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Chaotic dynamics and finitude of attractors

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CHAOTIC DYNAMICS AND FINITUDE OF ATTRACTORS

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Description of the asymptotic behaviour of orbits $\{f^n(x)\}$ when $f : M \to M$ is a diffeomorphism of a compact riemaniann manifold M.

Given a diffeomorphisms f, we could try:

1. to "describe"

$$L(f) = \text{Closure}(\cup_{x \in M} \omega(x) \cup \alpha(x))$$

2. to "understand" how this description changes under perturbation.

One tentative description to achive, motivated by the hyperbolic theory, is the following:

Topological description.

To decompose the limit set,

$$L(f) = \cup_i \Lambda_i$$

such that

1. is a finite union;
2. the sets Λ_i are pair disjoints;
3. each Λ_i is a transitive closed set;
4. for "any" $y \in M$, exists $x_y \in L(f)$: $\mathrm{dist}(f^n(y), f^n(x_y)) \to 0$

We may consider that:

1. the sets Λ_i are attractors and
2. $M = \cup B(\Lambda_i)$ at Lebesgue almost every point ($B(\Lambda_i)$ is the basin of attraction of the attractors).

Statistical description.

At Lebesge almost every point

$$\frac{1}{n}\Sigma_{i=0}^{n}\delta_{f^i(x)}$$

converges, i.e.: at almost every point the limit of the Birkhoff sums exists.

When are the limits decribed by a finite number of ergodic measures?

Remarks

- Is this a proper description?

  Consider the case of conservative (symplectic) systems. KAM theory and Aubry-Mather theory. Although for this case the description above does not hold (in fact it is false), the information availabe is rich and extremly satisfactory.

- Due to the fact that it is impossible to give this description for every system (Residual sets of diffeomorphisms exhibiting infinitely many sinks; KAM theory), so:

  1. we could try to describe a dense set of systems. Topological point of view in the set of diffeos.
  2. Or for generic parametrized family, we could try to describe the dynamic for a set of total Lebesgue measure of parameters. "Metric" point of view in the set of diffeos

Conjecture (Palis)

Every system can be C^r−approximated by another having only finitely many attractors, which are *nice*, and whose basins cover almost all of the ambient space M.

Nice attractors:

- support SRB measures, and have no holes in their basins of attraction;
- the dynamics restricted to each basin of attraction is stochastically stable.

Moreover,

for almost all small perturbations of such systems, 99% of M (in measure) is covered by the basins of a finite number of nice attractors. Statistical Stability.

For generic parametrized family, such descrition can be done for a set of total lebesgue measure of parameters.

We want to discuss the following:

1. Cases in which it is possible to give such description:
   (a) hyperbolic systems (Newhouse-Smale theorems of decomposition of the limit set), attracting invariant measure with uniform Lyapunov exponents(Pesin theory),
   (b) smooth one dimensional endomorphisms,
   (c) surface diffeomorphisms exhibiting dominated splitting; (partial hyperbolic systems in higher dimension?)
2. Systems in which this "description" fails:
   (a) Newhouse's phenomena, residual sets of diffeomorphisms exhibiting infinitely many sinks;
   (b) Conservative systems;
3. Arguments that could show why it is possible to obtain such a description,
4. Arguments to the contrary,
5. Some other related problems,
6. Are the "stable" systems interesting?

The problem in general is extremly wide and there are many other question that we should understand first.

Classicaly, the basic tool for the understanding of the dynamic from topological and statistical point of view, was the study of local stable and unstable sets.

$W^s_\epsilon(x) = \{y \in B_\epsilon(f^n(x))\, d(f^n(x), f^n(y)) \to 0 \text{ for any } n > 0\,\}$

$W^u_\epsilon(x) = \{y \in B_\epsilon(f^n(x))\, d(f^n(x), f^n(y)) \to 0 \text{ for any } n < 0\}$

maybe $\epsilon = \epsilon(x)$

In particular, we should understand:

The $\omega-$limit ($\alpha-$limit) of points which are "Lyapunov stable".

Lyapunov stable point

A point x is Lyapunov stable (in the future) if for any ϵ there is $\delta = \delta(\epsilon)$ such that

$$f^n(B_\delta(x)) \subset B_\epsilon(f^n(x))$$

for any positive integer n.

Lyapunov, A.M. "Problème général de la stabilité du mouvement" (1892).

This means:

What is the $\omega(U)$ ($\omega-$limit of U) if U is a set of small diameter and such that $f^n(U)$ remains with small diameter for positive iterates?

We will try to adress this problem in any of the situation that we will consider:

1. hyperbolic systems

2. one dimensional endomorphism
3. surface systems exhibiting dominated splitting
4. partial hyperbolic systems with onedimensional subbundles.

Hyperbolic systems

The Limit set is well characterized for hyperbolic sets.

$L(f)$ is hyperbolic if there is a continuous splitting:

$$T_{L(f)}M = E^s \oplus E^u$$

where $|Df^n|_{E^s}| < C\lambda^n \quad |Df^{-n}|_{E^u}| < C\lambda^n$ for some $\lambda < 1$ and $C > 0$.

Under the hypothesis of hyperbolicity, then $L(f)$ can be decomposed into finite pieces of disjoint compact invariant and transitive sets. Moreover, the asymptotic behaviour of any point in the manifold is represented by an orbit in $L(f)$.

In the hyperbolic case, the description of the dynamics follows from a fundamental tool:

Local Stable and Unstable sets are transversal manifolds of uniform size.

At each point there are transverse invariant manifolds of uniform size and these manifolds have a dynamic meaning (points in the "stable" one are asymptotic to each other in the future, and points in the "unstable" one are asymptotic to each other in the past).

What happens with this decomposition for perturbations of the initial system?

If $L(f)$ is hyperbolic and a non cycle condition holds, for g C^1 close to f then the dynamics are conjugated on the Limit set.

Statistical information

If f is C^2, considering just the attractors Λ_i of the spectral decomposition then

1. $M = \cup_i B(\Lambda_i)$ at lebesgue almost every point,
2. each attractor support an SBR- measure.

Non-uniform hyperbolic case

Let μ an invariant measure of a diffeo $C^{1+\beta}$.

Oseledet Theorem:

There is a set of total measure showing a mesurable splitting $E^s + E^c + E^u$ such that:

1. for any vector $v^s \in E^s$

$$\lim_n \frac{1}{n} \log |Df^n(x)_{|v^s|}| = \lambda^s(x) < 0;$$

2. for any vector $v^u \in E^u$

$$\lim_n \frac{1}{n} \log |Df^{-n}(x)_{|v^u|}| = -\lambda^u(x) < 0;$$

3. for any vector $v^c \in E^c$

$$\lim_n \frac{1}{n} \log |Df^n(x)_{|v^c|}| = 0;$$

If $E^c = 0$ then the measure μ is called hyperbolic.

If there is $\lambda > 0$ such that for almost every point $\lambda^s(x) < -\lambda < 0 < \lambda < \lambda^u(x)$ then it is said that the Lyapunov exponents are uniformly bounded.

Is it possible to decompose the measure in a finite (countable) number of ergodic components? $\mu = \cup_i \mu_i$ such that each μ_i is ergodic.

Not true in general, even in the case that the Lyapunov exponents are uniformed bounded.

Pesin theory

Existence of hyperbolic blocks.

There exists a set K, $C(K) > 0$, and $C'(K)$ such that

1. $\mu(K) > 0$;
2. almost every point has an iterate on K;
3. for $x \in K$, there is $\lambda(x) > 0$ such that for any vector $v^s \in E^s$ and for any vector $v^s \in E^u$
   (a) $\lambda^s(x) < -\lambda(x) < 0 < \lambda(x) < \lambda^u(x)$,
   (b) $|Df^n_{|v^s(x)}| < C(K)exp[\lambda(x)n]$,
   (c) $|Df^{-n}_{|v^u(x)}| < C(K)exp[\lambda(x)n]$
4. angle$(E^s(x), E^u(x)) > C'(K)$

Large local stable and unstable manifolds for points in the hyperbolic blocks.

There is $\epsilon(x)$ such that for every point $x \in K$ the local stable and local unstable sets are transversal C^r- manifold of size $\epsilon(x)$.

Absolute continuity of the stable and unstable lamination

The holonomy maps along these foliations are Lebesgue measurable and preserve the class of Lebesgue measure.

Pesin, Pugh-Shub

Let $f \in C^{1+\beta}(\beta > 0)$ and μ and $f-$invariant measure, which induce an absolutely continuous measure on almost unstable manifolds, then

1. the basin of attraction has positibe Lebesgue measure;
2. the basin of attraction of μ (up to a zero Lebesgue measure) is the countable union of the basin of attraction of ergodic measure.

The union is finite if we assume that the Lyapunov exponents are uniformly bounded.

One dimensional endomorphisms

Given f endomorphisms of the circle or the interval.

Questions:

What is the ω–limit of a Lyapunov stable set?
weaker versions:

- What is the $w(J)$ for an interval J such that $\{f^n(J)\}_{n>0}$ is pair disjoint and $f^n_{|J}$ is an homeo for any $n > 0$.

  Existence of Wandering Intervals. J is a wandering interval if $\{f^n(J)\}_{n>0}$ is pair disjoint, $f^n_{|J}$ is an homeo and J is not in the basin of attraction of a sink.

- What is the $\omega(J)$ such that $f^n_{|J}$ is an homeo and $|f^n(J)|$ is small for any $n > 0$.

C^1-topology

1. existence of endomorphisms exhibiting wandering intervals;
2. existence of endos exhibiting infinite sinks with unbounded period (even without critical points);
3. hyperbolic systems are generic (Jakobson).

The C^1-topology, is extremly local. This is showed by the existence of wanderings intervals. It is rich in the universe of the "hyperbolic systems", but maybe poor out of it.

How the dynamic of a system is affected as the smoothness of the system is improved?

C^r-topology $(r \geq 2)$ (rigidity)

1. non existence of wandering intervals:
   (a) circle diffeomorphisms (Denjoy),
   (b) non presence of critical points (Schwartz),
   (c) unimodal maps with negative Schwartzian derivative (Guckenheimmer),
   (d) unimodal maps with non flat criticalities (de Melo Van Strien),
   (e) endomorphisms with non flat critical points which are turnings (Blokh Lyubich)
   (f) general case, non flat critical points (Martens et al)

2. For endos without critical points, and under the hypothesis that all periodic points are hyperbolic then either $L(f)$ is hyperbolic or f is conjugated to an irrational rotation.

3. For unimodal maps such that the critical point is quadratic then we get either:

   (a) existence of one attracting periodic point,

   (b) infinitely renormalizable (i.e.: there is a sequence $J_{n+1} \subset J_n$ each interval containing the critical point and converging to it, a sequence $k_n \to \infty$ such that $f^{k_n}(J_n) \subset J_n$ and $\{f^i(J_n)\}_{0 \leq i < k_n}$ is pair disjoint),

   (c) it is induced a Markov map, i.e.: there exists neighborhood $I \subset J$ such that for almost any point there exists a positive integer m and a neighborhood T_x of x such that $f^m_{T_x}$ is monotone, $f^m(x) \in I$ and $J \subset f^m(T_x)$. This implies that we can change f by $\{f^{n_x}\}_x$ such that this maps is *Markov* and has bounded distortion.

Without the hypothesis of quadratic criticality, also we can get absorbing cantor set. A cantor set of measure zero, absorbing the hole interval (a.e.) and the map being transitive in the whole interval.

What happens about the periodic attractors?

What happens about the renormalizable systems? And what about the absorbing cantor sets?

1. Finiteness of attracting periodic points. Under smooth and non flat condition, the period of the attracting periodic points is bounded

2. In the quadratic family and in unimodal families, the measure of the renormalizable systems and the one exhibiting attracting cantor set is zero

Dominated splitting and partial hyperbolic systems

All the results are with Martin Sambarino.

Two ways to relax hyperbolicity:

Partial hyperbolcity, which allow the tangent bundle to split into Df-invariant subbundles $TM = E^s \oplus E^c \oplus E^u$, and such that the action of the tangent map in E^c may be neutral.

Non-uniform hyperbolicity (or Pesin theory), where the tangent bundle splits for points a.e. with respect to some invariant measure, and vectors are assympotically contracted or expanded in a rate that may depends on the base point.

Another category (which includes the partial hyperboliciy). The **Dominated Spliting.**

An f-invariant set Λ is said to have a *dominated splitting* if we can decompose its tangent bundle in two invariant subblundles $T_\Lambda M = E \oplus F$, and such that:

$$\|Df^n_{/E(x)}\| \|Df^{-n}_{/F(f^n(x))}\| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0.$$

with $C > 0$ and $0 < \lambda < 1$

Introduced independently by Mañé, Laio and Pliss, as a first step in the attemp of proving that structurally stable systems implies hyperbolicity.

Trivial examples of sets having dominated splitting wich are not hyperbolic:

1. invariant closed curves normaly hyperbolic (at least one direction is hyperbolic),
2. two non-hyperbolic periodic points homoclinic related (no hyperbolic direction),

3. a horseshoes where the hyperbolicity is relaxed along an infinite orbit (C^1-topology).

Question:
It is possible to describe the dynamics of systems having Dominated Decomposition?
Is Domination rich enough to obtain some dynamical consequences?

Considering Smoothness:

1. Yes, for surface diffeomorphisms,
2. Also, if all the subbundles are one dimensional,

Main Theorem: *Let $f \in Diff^2(M^2)$ and assume that $L(f)$ has a dominated splitting. Then $L(f)$ can be decomposed into $L(f) = \mathcal{I} \cup \tilde{L}(f) \cup \mathcal{R}$ such that*

1. *$\mathcal{I}$ is contained in a finite union of normally hyperbolic periodic arcs.*
2. *$\mathcal{R}$ is a finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.*
3. *$f/\tilde{L}(f)$ is expansive and admits a spectral decomposition (into finitely many homoclinic classes)*

The dynamic of C^2-diffeomorphism having dominated splitting can be descomposed into two parts:

one where the dynamic consists on *periodic* and *almost periodic motions* ($\mathcal{I}$, $\mathcal{R}$) and the diffeomorphism acts equicontinuously, and another one where the dynamics is expansive and similar to the hyperbolic case.

Key Theorem

Let f be a C^2-diffeomorphism on a compact surface, and let $\Lambda \subset \Omega(f)$ be a compact f-invariant set having a dominated splitting $T_{/\Lambda}M = E \oplus F$ and such that all the periodic points in Λ are hyperbolic. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1

is hyperbolic and Λ_2 consists of a finite union of periodic simple closed curves

$C_1, \ldots C_n$, normally hyperbolic and such that $f^{m_i} : C_i \to C_i$ is conjugated to an irrational rotation (m_i denotes the period of C_i).

Domination is a rich enough structure (improving smoothness) to obtain dynamical consequences and topological descriptions.

What happen in higher dimension, for systems having dominated splitting or at least partialy hyperbolic?

Remarks

- There are open sets of diffeomorphisms which are not hyperbolic but partially hyperbolic;
- also examples with multidimensional central directions.

Thm: Assume $TM = E \oplus F$ and F is one dimensional then, the central manifold tangent to the F direction is dynamicaly defined; i.e.:
$W^u_\epsilon(x) = \{y : f^n(y) \in B_\epsilon(f^n(x)) : d(f^n(x), f^n(y)) \to 0 \ n < 0\}$
is a C^2 manifold of large ϵ
(If also $E = E^s$ then F is hyperbolic.)

Thm:
Let $f \in Diff^r$ $(r \geq 2)$ and assume $TM^3 = E^s \oplus E^c \oplus E^u$, then either:

1. is an Anosov or,
2. there is a closed invariant curve tangent to the central direction, and f can be C^r approximated by another one showing points of different index.

Question:
Let $f \in Diff^r$ $(r \geq 2)$, assume that $TM^3 = E^s \oplus E^c \oplus E^u$ and all the periodic points are hyperbolic with the same index, then either:

1. is an Anosov (central direction is either espansive or contractive) or,
2. Skew product on $T^2 \times S^1$ over an Anosov on T^2, with dynamics on S^1 conjugated to irational rotations (central direction is an isometry) or,
3. the time map of an Anosov flow (central direction is an isometry).

- Skew products on $T^2 \times S^1$ over an Anosov on T^2 could be (robustly):
    1. hyperbolic (essencially Anosov on T^2 product Morse-Smale),
    2. non hyperbolic (presence of points of different index in a same transitive set; central direction for some points is expansive, and for other is contractive).

- Time one maps of Anosov flows. Can be approximated by hyperbolic systems?

First steps in the direction to prove previous results.

Analysis of the dynamic behaviour of the central manifolds.

- For dominated splitting (in surfaces), is showed that the manifolds tangent to E and F are dynamically defined.
- For partial hyperbolicity is showed that the manifold tangent to E^c is either: :
    1. dynamically defined (stable or unstable) or,
    2. there is a closed invariant curve.

Dynamically defined:

$W_\epsilon^{E,F,E^c}(x)$ is eiher $W - \epsilon^s(x)$ or $W_\epsilon^u(x)$.
To understan this, we have to understand:
The $\omega-$limit ($\alpha-$limit) of points which are "Lyapunov stable".

Denjoy anlysis

1. Dominated Splitting for surfaces: We try to understand $\omega(I)$, curve such that
    (a) $I \subset \Lambda_1^+$ and $\ell(f^n(I)) \leq \delta$ for all $n \geq 0$.
    (b) $f^n(I)$ is always transversal to the E-direction.
2. Partial hyperbolic systems with one dimensional central direction: We try to understand $\omega(I)$, curve such that
    (a) $I \subset \Lambda_1^+$ and $\ell(f^n(I)) \leq \delta$ for all $n \geq 0$.
    (b) $f^n(I)$ is always transversal to the $E^s \oplus E^u$-direction.

Residual sets of diffeos exhibiting infinitely many attractors

Newhouse:

1. Opens sets of stable tangencies

   Existence of open sets $\mathcal{U} \subset Diff^2(M)$ such that any $f \in U$ exhibits a tangency between the stable and the unstable manifolds of a hyperbolic set. As a consequences:

2. There exists open sets $\mathcal{U} \subset Diff^2(M^2)$ containing a residual set $\mathcal{R}$ in $\mathcal{U}$ such that any $f \in \mathcal{R}$ exhibits infinitely many sinks.

What happens in the C^1-topology? Are the hyperbolic systems dense?

In higher dimension, the are C^1- residual sets of diffeos exhibiting infinitely many sinks (Bonatti-Diaz).

Questions:

- It is possible to find a dense set $\mathcal{D} \subset \mathcal{U}$ such that for any $f \in \mathcal{D}$ the $L(f)$ is (almost every point) decomposed in a finite number of attractors? Or at least exhibiting just a finite number of sinks?
- In parametrized family, is it true that the lebesgue measure of the parameter corresponding to diffeos exhibiting infinitely many sinks is zero?
- Open sets of diffeos exhibiting infinitely many sinks?

Simple sink: given a sink p of period n we say that it is simple if for $n-1$ iterates the point remains in a neighborhood of a hyperbolic set.

Tedeschini-Lalli, Yorke:

The measure (in the parameter) of diffeos exhibiting infinitely many simple sinks is zero.

For each simple sink p let $J(p)$ be the set of the parameter where a simple sink "persist", then it is proved that $\Sigma_p|J(p)| < \infty$. Using a Borel-Cantelli argument's is proved the result.

There are much more than simple sinks. Sinks obtained by the unfolds of the iterates of the local stable and unstable sets. Moreover, the unfold of an homoclic tangency implies:

1. cascade of bifurcation (Alligood, Yorke)
2. Henon like attractors (Mora, Viana)
3. infinitely many Henon like attractors (Colli)
4. super exponential growth of periodic points (Related to the higher order of contact) (Kaloshin)
5. existence of wandering regiond not increasing the diameter (Colli, Vargas)

And each of the last 3 phenomenas imply the existence of tangencies.

On the other hand, even in the presence of stable tangencies, some kind of non-uniform hyperbolicity can be guarantee:

Palis, Yoccoz:

$$\frac{\mathrm{Leb}(\{\mu < \epsilon : f_u \text{ has not sinks}\})}{\epsilon} = 1$$

It is proved that some kind of non-uniform hyperbolicity is prevalent. It do not allow the existence of infinitely many attractors.

In this case, there is an strong dissipation and when the tangency is unfolded the dynamic generated is close to a one dimensional dynamic.

What happens for systems without strong dissipation?

For C^r-conservative systems, there exists open sets of diffeos exhibiting infinetily many elliptic points and invariant circles (KAM, Aubry Mather).

Is it possible to perturb the system to have just finite number of attractors? Or infinitely many attracting components persists even if the systems is perturbed in a non-conservative way?

What is the "structure" of the Limit set in the presence of the infinitely many sinks phenomena?

When the systems has infinitely many periodic attractors, what can we say about each sinks? Basin of attraction, eigenvalues.

In a one dimensional system, if there are infinitely many sinks $\{p_n\}$, then $|f^{k_n\prime}|^{\frac{1}{k_n}} \to 1$ (where k_n is the period).

It is not true in dimension 2. There is a residual set of diffeos such that any f has infinitely many sinks $\{p_n\}$, and $|Df^{k_n}|^{\frac{1}{k_n}}$ do not converge to 1 (where k_n is the period). And so, also the eigenvalues do not converge to 1.

Related questions:

1. Is it possibe to find a dense set of diffeos such that if f has infinitely many sinks then $|Df^{k_n}_{p_n}|^{\frac{1}{k_n}}$ converge to 1? ($\{p_n\}$ the sinks and k_n the period). The same questions for the eigenvalues. (results for parametrized families?).
2. Is it possible to find a dense set $\mathcal{D} \subset \mathcal{U}$ such that the "complex" sinks are finite? The same question for parametrized families.

Even in this case, we could have that this non uniform hyperbolicity is robust.

Sambarino, P-
There are C^2 surface diffeomorphism f such that:

1. f has infinitely many hyperbolic periodic points with eigenvalues going to 1 (there are not uniformly hyperbolic).
2. for any C^2-diffeo g close to f, these periodic points remain hyperbolic.

The analysis, maybe, should be more global.

For C^2-one dimensional endomorphisms, we get:
absence of critical points imply hyperbolicity.

Generically for C^2-surface diffeo we get:
Dominated Splitting imply hyperbolicity

Absence critical points <-> Dominated Splitting

For one dimensional endos, presence of critical point is "the" obstruction for hyperbolicity.

For diffeos on surfaces, tangencies is "the" obstruction for hyperbolicity.

For diffeos on compact surfaces we get that C^1-far from tangencies we get Dominated Splitting.

In same sense, tangencies work like critical points.

What happens when a set has not dominated splitting?

two alternatives:

1. there is a continuous splitting but without domination,
2. there is not splitting.

Can we describe the dynamic of a continuous splitting?
Alternatives:

1. saddle connection,
2. dominated splitting,
3. conjugated to a two dimensional rotation.

It is a problem related to the existence of Denjoy counterexample for surfaces diffeo. Do exist maps on T^2 semiconjugated to an irrational rotation on T^2 and exhibiting wandering regions?

Yes, for $C^{2+\epsilon}$ (McSwieggen)

But, if $f \in C^{2+Z}$ then the wandering region can not be all "square regions" (Norton, Sullivan)

What is the two dimensional phenomena just that its *absence* is enough to guarantee domination?
What are *Two dimensional critical points.*
It should be dynamically defined. (Benedix-Carlesson, L. Wen-Lai Sang Young),

We could have splitting but not uniform in angle. Recall the case of invariant measures.

Question:
Given an invariant measure with one negative Lyapunov exponents, is this enough to guarantee that the other one is positive?

Previous question:
Given an invariant measure with one negative Lyapunov exponents, can we guarantee that we have dynamically defined stable and unstable sets? i.e.: for almost every point can we have eventually defined the local stable and the unstable sets?

Not true in general. Obstructions:

1. invariant circle normally hyperbolic;

2. time one map of a Cherry flows;

3. two dimensional version of one dimension phenomenas like either infinitely renormalizable or absorbing cantor sets.

The one dimensional analysis is done using strongly the concept of critical point, an arguments of distortion.

What are the *critical points* and what kind of *distortion argument* should we consider in the two dimensional case?

Two dimensional baby models.

$$F : [-1, 1] \times [0, 1] \to [-1, 1] \times [0, 1]$$

$$F(x, y) = (1 - a(x)y^2, g(x, y))$$

where $a : [-1, 1] \to (1, 2)$, and
$g(x, y) = by$ if $x < 0$
$g(x, y) = by + (1 - b)$ if $x > 0$

The dynamic is concentrated on a cantor cross an interval.

Let us consider an invariant measure $\mu = \mu_{y_{y \in K}} \times \mu_K$, μ_K is a Cantor measure, and $\mu_y <<$ Lebesgue

Questions

Assuming that there exists almost every point negative lyapunov exponents,

- there are not wandering intervals;
- alternative as in one-dimensional theorems