# The Steady State of the Tapped Ising Model

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We consider a tapping dynamics, analogous to that in experiments on granular media, on the simple one dimensional ferromagnetic Ising model. When unperturbed the system undergoes a single spin flip falling dynamics where only energy lowering moves occur. With this dynamics the system has an exponentially large number of metastable states and gets stuck in blocked or jammed configurations as do granular media. When stuck, the system is tapped, in order to make it evolve, by flipping in parallel each spin with probability $p$ (corresponding to the strength of the tapping). Under this dynamics the system reaches a steady state regime characterized by an asymptotic energy per spin $E(p)$, which is determined analytically. Within the steady state regime we compare certain time averaged quantities with the ensemble average of Edwards based on a canonical measure over metastable states of fixed average energy. The ensemble average yields results in excellent agreement with the dynamical measurements.

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## I. INTRODUCTION

The Hamiltonian of the one dimensional Ising model is given by

$$H = -\sum_i S_i S_{i+1} \tag{1}$$

A thermal dynamics which evolves this system into the canonical ensemble is Glauber dynamics [1] and has the merit of being soluble (in the sense one can calculate the time dependence of the energy and certain correlation functions). At zero temperature spin flips which reduce the energy are accepted and those which leave it unchanged are accepted with probability $1/2$. The only blocked configurations are the two ground states (all spins up or down). The system coarsens and the average domains length grows as $\sqrt{t}$ [2].

Granular systems have a large number of blocked or jammed configurations and one expects that the entropy of these configurations is extensive [3]. It is well known [4] that spin glasses have an extensive entropy of metastable states, which are responsible for the glassy dynamics of these systems. Indeed spin glass type systems have been proposed as paradigms of granular material [5,6]. With this in mind the one dimensional Ising model does not look like an obvious candidate to model the behavior of granular systems. However, consider now the following modification of Glauber dynamics: spin flips are accepted if they lower the energy but not if they leave it unchanged. This is somewhat natural in the context of granular media as frictional forces must be overcome. With this dynamics for a general Ising spin system with Hamiltonian

$$H = -\frac{1}{2}\sum_{ij} J_{ij} S_i S_j$$

the total number of metastable states is given by

$$N_{MS} = \mathrm{Tr} \prod_{i=1}^{N} \theta\left(\sum_{j\neq i} J_{ij} S_i S_j\right) \tag{2}$$

$N$ is the number of spins, and $\theta$ is the Heaviside step function. Note, with our definition of metastability $\theta(0) = 1$. This equation may be understood as follows. The energy change $\Delta E_i$ due to the spin flip $S_i \to -S_i$ is given by $\Delta E_i = \sum_{j\neq i} J_{ij} S_i S_j$. Hence a configuration is single spin flip stable if all $\Delta E_i$ for that configuration are non negative.

It should be pointed out here that the definition of metastable states is of course dependent on the dynamics of the system, in contrast with micro-states in classical statistical mechanics.

The Edwards entropy [3] per spin of blocked or jammed configurations is then given by

$$s = \ln\left(N_{MS}\right)/N \tag{3}$$

In complex systems such as spin glasses $s$ is $O(1)$ and independent of $N$ in the thermodynamic limit [4]. The Edwards entropy per spin at fixed average energy per spin $E$ is given by

$$s(E) = \ln\left(N_{MS}(E)\right)/N \tag{4}$$

where

$$N_{MS}(E) = \mathrm{Tr}\prod_{i=1}^{N}\theta\left(\sum_{j\neq i}J_{ij}S_iS_j\right)\delta(H-NE) \tag{5}$$

One may calculate $s(E)$ for the Ising ferromagnet with the definition of metastability used here [7,8] and one finds

$$s(E) = \frac{(1-E)}{2}\log(1-E) - \frac{(1+E)}{2}\log(1+E) + E\log(2) + E\log(-E) \tag{6}$$

This function is shown in Fig. (1), its maximum value is at $E_m = -1/\sqrt{5}$, in the large $N$ limit this is where the metastable states are concentrated giving $s = \ln(\frac{1+\sqrt{5}}{2})$. Hence this rather simple modification of zero temperature Glauber dynamics gives us a simple one dimensional spin system with an extensive entropy of blocked configurations - as one would expect in a granular media.

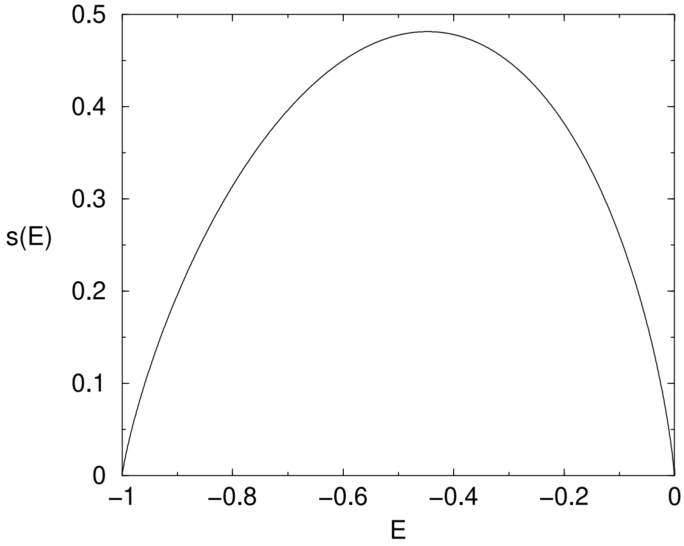


FIG. 1. The Edwards entropy per spin $s(E)$ of metastable states with respect to falling dynamics

We now proceed by constructing the simplest tapping dynamics imaginable on this system,

A falling or relaxational dynamics is defined as follows: a spin is chosen at random (random sequential update), if flipping this spin strictly lowers the energy of the system then the spin is flipped. If not another spin is chosen and the same criterion applied until all of the spins are locally stable. The system is now in a blocked or single spin flip metastable state.

Once the system is in a metastable state it is tapped, that is to say each spin is flipped in parallel with a probability $p$. Generally the system will no longer be in a metastable state and the falling dynamics is once again activated until again the system becomes blocked. Clearly the fact that the system is in a metastable state when tapped models a system where the relaxation process to a metastable state is much shorter than the time between taps. This is the simplest form of tapping dynamics as there is only one time scale (the time between taps) in the problem.

When this tapping dynamics is simulated one finds that the system, after some *ergodic* time, enters a regime where the energy per spin has becomes constant depending on $p$

$$E(t) \to E(p) \tag{7}$$

The simulation results are shown in Fig. (2), one sees here that $E(p)$ is a monotonically increasing function of $p$. This is rather reminiscent of the dependence of the asymptotic compactivity of vertically shaken granular media in the Chicago group's experiments [9] - as the tapping strength (parameterized by $\Gamma = a/g$ the ratio of the maximal vertical acceleration $a$ of the system through a sine cycle to the acceleration due to gravity $g$) is reduced, the compactivity increases (in the reversible part of the experiment). In numerical simulations of more realistic models of granular media, the enhancement of compactification on reducing the tapping strength has also been observed [10].

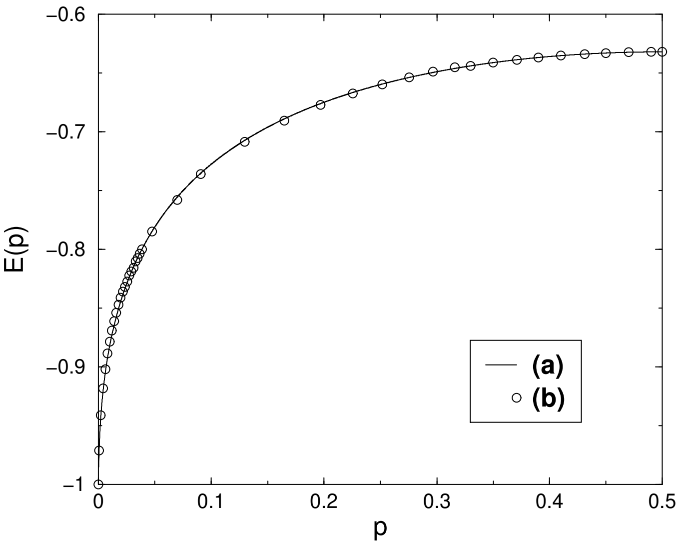


FIG. 2. Comparison between numerical simulations of tapping experiments (b) and the analytical result (a) obtained with (14).

## II. DETERMINATION OF THE STEADY STATE ENERGY

In order to determine $E(p)$ we shall consider a simple mean field theory for the falling dynamics of the ferromagnet on a Bethe lattice of connectivity $c$. If one concentrates on the bonds between connected sites $(i, j)$, we say that the bond is satisfied if it gives a negative contribution to the energy *i.e.* $-J_{ij}S_iS_j < 0$. In the case of the ferromagnet this contribution to the energy is clearly either 1 or $-1$. Hence, here a bond is satisfied/unsatisfied if its contribution to the energy is $-1/1$. For a given site define $x$ to be the difference between the number of unsatisfied and satisfied bonds. Hence $x$ is the local field on the spin at this site and $x \in -c,\ -c+2, \cdots, c-2,\ c$. If $x > 0$ then the spin can flip bringing about the change $x \to -x$. In addition we denote by $P(x, k)$ the probability that the site of interest has local field $x$ after a total of $k$ attempted random sequential spin flips under the zero temperature falling dynamics (that is to say the dynamics in between taps).

If we denote by $k$ the number of attempted spin flips during the falling dynamics, the possible changes at site $i$ between $k$ and $k + 1$ are

- The spin at site $i$ is chosen and $x > 0$, then the spin at site $i$ will flip and $x$ goes to $-x$.
- The spin at site $i$ is chosen and $x \leq 0$, then the spin at site $i$ can not flip and $x$ does not change.
- A neighbor of site $i$ with positive local field is chosen and so flips. In this case, $x$ goes to $x + 2$ or to $x - 2$ depending whether or not the bond with site $i$ was satisfied or not satisfied.
- A neighbor of site $i$ with negative or zero local field is chosen and so does not flip. In this case, $x$ does not change
- One chooses neither the spin at site $i$ nor any of its neighbors, and so $x$ stays $x$.

An schematic example of the falling dynamics for a system of connectivity $c = 3$ is shown in Fig. (3).

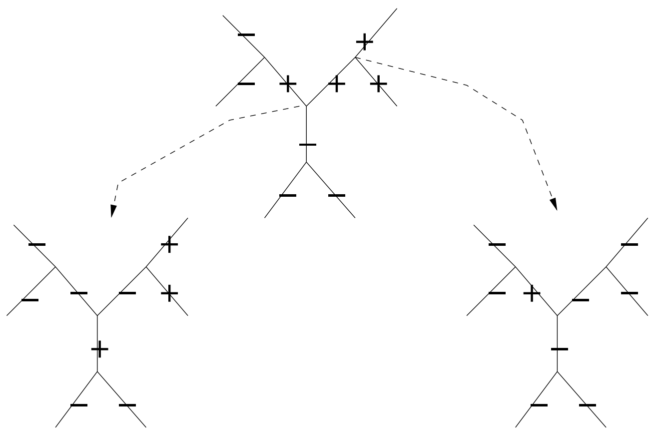

FIG. 3. An example of the falling dynamics for a spin system with $c = 3$ The symbols $+/-$ indicate unsatisfied/satisfied bonds. The broken arrowed lines start at the spin that is flipped and lead to the following configuration.

We define $f_+$ and $f_-$ the probabilities that a given spin can flip conditional on the fact that the bond with a given neighboring site is not satisfied or satisfied respectively, one can show that

$$f_\pm = \frac{\sum_{x>0} P(x)(c \pm x)}{\sum_x P(x)(c \pm x)} \tag{8}$$

Assuming that the distribution at every site is given by $P(x,k)$ and assuming independence between the values of $x$ from site to site (the mean field approximation) we obtain, and on taking the limit $N \to \infty$ and introducing the continuous time $\tau = k/N$,

$$\begin{aligned} \frac{dP(x)}{d\tau} &= \theta(-x)P(-x) + \theta(-x)P(x) - (c+1)P(x) \\ &+ P(x)\left(\frac{c+x}{2}(1-f_+) + \frac{c-x}{2}(1-f_-)\right) \\ &+ P(x+2)\,\frac{c+x+2}{2} f_+ + P(x-2)\,\frac{c-x+2}{2} f_- \end{aligned} \tag{9}$$

The case where $c = 2$ (the one dimensional case) is accessible to analytic solution and we proceed by defining

$$\begin{aligned} u(\tau) &\equiv P(-2,\tau) \\ v(\tau) &\equiv P(0,\tau) \\ w(\tau) &\equiv P(2,\tau) \end{aligned}$$

When $\tau \to \infty$ the system is in a metastable state and hence $w(\infty) = 0$. One finds also that

$$v(\infty) = (v(0) + 2w(0))\, e^{-\frac{4}{\lambda(0)+2}} \tag{10}$$

where $\lambda(0) = v(0)/w(0)$, and clearly $w(\infty) = 1 - v(\infty)$. If we consider an initial configuration where a spin differs from its neighbor with probability $a$, then the induced initial conditions are: $u(0) = (1-a)^2$, $v(0) = 2a(1-a)$ and $w(0) = a^2$.

The initial energy per spin $E_0$ is

$$E_0 = -1 + 2a \tag{11}$$

and one finds that the energy obtained after the first fall is

$$E_f = -1 + v(\infty) = -1 + 2ae^{-2a} \tag{12}$$

This mean field result for the one dimensional system can be shown to be exact by combinatorial methods [11].

We see that final value of the energy $E_f$ depends strongly on the initial configuration. Also the system does not fall into a state of energy corresponding to the maximum of $s(E)$ at $E_m = -1/\sqrt{5} \approx 0.44721$. Hence even if the total number of metastable states is dominated (in the statistical sense) by those of energy $E_m$, generic initial conditions always seem to lead to an energy lower than this [12]. In [13] the value of $E_f$ for a variety of zero temperature dynamics (sequential, greedy and reluctant) in the fully connected Sherrington Kirkpatrick (SK) spin glass model [14] was studied, similar behavior was found.

Tapping the system with tapping probability $p$, starting from the values $\{u(\infty), v(\infty), w(\infty)\}$, we obtain the new *tapped* values $\{u'(0), v'(0), w'(0)\}$. Defining $q \equiv (1-p)$, the relations between the old and *tapped* probabilities are:

$$\begin{aligned} u'(0) &= (1-3pq)\, u(\infty) + pq\, v(\infty) \\ v'(0) &= 2pq\, u(\infty) + (1-2pq)\, v(\infty) \\ w'(0) &= pq \end{aligned} \tag{13}$$

In the steady state regime under tapping, the probability $v_s(p)$ (the subscript $s$ indicating steady state) for sites to have zero local field is solution of the fixed-point equation:

$$\begin{aligned} v_s(p) &= (4pq + v_s(p)(1-4pq)) \\ &\quad \times \exp\left(-\frac{4pq}{4pq + v_s(p)(1-4pq)}\right) \end{aligned} \tag{14}$$

This equation can be solved numerically to calculate $E(p) = -1 + v_s(p)$, the result is shown in Fig. (2) in comparison with the numerical simulations which we see is excellent.

## III. THE STEADY STATE MEASURE

We now come to the question of what is the nature of the steady state regime under tapping. Edwards has proposed [3] that in the regime of slight tapping or shearing, the measure is a flat one over all blocked configurations satisfying the macroscopic constraints. Here the system is characterized by the average value of its energy and hence in its canonical form [3,15] the Edwards hypothesis suggests the measure

$$\langle O \rangle_{Edw} = \frac{\sum_{\mathcal{C}} O(\mathcal{C}) \exp\left(-\beta(p) H[\mathcal{C}]\right)}{Z} \tag{15}$$

where $\{\mathcal{C}\}$ is the ensemble of metastable configurations and

$$Z = \int dE N_{MS}(E) \exp(-\beta(p) N E) \tag{16}$$

There has been much recent study of the validity of this measure in non thermal systems and on balance has been shown to be quite successful (although there are examples where it does not hold). Clearly the Edwards' measure has the advantage of being the simplest and in a sense the most natural starting point. Unfortunately there is no general theory concerning its applicability and for the moment it has to be checked case by case.

In the context of the model here one may calculate a variety of observables with the Edwards measure analytically and also the numerical simulation of the tapping process is relatively straight forward [16].

The partition function for the system is then given by

$$Z = \mathrm{Tr} \prod_{i=1}^{N} \exp(\beta S_i S_{i+1}) \theta(S_{i-1} S_i + S_i S_{i+1}) \tag{17}$$

where the function $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x \geq 0$ enforces the metastability (each spin is stable or marginally stable in its local field). Performing a local change of variables to new Ising spins $\sigma_i = S_i S_{i+1}$ we obtain

$$Z = \mathrm{Tr} \prod_{i=1}^{N} \exp(\beta \sigma_i) \theta(\sigma_i + \sigma_{i+1}) \tag{18}$$

Hence we find that $Z = \mathrm{Tr}\ T^N$ where $T$ is the transfer matrix

$$T = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \tag{19}$$

with $a = \exp(\beta)$. From this we find that

$$E = \frac{a}{\sqrt{a^2+4}} \tag{20}$$

which simply determines $\beta$.

By a straight forward transfer matrix calculation one may calculate

- The internal energy fluctuation per spin $c = (\langle \mathcal{E}^2 \rangle - \langle \mathcal{E} \rangle^2)/N$ where $\mathcal{E}$ is the total internal energy, and consequently $\langle \mathcal{E} \rangle = NE$. One obtains

$$c = -E(1-E^2) \tag{21}$$

- The correlation functions $C(r) = \langle S_i S_{i+r} \rangle$ and $D(r) = \langle S_i S_{i+1} S_{i+r} S_{i+r+1} \rangle$. We find that the connected part of $D(r)$ is given by

$$D_c(r) = (1-E^2)\left(\frac{E+1}{E-1}\right)^r \tag{22}$$

and

$$C(r) = A\left(\frac{u}{t_0}\right)^r + (1-A)\left(\frac{1}{ut_0}\right)^r \tag{23}$$

with

$$A = \frac{u^2 t_0^2 - 1}{(u^2-1)(t_0^2+1)} \tag{24}$$

where $u = (a + \sqrt{a^2-4})/2$ and $t_0 = (a + \sqrt{a^2+4})/2$.

- The distribution of domain sizes. If $P(r)$ the probability that a given domain has length $r$ we find

$$\begin{aligned} P(r) &= \left(\frac{1+E}{1-E}\right)\left(\frac{-2E}{1-E}\right)^{r-2} & r \geq 2 \\ &= 0 & r < 2 \end{aligned} \tag{25}$$

Hence, the distribution of domain sizes is geometric for $n > 2$, the fact that $P(1) = 0$ is a consequence of metastability as a domain of length 1 would be a single spin surrounded by two antiparallel neighbors which is unstable.

One can compare the results of numerical simulations of tapping with the above theoretical ones. For a given value of the energy, let us say $E$, we have tapped the system with a strength $p$ such that in the steady state $E = E(p)$. The system is tapped for a sufficiently large number of times, say $t_s$, to ensure that the average of the internal energy $E(t)$ measured becomes stationary. Once in this steady state regime, the quantities of interest are measured over a measurement time (number of taps) $t_m = 10^5$ . The systems were of size of order $10^5$ spins and the results were also averaged over $N_s = 5000$ realizations. Hence, mathematically, the average value of a quantity $A$ is calculated, as one would in a Monte Carlo simulation of a thermal system, as

$$\langle A \rangle = \frac{1}{N_s}\sum_{i=1}^{N_s} \frac{1}{t_m} \sum_{t=t_s+1}^{t_s+t_{mes}} A_i(t) \tag{26}$$

In Fig.(4), we compare the fluctuation of the energy $c$ calculated using Edwards' measure, as a function of $E$, against those obtained from the simulations, the agreement is very good. For small values of $p$ the error bars in our measurements are very small and the agreement with Eq. (21) is excellent. For larger values of $p$ the error bars are large as the statistical fluctuations are larger, however from Fig.(4) we see that the value given by Eq. (21) is within the error bars.

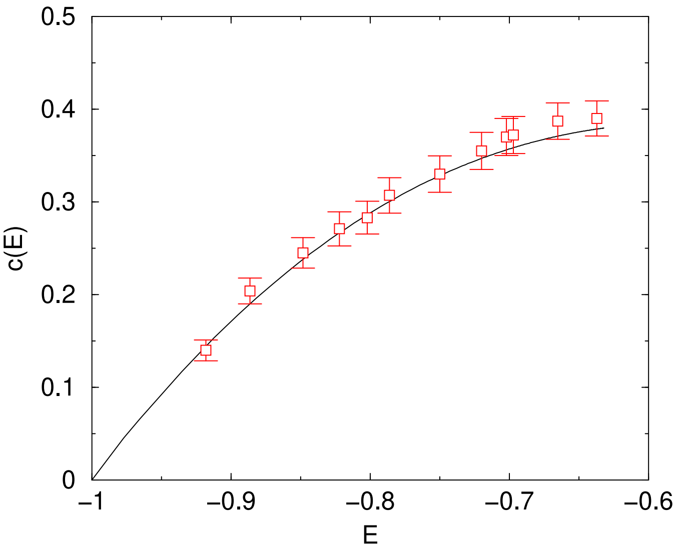


FIG. 4. The internal energy fluctuation $c(E)$ versus $E$. The solid line corresponds to the value obtained from Eq. (21) and the symbols are the results obtained from tapping simulations made on 5000 systems of 20000 spins

The correlation functions $C(r)$ and $D(r)$ have also been computed numerically. In Fig. (5) we have plotted the results in comparison with those expected from Eqs. (22) and (23). Here again, the comparison is excellent (remark that the agreement is better for lower energies, as again the statistical fluctuations due to the tapping are much smaller for low $p$ than for high $p$).

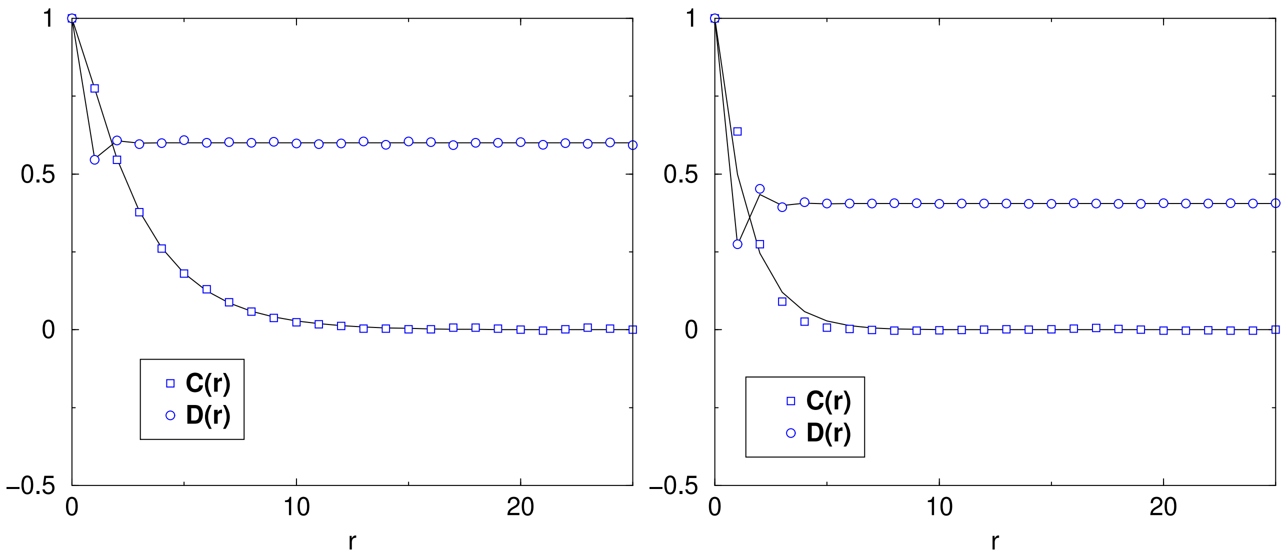


FIG. 5. Comparison between the expected $C(r)$ and $D(r)$ from the theoretical calculation with the results from numerical simulations for $E = -0.78$ (left) and $E = -0.63$ (right). The symbols are the results of the tapping experiments and the solid lines correspond to the one predicted by Eq.(23) and Eq.(22).

The distribution $P(r)$ of domain sizes, is shown in Fig.(6) and has a perfect exponential decay for $r \geq 2$. This guarantees that the result Eq. (25) is in perfect agreement with the simulations (as the energy $E$ is fixed).

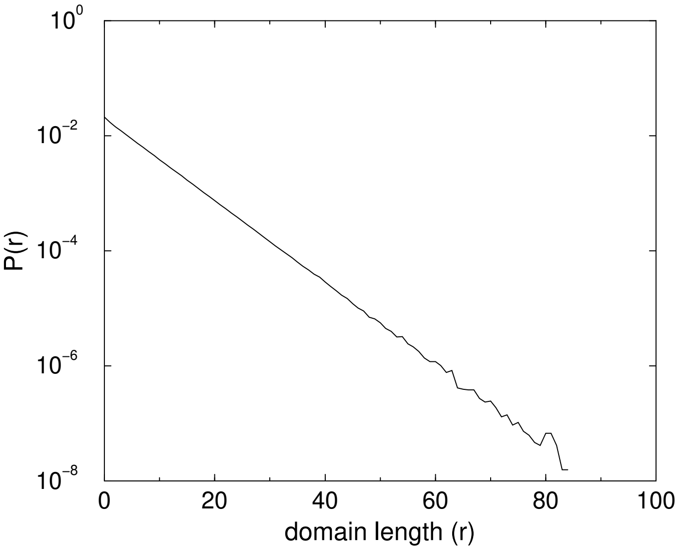


FIG. 6. Distribution of domain lengths from tapping simulations for $p = .1$. The vertical scale is logarithmic. The slope is $b = 0.165 \pm 0.001$, in excellent agreement with that obtained from Eq. (25).

## IV. CONCLUSION

We have seen that a slight and natural modification of the zero temperature dynamics of the one dimensional Ising model gives one a system with an extensive entropy of metastable states as one would expect in a granular system. A natural tapping dynamics yields steady state behavior similar to that of vertical tapping on granular media [9], hence this system is a simple testing ground to explore the behavior of mechanically perturbed complex systems. In the steady state regime we have seen that the flat measure proposed by Edwards [3] predicts certain macroscopic quantities of the steady state with a high degree of precision, even though no obvious form of detailed balance or ergodicity is present in the system. However it should be noted that the fact that the mean field calculation (14) appears to be exact also supports the validity of the geometric distribution of domains sizes Eq. (25) predicted by the Edwards measure.

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