The effect of avalanching in a two-species ripple model

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Abstract

This paper introduces a simple two-species ripple model with avalanching. The effect of the avalanching term is investigated numerically.

1 Introduction

Aeolian sand ripples are formed by the action of the wind on the sand bed in the desert or at the seashore. They are a few centimetres in wavelength with crests perpendicular to the prevailing wind direction. Early theoretical work on ripple formation (Bagnold 1941, Anderson 1987, 1990) has been followed more recently by models (Hoyle and Mehta 1999, Prigozhin 1999, Terzidis, Claudin and Bouchaud 1998, Valance 1999) that treat these ripples as being composed of two layers of sand grains: the 'bare surface' made up of relatively immobile grain clusters, and a layer of mobile grains moving on top. There are important differences in these approaches; in those of Claudin and Bouchaud (1998) and Valance (1999), for example, the nature of the interaction between the flowing and stuck layers differs significantly from that described in our approach (Hoyle and Mehta 1999), particularly in the area of *nonlocality*, which forms an important ingredient of our model. Recently, these ideas have been used to investigate the formation of sand dunes via continuum approaches very similar to our own (Sauermann, Kroy and Herrmann 2001). Finally, laboratory experiments on ripple formation (Stegner and Wesfreid 1999, Hansen et al 2001) have recently been devised, which enable the testing of theoretical hypotheses on systems more manageable than those provided by nature.

One of the important ingredients of our earlier model was its inclusion of bistability at the *angle of repose*. As is well known, a sandpile can either be static or flowing if its angle of repose is within a given range; the upper bound of this range is the *maximum angle of stability*, after which the sandpile avalanches - that is, spontaneous flow sets in. This phenomenon has also been represented in a discrete model of sandpile avalanches (Mehta and Barker, 2000), where it has been used for the interpretation of avalanche shapes found in experiment (Daerr and Douady 1999). Here we consider the effect of 'avalanching' in a two-species continuum model of sand ripples based on our earlier model (Hoyle and Mehta 1999). As explained above we use the term 'avalanching' to describe spontaneous flow, when sand grains are shed very rapidly from the immobile layer into the flowing layer, as the surface slope approaches the *maximum angle of stability* γ. Though this process is less dramatic in ripples than in sandpiles or dunes, and we do not expect to see discrete avalanche events with large sections of the sand surface falling away, this rapid grain shedding nonetheless turns out to have important consequences for the development and shaping of ripples.

2 Ripple equations

We consider two-dimensional sand ripples comprising a surface defined by the local height of clusters, $h(x,t)$, covered by a thin layer of flowing mobile grains whose local density is $\rho(x,t)$, where x is a horizontal space coordinate and t is time. The ripples evolve under the influence of a constant flux of saltating grains, which impact the sand bed at an angle β to the horizontal, knocking grains out of the bare surface, causing them to hop along the ripple surface and land in the layer of flowing grains. Granular relaxation mechanisms then smooth the ripple surface. We aim for a minimal model capturing the essential physics of ripple formation; thus the model equations used here are a simplification of those studied in Hoyle and Mehta (1999). They take the following form:

$$
\begin{aligned}
h_t &= D_h h_{xx} - \lambda\rho\left(|h_x| - \tan\delta\right) - \nu\left(|h_x| - \tan\delta\right)\left(\tan^2\gamma - h_x^2\right)^{-1/2} \\
&\quad - f(x,t), \\
\rho_t &= D_\rho \rho_{xx} + \lambda\rho\left(|h_x| - \tan\delta\right) + \nu\left(|h_x| - \tan\delta\right)\left(\tan^2\gamma - h_x^2\right)^{-1/2} \\
&\quad + \chi(\rho h_x)_x + \int_{-\infty}^{+\infty} p(a) f(x-a,t)\,da,
\end{aligned}
\tag{1}
$$

where D_h, D_ρ, λ, ν and χ are positive constants, δ is the *angle of repose* and $p(a)$ is the distribution of hop lengths a for grains knocked out of the ripple surface by the saltation flux, and where $f(x,t) = a_p J(\sin\beta + h_x \cos\beta)$ with a_p the average cross-sectional area of a sand grain and $J > 0$ a measure of the saltation flux intensity.

Naturally the flowing grain density can never be less than zero, so we also impose $\rho_t \geq 0$ anywhere that we have $\rho = 0$.

The rate of knocking out of grains by the saltation flux is assumed proportional to the component of the saltation flux perpendicular to the ripple surface (Hoyle and Woods 1997). The hopping out of the layer of clusters is modelled by the term $-f(x,t)$ in the equation for h_t, and grains landing in the flowing layer are modelled by the term $\int_{-\infty}^{+\infty} p(a) f(x-a,t)da$ in the equation for ρ_t (Hoyle and Mehta 1999). The hop length distribution $p(a)$ can be measured experimentally (Mitha et al 1986, Ungar and Haff 1987). Here we assume a normal distribution with mean $\bar{a}$ and variance σ^2.

Where the sand bed is shielded from the saltation flux by upwind ripple peaks it is said to be in *shadow*. No grains are knocked out of the surface in these regions and the equation for h_t is modified by neglecting the term $-f(x,t)$.

The remaining terms (Hoyle and Mehta 1999) describe the granular relaxation mechanisms that smooth the ripple surface (Mehta, Luck and Needs 1996). The term $D_h h_{xx}$ represents the diffusive rearrangement of clusters while the term $D_\rho \rho_{xx}$ represents the diffusion of the flowing grains. The flux-divergence term $\chi(\rho h_x)_x$ models the flow of surface grains under gravity. The current of grains is assumed proportional to the density of flowing grains and to their velocity, which in turn is proportional to the local slope to leading order (Hoyle and Woods, 1997). The term $\lambda\rho(|h_x| - \tan\delta)$ represents the tendency of flowing grains to stick onto the ripple surface at slopes less than the angle of repose, δ. The tilt and avalanching term $\nu(|h_x| - \tan\delta)(\tan^2\gamma - h_x 2)^{-1/2}$ models the tendency of clusters in the bare surface to shed grains into the flowing layer when tilted beyond the angle of repose, δ. The rate of shedding of grains becomes very large as slopes approach the maximum angle of stability γ - this is our representation of the phenomenon of *avalanching*.

The model is renormalised by setting $x \to x_0\tilde{x}$, $t \to t_0\tilde{t}$, $a \to x_0\tilde{a}$, $\rho \to \rho_0\tilde{\rho}$, $h \to h_0\tilde{h}$, where $x_0 = D_h/a_pJ\cos\beta$, $t_0 = D_h/(a_pJ\cos\beta)^2$, $h_0 = D_h\tan\gamma/a_pJ\cos\beta$, $\rho_0 = a_pJ\sin\beta/\lambda\tan\delta$. The renormalised equations are

$$\begin{aligned} h_t &= h_{xx} - \rho\frac{\tan\beta}{\tan\delta}\left(|h_x| - \frac{\tan\delta}{\tan\gamma}\right) - \hat{\nu}\left(|h_x| - \frac{\tan\delta}{\tan\gamma}\right)(1-h_x^2)^{-1/2} \\ &\quad -f(x,t), \\ \rho_t &= \frac{D_\rho}{D_h}\rho_{xx} + \frac{h_0}{\rho_0}\left(\rho\frac{\tan\beta}{\tan\delta}\left(|h_x| - \frac{\tan\delta}{\tan\gamma}\right) + \hat{\nu}\left(|h_x| - \frac{\tan\delta}{\tan\gamma}\right)(1-h_x^2)^{-1/2}\right) \\ &\quad +\hat{\chi}(\rho h_x)_x + \frac{h_0}{\rho_0}\int_{-\infty}^{+\infty} p(a)f(x-a)da, \end{aligned} \tag{2}$$

where the tildes have been dropped and where $f(x,t) = h_x + \tan\beta/\tan\gamma$, $\hat{\nu} = \nu t_0/h_0$ and $\hat{\chi} = \chi h_0/D_h$.

A steady solution $h_t = \rho_t = 0$ to these equations is $h = h_c$, where h_c is any constant, and $\rho = \rho_c \equiv 1 - \hat{\nu}\tan\delta/\tan\beta$. The stability of this solution can be investigated by setting $h = h_c + \hat{h}e^{\eta t + ikx}$ and $\rho = \rho_c + \hat{\rho}e^{\eta t + ikx}$, where $\hat{h}, \hat{\rho} \ll 1$ are constants and linearising in $\hat{h}$ and $\hat{\rho}$. To leading order the growth rate eigenvalues are $\eta = -h_0\tan\beta/\rho_0\tan\gamma$, which is associated with the relaxation of the flowing grain density ρ to its equilibrium value, and $\eta = \alpha k^2$, where

$$\alpha = -1 + \bar{a} - \frac{\rho_0}{h_0}\hat{\chi}\left(1 - \hat{\nu}\frac{\tan\delta}{\tan\beta}\right), \tag{3}$$

which is associated with ripple growth. For ripples to emerge, we must have $\alpha > 0$.

3 Numerical results

The effect of avalanching on ripple profile was studied numerically. The equations were integrated numerically using periodic boundary conditions and compact finite differences (Lele 1992, Hurlburt and Rucklidge 2000). Both *shadowing* and the requirement for ρ not to be less than zero were taken account of in the code. For the run with avalanching behaviour we chose $\bar{a} = 3.1$, $\sigma = 0.1$, $D_\rho/D_h = 1.0$, $h_0/\rho_0 = 20.0$, $\hat{\chi} = 0.1$, $\hat{\nu} = 0.25$, $\beta = 10^\circ$, $\delta = 30^\circ$, $\gamma = 35^\circ$. The length of the integration domain was $x_{max} = 166.6$. The angles were chosen to agree with observational evidence (Bagnold 1941, von Burkalow 1945, Sharp 1963), the ratio h_0/ρ_0 to ensure a thin layer of flowing grains in comparison to the ripple height, and the remaining parameters to allow ripple growth. The initial conditions for the dimensionless variables were $h = 1.0 + 0.1\eta_h$ and $\rho = 1 - \hat{\nu}\tan\delta/\tan\beta + 0.1(\eta_\rho - 0.5)$, where η_h and η_ρ represent white noise generated by random variables on $[0, 1)$ as a model of surface roughness. The output was rescaled back into physical coordinates using $D_h = 1.0$ and $\lambda = 10.0$. We also performed a run with avalanching switched off, $\hat{\nu} = 0$, but with all other parameters the same.

In the case where avalanching is included ripples developed from the initial surface roughness, and a process of ripple merger ensued (similar to that described in previous ripple models - e.g. Prigozhin 1999, Hoyle and Mehta 1999), leading to a final state with one large ripple in the periodic box. The final large ripple, shown in figure 1, has a fairly pointed crest with slope 30.9° on the windward side and -27.5° on the lee side. The stoss (windward) slope is long, dipping down low before building up to the crest, whereas the lee slope is short and relatively straight. The density of flowing grains, also shown in figure 1, has a maximum just before the ripple crest where grains have avalanched out of the ripple surface.

In the case where avalanching is suppressed ripples take much longer to grow, but develop somewhat higher surface slopes on the lee side of the crest (-28.5° compared with -27.5° for the avalanching ripple), with the high slope being maintained over a greater distance from the crest. The slopes on the stoss side are almost identical (30.8° compared with 30.9° for the avalanching ripple) as behaviour on this slope is dominated by the effect of the saltation flux impacting the sand bed. The higher slopes on the lee side are to be expected, since without avalanching, grains on steep slopes are shed much less readily into the flowing layer. Note also that in accordance with this the maximum density of flowing grains is slightly lower when avalanching is suppressed. Some shedding does still occur via the sticking term $-\lambda\rho(|h_x| - \tan\delta)$ in the equation for h_t, but this is much less efficient than the avalanching term and also depends on the local density of flowing grains, so that very few grains will be shed when the flowing grain density is low. The rapid shedding of grains during avalanching also leads to a more rapid evolution of the surface profile, and hence ripples grow more quickly when avalanching is allowed. The final ripple shape (time $t = 167.0$) of the nonavalanching ripple is shown in figure 2. Note that shallow waves also remain on the sand surface in contrast to the case of the avalanching ripple.

4 Conclusion

It is observed in nature (Sharp 1963) that ripple slopes are generally quite shallow. Our results show that even the subtle avalanching that occurs in ripples is enough to decrease surface slopes, and the avalanching term in our model could account for shallow ripples by maintaining surface gradients away from the maximum angle of stability γ.

In the current model, even when avalanching is turned off some shedding still occurs via the 'sticking' term that models the tendency of flowing grains to stick onto the ripple surface when the slope is less than the angle of repose. For simplicity we have retained this term even when the slope exceeds the angle of repose, which has the effect of turning the sticking term into a shedding term. It would be interesting to see what would happen if in addition to switching off avalanching we also switch this term off for slopes greater than the angle of repose. It is likely that we would then see an even greater divergence between avalanching and non-avalanching ripples.

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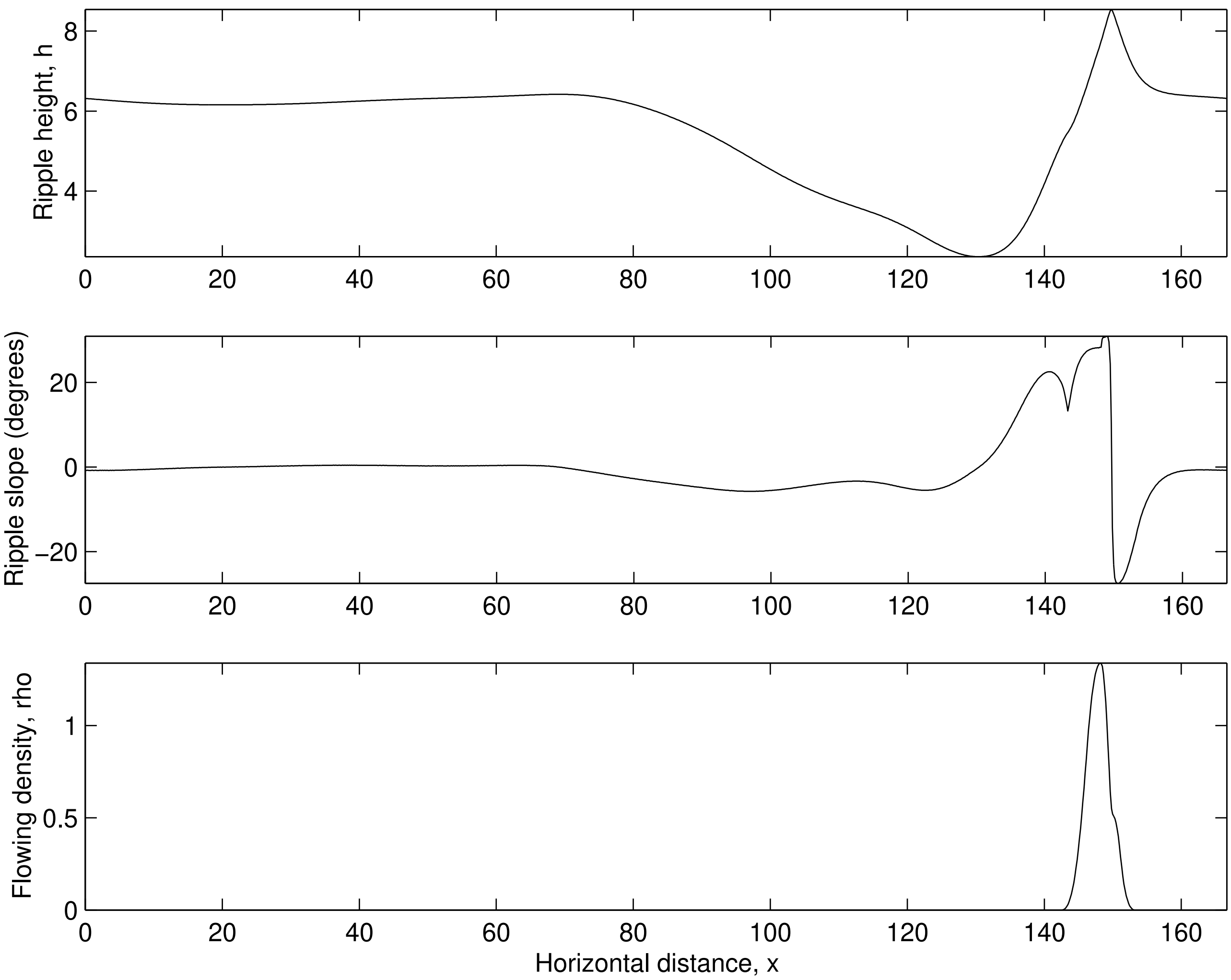

Figure 1: Profiles of the ripple surface height h, ripple surface slope $tan^{-1}(h_x)$ and flowing grain density ρ against horizontal distance x at time $t = 107.4$ for a ripple subject to avalanching. Parameter values are given in the text.

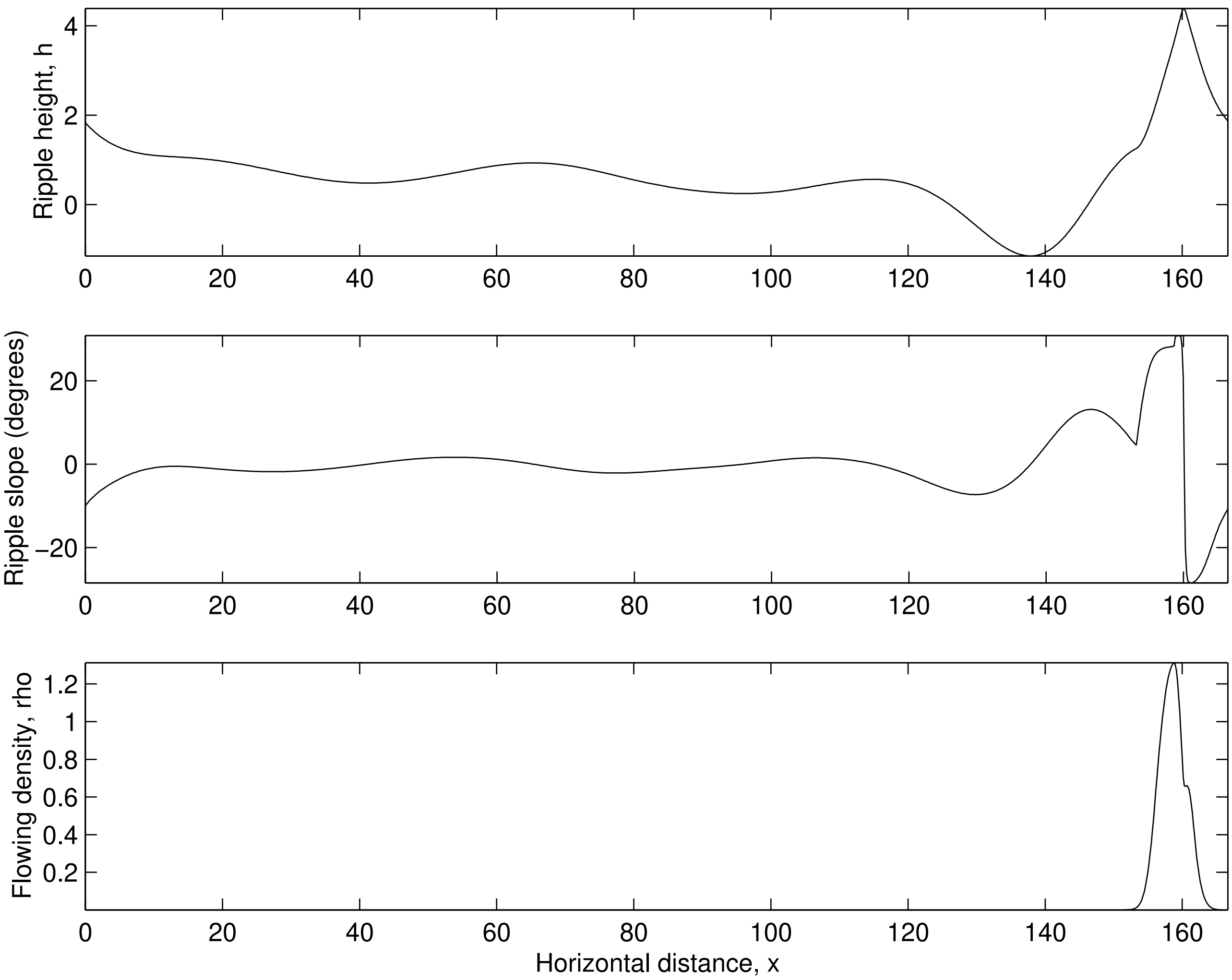

Figure 2: Profiles of the ripple surface height h, ripple surface slope $tan^{-1}(h_x)$ and flowing grain density ρ against horizontal distance x at time $t = 167.0$ for a ripple **not** subject to avalanching. Parameter values are the same as those for the ripple with avalanching except that the coefficient of avalanching ν is set to zero.