text{def}}{=} \mathrm{Aff}(\mathbb{R}^{m+n}) = G \ltimes \mathbb{R}^{m+n} \text{ and } \hat{\Gamma} \stackrel{\text{def}}{=} \Gamma \ltimes \mathbb{Z}^{m+n}.$$

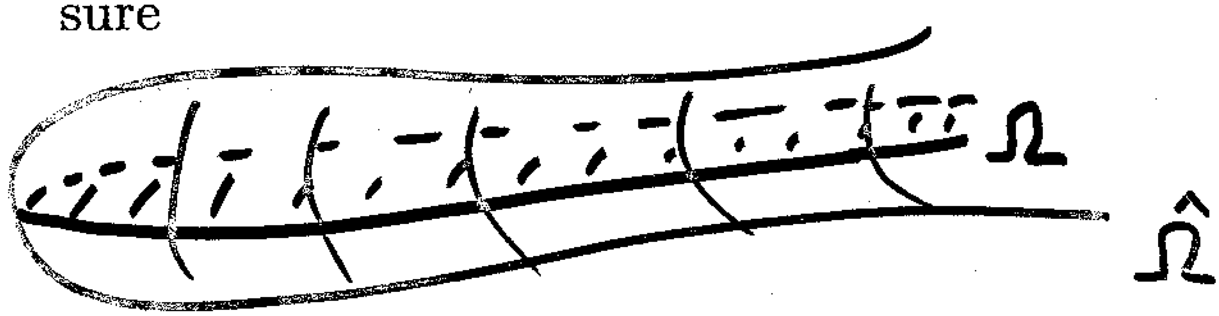
That is,



$$\hat{\Omega} \cong \{\Lambda + \mathbf{w} \mid \Lambda\in\Omega, \ \mathbf{w}\in\mathbb{R}^{m+n}\}.$$

Note that:

- the quotient topology on $\hat{\Omega}$ coincides with the natural topology on the space of affine lattices; that is, $\Lambda_1+\mathbf{w}_1$ and $\Lambda_2+\mathbf{w}_2$ are close to each other if so are $\mathbf{w}_i$ and the generating elements of Λ_i
- $\hat{\Omega}$ is non-compact and has finite Haar measure



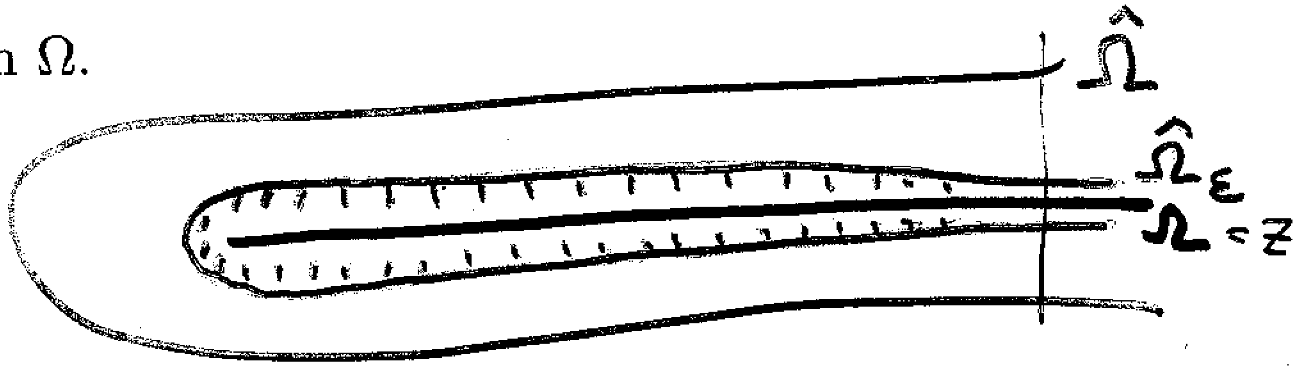
- Ω (the set of *true lattices*) can be identified with a subset of $\hat{\Omega}$ $\quad (\Omega \cong \{\Lambda \in \hat{\Omega} \mid 0 \in \Lambda\})$
- g_t as in $(*)$ acts on $\hat{\Omega}$, and the expanding horospherical subgroup corresponding to g_1 is exactly the set of all elements of $\hat{G}$ with linear part L_A and translation part $\begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}$, $A \in M_{m\times n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$.

$$\Big(g_t\big(L_A \mathbb{Z}^{m+n} + \tbinom{b}{0}\big) = (g_t L_A g_{-t})\, g_t \mathbb{Z}^{m+n} + g_t \tbinom{b}{0} \Big),$$

For $\varepsilon > 0$, define

$$\hat{\Omega}_\varepsilon \stackrel{\text{def}}{=} \left\{ \Lambda \in \hat{\Omega} \mid \|\mathbf{x}\| < \varepsilon \text{ for some } \mathbf{x} \in \Lambda \right\}.$$

Then $\hat{\Omega} \smallsetminus \hat{\Omega}_\varepsilon$ is a closed (non-compact) set disjoint from Ω.



Theorem 5. *Let $F = \{g_t \mid t \geq 0\}$ be as in $(*)$. Then*

$$F\left(L_A \mathbb{Z}^{m+n} + \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}\right) \quad \left\{ \begin{array}{l} \textit{is bounded and} \\ \textit{stays away from } \Omega \end{array} \right.$$

$$\text{a)} \Downarrow$$

$$F\left(L_A \mathbb{Z}^{m+n} + \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}\right) \subset \hat{\Omega} \smallsetminus \hat{\Omega}_\varepsilon \textit{ for some } \varepsilon > 0$$

$$\text{b)} \Downarrow$$

$\langle A, \mathbf{b} \rangle$ *is badly approximable*

Proof. a) Otherwise $\exists\ \Lambda_k \in F\left(L_A \mathbb{Z}^{m+n} + \binom{b}{0}\right)$ and $x_k \in \Lambda_k$ with $\|x_k\| \to 0$; since the orbit is relatively compact, $\{\Lambda_k\}$ has a limit point Λ which must contain 0, i.e. belong to Ω.

b) See the picture, p. 17, and shift the lattice

From a general version of Theorem 4, one deduces

Theorem $\hat{4}$. *Let $F = \{g_t \mid t \geq 0\}$ be a one-parameter subsemigroup of $\hat{G}$, $H = H_+(g_1)$ the expanding horospherical subgroup. Then for any closed F-invariant null subset Z of $\hat{\Omega}$ and any $\Lambda \in \hat{\Omega}$, the set*

$$\{h \in H \mid Fh\Lambda \text{ is bounded and } \overline{Fh\Lambda} \cap Z = \varnothing\}$$

has full Hausdorff dimension.

Corollary. *The set of badly approximable $\langle A, \mathbf{b}\rangle \in M_{m\times n}(\mathbb{R}) \times \mathbb{R}^m$ has full Hausdorff dimension.*

Back to the homogeneous approximation.

One can generalize Dani's correspondence as follows:

Given a non-increasing function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, there is a unique function $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following holds:

$$A \in M_{m\times n}(\mathbb{R}) \text{ is } \psi\text{-approximable}$$

$$\Updownarrow$$

$$\boxed{g_t L_A \mathbb{Z}^{m+n} \in \Omega_{\varepsilon(t)} \text{ for infinitely many } t \in \mathbb{N}}$$

Thus Theorem 2 is equivalent to the following

Theorem 2′. *For almost all (resp. almost no) $\Lambda \in \Omega$ one has $\underline{g_t\Lambda \in \Omega_{\varepsilon(t)} \text{ for infinitely many } t \in \mathbb{N}}$, provided the sum*

$$\sum_{t=1}^{\infty} \varepsilon(t)^{m+n} \qquad \left(\sim \sum_{t=1}^{\infty} \mu(\Omega_{\varepsilon(t)}) \right)$$

diverges (resp. converges).

The above theorem can be proved using ergodic theory (in particular, exponential decay of correlations) and can be generalized to

- any partially hyperbolic g_t, not necessarily of the form $(*)$
- other Lie groups G and lattices $\Gamma \subset G$
- more general than Ω_ε subsets of G/Γ (with "uniformly regular boundaries")
- multi-parameter actions

See [K-Margulis, Inv. Math. 1999]

$$\Downarrow$$

- a new (dynamical) proof of Theorem 2
- logarithm laws for geodesics and flats in noncompact finite volume loc.sym.spaces

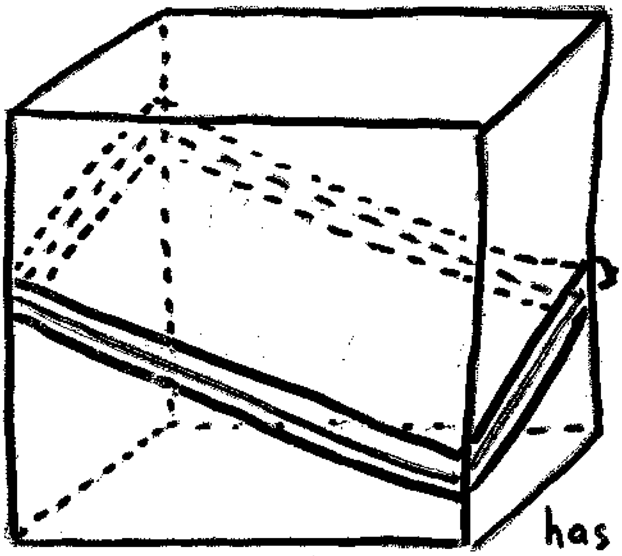
Recall: the Khintchine-Groshev Theorem, $m=1$, easy part:

$$\boxed{\sum_{k=1}^{\infty} \psi(k) < \infty} \Rightarrow \text{for a.e. } y \in \mathbb{R}^n$$

(∗) $\boxed{|y_1 q_1 + \dots + y_n q_n + p| < \psi(\|q\|^n)}$

has at most finitely many solutions.

Proof: the Borel-Cantelli Lemma



For fixed p, q, the set of $y \in [0,1]^n$ satisfying (∗) has measure $\leq$ const $\cdot \frac{1}{\|q\|} \psi(\|q\|^n)$,

there are at most const $\cdot \|q\|$ values of p,

and $\displaystyle\sum_{q\in\mathbb{Z}^n} \psi(\|q\|^n) \asymp \sum_{k=1}^{\infty} k^{n-1}\psi(k^n) \asymp \sum_{k=1}^{\infty} \psi(k)$. ∎

Example $\psi_\beta(k) \stackrel{\text{def}}{=} k^{-(1+\beta)}$, $\beta > 0$, is OK

Very well approximable $\stackrel{\text{def}}{=}$ ψ_β-approximable for some $\beta > 0$

(Diophantine $\stackrel{\text{def}}{=}$ not ψ_β-approximable for some $\beta > 0$)

<u>Problem</u>. (Mahler's conjecture, 1932)

Is it true that for almost all $x \in \mathbb{R}$ the inequality

$$|p + q_1 x + q_2 x^2 + \cdot + q_n x^n| \leq \|\mathbf{q}\|^{-n(1+\beta)}$$

has at most finitely many solutions for every $\beta > 0$?

(for a.e. x, the n-tuple $(x, x^2, \ldots, x^n)$ is not VWA)

<u>Why the same proof does not work</u>:

the measure of solutions sets near tangency points is bigger than it should be.

Solved in 1964 by V. Sprindžuk

Gave rise to a new branch of number theory,

"Diophantine approximation

with dependent quantities"

(Crucial: dependence relations between

$y_1 = x,\ y_2 = x^2,\ \ldots\ ,\ y_n = x^n$)

Why this is important:

1. Mahler's motivation:

$(x, \dots, x^n)$ not VWA

$$\Updownarrow$$

$\forall \beta > 0$ there are at most finitely many polynomials $P \in \mathbb{Z}[x]$, $\deg P \le n$, with

$$\boxed{|P(x)| < h(P)^{-n(1+\beta)}}$$

height of P

$$\Updownarrow$$

x is "not very algebraic"

2. Diophantine conditions in KAM

If coefficients of a diff. equation are restricted to lie on a submanifold of $\mathbb{R}^n$, it may be important to know that almost all values of the coefficients have certain approximation properties

3. Potential generalizations

Say that a C^m submanifold $M \subset \mathbb{R}^n$ is non-degenerate at $y_0 \in M$ if "planes cannot have a higher order tangency to it at y_0"

($\Updownarrow$ $\mathbb{R}^n$ is spanned by partial derivatives of $f = (f_1, \dots, f_n)$ at x_0, where $M = f(U)$, $U \subset \mathbb{R}^d$, and $y_0 = f(x_0)$)

Meta-conjecture Let $M \subset \mathbb{R}^n$ be a C^m submanifold nondegenerate at almost every point. Then "any Diophantine property" of $y \in \mathbb{R}^n$ which holds for a.e. $y \in \mathbb{R}^n$, holds for a.e. $y \in M$.

Conjecture (Sprindžuk 1980) For M as above, a.e. $y \in M$ is not VWA ($\overset{\text{def}}{\Longleftrightarrow}$ M is extremal)

$n=2$ (M = a planar curve with nonzero curvature a.e.) W. Schmidt 1964

General case: K-Margulis 1998

using the dynamical approach

More generally: Khintchine-type theorems on manifolds

Start with $M = \{(x, x^2, \dots, x^n) \mid x \in \mathbb{R}\}$

Bernik 1984: $\sum_{k=1}^{\infty} \psi(k) < \infty$

$\Downarrow$

$(x, x^2, \dots, x^n)$ is NOT ψ-approximable for a.e. x

Beresnevich 1998: $\sum_{k=1}^{\infty} \psi(k) = \infty$

$\Downarrow$

$(x, x^2, \dots, x^n)$ IS ψ-approximable for a.e. x

A combination of traditional technique with the "method of lattices":

> Let $M \subset \mathbb{R}^n$ be a C^m a.e. non-degenerate submanifold. Then <u>almost all</u> (resp. <u>almost no</u>) $y \in M$ are ψ-approximable provided the sum $\sum_{k=1}^{\infty} \psi(k)$ <u>diverges</u> (resp. <u>converges</u>)

Bernik, K, Margulis 1999 (& Beresnevich 1999)

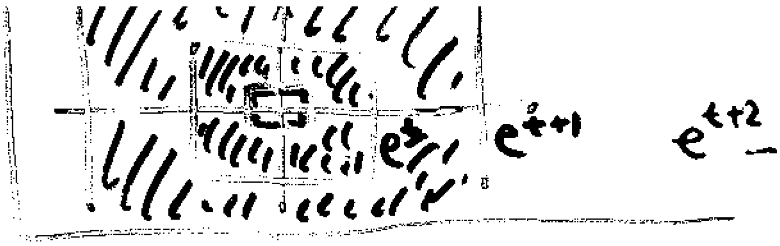
— " — + Beresnevich (2001?)

<u>In this talk:</u>

> $M = \{(f_1(x), \dots, f_n(x)\}$ is a.e. nondegenerate (think $(x, \dots, x^n)$) $(x \in I \subset \mathbb{R})$
>
> $\Downarrow$
>
> for a.e. x, $(f_1(x), \dots, f_n(x))$ is not VWA

(i.e. M is extremal)

<u>Recall</u>: $\underline{y \in \mathbb{R}^n \text{ is VWA}}$



$$\Updownarrow$$

$\exists\ \beta > 0$ such that for infinitely many $q \in \mathbb{Z}^n$ one has

$$\boxed{|q_1 y_1 + \dots + q_n y_n + p| < \|q\|^{-n(1+\beta)}}, \text{ some } p \in \mathbb{Z}$$

$$\Downarrow$$

$\exists$ infinitely many $t \in \mathbb{N}$ such that for some $q \in \mathbb{Z}^n$, $p \in \mathbb{Z}$ one has

$$\boxed{\begin{cases} |q_1 y_1 + \dots + q_n y_n + p| < e^{-t(1+\beta)} \\ \cancel{e^t \le} \|q\|^n < e^{t+1} \end{cases}}$$

$$\Downarrow$$

$$\begin{cases} |q_1 y_1 + \dots + q_n y_n + p| < e^{-t} \cdot e^{-\beta t} \\ |q_i| < e^{t/n} \cdot e^{1/n}, i = 1, \dots, n \end{cases} \iff \begin{cases} e^t |q_1 y_1 + \dots + q_n y_n + p| < e^{-\beta t} \\ e^{-t/n} |q_i| < e^{1/n}, i = 1, \dots, n \end{cases}$$

$$\Updownarrow \qquad \Updownarrow$$

$$\begin{cases} e^{(t + \frac{n\beta}{n+1} t + \frac{1}{n+1})} |q_1 y_1 + \dots + q_n y_n + p| < e^{\frac{1}{n+1}} e^{-\frac{\beta}{n+1} t} \\ e^{-\frac{1}{n}(t + \frac{n\beta}{n+1} t + \frac{1}{n+1})} |q_i| < e^{\frac{1}{n+1}} e^{-\frac{\beta}{n+1} t}, i = 1, \dots, n \end{cases}$$

$$\Updownarrow$$

$\exists$ a sequence $t_i \to +\infty$ and $c, \gamma > 0$

such that $\boxed{g_{t_i} L_y \mathbb{Z}^{n+1} \in \Omega_{c e^{-\gamma t_i}}}$

Also recall:

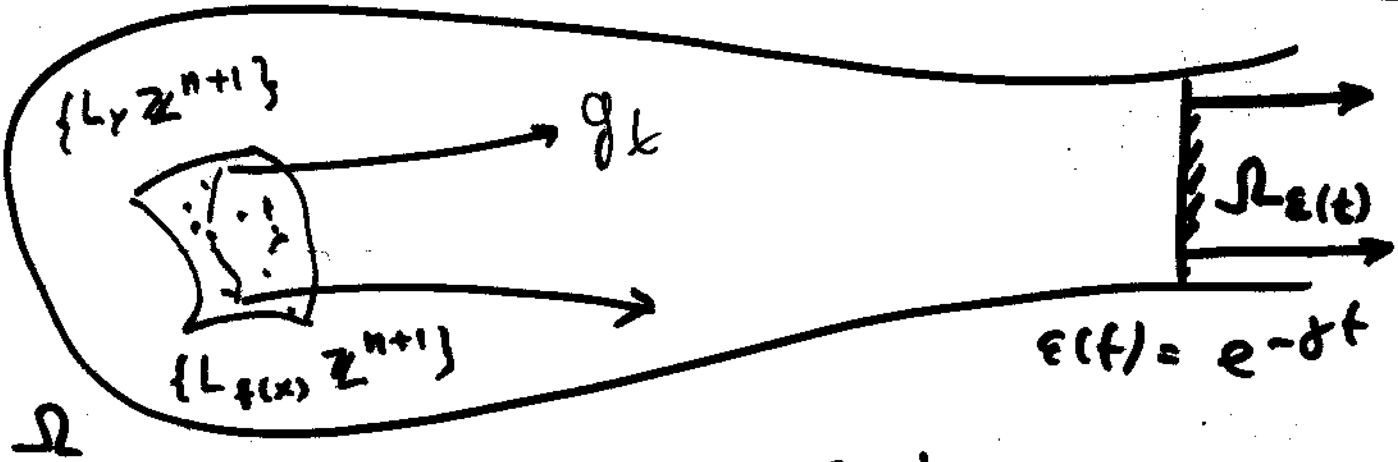
$$L_y = \begin{pmatrix} 1 & y_1 & \cdots & y_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Ω = space of unimodular lattices in $\mathbb{R}^{n+1}$

$$g_t = \mathrm{diag}(e^t, e^{-t/n}, \ldots, e^{-t/n})$$

$$\Omega_\varepsilon = \{\Lambda \in \Omega \mid \Lambda \cap B(0,\varepsilon) \neq \{0\}\}$$

$H = H_+(g_1) = \{L_y \mid y \in \mathbb{R}^n\}$



In fact, $\mathrm{dist}(\mathbb{Z}^{n+1}, \Omega_\varepsilon) \asymp \log \frac{1}{\varepsilon}$, so

y is VWA $\Rightarrow$ $g_t L_y \mathbb{Z}^{n+1}$ grows "super-linearly"
(in fact, $\Leftrightarrow$)

$\otimes \Leftrightarrow$ $\boxed{\text{for a.e. } x \ \{g_t L_{f(x)} \mathbb{Z}^{n+1}\} \text{ grows not too fast}}$

$$|\{x \in I \mid g_t L_{f(x)} \mathbb{Z}^{n+1} \in \Omega_{ce^{-\gamma t}} \text{ for inf. many } t \in \mathbb{N}\}| = 0$$

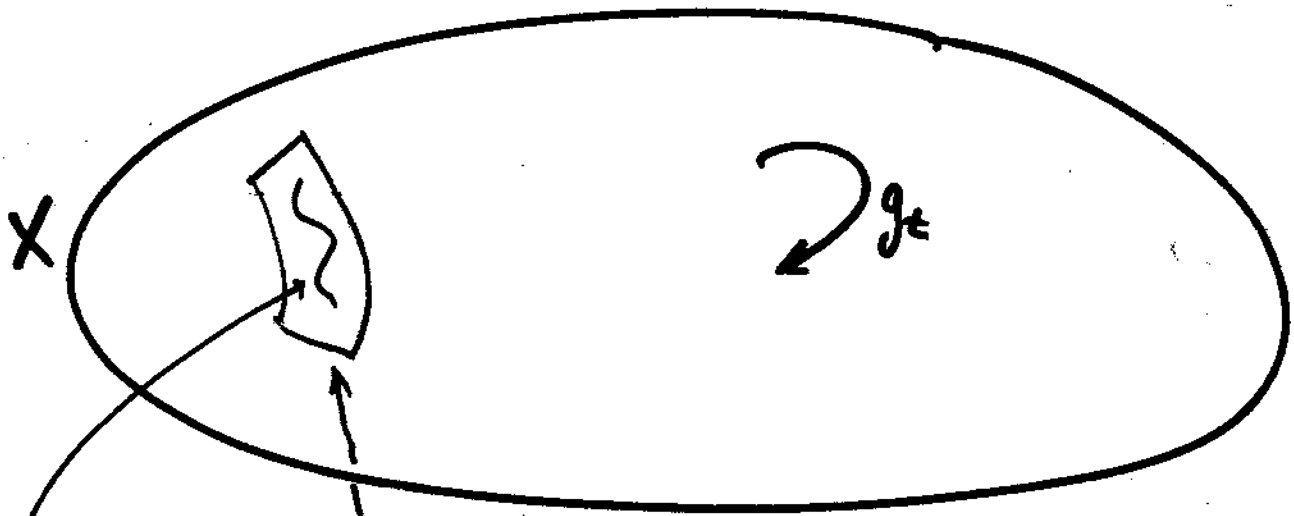
Note: if $\varepsilon(t) = ce^{-\gamma t}$, $\sum_{t=1}^{\infty} \mu(\Omega_{\varepsilon(t)}) \asymp \sum_{t=1}^{\infty} (ce^{-\gamma t})^{n+1} < \infty$

$$\Downarrow$$

a.e. $\Lambda \in H\mathbb{Z}^{n+1}$ has "not very fast g_t-escape rate"

Need the same for a non-degenerate submanifold of $H\mathbb{Z}^{n+1}$

General situation:



a region U where some property of orbits of $x \in U$ holds generically (a.e. x)

a curve (submanifold) M non-degenerately imbedded in U

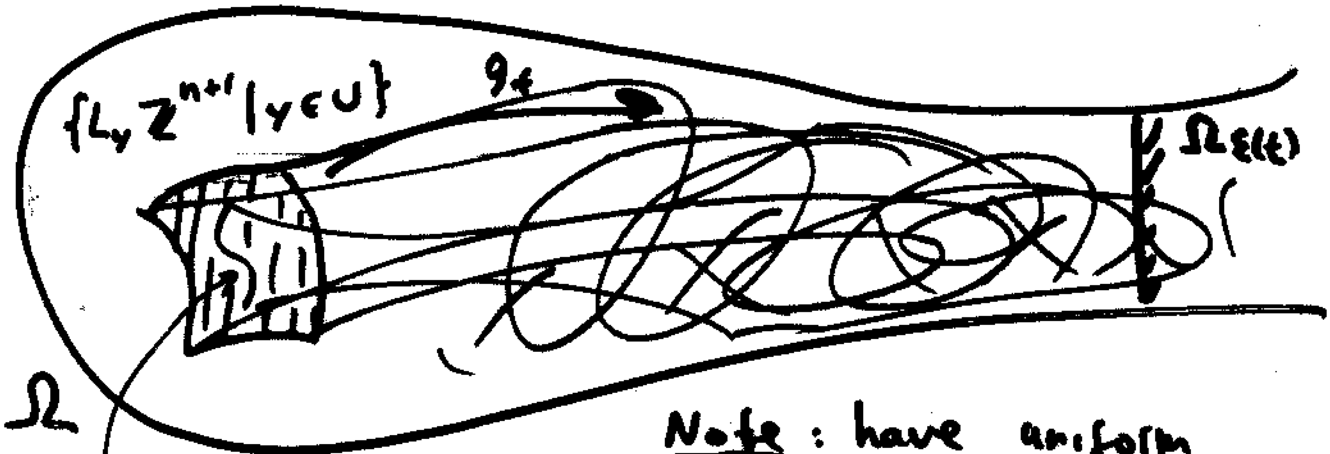
Challenge: to show that this property (or its weaker form) holds for a generic point of M

How to prove $\bigotimes$:

fix $I \subset \mathbb{R}$ and for any $t \in \mathbb{N}$ consider

$$A_t \stackrel{\text{def}}{=} \{ x \in I \mid g_t L_{f(x)} \mathbb{Z}^{n+1} \in \Omega_{\varepsilon(t)} \}$$

Need $\sum_{t=1}^{\infty} |A_t| < \infty \Leftarrow$ an upper estimate for $|A_t|$



Note : have uniform distribution for the whole expanding leaf :

$$|\{ y \in U \mid g_t L_y \mathbb{Z}^{n+1} \in \Omega_{\varepsilon(t)} \}| \underset{t \to \infty}{\sim} |U| \cdot \mu(\Omega_{\varepsilon(t)}) \asymp |U| \cdot \varepsilon(t)^{n+1}$$

$$\boxed{\text{Claim : } |A_t| \leq \text{const} \cdot |I| \cdot \varepsilon(t)^{1/n}} \qquad \varepsilon(t) = c e^{-\gamma t}$$

$\Rightarrow \sum_{t=1}^{\infty} |A_t| < \infty$, which proves $\bigotimes$

<u>To prove the claim</u>: think of $g_t L_{f(x)} \mathbb{Z}^{n+1}$ as of a "dynamical system" with time x (t is fixed)

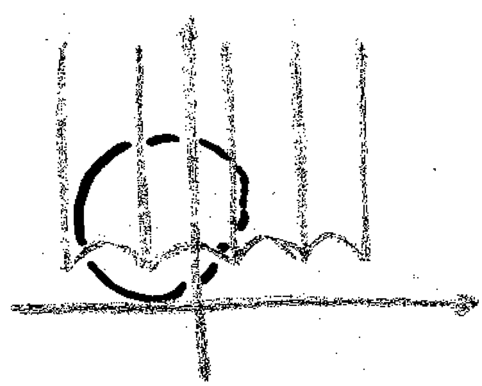
$$\left(f(x) = (x, .., x^n) \Rightarrow g_t L_{f(x)} = \begin{pmatrix} e^t & e^t x & \cdots & e^t x^n \\ & e^{-t/n} & & \\ & & \ddots & \\ & & & e^{-t/n} \end{pmatrix} \right)$$

Get a "flow" on Ω which looks very much like unipotent flow

$$u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Recurrence of unipotent trajectories.

<u>An elementary observation:</u>
<u>horocyclic trajectories</u>
on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$
do not run off to infinity.



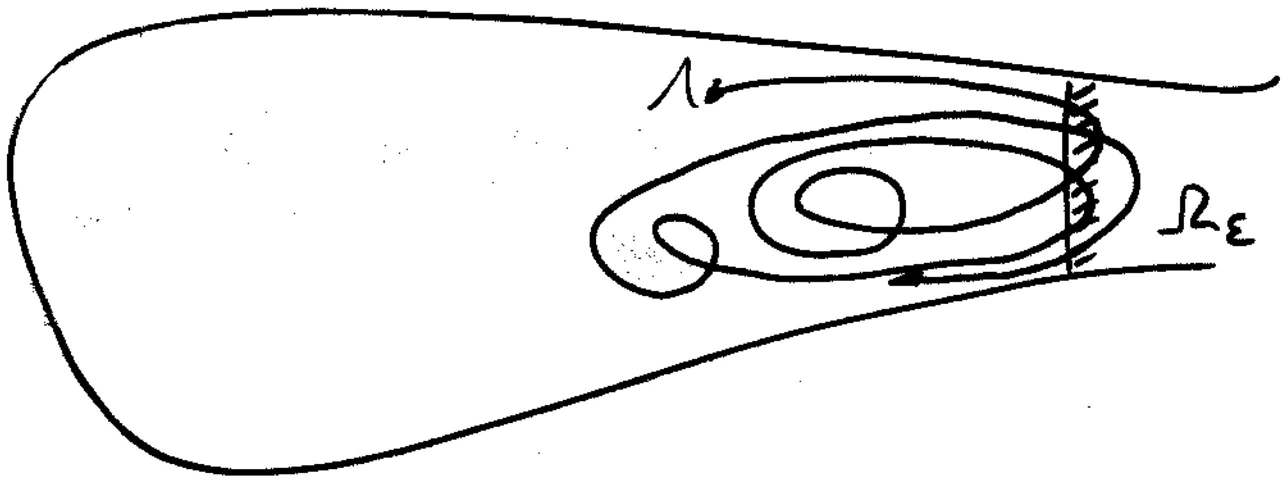
<u>Much harder to prove</u>: the same holds for any <u>unipotent flow</u> on Ω for $k \geq 3$ (Margulis 1971, Dani 1986)

Margulis 1971: for any $\Lambda \in \Omega$

and any unipotent $\{u_x\} \subset G$

$\exists\, \varepsilon > 0$ such that $\forall T > 0$, $\{u_x \Lambda \mid x \geq T\} \not\subset \Omega_\varepsilon$

(all unipotent orbits return to some compact subset of Ω)



Dani 1986: for any $\Lambda \in \Omega$

any unipotent $\{u_x\} \subset G$ and any $\delta > 0$

$\exists\, \varepsilon > 0$ such that $\forall T > 0$

$$\left|\{0 \leq x \leq T \mid u_x \Lambda \in \Omega_\varepsilon\}\right| \leq \delta T$$

<u>Need to</u>: 1) express δ as a function on ε

2) replace unipotent subgroups by more general polynomial or "polynomial-like" maps.

Notation: : $\Delta \subset \mathbb{R}^k$ a discrete subgroup

$\Downarrow$

$d(\Delta)$ "=" volume of $\Delta_{\mathbb{R}}/\Delta$

$= \| x_1 \wedge \dots \wedge x_m \|$, where $\{x_i\}$ are linearly independent generators of Δ

Theorem $I \subset \mathbb{R}$ an interval, $\rho > 0$, $r \in \mathbb{N}$

$h : I \to GL_n(\mathbb{R})$ such that $\forall \Delta \subset \mathbb{Z}^k$, $\Delta \neq 0$,

(i) $d(h(x)\Delta)$ is a polynomial of degree $\leq r$

(ii) $\sup_{x \in I} d(h(x)\Delta) \geq \rho$

Then for any positive ε one has

$$\boxed{\left| \{ x \in I \mid h(x)\mathbb{Z}^k \in \Omega_\varepsilon \} \right| \leq 2^k k \, C_r \left(\frac{\varepsilon}{\rho}\right)^{1/r} |I|}$$

where $C_r = 2r(r+1)^{1/r}$

To prove the claim, need to check (i) and (ii)

Straightforward → (i)

write down action of $g_t L_{f(x)}$ on exterior powers → (ii)

Also can derive

<u>Corollary</u> (quantitative strengthening of Margulis-Dani)

For any $\Lambda \subset \Omega$ there exists $\rho = \rho(\Lambda) > 0$ such that for any unipotent $\{u_x\} \subset G$ and any $\varepsilon > 0$ one has

$$\boxed{\left|\{0 < x < T \mid u_x \Lambda \in \Omega_\varepsilon\}\right| \leq 2^k k\, C_{k^2} \left(\frac{\varepsilon}{\rho}\right)^{1/k^2} |I|}$$

(here $\rho(\Lambda) = \min_{\Delta \subset \Lambda} d(\Lambda)$, and

$d(u_x \Lambda)$ = a polynomial of degree $\leq k^2$)

To replace $(x, \dots, x^n)$ by a non-degenerate curve $(f_1(x), \dots, f_n(x))$, need to understand what is nice about polynomials

Definition Say that f is (C,α)-good on I $(C, \alpha > 0)$

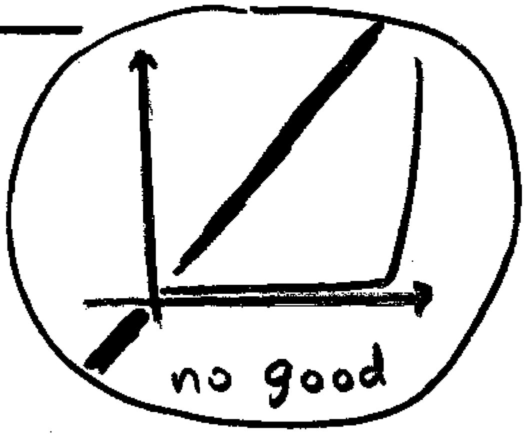
if for any subinterval $J \subset I$ and any $\varepsilon > 0$,

$$\boxed{\left|\{x \in J \mid |f(x)| < \varepsilon \cdot \sup_{y \in J} |f(y)|\}\right| \le C \cdot \varepsilon^{\alpha} |J|}$$

Fact 1 $f \in \mathbb{R}[x]$, $\deg(f) \le r$

$$\Downarrow$$

$\left(2r(r+1)^{1/r}, \frac{1}{r}\right)$-good on $\mathbb{R}$



Fact 2 $(f_1(x), \dots, f_n(x))$ is non-degenerate at $f(x_0)$

$$\Downarrow$$

$\exists$ a neighborhood I of x_0 and $C > 0$ such that any linear combination of $1, f_1, \dots, f_n$ is $\left(C, \frac{1}{n}\right)$-good on I

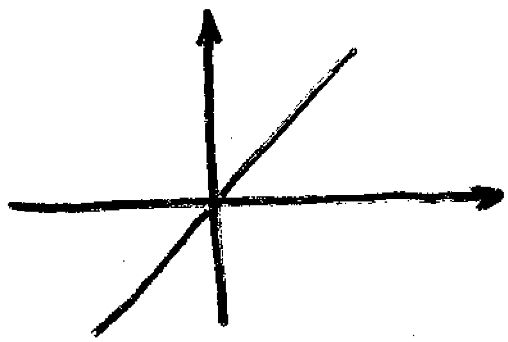
Then in the Theorem (p. 39) one replaces "polynomial" by (C,α)-good, C_r by C and $\frac{1}{r}$ by α (and then back to $\frac{1}{n}$ by Fact 2), and the same proof works

What about degenerate submanifolds?

The simplest class of examples:

proper affine subspaces $L \subset \mathbb{R}^n$.

It has been known for a long time (Schmidt 1964) that some of them are extremal, and that it depends on Diophantine properties of coefficients of parametrizing affine maps.



$\left(\begin{array}{l} \text{clearly } (x, x) \text{ is} \\ \text{VWA for all } x \text{ !} \end{array} \right)$

A modification of the method described above allows one to:

- write down the criterion for extremality of L

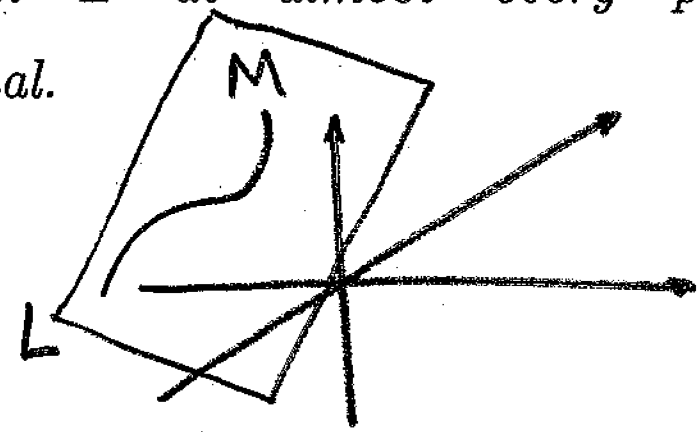
[not hard to obtain by standard (Sprindžuk's) methods, but still unpublished]

- as a consequence, show that the set of <u>non-extremal r-dimensional</u> affine subspaces of $\mathbb{R}^n$ has <u>Hausdorff codimension r</u>.

[$\Leftrightarrow$ has Hausdorff dimension $(n-r)(r+1)-r$]

- prove the following generalization of  :

<u>Theorem</u>. *Let L be an extremal affine subspace of $\mathbb{R}^n$, and let M be a smooth submanifold of L which is non-degenerate in L at almost every point. Then M is extremal.*



($M \subset L$ is <u>nondegenerate in L at</u> $\mathbf{y}_0$ if $T_{\mathbf{y}_0} L$ is spanned by partial derivatives of $\mathbf{f}$ at $\mathbf{x}_0$, where $M = \mathbf{f}(U)$, $U \subset \mathbb{R}^d$, and $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$.)

Multiplicative approximation

Let $\psi(x)$ be a non-increasing function $\mathbb{R}_+ \mapsto \mathbb{R}_+$.

Definitions. Say that $\mathbf{y} \in \mathbb{R}^n$ is

ψ-approximable	*ψ-mult.approximable*

if there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$|\mathbf{y} \cdot \mathbf{q} + p|$ is not greater than

$\boxed{\psi(\|\mathbf{q}\|^n)}$	$\boxed{\psi(\prod_{q_i \neq 0} |q_i|)}$

for some $p \in \mathbb{Z}$.

Clearly ψ-approximable $\Rightarrow$ ψ-mult.approximable, hence every $\mathbf{y} \in \mathbb{R}^n$ is $\frac{1}{x}$-multiplicatively approximable.

Theorem 2M. *Almost every (resp. almost no) $\mathbf{y} \in \mathbb{R}^n$ is ψ-mult.approximable, provided the integral* $\boxed{\int_1^\infty (\log x)^{n-1} \psi(x)\, dx}$ *diverges (resp. converges).* (W. Schmidt 1960)

Definition. $\mathbf{y} \in \mathbb{R}^n$ is *badly mult.approximable* (*BMA*) iff it is not $\frac{c}{x}$-mult. approximable for some $c > 0$; that is, if

$$\boxed{\inf_{\substack{p\in\mathbb{Z},\\ \mathbf{q}\in\mathbb{Z}^n\smallsetminus\{0\}}} |\mathbf{y}\cdot\mathbf{q}+p| \prod_{q_i\neq 0} |q_i| > 0}$$

$$\Updownarrow$$

$$\boxed{\inf_{\substack{\mathbf{p}\in\mathbb{Z}^n,\\ q\in\mathbb{Z}\smallsetminus\{0\}}} \prod_i |y_i q + p_i||q| > 0}$$

Facts. The set of BMA $\mathbf{y} \in \mathbb{R}^n$ is

- of measure zero (Theorem 2M)

? empty if $n \geq 2$ (Littlewood's Conjecture)

Note: the validity of the conjecture for $n = 2$ implies the general case $\Rightarrow$ will assume $n = 2$.

Repeat: (y_1, y_2) is BMA iff $\left\{ \begin{pmatrix} e^{t_1+t_2} & & \\ & e^{-t_1} & \\ & & e^{-t_2} \end{pmatrix} L_y \mathbb{Z}^3 \right\}$ is bounded

$$\boxed{\inf_{\substack{p \in \mathbb{Z}, \\ q_1, q_2 \in \mathbb{Z} \setminus \{0\}}} |y_1 q_1 + y_2 q_2 + p| \cdot \max(1, |q_1|) \cdot \max(1, |q_2|) > 0}$$

$$\Updownarrow$$

$$\boxed{\inf_{\substack{p_1, p_2 \in \mathbb{Z}, \\ q \in \mathbb{Z} \setminus \{0\}}} |y_1 q + p_1| \cdot |y_2 q + p_2| \cdot |q| > 0}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$\left\{ \begin{pmatrix} e^{t} & & \\ & e^{-t} & \\ & & 0 \end{pmatrix} L_y \mathbb{Z}^3 \right\}$ and $\left\{ \begin{pmatrix} e^{t} & & \\ & 0 & \\ & & e^{-t} \end{pmatrix} L_y \mathbb{Z}^3 \right\}$ are bounded

both y_1 and y_2 are BA

$\left\{ \begin{pmatrix} e^{t} & & \\ & e^{-t/2} & \\ & & e^{-t/2} \end{pmatrix} \right\}$ is bounded

(y_1, y_2) is BA

- y_1, y_2 are cubic irrational $\Rightarrow$ (y_1, y_2) is not BMA

  (Cassels and Swinnerton-Dyer 1955)

- for any $y_1 \in \mathbb{R}$,

$$\dim\left(\left\{ y_2 \mid y_2 \text{ is BA, } (y_1, y_2) \text{ is not BMA} \right\}\right) = 1$$

(Pollington and Velani 2000)

An elementary observation: $\mathbf{y} \in \mathbb{R}^2$ is BMA iff the trajectory $D_+ L_{\mathbf{y}} \mathbb{Z}^3$, with $D_+ \stackrel{\text{def}}{=} \{g_{\mathbf{t}} \mid \mathbf{t} \in \mathbb{R}^2_+\}$ and

$$g_{\mathbf{t}} = \operatorname{diag}(e^{t_1+t_2}, e^{-t_1}, e^{-t_2}),$$

is bounded in the space $\Omega = SL_3(\mathbb{R})/SL_3(\mathbb{Z})$.

Recall: $$L_{\mathbf{y}} = \begin{pmatrix} 1 & y_1 & y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ so that}$$

$$\boxed{g_{\mathbf{t}} L_{\mathbf{y}} \begin{pmatrix} p \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} e^t(q_1 y_1 + q_2 y_2 + p) \\ e^{-t_1} q_1 \\ e^{-t_2} q_2 \end{pmatrix}}.$$

(here $t = t_1 + t_2$)

Moreover: for $\mathbf{s} = (s_1, s_2)$, with $s_i > 0$ and $s_1 + s_2 = 1$ (*weight vector*), define the **s**-*quasinorm* on $\mathbb{R}^2$ by

$$\|(x_1, x_2)\|_{\mathbf{s}} \stackrel{\text{def}}{=} \max\left(|x_1|^{1/s_1}, |x_2|^{1/s_2}\right),$$

and say that $\mathbf{y}$ is **s**-*badly approximable* if

$$\boxed{\inf_{\substack{p\in\mathbb{Z},\\ \mathbf{q}\in\mathbb{Z}^2\smallsetminus\{0\}}} |y_1q_1 + y_2q_2 + p|\|\mathbf{q}\|_{\mathbf{s}} > 0}$$

$$\Updownarrow$$

$$\left\{ \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-s_1t} & 0 \\ 0 & 0 & e^{-s_2t} \end{pmatrix} L_{\mathbf{y}}\mathbb{Z}^3 \,\middle|\, t \geq 0 \right\} \text{ is bounded.}$$

Clearly BMA implies **s**-BA for every **s**.

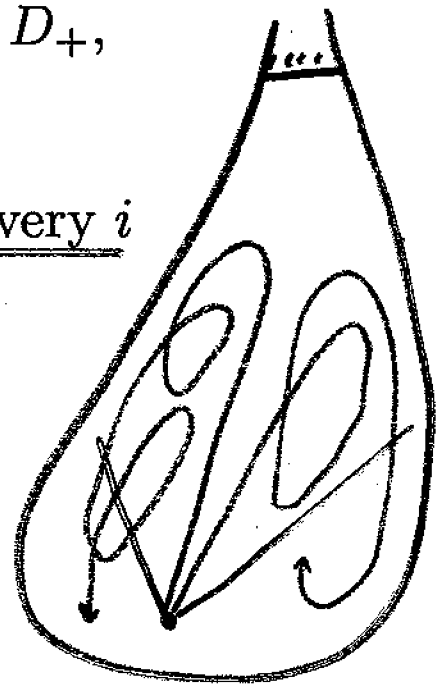
One can prove: the set of **s**-BA pairs has Hausdorff dimension 2 for every **s**.

Conjecture. (W. Schmidt 1982) *There exists a pair* (y_1, y_2) *which is both* $(\frac{1}{3}, \frac{2}{3})$*-BA and* $(\frac{2}{3}, \frac{1}{3})$*-BA.*

Quoting Schmidt: "If this conjecture is false, then Littlewood's conjecture is true."

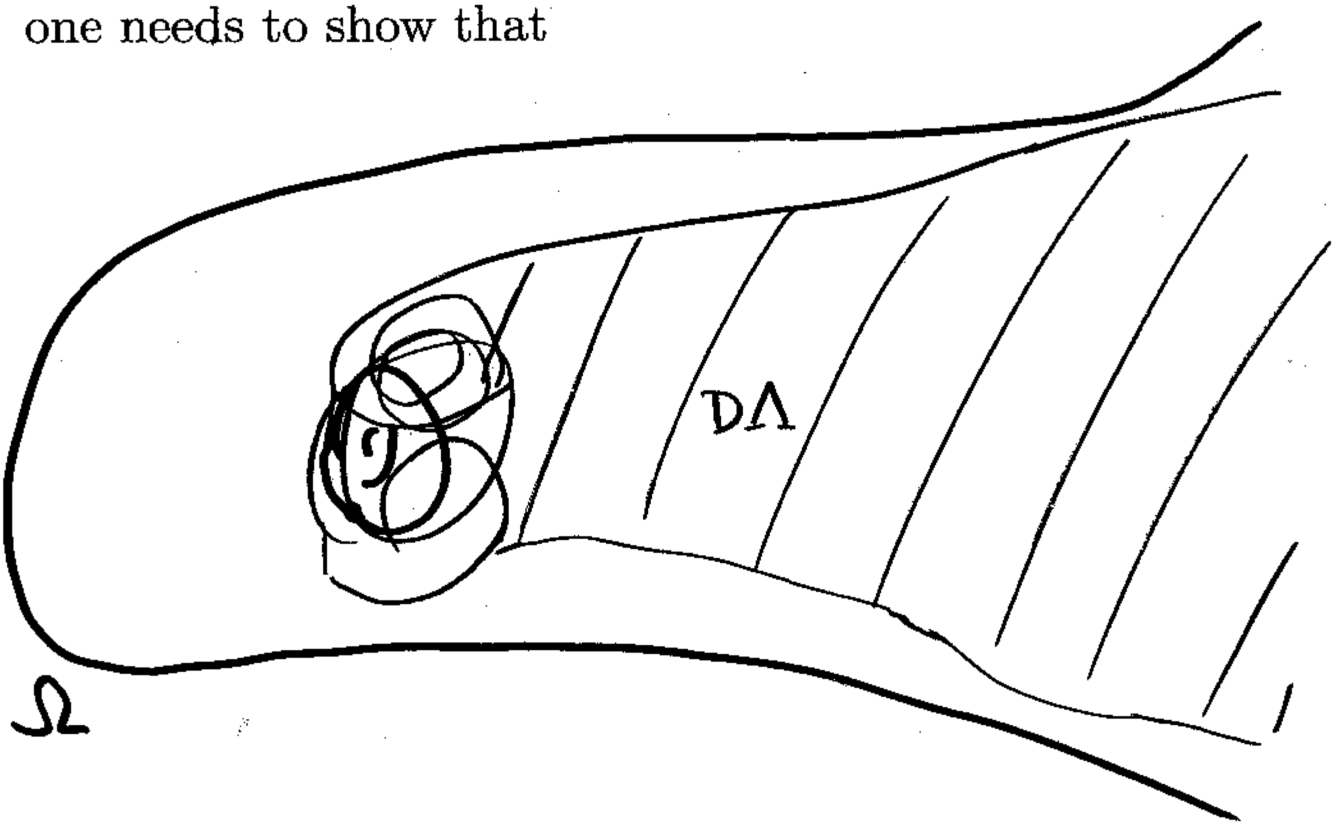
It seems plausible to conjecture that:
for any choice of finitely many weight vectors $\mathbf{s}_1, \ldots, \mathbf{s}_k$, the set of pairs (y_1, y_2) which are $\mathbf{s}_i$-BA for every i is non-empty (and maybe even has full Hausdorff dimension).

A more general dynamical conjecture:
for any choice of finitely many partially hyperbolic one-parameter subsemigroups F_i of D_+, the set of $\Lambda \in \Omega$ such that the trajectory $F_i\Lambda$ is bounded for every i is non-empty (and maybe even has full Hausdorff dimension).

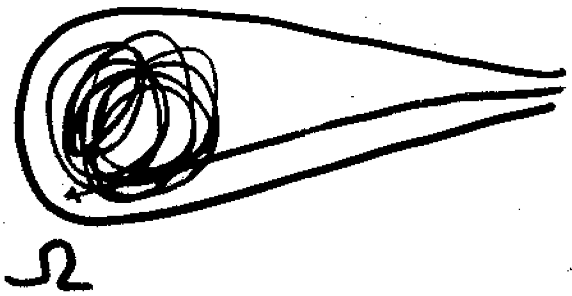


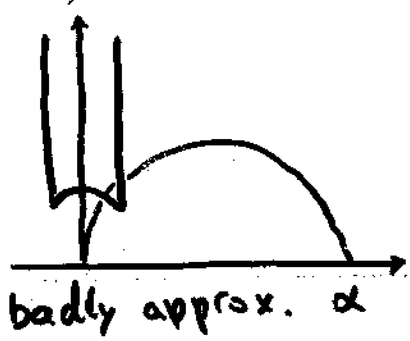
Note: the orbit of the full diagonal group $D \overset{\text{def}}{=} \{g_{\mathbf{t}} \mid \mathbf{t} \in \mathbb{R}^2\}$ is obviously unbounded (moreover, any sequence $g_{\mathbf{t}^{(k)}} L_{\mathbf{y}} \mathbb{Z}^3$ with $t_1^{(k)} + t_2^{(k)} \to -\infty$ tends to infinity in Ω).

So to prove Littlewood's Conjecture one needs to show that



cannot happen (no problem if $n = 1$).





Another elementary observation: if it does happen, then the closure of $D_+ L_{\mathbf{y}}\mathbb{Z}^3$ contains a lattice Λ such that the full orbit $D\Lambda$ is relatively compact in Ω.

(take a limit point of $g_{k\mathbf{t}} L_{\mathbf{y}}\mathbb{Z}^3$, $k \to +\infty$)

A non-elementary observation: there are very strong reasons which rule out a possibility for such an orbit $D\Lambda$ to be compact!

Thus Littlewood's Conjecture is reduced to

Conjecture (CSM).

(Cassels and Swinnerton-Dyer 1955, Margulis 1999)

Any relatively compact orbit $D\Lambda$, $\Lambda \in \Omega$, is compact.

Theorem. (Cassels and Swinnerton-Dyer 1955)

("isolation theorem" or

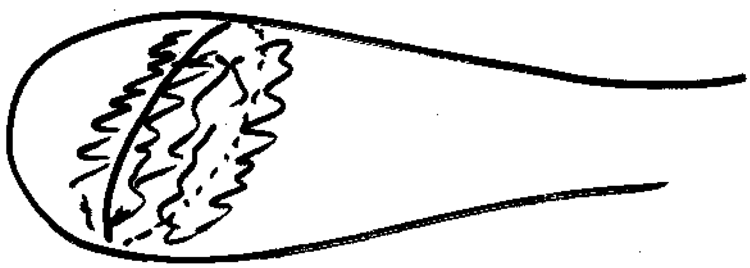
"local rigidity of compact orbits")

Define $\boxed{B(\mathbf{x}) = x_1 x_2 x_3}$ *(a cubic form on* $\mathbb{R}^3$ *stabilized by D), and let* $\Lambda \in \Omega$ *be such that* $D\Lambda$ *is compact. Then for every* $0 \leq a < b$ *there exists a neighborgood* U *of* Λ *such that*

$$\forall \Delta \in U \setminus D\Lambda \quad \exists \mathbf{x} \in \Delta \quad \textit{with} \boxed{a < |B(\mathbf{x})| < b}$$

In particular, $\forall\, \varepsilon > 0$ *there exists* $U \ni \Lambda$ *such that*

$$\boxed{D\Delta \cap \Omega_\varepsilon \neq \varnothing} \ \textit{for any} \quad \Delta \in U \setminus D\Lambda\,.$$

(not true in rank 1, )

Reduction of Littlewood's Conjecture to (CSM):

If $\mathbf{y}$ is a counterexample, then one finds a sequence of lattices $\Delta_k = d_k L_{\mathbf{y}} \mathbb{Z}^3$, $d_k \in D$, converging to a lattice Λ with a relatively compact D-orbit.

By (CSM), $D\Lambda$ is compact.

By the Isolation Theorem, no gap in the values of $|B(\mathbf{x})|$, $\mathbf{x} \in \Delta_k$, is possible.

However

$$\boxed{B(\mathbf{x}) = B\left(L_{\mathbf{y}}\binom{p}{\mathbf{q}}\right) = (q_1 y_1 + q_2 y_2 + p) q_1 q_2}$$

is equal to zero if $q_1 q_2 = 0$,

and is bounded away from 0 otherwise. $\square$

Lemma. *Let* $\Lambda \in \Omega$ *be such that* $D\Lambda$ *is compact. Then any root subgroup of* D *(for example,* $F = \{f_t = \mathrm{diag}(e^t, e^{-2t}, e^t)\}$*) acts topologically transitively on* $D\Lambda$.

Explanation. $Dg\mathbb{Z}^3$ is compact

$$\Updownarrow$$

$$g^{-1}Dg \cap SL_3(\mathbb{Z}) \text{ is Zariski dense in } g^{-1}Dg$$

$$\Updownarrow$$

$$g^{-1}Dg \text{ has no non-trivial rational characters.}$$

Similarly, if χ is a character on D and $F = \mathrm{Ker}(\chi)$,

$Fg\mathbb{Z}^3$ is compact $\Leftrightarrow g^{-1}fg \in SL_3(\mathbb{Z})$ for some $f \in F$
$\Rightarrow h \mapsto \chi(ghg^{-1})$ is rational, a contradiction. $\square$

<u>Proof of Isolation Theorem</u> Assume that there exist $\Delta_k = g_k \Lambda$, $g_k \to I$, such that

there is a gap in values of B on $x \in \Delta_k$

$\Downarrow$ $(B(x) \notin (a,b)$ for some $a<b)$

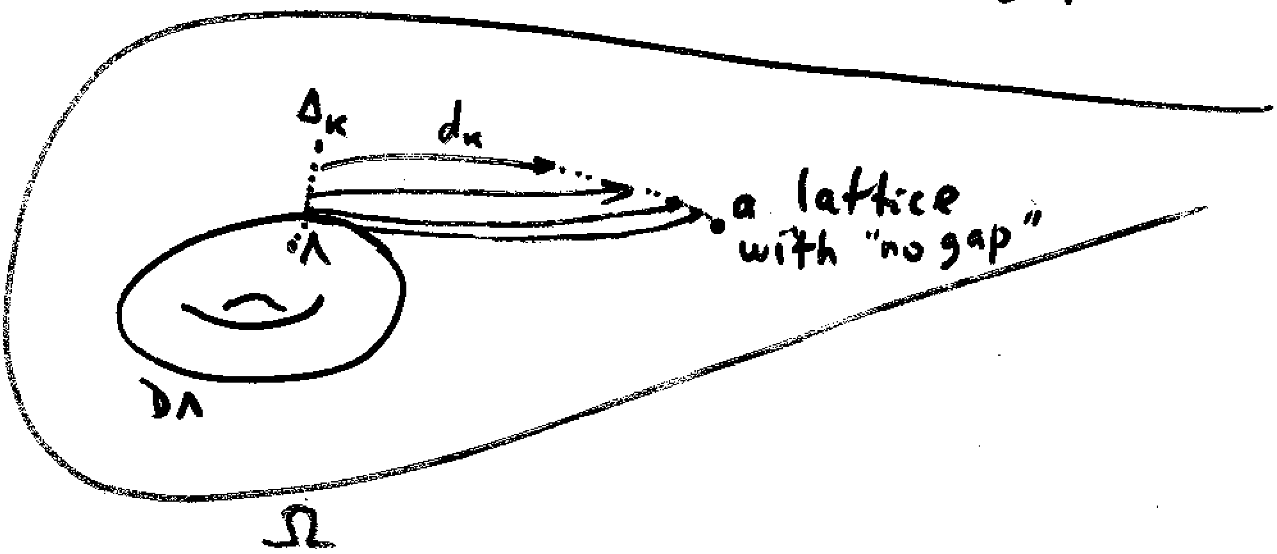
there is a gap in values of B on $x \in d\Delta_k$, $d \in D$

$\Downarrow$

there is a gap in values of B

on $x \in \lim_{k\to\infty} d_k \Delta_k$, $d_k \in D$

<u>Strategy</u>: for any $a<b$, find $d_k \in D$ such that the limit lattice of $\{d_k \Delta_k\}$ does not have a gap at (a,b)



<u>Step 1</u> WLOG can assume that

- g_k is transversal to D
- $(g_k)_{13} = \max_{i,j} |(g_k)_{ij}|$

$$\begin{pmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{pmatrix}$$

<u>Claim</u> Given any $r > 0$ one can choose $t_k \to \infty$ such that

$$\underbrace{\begin{pmatrix} e^{t_k} & & \\ & 1 & \\ & & e^{-t_k} \end{pmatrix}}_{h_{t_k}} g_k \underbrace{\begin{pmatrix} e^{-t_k} & & \\ & 1 & \\ & & e^{t_k} \end{pmatrix}}_{h_{t_k}^{-1}} \to \underbrace{\begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{u_r}$$

Then

$$h_{t_k} \Delta_k = h_{t_k} g_k \Lambda = \underbrace{h_{t_k} g_k h_{t_k}^{-1}}_{\downarrow \; u_r} \; \underbrace{h_{t_k} \Lambda}_{\downarrow \; \Lambda' = \Lambda'(r) \in D\Lambda} \to u_r \Lambda'$$

I.e. $h_{t_k} \Delta_k = \gamma_k u_r \Lambda'$, where $\gamma_k \to I$

(WOW! It looks like the closure contains a <u>unipotent orbit</u>!)

(although Λ' depends on r)

<u>Step 2</u> Now use <u>another direction</u>,

that is, $f_s = \begin{pmatrix} e^s & & \\ & e^{-2s} & \\ & & e^s \end{pmatrix}$

By Lemma, can choose $s_k \to \infty$ such that $f_{s_k}\Lambda' \to \Lambda$

Passing to a subsequence of $\{t_k\}$,

can assume $f_{s_k}\gamma_k f_{s_k}^{-1} \to I$ Then

$$f_{s_k} h_{t_k} \Delta_k = f_{s_k}\gamma_k f_{s_k}^{-1} f_{s_k} u_r \Lambda'$$

$\boxed{f_s \text{ and } u_r \text{ commute!}}$

$$= \underbrace{f_{s_k}\gamma_k f_{s_k}^{-1}}_{\downarrow \atop I} u_r \underbrace{f_{s_k}\Lambda'}_{\downarrow \atop \Lambda} \to u_r \Lambda$$

(now it is a true unipotent orbit!!!)

<u>But</u>: for any $x = (x_1, x_2, x_3) \in \Lambda$

with $x_1 x_3 < 0$ and $x_2 \neq 0$,

$\{B(u_r x) = (x_1 + r x_3) x_2 x_3\}$ has no gap!

(i.e. $\forall a<b \ \exists r$ such that $a < B(u_r x) < b$)

$\Rightarrow$ a contradiction ∎

<u>Corollary</u> (from the theorem)

Any relatively compact orbit $D\Lambda$, $\Lambda \in \Omega$,
such that its closure $\overline{D\Lambda}$ contains a compact orbit
is <u>compact</u>

In fact, one can apply Ratner's Theorem to the unipotent subgroup U_r constructed in the course of the proof to establish

<u>Corollary</u> (from the proof)

Let $\Lambda \in \Omega$ be such that
the closure $\overline{D\Lambda}$ contains a compact orbit.

Then either $\overline{D\Lambda} = D\Lambda$ (periodic) or $\overline{D\Lambda} = \Omega$ (dense)

A generalization of the above argument gives

Theorem (Barak Weiss + Elon Lindenstrauss) $k \geq 3$,

$\Omega = SL_k(\mathbb{R})/SL_k(\mathbb{Z})$, D = diagonal sbgp of $SL_k(\mathbb{R})$

Let $\Lambda \in \Omega$ be such that

the closure $\overline{D\Lambda}$ contains a compact orbit

Then there exists a closed subgroup L of $SL_k(\mathbb{R})$ containing D such that $\boxed{\overline{D\Lambda} = L\Lambda}$ and

$L\Lambda$ carries an L-invariant probability measure

Moreover, L has the form (k is prime $\Rightarrow$ [closed / dense)

$$L = \left\{ P \begin{pmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & & & \boxed{A_\ell} \end{pmatrix} P^{-1} \;\middle|\; \begin{array}{l} A_i \in GL_d(\mathbb{R}) \\ \det(A_1) \cdots \det(A_\ell) = 1 \end{array} \right\}$$

where $k = d\ell$ and P is a permutation matrix

<u>Conjecture</u> (Margulis 1999) $(k \geq 3)$

For any $\Lambda \in \Omega$ one of the following holds:

- either $\overline{D\Lambda}$ is homogeneous, i.e.

  there exists a closed <u>subgroup</u> $L \subset SL_k(\mathbb{R})$ containing D such that $\boxed{\overline{D\Lambda} = L\Lambda}$

- or there is an algebraic factor map onto a rank-one action, i.e.

  (M. Rees)

  there exist $D \subset L \subset G$ and an epimorphism $\varphi : L \to H$ such that

  $L\Lambda$ is closed in Ω,

  $\varphi(\{g \in L \mid g\Lambda = \Lambda\})$ is discrete in H,

  and $\boxed{\dim \varphi(D) = 1}$

(Conjecture (CSM) is a special case)

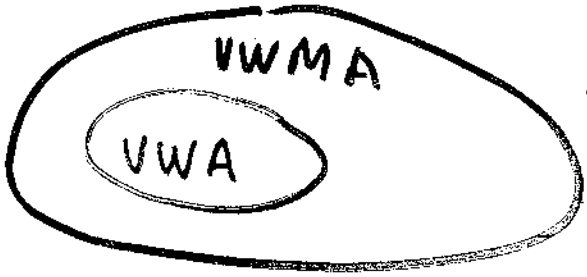
Multiplicative approximation on manifolds

$y \in \mathbb{R}^n$ is very well mult. approximable (VWMA)

if it is ψ_β-mult. approximable for some $\beta > 0$,

i.e. for infinitely many $q \in \mathbb{Z}^n$ one has

$$\boxed{|y \cdot q + p| \le \left(\prod_{q_i \neq 0} |q_i|\right)^{-(1+\beta)} \quad \text{for some } p \in \mathbb{Z}}$$



– a bigger null subset of $\mathbb{R}^n$

(for a submanifold $M \subset \mathbb{R}^n$, it is much harder to prove that a.e. $y \in M$ is not VWMA)

Conjecture (A. Baker 1975)

For a.e. $x \in \mathbb{R}$, $(x, x^2, \dots, x^n)$ is not VWMA

Proved for $n \le 4$ (Bernik–Borbat)

Conjecture (V. Sprindžuk 1980)

$M \subset \mathbb{R}^n$ non-degenerate (analytic) $\Rightarrow$ a.e. $y \in M$ is not VWMA

Proved for $n = 2$ (Sprindžuk)

$\Updownarrow$ def

M is strongly extremal

<u>Results</u>: $M \subset \mathbb{R}^n$ non-degenerate at a.e. point

$\Rightarrow$ a.e. $y \in M$ is not VWMA

(K-Margulis 1998)

$\Rightarrow$ a.e. $y \in M$ is not ψ-MA

if $\int_1^\infty (\log x)^{n-1} \psi(x)\, dx < \infty$

(Bernik-K-Margulis 1999)

<u>Based on</u>: $y \in \mathbb{R}^n$ is VWMA

$\Updownarrow$

the orbit $D_+ L_y \mathbb{Z}^{n+1}$ grows not too fast

where $D_+ = \{ \mathrm{diag}(e^{t_0}, e^{-t_1}, \dots, e^{-t_n}) \mid t_i > 0, \sum_1^n t_i = t_0 \}$

Then $t = (t_0, t_1, \dots, t_n)$ becomes fixed, and the main yesterday's estimate applies.

Books:

Starkov Dynamical Systems on Homogeneous Spaces AMS, 2000

Bekka – Meyer Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces LMS, 2000

Surveys:

Dani in: Dynamical Systems, Ergodic Theory and Applications EMS, 2000

K-Shah-Starkov Elsevier, to appear

Topics omitted:

- Integer points on algebraic varieties
  Eskin-Mozes-Shah, Clozel-Oh-Ullmo, Oh-Gan, Oh
- Quantitative Oppenheim Eskin-Margulis-Mozes
- Error term for lattice points in polyhedra
  Skriganov, Skriganov-Starkov
- n-point correlations for values of linear forms
  Marklof
- Multidimensional continued fractions
  Korkina, Lagarias, Kontsevich-Suhov
- Homogeneous dynamics $\longleftrightarrow$ Teichmüller theory
  $\updownarrow$
  Diophantine approximation $\longleftrightarrow$ Teichmüller theory

  Veech, Masur, Eskin, Minsky, Weiss