Critical distribution of Green functions in disordered isostatic networks: applications to granular matter

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The contact network of a cohesionless polydisperse granular packing is isostatic in the limit of low applied pressure(PRL V81, 1998, p. 1634). This property explains some of the peculiar static properties of granular materials, namely: instability to perturbation and exponential tails in the distribution of stresses. Both are due to an anomalous (truncated power-law) distribution of displacement-displacement and stress-stress Green functions. It is argued in this work that, on disordered isostatic systems, displacement-displacement and stress-stress Green functions are random multiplicative variables (with a coupling mechanism similar to diffusion). Thus they have a truncated power-law distribution, with a cutoff that grows exponentially with distance. If the external pressure is increased sufficiently, excess contacts are created, the packing becomes hyperstatic, and the abovementioned anomalous properties dissappear. These ideas are discussed in this work and illustrated with the help of simple two-dimensional numerical models.

I. INTRODUCTION

It has been recently shown [1–3] that disordered packings of frictionless, cohesionless granular materials, idealized as spheres, have, in any dimension d, an *isostatic* contact network when the external pressure is small with respect to the stiffness of its particles. Isostaticity means minimal rigidity, that is, that the network is rigid, but would cease to be so if any of its contacts where removed. As is well-known in the area of Structural Rigidity, this is means the system of equilibrium equations for interparticle stresses is exactly determined, and therefore stresses do not depend on the rheology of the material, but are entirely determined by the applied load and the network geometry. Isostaticity clearly implies that the number of unknowns (stresses) is exactly equal to the number dN of equilibrium equations. Thus the number of interparticle contacts (which is equal to the number of unknown stresses) is exactly dN. As mentioned in [1,3], this explains why the mean coordination number of stiff sphere-packings is 6 (four for disks).

A second relevant observation was the fact that because of isostaticity, the load-stress response (or Green) function G^i_b, which measures the stress induced in bond b by a vertical load applied at site i, is exactly equal to the vertical displacement Δ^i_b that site i suffers when the interparticle bond b is stretched. In other words, on isostatic systems there is a symmetry between force-force and displacement-displacement response functions. This observation is a simple consequence of the well-known Virtual Work Principle [4], and links together two apparently unrelated phenomena observed in granular systems; a) large-scale rearrangements under small perturbations [5–7] and b) stress-distributions which are not Gaussian as in "usual" disordered media, but much wider, perhaps with an exponential tail [8–16,7,17–19,6,20]. Both phenomena are due to an "anomalous" probability distribution (to be discussed later) of Green functions on isostatic systems, and disappear when the packing is strongly compressed, making its contact network become hyperstatic.

In a disordered system, Green functions are random variables. I argue here that on generic disordered isostatic networks, Green functions are *random multiplicative variables* [21,22], i.e. stresses and displacements propagate multiplicatively. This property explains the third important observation reported in [1–3], namely that the probability distribution $P(G)$ of Green functions on disordered isostatic networks is a truncated power-law with a cutoff that grows exponentially with distance [22].

This picture also suggests that when a packing is compressed, introducing excess contacts and thus eliminating isostaticity, the observed abnormal properties of granular materials should disappear, giving place to a "normal" elastic behavior. Here I present in some detail the arguments described above and discuss numerical results form a model 2d system which allows one to mimic the effects of external pressure by controlling the amount of departure from the isostatic limit.

This paper is organized as follows: In Section II, the contact network is defined and the concepts of network rigidity, isostaticity and hyperstaticity are introduced. In Section II E the isostaticity property of stiff cohesionless packings is briefly recalled, and its immediate consequences are discussed. In Section III the symmetry in response functions of isostatic systems is derived. In Section IV it is argued that the static response functions (load-load and displacement-displacement Green functions) on disordered isostatic lattices are the result of a certain kind of random multiplicative process, and therefore have a truncated power-law distribution, with a cutoff that grows exponentially with distance. Section V describes numerical models that illustrate the non-trivial properties of isostatic systems, and how they change when the external pressure increases. Numerical results from these models are presented in Section VI. Finally Section VII contains a brief discussion of the main ideas in this work.

II. RIGIDITY CONSIDERATIONS FOR CONTACT NETWORKS

A. Formulation

We deal with a d-dimensional packing of N spheres of radii R_i, interacting in pairs via an arbitrary central-force Hamiltonian $E_{ij} = k_{ij} U^{(ij)}(d_{ij})$, where d_{ij} is the distance between particle centers, and k_{ij} are stiffness constants. We will take $U^{(ij)}(d)$ to be constant for $d \geq R_i + R_j$. Therefore elastic forces are nonzero only if particles i and j are "in contact" ($d_{ij} < R_i + R_j$). We will also allow for an arbitrary set of external forces $\vec{F}_i$ acting on the sphere centers, e.g. gravity or confining pressure, and represent it by means of a vector of dN components ("the load") $\vec{F} = \{\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_N\}$. We assume that the system is in static equilibrium and that the vector of particle positions $\vec{X}$ is known.

Let $\vec{f}_{ij}$ be the elastic force acting on particle i due to its contact with particle j. We write it as $\vec{f}_{ij} = \lambda_{ij}\vec{\mu}_{ij}$, where $\vec{\mu}_{ij}$ is the unit vector pointing from j to i, and

$$\lambda_{ij} = \lambda_{ji} = -k_{ij}\partial U^{(ij)}(d_{ij})/\partial d_{ij}, \qquad (1)$$

Using this convention, the "stress" λ is positive for compressive forces. Although granular packings only have compressive forces, notice that our considerations apply for stresses of any sign.

The stiffness constants k_{ij} are assumed to be large enough with respect to the load so that versors $\vec{\mu}_{ij}$ can be taken to be constant (load-independent). This is the so-called *linear approximation* of elasticity. Notice that we are not requiring *linear elasticity*, i.e. that the potential U be quadratic in d.

B. Network rigidity

The structure obtained by associating a site to each (hyper)sphere center i and a bond to each pair of spheres (ij) in contact is called *contact network*. Let B be the total number of contacts, and let $\vec{\lambda}$ be the vector of B bond stresses.

The equations for force equilibrium at each site take the form

$$\vec{F}_i + \sum_{j(i)} \lambda_{ij}\vec{\mu}_{ij} = \vec{0} \qquad i = 1, \ldots, N \qquad (2)$$

where the sum goes over sites j in contact with site i. We take λ_{ij} to be our unknowns. We thus have a set of dN equations in B unknowns. These equations can be written in matricial form as

$$\tilde{M}\vec{\lambda} = \vec{F} \qquad (3)$$

Because of our assumption of large k_{ij}'s, the *rigidity matrix* $\tilde{M}$, which has B columns and dN rows, does not depend on $\vec{F}$.

From the point of view of structural rigidity [23–25] a network can be classified into *rigid* and *flexible*. A network is rigid if a solution $\vec{\lambda}(\vec{F})$ of Eq. (3) exists for any load $\vec{F}$, and flexible (or deformable, or *floppy*) otherwise.

A note may be worth here. The classification we are dealing with only refers to the geometrical properties of the network, i.e. if it has, or does not have, the connectivity required to sustain a load. In this sense rigidity does not mean that the network is absolutely undeformable. The main difference between these two groups is that a rigid network can only be deformed if some energy is spent, while a flexible network can be deformed without any energy cost.

We are not, and will not be in this work, concerned with the elastic properties (i.e. elastic moduli) of the network, which eventually depend on the rheological properties of its bonds. In a similar sense and considering the equivalent scalar problem (each site has one degree of freedom instead of d) a network can be classified into connected and disconnected. Connectivity is a purely topological property (and can also be related the rank of a "connectivity matrix"), while conductivity is a transport property that depends on the transport properties of its bonds. For further discussion of the similarities and differences between connectivity and rigidity see for example [26–29].

It follows from basic theory of linear equations that the rigidity of the network is determined solely by the rank K of $\tilde{M}$.

- **a)** If $K < dN$, the network is flexible (or *hypostatic*). In this case $h = dN - K$ is its degree of hypostaticity (or number of *floppy modes*), which is the dimension of the subspace of non-resolvable loads, or equivalently the dimension of the subspace of infinitesimal deformations that cost no energy.
- **b)** If $K = dN$ (it cannot be larger) the network is rigid.

Because the rank K of an $N \times M$ matrix satisfies $K \leq \min(N, M)$ we have that $K \leq B$. Thus $B \geq dN$ is a necessary (but not sufficient) condition for rigidity.

If the network is rigid a solution $\vec{\lambda}(\vec{F})$ of Eq. (3) exists for any $\vec{F}$. The linearity of the force-equilibrium equation implies then that $\vec{\lambda}(\vec{F})$ can in general be written as

$$\vec{\lambda} = \vec{\lambda}^{(load)} + \vec{\lambda}^{(self)} \qquad (4)$$

where $\vec{\lambda}^{(self)}$ is a solution of

$$\tilde{M}\vec{\lambda}^{(self)} = \vec{0} \qquad (5)$$

For later use let us notice that $\vec{\lambda}^{(self)}$ depends on the stiffnesses k_{ij} but not on the load, while $\vec{\lambda}^{(load)}$ depends on the load but not on the stiffnesses. More specifically if all stiffnesses are rescaled by γ_k and all loads by γ_F, then $\vec{\lambda}^{(self)} \to \gamma_k \vec{\lambda}^{(self)}$, while $\vec{\lambda}^{(load)} \to \gamma_F \vec{\lambda}^{(load)}$.

Rigid networks in turn admit a subclassification into two classes: *hyperstatic* and *isostatic*.

b.1) If a rigid network has $B > K = dN$, in other words, if it has more contacts than necessary for rigidity, it is said to be **hyperstatic**, and the problem of finding the stresses is underdetermined. In this case the null-subspace $\mathcal{N}(\tilde{M})$ of $\tilde{M}$ has nonzero dimensionality $H = B - K$ (H is the the degree of hyperstaticity of the network), and thus Eq. (5) admits non-trivial solutions $\lambda^{(self)}$.

b.2) Finally, if the network is rigid but $H = 0$, i.e. if $B = K = dN$, there is a unique solution $\vec{\lambda}(\vec{F})$ for each load $\vec{F}$. Equivalently, $\lambda^{(self)} = \vec{0}$ is the only solution of Eq. (5). In this case the network is said to be **isostatic**, or minimally rigid, and $\tilde{M}$ is invertible.

For an isostatic network one can write Eq. (3) as

$$\vec{\lambda} = \vec{F}\tilde{M}^{-1} \tag{6}$$

This last expression makes it explicit that, for isostatic networks, stresses are determined by the load and geometry (which is the only information contained in $\tilde{M}$) alone, and are independent on the rheology of the bonds.

Let us define the load-stress Green function $G_{i,\alpha}^{(ab)}$ to be the excess stress induced in bond (ab) by a unit load acting on site i in the direction of axis α ($\alpha = 1, 2, ..d$ in d dimensions). Taking α as a vector index we can write all d Green functions for a given site and bond as a vector $\vec{G}_i^{(ab)}$. Load-induced stress on bond (ab) can thus be written as

$$\lambda_{ab}^{(load)} = \sum_i \vec{F}_i \cdot \vec{G}_i^{(ab)} \tag{7}$$

Comparison between Eq. (6) and Eq. (7) shows that $\tilde{M}^{-1}$ is a matrix of Green functions. We will discuss its properties later in Section III and Section IV.

C. Alternative formulations of the rigidity problem

Thinking in terms of stresses as unknowns, hyperstatic networks are "underdetermined". They have more unknowns λ than can be determined by equilibrium equations alone. What one does to handle this situation, is to bring in rheology, which provides a relationship (Eq. (1)) between deformation and stress. This allows one to use site positions $\vec{x}_i$ as unknowns. Given that there are now dN unknowns and exactly dN equilibrium equations, the resulting system of equations always has at least one solution (which may however be non-unique).

This transformation is related to a complementary formulation of the network rigidity problem in which the sites positions x_i are the unknowns. In this formulation, each bond represents a constraint in the form of an equation fixing the distance between two points (See e.g. [23–25]). Hyperstatic subgraphs, i.e. subgraphs with "too many" bonds are therefore called overconstrained in this formulation (although they are underdetermined from the point of view of their stresses).

Conversely flexible structures are "underdetermined" in terms of positions because they can be deformed while satisfying all constraints. However, they are "overdetermined" if one thinks in terms of stresses as unknowns (there are more equilibrium equations than stresses to satisfy them). Thus there are two complementary formulations of the rigidity problem. Underdetermined in one formulation is overdetermined in the other. Isostatic networks are exactly determined in both formulations. Bond-stresses and site-positions are uniquely determined by load and geometry.

Let us mention in passing that elastic continuous media are hyperstatic in $d > 1$. Equilibrium equations alone (a total of $d(d+1)/2$: d force- plus $d(d-1)/2$ torque-equilibrium equations) are not enough to determine the d^2 components of the stress tensor $\sigma_{\alpha\beta}$. After using the $d(d-1)/2$ momentum equilibrium equations to find that $\sigma_{\alpha\beta}$ is symmetric, we are left with $d(d+1)/2$ independent stress components, and only d additional force-equilibrium equations. In order to handle continuum elasticity, one has to specify a relationship between stress and strain, and then solve for the strain field $\epsilon_{\gamma\delta}$, whose number of independent components is d because of the existence of "compatibility conditions" [30,31]. Therefore the elasticity problem on a continuous medium in $d > 1$ is hyperstatic and thus requires the specification of the rheological properties of the medium.

D. A closer look at overconstrained subgraphs

As said, a network is hyperstatic when it has more contacts than needed to be rigid. Such "excess" contacts are called redundant bonds. Each added redundant bond establishes a new linear dependency involving some of the columns of $\tilde{M}$ (otherwise its rank would increase when adding this new column, which is not possible because $\tilde{M}$ already has its maximum rank dN). The bonds associated to these dependent columns are said to form an overconstrained subgraph. These bonds are the loci of self-stresses induced by the redundant bond. We now illustrate these concepts with a simple example.

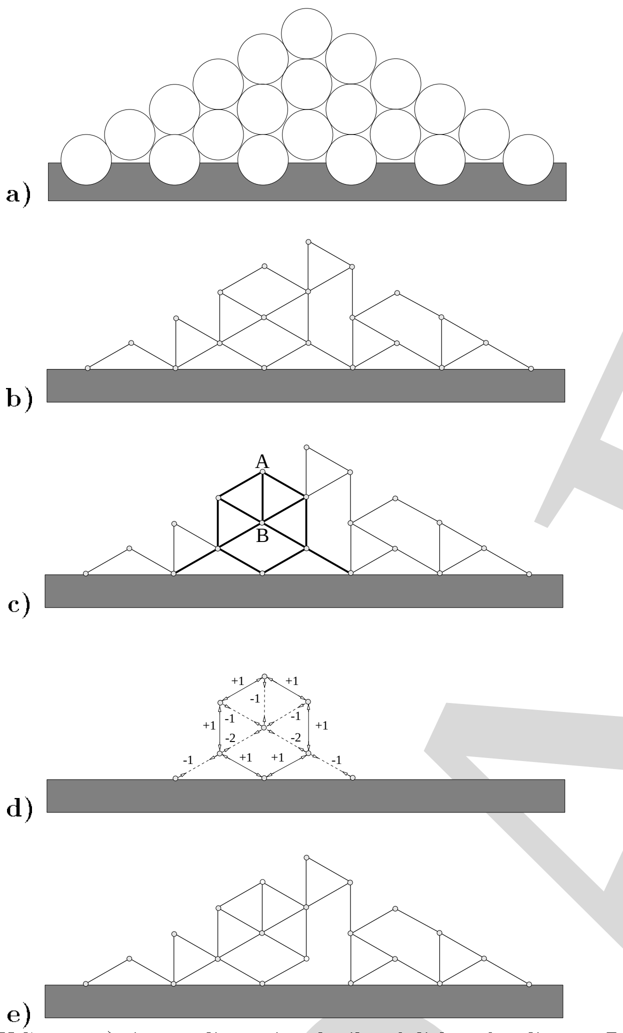


FIG. 1. **a)** A two-dimensional pile of disks of radius $\approx R$ with extremely small polydispersity. **b)** A possible isostatic contact network. **c)** The addition of one redundant contact AB turns the network hyperstatic. Now stresses are no longer uniquely determined by the load. Thick lines indicate the overconstrained subgraph, i.e. bonds that may sustain self-stresses. **d)** A possible set of self-stresses. Any multiple of these is of course also a solution of Eq. (5). This is the origin of the indeterminacy in stresses. The overconstrained subgraph in **c)** is 1-redundant or *minimally overconstrained*. The removal of any of its bonds makes the network again isostatic (**e)**).

Fig. 1 shows a small two dimensional pile, together with some possible contact networks representing sets of active contacts. All networks are rigid, but b) and e) are isostatic while c) is hyperstatic. The hyperstaticity of the second network is due to the existence of an additional (redundant) contact (AB). For this case, equation Eq. (5) has non-trivial solutions, which means that it can support certain sets of self-equilibrated bond forces, or self-stresses. The thick lines in the Fig. 1c indicate the subset of bonds which carry self stresses in this particular example, and a possible set of self-stresses is indicated in Fig. 1d. This subset of bonds is the overconstrained subgraph associated to, or induced by, the redundant bond AB. For later use we remark here that any set of self-equilibrated stresses must have at least one tensile stress [1,2], in any dimension, if the overconstrained graph is bounded. The bonds in the overconstrained graph are exactly those involved in the linear dependence introduced among columns of $\tilde{M}$ by the addition of AB.

Notice that stresses on other parts of the network (on bonds whose columns are not linearly dependent with respect to these) remain unaffected by the addition of AB; they are still uniquely defined by the load and geometry. Stresses on thick bonds, on the other hand, cannot be determined if we do not know the rheology of the bonds involved in the overconstrained subgraph. In other words, there is an infinity of sets of self-stresses that satisfy the equilibrium equations Eq. (3), which is simply a trivial consequence of the homogeneous character of Eq. (5).

The particular example in Fig. 1c shows a “cycle”, or *1-overconstrained* subgraph. The removal of just one of its bonds would make the graph again isostatic. The removal of any of the bonds in the overconstrained subgraph will do the job, therefore the network in Fig. 1e is again isostatic. A hyperstatic network with degree of hyperstaticity H contains exactly H redundant bonds, and their associated overconstrained subgraphs. The degree of hyperstaticity H of a network is the number of redundant bonds one has to take out from it in order to turn it isostatic.

E. Isostaticity in the limit of large stiffness to load ratio

Let us now assume that all interactions are purely repulsive, that is, that $U(d)$ is monotonously decreasing function of d. This means that the system *can only sustain compressive forces* between particles. I have shown in [1,2] that under these conditions, when the “stiffness to external load” ratio $I = \gamma_k/\gamma_F$ grows, the contact network has to be isostatic. The sketch of the proof goes as follows: under large enough pressure (low I), granular packings are hyperstatic (rigid and with excess contacts), so they have overconstrained subgraphs. Any overconstrained subgraph must have at least one of its bonds subject to tensile *self-stress*. This is not in contradiction with the requirement of compressive *total* stress if the applied pressure is large, because the superposition of both contributions in Eq. (4) is still compressive. But when I is increased, either by reducing the external pressure (γ_F decreases) or increasing the stiffness (γ_k increases), self-stresses become dominant in Eq. (4), and therefore if an overconstrained subgraph continued to exist, at least one of its bonds would have a tensile total stress, which is not allowed by assumption. Therefore when the external pressure is lessened, overconstrained graphs must cease to exist. This comes about by the “opening” of those contacts that would otherwise develop a tensile total stress. Therefore on reducing the external pressure the system must finally become isostatic. One sees then that granular packings become isostatic in the limit of low pressure because tensile stresses are forbidden. As mentioned in [2], a similar result (i.e. isostaticity) is obtained by imposing any constraint on stresses. For example if we think of a disordered network of strings (which oppose traction but not compression) connected at point-nodes and under homogenous tension, when the stiffness of the strings is large, only an isostatic subset of the strings is stretched.

The main point in [1–3] and in this work is that "isostaticity is responsible for some of the unusual static properties of granular packings", and not which mechanism produces isostaticity in these systems. The isostaticity property was rederived by Tkachenko and Witten (TW) [32], using constraint-counting arguments of the type previously put forward by Alexander [33]. TW however claim that the sign-constraint on stresses is irrelevant, i.e. that their conclusions would still hold if contacts where replaced by "stiff springs that may exert strong positive or negative force" [32]. Taken at face value this claim appears to be misleading. To see how tensile forces would eliminate isostaticity, think for example of the process by which sintered materials are produced. A powder is first subjected to a large confining pressure, large enough to produce substantial deformation of its particles and thus to make its contact network hyperstatic. Then, or simultaneously, the temperature is increased. This produces micro-welding of the particles. Because of this, tensile stresses are possible between particles, although at this stage there are no tensile forces because of the strong external compression. But when the external pressure is released, redundant bonds will not open, thus overconstrained subgraphs will continue to exist and internal stresses will be developed, some of them tensile. The final scale of these self-stresses will depend on the stiffness of the material, and on the disorder of the packing. The resulting product is highly self-stressed and in no way isostatic, because it is able to sustain tensile stresses.

The only, and inessential, difference between the work reported here and that by TW is the fact that TW consider undeformable particles, i.e. particles which are mathematically defined to have a fixed shape, instead of considering the process in which the stiffness-to-load ratio I diverges. If constraints cannot be violated, simple constraint-counting arguments [33,32] imply that the contact network must be isostatic if the particles have polydispersity. The resulting isostatic structure, considered as a network of stiff springs, can sustain stresses of any sign. This is of course a trivial truth for any isostatic structure, but does not clarify the process by which isostaticity appears in cohesionless granular materials. For granular packings, isostaticity is a consequence of the lack of tensile forces. The specific differences between networks with compressive-only stresses and networks with stresses of any sign will be discussed in Section VI A and Section VI B. However notice that most of the conclusions in this work apply to isostatic structures with stresses of any sign.

Isostaticity trivially means that $B = dN$ (See b.2 in Section II B), and therefore the average coordination number Γ of a stiff cohesionless packing is $2d$ [1,33,2,32]. But also and more importantly, the isostaticity property implies that bond stresses are uniquely determined by the geometry and the load (since $\tilde{M}$ only depends on the geometry). In other words, they do not depend on the specific form of the elastic potential $U(d)$ between particles, as is the case for hyperstatic networks. An important result reported in [32] is the fact that the isostaticity property provides the main closure relation needed to obtain the stress propagation equations of hyperbolic type proposed by Bouchaud and coworkers [34,35]. Roux [36] recently provided a detailed formal discussion of isostaticity in granular materials. Using tools from convex optimization theory he was able to extend the isostaticity property to packings of particles of arbitrary types and with friction forces.

In the following sections we discuss further fundamental properties associated with isostaticity, namely a) on isostatic networks the load-load and stress-stress response functions are the equal, and b) on disordered isostatic networks, Green functions G have a truncated power-law distribution, with a cutoff that grows exponentially with distance. I will argue that property b) is due to the fact that the response functions G are random multiplicative variables.

III. GREEN FUNCTIONS AND THE VIRTUAL WORK PRINCIPLE

The virtual work principle (VWP) is well known in Classical Mechanics [4]. In its simplest form it establishes that if an equilibrated system suffers a virtual displacement $\delta\vec{x}_i$ consistent with all constraints, the total work done by the intervening forces adds up to zero.

$$\sum_i \vec{F}_i \delta\vec{x}_i = 0 \tag{8}$$

Assume now we are dealing with a rigid packing (it can resolve any load). Given that interparticle contacts are "constraints" that fix the distance between two points, a rigid system cannot be displaced while satisfying all constraints, i.e. the only displacement consistent with all constraints would be $\delta\vec{x}_i = 0, i = 1, \ldots, N$. So that the simple form of the VWP above would seem in principle useless.

But if the packing is isostatic (rigid and minimally so) there is a simple trick that allows us to use this form of the VWP with profit. The trick consists in removing one bond mn from the original equilibrated system, and replacing it by its stress λ_{mn} acting on sites m and n, as shown in Fig. 2). We are now left with a structure which is no longer rigid, but still in equilibrium. This system is a *mechanism* (a structure with one degree of freedom) and a deformation is possible without violation of any constraint (i.e. maintaining all other bond lengths constant). Consider now changing the distance d_{mn} between sites m and n by an infinitesimal amount δd_{mn}.

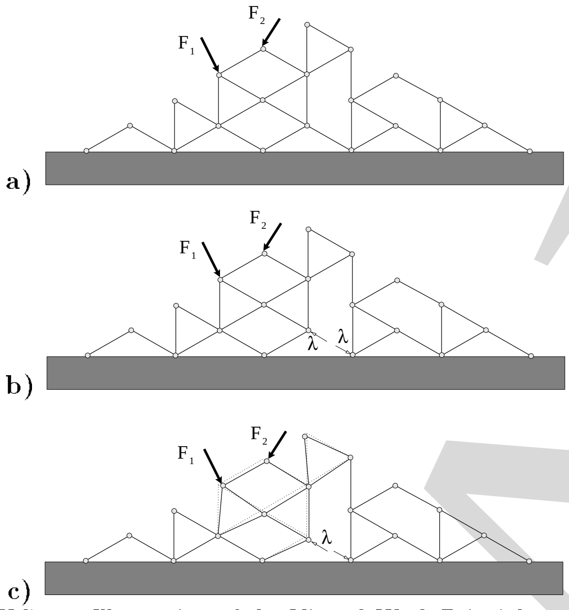


FIG. 2. Illustration of the Virtual Work Principle: **a)** An isostatic system in equilibrium under external forces $\vec{F}_1$ and $\vec{F}_2$ remains in equilibrium if one of its bonds is removed (**b)**) and replaced by a couple of forces λ that this bond provided. **c)** The system is now a *mechanism*, and can be deformed. Under small displacements, the work done by all intervening forces (i.e. $\vec{F}_1$, $\vec{F}_2$ and λ) adds up to zero.

We can now apply Eq. (8), which for this particular situation reads

$$\sum_i \vec{F}_i \delta \vec{x}_i - \lambda_{mn} \delta d_{mn} = 0 \tag{9}$$

and now calling $\vec{\Delta}_i^{(mn)} = \delta \vec{x}_i / \delta d_{mn}$ one has that

$$\sum_i \vec{F}_i \vec{\Delta}_i^{(mn)} = \lambda_{mn} \tag{10}$$

Comparison with Eq. (7) then allows us to conclude that

$$\vec{G}_i^{(mn)} = \vec{\Delta}_i^{(mn)} \tag{11}$$

Thus on an isostatic system, the stress λ_{mn} induced in a contact (mn) by a unit force acting on site i in the direction $\vec{e}_\alpha$, is equal to the component in the direction $\vec{e}_\alpha$ of the displacement induced in site i by a unit stretching of bond mn. This is the "symmetry" in response functions mentioned in the introduction. The force-force and displacement-displacement response functions are equal on isostatic systems (but notice that they were defined here in such a way that propagation of forces and displacements occurs in opposite directions).

IV. GREEN FUNCTIONS ARE RANDOM MULTIPLICATIVE VARIABLES ON DISORDERED ISOSTATIC NETWORKS

A. Regular isostatic networks

Before introducing disorder, let us consider the simplest regular two dimensional isostatic network, namely a square lattice. Because of our discussion in Section III the vertical stress Green function G_y is obtained by considering the vertical displacement Δ_y induced on a given site by a vertical displacement of a site below it. Consider then displacing one of the bottom sites upwards by an infinitesimal amount as shown in Fig. 3a. Because our system is originally isostatic (minimally rigid), this displacement produces a deformation without any change in the bond lengths (no elastic energy must be expended in order to produce this deformation). Clearly only those sites along the diagonals (characteristics) stemming from the perturbation point will be displaced. Consequently G is some constant on the sites of the diagonals and zero everywhere else. In other words, forces propagate independently along the diagonals. Such a network cannot have an exponential distribution of stresses. Clearly some ingredient is missing, and this is disorder.

B. Isostatic networks with positional disorder

Let us first consider positional disorder, as in Fig. 3b. Because the bond angles are now random, some of the displacement diffuses inside the cone defined by the two characteristics. One notable feature is that the displacement of some sites inside this cone may now be negative, meaning that an overweight there would reduce the stress on the bottom site. Moreover, it is easy to see that the propagation of displacements δ is a *random multiplicative process*, as we now show.

Consider for example propagation along one of the characteristics. Because the sites outside the cone (shaded in Fig. 3b) do not move, it suffices to consider propagation of displacements along a one-dimensional chain of sites as depicted in Fig. 3c. Let $\vec{\mu}_i$ be the unit vector normal to the bond supporting site i. Since displacements preserve bond lengths, site i can only move along the direction of $\vec{\mu}_i$, and let δ_i be the amplitude of its displacement. Conversely site $i+1$ moves by an amount $\delta_{i+1}\vec{\mu}_{i+1}$. Now let $\vec{e}_i$ be the unit vector from i to $i+1$. The condition that the length of the "horizontal" bond be unchanged reads $(\delta_i \vec{\mu}_i - \delta_{i+1}\vec{\mu}_{i+1}) \cdot \vec{e}_i = 0$, from which we get for the scalar amplitudes δ

$$\delta_{i+1} = \delta_i \frac{\vec{\mu}_i \cdot \vec{e}_i}{\vec{\mu}_{i+1} \cdot \vec{e}_i} \tag{12}$$

And therefore after k steps, $\delta_k = G_k \delta_0$ with

$$G_k = \prod_{i=0}^{k-1} \frac{\vec{\mu}_i \cdot \vec{e}_i}{\vec{\mu}_{i+1} \cdot \vec{e}_i} \tag{13}$$

showing that G is the result of a random multiplicative process [21] along the two characteristics, if the angles between the "legs" and the horizontal bonds have any amount or randomness. The multiplier $\frac{\vec{\mu}_i \cdot \vec{e}_i}{\vec{\mu}_{i+1} \cdot \vec{e}_i}$ can take values smaller and larger than one. Therefore, values of G_k much larger and much smaller than one will appear for large k. Notice however that, because of the left-right symmetry of the problem, after disorder average one must have $< G_k >= 1$, but for each realization of disorder G_k may take very large (of order $e^{c_1 k}$) and very small (of order $e^{-c_2 k}$) positive values. In other words the distribution $P(G_k)$ at each site is such that its first moment is one, but higher moments grow exponentially with distance [21].

Of course if the network were regular (with any constant angles) instead of disordered, then $\vec{\mu}_i \cdot \vec{e}_i = \vec{\mu}_{i+1} \cdot \vec{e}_i$ and $G = const$ is recovered along the diagonals.

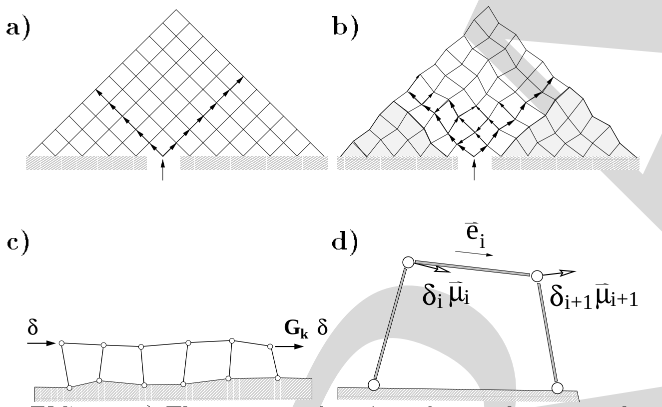


FIG. 3. **a)** The response function of a regular square lattice is nonzero only along the diagonals stemming from the perturbation point. **b)** If positional disorder is introduced, displacements propagate multiplicatively and diffuse inside the cone. **d)** Propagation of displacements along the diagonals. **e)** Site motions preserve bond lengths, thus the propagation of displacement is a random multiplicative process (see text).

Next consider any path inside the cone. Assume for a moment that the "supporting sites" (sites not in the path but connected to it by an upwards-pointing bond) did not move. Under this approximation, propagation of displacements along any path would be a RMP of exactly the same kind as described by Eq. (13). Roughly speaking, the contribution of the motions of the supporting sites can be thought of as "additive noise" on top of a multiplicative process. It is known that random multiplicative processes with additive noise give rise to truncated power-law distributions [37,38,22,39,40]. A more accurate description of the propagation of displacements inside the cone would be provided by a model of a $(d-1)$-dimensional multiplicative process with diffusive coupling, which is also known to give rise to power-law distributions [22]. We will later see that these features are indeed present in the distribution of Green functions of disordered isostatic networks.

C. Isostatic networks with contact disorder

We may now think of a network whose sites are located on a regular triangular lattice, but its bonds are an isostatic subset of the triangular lattice [1–3]. This model will be presented in detail and discussed in Section V. Considering now the propagation of displacements along arbitrary paths on such networks, we again find that they can be roughly though of as a random multiplicative process with additive noise. Here the multipliers take discrete values which include zero and negative values, instead of being continuously distributed as in the previous example.

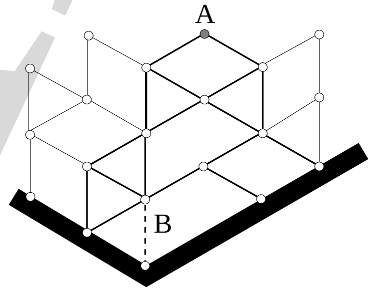


FIG. 4. An example of a pantograph. When bond B (dashed) is stretched by an infinitesimal amount δ, site A (shaded) moves upwards by an amount 2δ. The bonds drawn with thick lines represent the minimal set that compose the pantograph itself. Sites below the dark band do not move when bond B is stretched.

The multiplication of displacements is in this case due to the existence of particular configurations of contacts (called "pantographs" [1,2], see Fig. 4) which multiply displacements by an integer number. Here it suffices to notice that the existence of random multiplicative processes in the propagation of stresses and displacements is a generic property of isostatic disordered networks. As a general feature of disordered isostatic networks we will find in later sections that the Green function has an approximately symmetric, truncated power-law distribution, with a cutoff that grows exponentially with distance.

D. Considerations about hyperstatic networks

We have argued already that strongly compressed granular packings have a hyperstatic contact network. When the contact network is hyperstatic, i.e. it has more contacts than needed for minimal rigidity, the equivalence between stress-stress and displacement-displacement Green functions is no longer valid, thus one must distinguish between G^{stress} and G^{displ}. Moreover on hyperstatic networks, displacements and stresses cannot be calculated by local propagation. As discussed here and in previous work [1,2], for hyperstatic networks one needs to specify the rheology of the "bonds" and solve the elastic equations all over the system, usually taking the sites positions as unknowns. This formally requires the inversion of a large matrix, which is a difficult task even for moderate systems, so the usual choice for numerical studies is the use of iterative methods [41]. Leaving the detailed numerical study of how Green functions are modified by hyperstaticity for later sections, we can argue in simple terms that the presence of "overconstraints" removes the anomalous properties of $P(G)$ (i.e. higher moments which grow exponentially with distance).

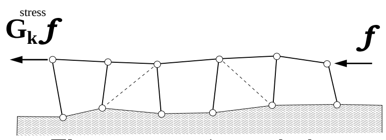


FIG. 5. The propagation of forces in a quasi one-dimensional overconstrained network is dissipative, because the system is rigid and opposes deformation by storing elastic energy. Thus the response functions decay with distance. We are considering all bonds to be "springs" with the same stiffness constant.

For this purpose consider for example the one-dimensional propagation of forces along one of the characteristics on the distorted square lattice, as shown in Fig. 5. In the absence of excess contacts (dashed lines) the system is a mechanism (it can deform without storing elastic energy) and $G_k^{stress} = G_k^{displ.}$ as required by the virtual work principle (Section III). As argued previously, $G_k^{displ.}$ can take exponentially large values. But if the system is hyperstatic, the excess constraints make the system rigid: its bonds store elastic energy when the network is deformed, and this introduces a dissipative term in the propagation of f. Therefore the average perturbation (and all its higher moments) due to the force f acting on the left side will decay with distance. In this sense, overconstraints introduce a sort of "damping term" in the (discrete) equations which rule the propagation of f along the chain.

A similar mechanism operates on $d-$dimensional hyperstatic networks. Overconstraints introduce an effective damping term and as a consequence all moments of the response functions decay with distance, instead of increasing exponentially as on isostatic networks (See Section VI D).

V. NUMERICAL MODEL OF GRANULAR PACKINGS WITH CONTACT DISORDER

This section provides numerical tests for the ideas developed above. I use a simple numerical model intended to capture what I think are the important physical ingredients: isostaticity (or the lack thereof) and disorder. As discussed above, starting from a lightly loaded, isostatic packing, increasing the external pressure creates excess contacts, or overconstraints. The models here used contain a tunable parameter O_v which controls the amount of overconstraints, i.e. the departure from the isostatic limit. We will work in 2d for simplicity, but the model is trivially generalized to higher dimensions, and the results here reported are expected to be qualitatively the same.

I consider a two dimensional triangular packing of $N = L^2$ slightly polydisperse disks of radius $\approx R$ (Fig. 6a). All particles have a unit weight. Boundary conditions are periodic in the horizontal direction. The width of the system is twice its height, therefore force lines stemming from one particle do not "interfere" with each other around the packing.

Each particle has three lower neighbors on which it can eventually "rest", i.e. establish a contact with. If all particles had exactly the same radius R, then all three contacts would be "active" for each disk, thus the contact network would always be a complete triangular lattice, no matter the applied pressure. The coordination number would then be 6 and the packing would be hyperstatic. This is the case considered by Oron and Herrmann [42].

We will on the contrary assume that our disks have a very small polydispersity (random variations in radii), small enough such that the particle centers can still be considered to be located on the sites of the regular triangular lattice. The existence of polydispersity produces contact disorder. Thus not all pairs of neighbors will be in contact. The total number of contacts will of course depend on the "applied pressure", being $2N$ for low pressure and $3N$ for large enough pressure. The applied pressure is not an explicit parameter in our model, we only mimic its effect by controlling the density O_v of overconstraints in the contact network, in a way that will be specified later.

A. Isostatic networks

From the theoretical considerations discussed in the previous sections, we know that the contact network is isostatic in the limit of low pressure. This means that it will have a total of $2N$ contacts (in two dimensions). Let us first discuss how isostatic piles are constructed and analyzed.

1. *Random-stress isostatic networks (RIM)*

The pile is built by adding layers *from top to bottom*. All particles in a layer have the same height. For each particle, we choose one among the three configurations of supporting contacts labeled L (left), S (symmetric), and R (right) in Fig. 6b.

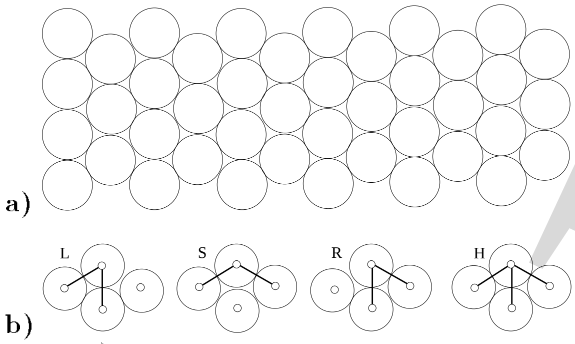


FIG. 6. **a)** The triangular piles that our model represents. Gravity acts downwards, and horizontal boundary conditions are periodic. **b)** A present leg means that two particles are in contact. The first three configurations give rise to isostatic networks. The density of configurations of type H is what we call "density of overconstraints" O_v and is a tunable parameter in our model.

The type of contact disorder is determined by a triplet of numbers $\{p_L, p_S, p_R\}$ adding up to one, which give the *a priori* probability to choose the configurations L, S and R respectively (See Fig. 7). The case $p_S = 1$ is the square lattice. In the case $p_S = 0$ we have that stresses propagate only vertically along independent columns, and diagonal bonds suffer zero stress (because we only have vertical gravitational forces). In both these limits there is effectively no contact disorder for the loading conditions described here. The preferred choices in this work were usually $p_L = p_S = p_R = 1/3$, although other combinations were also explored (see later). Notice that this model has no positional disorder, therefore all contacts are either vertical or form an angle of 60 degrees with the vertical. This has the advantage that all stresses and induced displacements are integer numbers, and allow the study of large systems. We will present in this work results on systems of up to 4×10^6 particles, taking disorder averages over several million samples.

Once the supporting configuration (L, S or R) of a particle is chosen, we can calculate stresses on both supporting legs, because we already know the "incoming" forces due to contacts with the previous (upper) layer. Thus when the bottom is reached and the pile is finally built, all stresses are already known.

The procedure described above produces isostatic networks, but has *no built-in sign constraint* for stresses, and therefore tensile stresses appear. We call it the "random-stress" isostatic model (RIM). Although most of the results discussed in this work apply to isostatic networks with stresses of any sign, we also want to be able to produce isostatic networks which only have compressive stresses, and thus can be compared with experiments in granular packings.

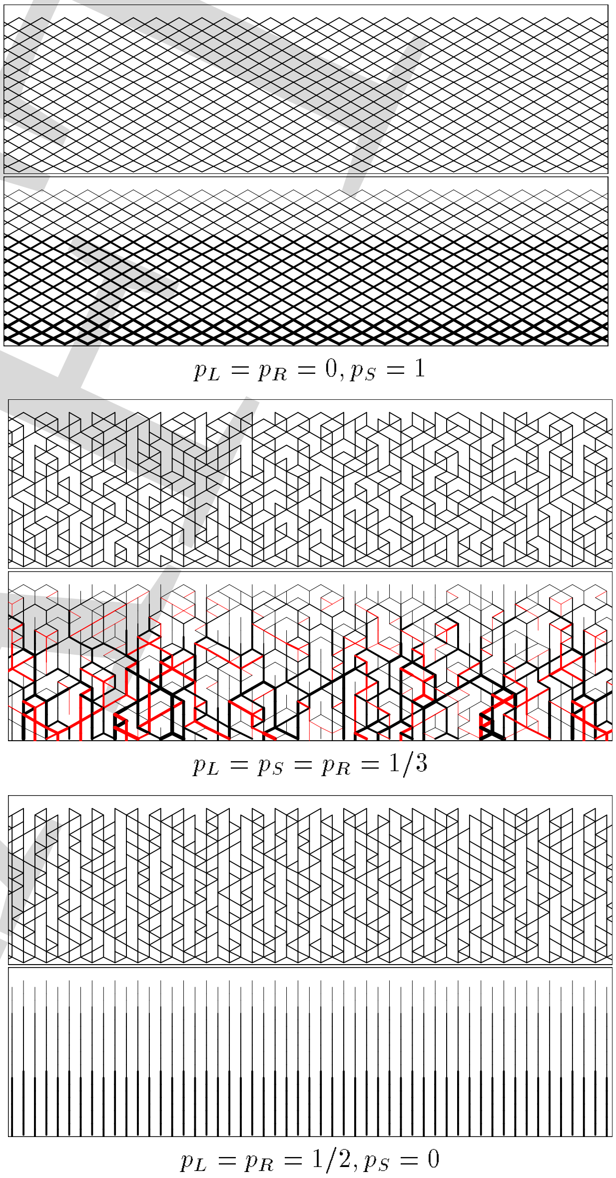


FIG. 7. Isostatic packings built by choosing configurations L,S and R at random, with the different sets of probabilities indicated below each group. In each of these groups, the upper drawing shows the configuration of contacts, the lower one shows stresses. For the latter line thickness is proportional to $\log(|\lambda|)$. Black lines are compressive, dashed tensile stresses (unstressed bonds are not shown).

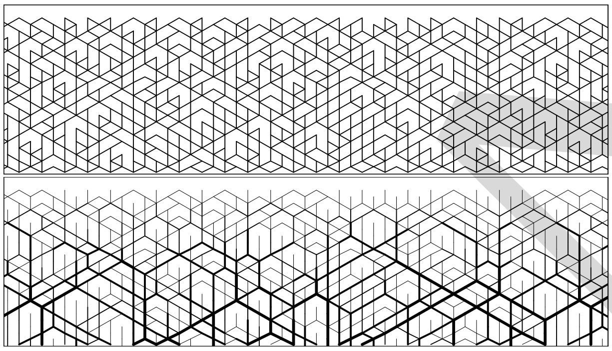
2. *Compressive-stress isostatic networks (CIM)*

It is easy to introduce a modification in order to only obtain compressive stresses. This modified, "compressive-stress" isostatic model (CIM), is constructed as follows:

- If configuration S is chosen at a given site (which happens with probability p_S), and since the "incoming forces" are also compressive by construction, we can rest assured that the supporting stresses will also be compressive.

- If configuration S is not chosen (which happens with probability $1 - p_S$), then we look at the horizontal component of the incoming forces. If it points leftwards then we choose configuration L. If it points rightwards we chose configuration R, and if it is exactly zero we choose any of those.

It is easy to see that this simple recipe completely eliminates tensile stresses (Fig. 8). This trick only succeeds to avoid tensile stresses for purely isostatic networks with no positional disorder and with the specific loading conditions defined here. If strong horizontal forces were present, it would not work.



$p_L = p_S = p_R = 1/3$

FIG. 8. CIM Networks. Same as Fig. 7 but for isostatic networks with compressive-only stresses, built as described in Section V A 2.

Clearly, this is a simplified model of isostatic packings. On general isostatic networks, there could be in principle some sites supported by only one (three) lower neighbors, provided that this is compensated by an extra (less) bond located at some other place. An example of this is shown in Fig. 1e, where the network is exactly isostatic, but there are two sites supported by one and three legs respectively. In other words, isostatic packings in which each site is supported by exactly two lower neighbors only cover a subset of the possible universe of isostatic networks (I thank J. Socolar for discussions on this point). It is not difficult to generate more general isostatic networks, but they would pose a problem we do not want to cope with at this moment: the rigidity matrix will in general not be upper diagonal, and thus, after the network is constructed, a large system of linear equations would have to be solved. This is not necessary in our CIM and RIM models, which are "propagative" ($\tilde{M}$ is upper diagonal) by construction.

In a realistic model of polydisperse frictionless packings, the contact network would be determined by the specific values of the radii. Obtaining a contact network without tensile stresses for a given set of radii and a given set of external forces is a complicate problem in general. A realistic contact network can be obtained for example by using an iterative procedure proposed by Roux and coworkers [43], but it is very CPU intensive and does not allow the study of large systems. Another alternative, which however suffers from the same limitations is the use of molecular dynamics as for example in [44,45].

B. Overconstrained networks

According to our discussion in previous sections, a polydisperse packing, which for low pressures is isostatic, develops excess contacts (overconstraints) when the pressure is increased. As argued here and in previous work [1–3], overconstrained networks should approximate the elastic behavior of disordered continuum media. In order to simulate the effect of increasing the external pressure and be able to check these expectations, we introduce a variable density O_v of overconstraints. This is done by letting each site be supported by three legs (configuration labeled H (hyperstatic) in Fig. 6b) with probability O_v. This parameter measures the amount of overconstraints in the network, or, in other words, the amount of departure from isostaticity. The exact relationship between O_v and the applied pressure will in general depend on the exact rheology of the material, the distribution of interparticle gaps, etc, and will not be discussed in this work. Here we control O_v and content ourselves with the fact that it "mimics" the effect of external pressure. In order to obtain a network with a density of overconstraints O_v, a fully constructed CIM or RIM network is taken and, in a second phase, each site is given three legs (configuration H in Fig. 6b) with probability O_v. Each particle with three lower contacts in the networks so built, creates an overconstrained subgraph (See e.g. Fig. 1c).

For any $O_v \neq 0$, the existence of overconstrained graphs destroys the nice propagativity property of stresses. Stresses are no longer determined by local conditions as in the isostatic ($O_v = 0$) case, but depend on the rheology and loading conditions all over the material [1,2]. Thus the stresses calculated before adding the three-legged sites are no longer valid, and we must recalculate them. For this purpose I assume linear elastic behavior of all bonds and use a Conjugate Gradient [41] iterative procedure for solving the elastic equations (Eq. (3)). Because of the increased computational complexity, only relatively small systems were studied (100×100 particles) when $O_v > 0$.

Clearly $O_v = 1$ gives the full triangular lattice studied by Oron and Herrmann [42], which according to the theory developed above is a valid model only if there is no polydispersity, or if the applied pressure is very large. The fact that a small amount of polydispersity modifies the nature of stress-distributions for low pressures has been observed in numerical experiments [46,47]. Our theory explains why [1,2]: for low applied pressures, a small amount of polydispersity is enough the make the contact network isostatic ($O_v = 0$). We will see later in Section VI D that CIM networks have an exponential tail in their distribution of stresses, while overconstrained networks display a quasi-Gaussian stress distribution.

Furthermore, after adding overconstraints to a CIM network there is no longer any warranty that stresses will be compressive. In fact, some amount of tensile stresses do appear as we shall see later, but these are small if the density of overconstraints is low. Anyway, it would be desirable to think of a fast method to build overconstrained networks which only have compressive stresses. It is not difficult to think of several ways to do this by recurrent modification of the structures obtained with the model described here, but their iterative nature makes them relatively time-consuming. Several types of iterative methods have been previously used for stress distributions of packings [47–49,43,50,45,51]. Regretfully they only allow the study of relatively small systems.

C. Green function measurements

Apart from stress distributions we also want to measure the displacements Δ_b^i induced in site i by the stretching of a bond b at the bottom, which we showed are the load-stress Green functions of the system. After the network has been built and its equilibrium stresses calculated, a randomly chosen particle in the bottom layer is shifted vertically. The size of this displacement is not relevant since our system is linear in the limit of large stiffness k. Now the new equilibrium positions are calculated. If the network is isostatic, this is trivially done by *upwards* propagation. The trick is that, on an isostatic system, a deformation of the type described here does not modify the repose length (or stress) of any of the bonds in the system. The sole condition that bond lengths be constant is enough to find how much a particle is displaced, knowing how much its lower (supporting) particles move. The resulting procedure is very similar to the one by which stresses are calculated, but now we proceed upwards.

If the system is overconstrained this simple procedure no longer works, because deformations of an overconstrained system *do produce* stress changes, and bond-length changes. Thus in this case what I do is to recalculate the equilibrium positions using the CG with the new boundary conditions (one of the lower particles shifted upwards). I checked that both procedures (local propagation and use of CG) give the same result for the isostatic ($O_v = 0$) networks. Notice as well that the induced displacements so calculated are no longer equal to the load-stress Green functions if the network is overconstrained. However they still provide useful information, since they measure how a perturbation propagates through the system.

VI. NUMERICAL RESULTS

A. Compressive-stress Isostatic networks (CIM)

We first present results from the CIM model (Isostatic with only compressive stresses) when the fraction p_s of symmetric configurations S is varied. In Fig. 9 we show stress distributions, and in Fig. 10 Green function distributions, for several values of p_s.

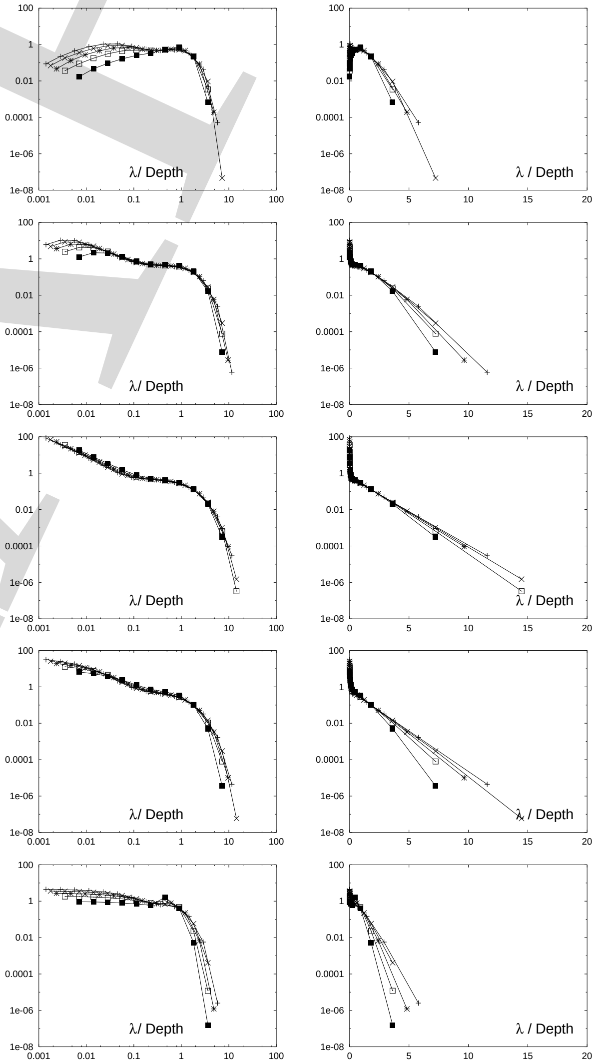


FIG. 9. Compressive Isostatic Model (CIM): Distribution of normalized stress λ/depth, in log-log (first column) and log-lin (second column) scales, at a depth of 200 (filled squares),400 (squares),600 (asterisks),800 (crosses), and 1000 (pluses) layers. The lines are only for guiding the eye. Averages were done over 10^4 to 10^5 realizations, each containing 10^6 disks. In all cases $p_R = p_L = (1 - p_S)/2$. From top to bottom, the density p_S of symmetric configurations is:$0.99, 0.90, 0.333, 0.05$ and 0.005

Let us first pay attention to the case $p_S = p_L = p_R = 1/3$ (middle row plots), which has the most disorder. Three salient characteristics can be observed in the distribution of normalized stresses: **a)** an exponential tail for stresses larger than average, **b)** a plateau near the average value 1, and **c)** a power law divergence $(\lambda/\text{depth})^{-\theta}$ for small stresses.

A lot of work has been done previously, regarding the existence of an exponential decay of $P(\lambda)$ for large forces [8–16,7,17–19,6,20]. The plateau around average forces has also been reported previously [15,11]. The power law divergence of $P(\lambda)$ for small-stresses has been reported in numerical simulations [18] of polydisperse disks with friction, giving an exponent $\theta \approx 0.3$ (But see also [52]). In experimental measurements [6,12] a depletion of small forces is instead reported. Notice however that the small-force behavior is difficult to measure in experiments.

The numerical results here reported give an exponent θ very close to 3/2, when measured on packings of 2000×2000 particles with $p_S = 1/2$, but a systematic study has not been performed yet.

The existence of many lightly loaded particles is, at least in our model, not surprising but a reasonable consequence of "arching" [7]. Additionally, the power-law character of the small-stress behavior would seem to suggest some kind of self-similar structure in the geometrical distribution of line-forces (e.g. smaller arches supporting larger arches and so on).

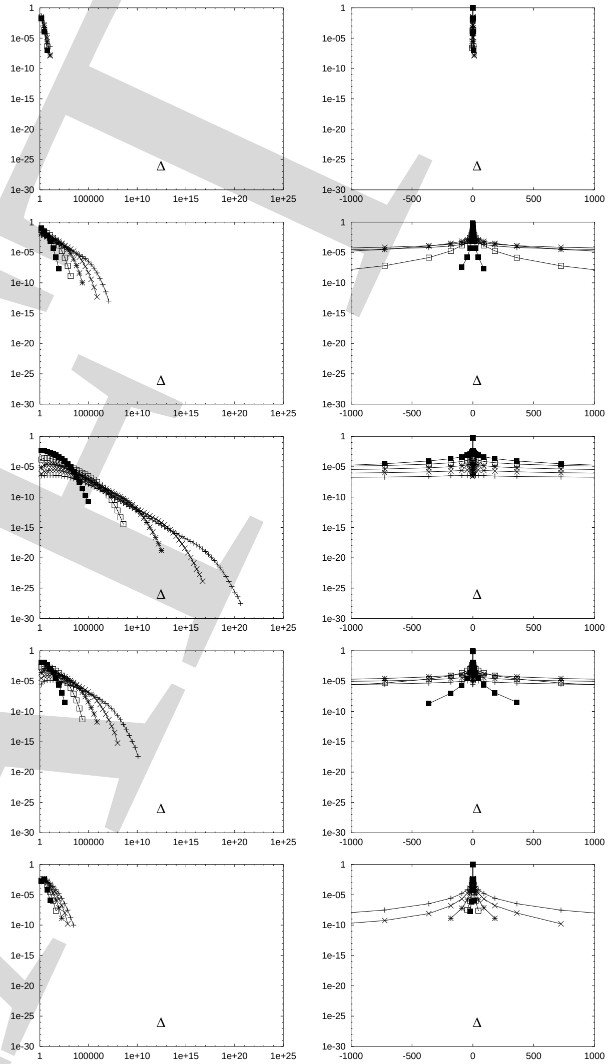


FIG. 10. Compressive Isostatic Model (CIM): Distribution of Stress-Stress Green functions Δ (the excess stress induced by a localized extra weight on the top) in log-log (first column) and log-lin (second column) scales, measured at a depth of 200 (filled squares),400 (squares),600 (asterisks),800 (crosses), and 1000 (pluses) layers. The lines are only for guiding the eye. Averages were done over 10^4 to 10^5 realizations, each containing 10^6 disks. In all cases $p_R = p_L = (1 - p_S)/2$. From top to bottom, the density p_S of symmetric configurations is:$0.99, 0.90, 0.333, 0.05$ and 0.005. A common feature is that $P(\Delta)$ is of a power-law type with a cutoff that grows *exponentially* with depth. This feature however only appears at large distances if p_S is close to either zero or one (resp. lowermost and uppermost graphs), which are the non-disordered limits.

We see that the same characteristics also appear when $p_s >> 1/3$ or $p_s << 1/3$, although they begin to be observable only at greater depths. Remember that the limits $p_S = 0$ and $p_S = 1$ have no disorder as already discussed. From these results, it seems that any departure from the ordered limits $p_S = 0$ or $p_S = 1$ will give something at least qualitatively equivalent to what is seen for

$p_S = 1/3$, if the depth (or height, for Green functions) is large enough. Of course, the sizes needed to observe the abovementioned characteristics are larger the closer one is to the trivial cases ($p_S = 0$ and $p_S = 1$). In some sense, these trivial limits appear to be "repulsive fixed points". Based on these simulations we cannot however say whether $P(\lambda/depth)$ is universal. It appears to be so at a qualitative level at least, but some details might depend on the disorder variables.

The distributions of vertical displacements Δ (Green functions) have already been discussed previously [1,2]. It was shown that its second moment grows *exponentially fast* with height, and the mechanism responsible for this phenomenon explained for contact-disordered networks. According to our discussions in Section IV we see that both features of the distribution of Green functions, namely a truncated-power law form and and exponentially growing cutoff, are due to the existence of random multiplicative processes on disordered isostatic networks.

Notice that $P(\Delta)$ is approximately symmetric, i.e. large negative and positive values appear. At large distances, Δ is enormously large sometimes (though this happens with a small probability). Now remember that because each particle has a unitary vertical force acting on it, the stress λ on a given bond b is given by

$$\lambda = \sum_i \Delta_i^b \tag{14}$$

Considering this formula, and given that Green functions have an exponentially broad distribution, one could naively expect the stresses to behave in the same way. However the scale of stresses grows only linearly with depth and not exponentially as for Green functions.

It is therefore evident that strong correlations exist, so that Green functions of different signs (and orders of magnitude larger than the stresses themselves) cancel each other exactly at each site. The origin of these correlations is easily discovered. They are a consequence of the fact that we chose the local supporting configurations at each site in such a way that negative stresses were avoided. This is equivalent to requiring that $\sum_i \Delta_i^b$ never be negative, and thus Green functions cannot be uncorrelated random variables.

Let us now see what happens if no such correlations exist, i.e. if local supporting configurations are chosen entirely at random.

B. Isostatic networks with stresses of any sign (RIM)

When the isostatic configurations are chosen entirely at random (RIM), stresses of both signs appear. The stress distribution $P(\lambda)$ is approximately symmetric around zero. Not totally symmetric because its first moment is of course still proportional to the depth as it should, but $P(\lambda) - P(-\lambda)$ is nonzero only for small stresses. The large-stress behavior is symmetric. The cutoff in $P(\lambda)$ now *grows exponentially with depth*(Fig. 11). The second moment of $P(\lambda)$ grows exponentially with depth (not shown).

These are the expected consequences of randomly superposing (See Eq. (14)) Green functions with an exponentially broad distribution as discussed above. The fact that the stress-scale is exponentially amplified with distance is a generic property of randomly disordered isostatic structures. As we can see, the sign constraint on stresses, and the resulting correlations induced by it, are crucial to avoid this behavior.

The distribution of Green functions $P(\Delta)$ in RIM is qualitatively similar to that in CIM, except for the fact that the cutoff grows much faster. At height h, one observes in RIM a distribution $P(\Delta)$ which is roughly that of CIM at height $h/2$.

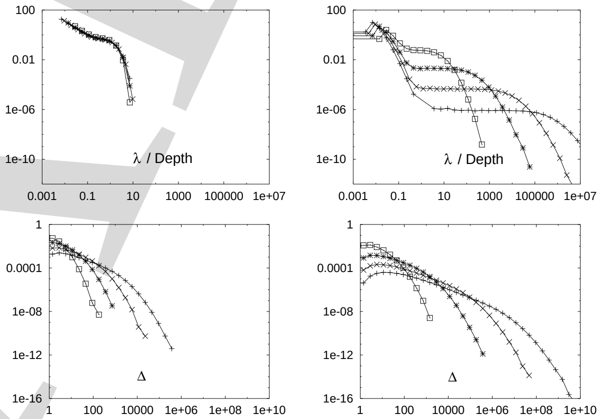


FIG. 11. Distribution of normalized stress λ/depth (upper plots) and of Green functions Δ (lower plots), for isostatic packings with compressive-only stresses (CIM, left) and unrestricted stresses (RIM, right), at a depth of 50 (squares), 100 (asterisks), 150 (crosses) and 200 (pluses) layers. Only positive stresses are shown in the upper right plot. Notice that in this case the stress-cutoff grows *exponentially* with depth, instead of being proportional to depth as in the compressive-stress case (upper left).

C. Square networks with positional disorder

In order to show that multiplicative processes are a generic property of disordered isostatic networks, let us now briefly consider square lattices with positional disorder (Fig. 3b). Each disk has a unit weight and rests on two neighbors, but their centers are slightly displaced from the sites of a regular lattice. Our results for Green function distributions and stress distributions are shown in Fig. 12.

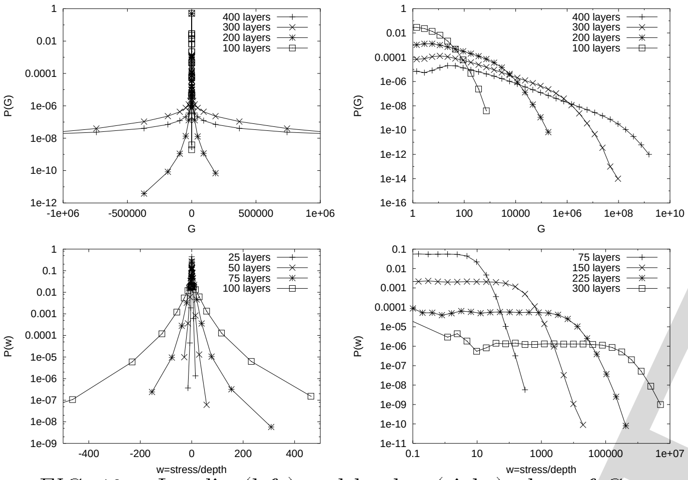


FIG. 12. Log-lin (left) and log-log (right) plots of Green function distributions (upper row) and stress distributions (lower row) for position-disordered square lattices, measured at several heights.

For these systems no constraint on stresses was imposed, so that stresses of both signs appear, even when the gravitational forces act downwards. Again the higher moments of the stress-distribution are seen to grow exponentially with depth, a result which follows from the exponentially broad distribution of response functions together with Eq. (14).

D. The effect of increasing the applied pressure

As argued in [1–3] and here, when the pressure applied on a packing is increased, excess contacts appear. The resulting hyperstatic structures that appear now damp the multiplicative effects, thus eliminating the unusual behaviors observed in isostatic networks. This is illustrated in Fig. 13 by showing that in the contact-disordered numerical model with $O_v \neq 0$, the distribution of induced displacements Δ is bounded, and its moments decay with distance instead of increasing exponentially as in the isostatic case.

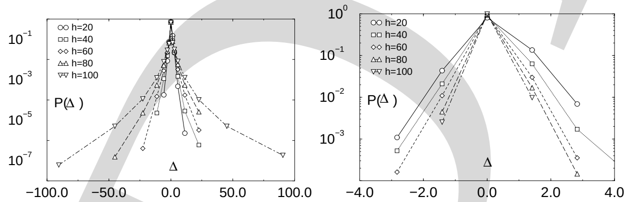


FIG. 13. On isostatic lattices (left), $P(\Delta)$, which measures the distribution of load-load and displacement-displacement Green functions, gets broader when the distance h increases. It eventually becomes a truncated power-law distribution (See Fig. 10) with an exponentially large (of order $e^{\alpha h}$) cutoff. However a density of overconstraints $O_v = 0.02$ is enough to revert this situation (right). In the overconstrained case, $P(\Delta)$ gets narrower with increasing distance h.

We now present results showing how stress distributions are modified by the introduction of overconstraints. In Fig. 14 we see how overconstraints induced by external pressure make $P(\lambda)$ become similar to a Gaussian.

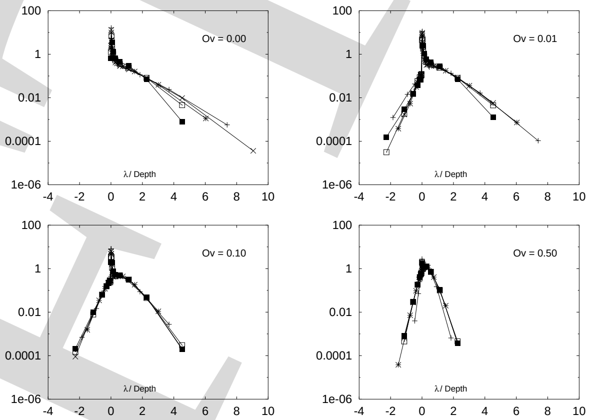


FIG. 14. Normalized stress distributions on a CIM model with increasing densities O_v of overconstraints, measured at a depth of 20 (filled squares), 40 (squares), 60 (asterisks) 80 (crosses) and 100 (pluses) layers.

VII. DISCUSSION

It has been shown that the load-load and stress-stress Green functions of disordered isostatic networks are distributed according to a truncated power-law, with a cutoff $\Delta^{max}(h)$ that grows exponentially with distance h. The reason for this surprising behavior is that, on disordered isostatic networks, response functions are the result of coupled random multiplicative processes, and these processes are know to give rise [22] to truncated power-laws. At large distances, exponentially large values of G may appear. Notice that these results only apply for *infinitesimal* perturbations, and say nothing about the properties of response functions when rearrangements of the contact network are allowed.

It has been previously shown [1] that stiff granular packings have an isostatic contact network, so these results should apply to granular packings, which by definition have no tensile stresses. However we studied here also the case of isostatic structures with unconstrained stresses. Our numerical results suggest the following picture: On disordered isostatic networks the distribution $P(G)$ of Green functions is a truncated power-law, and its cutoff grows exponentially with distance. If no sign constraint exists for stresses, the stress distribution $P(\lambda)$ presents surprising characteristics: all moments of the stress distribution of order larger than one grow exponentially with depth. In other words, the distribution of stresses is exponentially broad, but its first moment only grows linearly with depth because of cancellations among stresses of different signs.

If on the other hand only compressive stresses are allowed, nontrivial correlations are introduced among different Green functions at a given point. In this case the resulting stresses, which are the superpositions of Green functions, are "well behaved", i.e. its n-th moment $< \lambda^n >$ scales as depth^n.

When the external pressure is increased, excess contacts appear, the network is no longer isostatic but hyperstatic (modeled here by letting ($O_v > 0$). This restores the "normal" properties of Green functions [2], and of course also of stress-distributions (Section VI D). This transition from "anomalous" (exponential tails in $P(\lambda)$) to "normal" (Gaussian tails) elastic behavior under increasing pressure was observed in recent experiments [13] and numerical simulations [45], in which the packing fraction γ is controlled. The mechanism by which this transition happens is, I argue, that excess contacts eliminate the multiplicative effect by introducing an effective damping term in the equations which rule the propagation of Green functions. This damping term is due to the fact that an overconstrained network opposes deformation, requiring a finite expenditure of elastic energy to be deformed.

A continuum description of stress-transmission on granular media under varying pressure conditions should thus be able to encompass both the isostatic and the fully hyperstatic limiting behaviors (resp. described by stress equations of the hyperbolic and elliptic type [34,32]), perhaps by means of an internal parameter which controls the amount of hyperstaticity. Such a description could be similar to the one that was recently put forward N. Kenkre [53,54], which has as limiting cases the q-model [20,19] (described by parabolic equations) and the wave-equation [34,35] (described by hyperbolic equations).

I wish to thank H. Herrmann, J. Socolar, J. Goddard, J. P. Bouchaud and A. Tkachenko for useful discussions and correspondence about the topics here discussed. I also thank FAPERJ (Rio de Janeiro,Brazil) and CONACYT (Mexico) for financial support.

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