Force chain splitting in granular materials: from isotropic to anisotropic media

Matthias Otto[1], Jean-Philippe Bouchaud[1], Philippe Claudin[2], Dov Levine[3]

[1] *Service de Physique de l'Etat Condensé, CEA-Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France*
[2] *Laboratoire des Milieux Désordonnés et Hétérogènes, 4 place Jussieu - case 86, 75252 Paris Cedex 05, France*
[3] *Technion - Israel Institute of Technology, Physics department, Haifa 32000, Israel*

May 4, 2001

Abstract

We propose a mesoscopic model to investigate the influence of packing disorder and a global texture on the hyperbolic equations proposed in [1, 2]. The basic physical mechanism is the splitting of force chains (the characteristics of the hyperbolic equations) by packing defects, which is modelled analytically by a Boltzmann equation in analogy to light scattering in a turbid medium. We generalize the model previously presented for isotropic packings to anisotropic ones which are characterized by a global director $\mathbf{N}$, which may be constructed experimentally by avalanching. For a specific choice of the director $\mathbf{N}$ in a two-dimensional granular layer, we discuss the equations for the stress tensor, and derive a 'phase diagram' which contains regions corresponding to *i)* hyperbolic,*ii)* elliptic, and *iii)* a mixed type (corresponding to a coexistence of hyperbolic and elliptic behavior) of equations. Preliminary results indicate that the actual location of points in the phase diagram depends on the strength of the bias imposed on the scattering process by the director $\mathbf{N}$ as well as on the cone of scattering angles allowed.

1 Introduction

The study of hard, cohesionsless grains remains a subject of debate [3, 4, 5, 6]. A traditional approach, particularly useful in engineering applications, is based on the assumption that a granular material behaves on large enough length scales as an elastic (or elastoplastic) body. This hypothesis does not appear to be obvious, given the fact that microscopic grains may be considered as undeformable in the context of dense granular systems such as sandpiles. The latter implies that they can only be deformed by applying a compression larger than their Young modulus. The methods of conventional elasticity theory used in the former approach rest on notions such as a deformation field and an elastic energy [7], whose definitions appear to be problematic.

Therefore, in recent years, an alternative, 'stress-only' approach has been developed. Its fundamental assumption is a history-dependant relation between components of the stress tensor [1, 2, 5, 9]. The fundamental consequence is that the stress tensor obeys rather a *hyperbolic* equation than the elliptic equations of classical elasticity theory. Physically speaking, the stresses propagate along lines which are the characteristics of the hyperbolic equation. These lines have been argued to be related to the force chains which have indeed been observed in granular materials [12]. The 'stress-only' approach has been able to reproduce the pressure 'dip' underneath the apex of a sandpile [13, 14], and the pressure profile in silos [11].

An experimentum crucis to decide between hyperbolic and elliptic behavior in granular materials is to study the response function, i.e. the reponse of the stress of a localized force applied e.g at the top of layer of grains of height h. The hyperbolic equation predicts a response function (as a function of the horizontal distance away from the point where the force is applied) that show two peaks (in two dimensions) or a circular annulus (in three dimensions) [1]. By contrast, the elliptic equation gives rise to bell-shaped curve for the response function where the width scales as $\propto h$, the height of the layer. (See [20] for a more detailed discussion of experiments).

The present contribution is based on recent work of the authors [15] where a more thorough introduction to the subject is given than here. In this work, a numerical simulation and a model were presented which are both based on the stochastic propagation of force chains in a layer of granular material of finite height. In fact, force chains are supposed to scatter at packing defects which arise as the granular layer has been formed, according to the following rules: 1. an incoming force chains splits upon on arrival at a defect into two new outgoing force chains, 2. two incoming force chains merge at a defect two become one outgoing force chain. The simulations, which are discussed in detail by Dov Levine in the present volume, for layers of different heights, give the averaged response at the bottom of the layer to a point force on top of the layer as a function of the horizontal distance. It is remarkably close to the pure elastic response of a semi-infinite medium [7]. The analytical model transcribes rule 1.) while neglecting 2.) into a Boltzmann type equation for the probability distribution $P(f, \mathbf{n}, \mathbf{r})$ to have a force chain of value f at position $\mathbf{r}$ into direction $\mathbf{n}$. Complementing the numerical results, the analytical model may be cast at large scales into equations which have exactly the same structure as the equations for the stresses of an elastic body. Moreover, as the stress equations have not been derived from a free energy, the bound for the

Poisson ratio, $\nu_{el.} \leq 1/(D-1)$ in D dimensions, for elastic bodies is violated and is replaced by $\nu \geq 1/(D-1)$. The latter result might however be modified if rule 2.) is included again in the analytical model. An important assumption both of the simulation and the analytical theory presented in [15] is that the scattering of force chains is isotropic.

A remark on the formal equivalence of the equations for stresses derived from the force chain splitting model with the equations of isotropic elasticity theory is in order here. An important difference of the model proposed in [15] concerns the issue of boundary conditions. Taking e.g. a granular layer of infinite horizontal extension, in contrast to elasticity theory, the boundary conditions for the stress on the top and the bottom of the layer cannot be chosen independently, but have to chosen in a self-consistent manner. This fact is crucial in order to conserve the propagative character of the model being manifest in the orientation of force chains.

The main focus of the present work will be the analytical model presented in [15] and its extension to granular materials where the force chains splitting process is biased: force chains will be preferably scattered into a specific global direction (characterized by a director $\mathbf{N}$). This direction is a coarse grained description of a preferred texture that has been imposed on the granular material by the way the layer was constructed, e.g. by avalanches from a horizontally moving orifice. An important problem to address by a biased force chain splitting model will be eventually the pressure 'dip' underneath the apex of sandpiles which encode at least locally a preferred direction if constructed by an avalanching process.

In the next section the analytical model for force chain splitting in isotropic and anisotropic granular material is presented. In section 3, the hydrodynamic equations for the anisotropic case with a limit procedure reproducing the isotropic case are discussed. Next, in section 4, the characteristics of the stress equations for the anisotropic case are presented, and a 'phase diagram' is derived with regions corresponding to hyperbolic, elliptic, and a 'mixed' behavior (to be explained below) of the solutions for the stress tensor, before finally a conclusion is reached.

2 The Boltzmann equation for force chain splitting: biased and unbiased scattering

In [15], a Boltzmann equation for the chain-splitting model was derived for a granular medium which is strongly disordered. In the present work, we suppose that the scattering of force chains by defects is biased by a preferred orientation of the material.

The fundamental quantity considered will be $P(f, \mathbf{n}, \mathbf{r})$, the probability density to find an oriented force chain of intensity f in the direction $\mathbf{n}$ at point $\mathbf{r}$. This probability is defined with respect to an ensemble of different realizations of force chains for an assumed uniform spatial distribution of point defects (of density ρ_d), with same boundary conditions. In the spirit of previous models [1, 2] that give hyperbolic equations for the stresses, a mechanism of propagation is implemented, but now on the local level of force chains. In the analytical model, a pairwise merger of force chain to a single one will be neglected for simplicity. Then probability density

$P(f, \mathbf{n}, \mathbf{r})$ obeys the following equation

$$
\begin{aligned}
P(f_1, \mathbf{n}_1, \mathbf{r} + \mathbf{n}_1 dr) &= \left(1 - \frac{dr}{\lambda}\right) P(f_1, \mathbf{n}_1, \mathbf{r}) \\
&+ 2\frac{dr}{\lambda} \int d f' \int d f_2 \int d\mathbf{n}' \int d\mathbf{n}_2 P(f', \mathbf{n}', \mathbf{r}) \Psi(\mathbf{n}' \to \mathbf{n}_1, \mathbf{n}_2 | \mathbf{N}) \\
&\times \delta(f_1 \cos\theta_1 + f_2 \cos\theta_2 - f') \delta(f_1 \sin\theta_1 + f_2 \sin\theta_2) | \sin(\theta_1 - \theta_2)|, \qquad (1)
\end{aligned}
$$

where λ is the mean free path of force chains, and is of the order of $1/(\rho_d a^{D-1})$ in D dimensions. The length a represents the average size of a grain. The equation means the following: a force chain at some point $\mathbf{r} + \mathbf{n}_1 dr$ is either due to an unscattered force chain, which occurs with the probability that no scattering occurs times the probability that the same force chain existed at point $\mathbf{r}$ (given by the first term on the r.h.s. of the equation), or to a scattered force chain. The latter occurs with the probability given by the second term on the r.hs. of the equation: it is the sum with repect to all intensities and directions of the incoming (labelled by a prime) and the second outgoing force chains of the product of the probability for the incoming force chain to arrive at $\mathbf{r}$ times the probability of scattering $\frac{dr}{\lambda}\Psi(\mathbf{n}' \to \mathbf{n}_1, \mathbf{n}_2|\mathbf{N})$. The delta functions impose conservation of forces, the factor 2 accounts for the number of outgoing force chains, and the factor $|\sin(\theta_1 - \theta_2)|$ is a jacobian enforced by probablity conservation. The dependence of the scattering probability on $\mathbf{N}$ requires to consider the outgoing force chains separately. In the absence of $\mathbf{N}$ the outgoing force chains may be treated symmetrically, and one recovers the result for an isotropic medium [15].

Instead of solving Eq.(1), we multiply it by f and derive the following equation for a scalar local average force field $F(\mathbf{n}_1, \mathbf{r})$ per unit volume,

$$
\begin{aligned}
\lambda \mathbf{n}_1 \cdot \nabla_r F(\mathbf{n}_1, \mathbf{r}) &= -F(\mathbf{n}_1, \mathbf{r}) \\
&+ 2 \int d\mathbf{n}' \int d\mathbf{n}_2 F(\mathbf{n}', \mathbf{r}) \Psi(\mathbf{n}' \to \mathbf{n}_1, \mathbf{n}_2 | \mathbf{N}) \\
&\times \frac{1}{\cos\theta_1 - (\sin\theta_1 / \sin\theta_2)\cos\theta_2} + \frac{\lambda}{a} \mathbf{n} \cdot \mathbf{F_0}(\mathbf{r}), \qquad (2)
\end{aligned}
$$

where an extra term for an external body force density $\mathbf{F_0}(\mathbf{r})$ has been added. This equation is identical to the Schwarzschild-Milne equation for radiative transfer [29].

Now, we introduce physically relevant angular averages

$$
\begin{aligned}
p(\mathbf{r}) &= a \int d\Omega\, F(\mathbf{n}, \mathbf{r}) \qquad (3) \\
J_\alpha(\mathbf{r}) &= a \int d\Omega\; n_\alpha\, F(\mathbf{n}, \mathbf{r}) \qquad (4) \\
\sigma_{\alpha\beta}(\mathbf{r}) &= aD \int d\Omega\; n_\alpha n_\beta\, F(\mathbf{n}, \mathbf{r}), \qquad (5)
\end{aligned}
$$

where $\int d\Omega$ is a normalized integral over the unit sphere. The field p is the isostatic pressure, while $\mathbf{J}$ may be interpreted as the local average force chain intensity per unit surface, and σ is the stress tensor. Unlike the local average force which vanishes everywhere, $\mathbf{J} \neq \mathbf{0}$ due to the possibility to *orient* the force chains.

3 The hydrodynamic equations

We now proceed to obtain the equations governing the physically relevant fields introduced above, by calculating the zeroth, first, and second moment with respect to n_α of Eq.(2). The equations read as

$$\lambda \nabla \cdot \mathbf{J} = (c_1 - 1)p + c_2\, \sigma_{NN}, \tag{6}$$

$$\nabla_\beta \sigma_{\alpha\beta} = F_{0\alpha}. \tag{7}$$

$$\begin{aligned} \frac{\lambda}{(D+2)} & \left(\delta_{\alpha\beta} \nabla \cdot \mathbf{J} + \nabla_\alpha J_\beta + \nabla_\beta J_\alpha\right) = B_0\, \sigma_{\alpha\beta} \\ + & \quad \delta_{\alpha\beta} \left(B_1 \nabla \cdot \mathbf{J} + B_2\, \sigma_{NN}\right) + \mathbf{N}_\alpha \mathbf{N}_\beta \left(B_3 \nabla \cdot \mathbf{J} + B_4 \sigma_{NN}\right) \\ + & \quad B_5 \left(N_\alpha \sigma_{\beta\gamma} N_\gamma + N_\beta \sigma_{\alpha\gamma} N_\gamma\right) \end{aligned} \tag{8}$$

The notation $\sigma_{NN} = \mathbf{N} \cdot \sigma \cdot \mathbf{N}$ is used. The first two Eq.s, (6) and (7), are readily obtained upon averaging, while the third one, Eq.(8) is obtained using an Chapman-Enskog-type expansion of the local average force density $F(\mathbf{n}, \mathbf{r})$ in terms of the fields p, $\mathbf{J}$, and σ already given in [15]. Let us remark that Eq.(7) gives mechanical equilibrium as expected and is independent on the specific form of $\Psi(\mathbf{n}' \to \mathbf{n}_1, \mathbf{n}_2 | \mathbf{N})$.

The constants c_i and B_j appearing in Eq.(6) and (8) respectively are angular integrals involving the microscopic model for the scattering process, i.e. a specification of $\Psi(\mathbf{n}' \to \mathbf{n}_1, \mathbf{n}_2 | \mathbf{N})$, and are left for future publication. The present work does not depend on these details, but work is in progress that uses a particular model which is likewise used in simulations. If one neglects the dependance on $\mathbf{N}$ in the equations above, one recovers the simpler equations for force chain splitting in an isotropic granular medium [15].

As in the isotropic case, one would like to see if Eq.(8) can be cast into a form where the stress tensor $\sigma_{\alpha\beta}$ is a function of a pseudo-deformation tensor

$$u_{\alpha\beta} \propto \frac{1}{2} \left(\nabla_\alpha J_\beta + \nabla_\beta J_\alpha\right). \tag{9}$$

Eventually, one would then like to identify the anisotropic pseudo-elastic modulus tensor, using the relation

$$\sigma_{\alpha\beta} = \lambda_{\alpha\beta\gamma\delta} u_{\gamma\delta} \tag{10}$$

Similarly to conventional elasticity theory, the tensor $\lambda_{\alpha\beta\gamma\delta}$ obeys some symmetries such as

$$\lambda_{\alpha\beta\gamma\delta} = \lambda_{\beta\alpha\gamma\delta} = \lambda_{\alpha\beta\delta\gamma} = \lambda_{\beta\alpha\delta\gamma} \tag{11}$$

Please note however that unlike in classical anisotropic linear elasticity theory, in the present case,

$$\lambda_{\alpha\beta\gamma\delta} \neq \lambda_{\gamma\delta\alpha\beta} \tag{12}$$

except for certain cases imposed by the details of the scattering process. In the next section, a special case will studied which allows to derive a simple, but non-trivial equation for the stresses which supplemented by the mechanical equilibrium condition Eq.(7) opens a way to determine the stress tensor, or, put differently, the response function.

4 Towards the response function in anisotropic granular materials in D=2

Let us now illustrate the consequences of the force chain splitting model with biased scattering by studying the characteristics of stress equations in $D = 2$. In this case, the relation

$$u_{xx,zz} + u_{zz,xx} - 2u_{xz,xz} = 0 \tag{13}$$

holds for the pseudo-deformation tensor, which results simply from the symmetry $\alpha \leftrightarrow \beta$ of $u_{\alpha\beta}$ and the commutativity of partial derivatives. This relation is known as the compatibility condition in elasticity theory for the deformation tensor [7], but does not depend on a specific interpretation of the tensor in terms of real deformations. It will be used in the following to derive the missing equation (complementing the 2 Eq.s (7)) to complete the problem for the 3 independent entries of the stress tensor.

For simplicity we will restrict to the special case where the director is

$$\mathbf{N} = \frac{1}{\sqrt{2}} \left(\hat{x} + \hat{z} \right). \tag{14}$$

where $\hat{z}$ is the unit vector in the vertical direction pointing downwards from the top of the layer and $\hat{x}$ is the unit vector in the horizontal direction.

In addition to the symmetries mentioned above, one has in the case of Eq.(14) the following relations

$$\begin{aligned} \lambda_{zzzz} &= \lambda_{xxxx} && (15)\\ \lambda_{zzxx} &= \lambda_{xxzz} && (16)\\ \lambda_{zzxz} &= \lambda_{xxxz} && (17)\\ \lambda_{xzzz} &= \lambda_{xzxx} && (18) \end{aligned}$$

Using these symmetries, the compatibility relation is transformed into the following equation for the stress tensor:

$$\lambda_1 \Delta Tr(\sigma) - 2\lambda_2 \Delta \sigma_{xz} + \lambda_3 \left(\partial_x^2 \sigma_{xx} + \partial_z^2 \sigma_{zz} \right) + \lambda_4 \left(\partial_z^2 - \partial_x^2 \right) \left(\sigma_{xx} - \sigma_{zz} \right) = 0 \tag{19}$$

The coefficients are expressed in terms of the pseudo-elastic modulus tensor as follows:

$$\begin{aligned} \lambda_1 &= \lambda_{xzxz} && (20)\\ \lambda_2 &= \lambda_{xxxz} + \lambda_{xzxx} && (21)\\ \lambda_3 &= \lambda_{xxxx} + \lambda_{xxzz} && (22)\\ \lambda_4 &= \frac{\lambda_3 \lambda_1 - 2\lambda_{xxxz}\lambda_{xzxx}}{(\lambda_{xxxx} - \lambda_{xxzz})} && (23) \end{aligned}$$

The problem is now to solve Eq.(19) together with the equation for the equilibrium of forces Eq.(7) setting external body forces to zero for simplicity. This can be done going from the real space (z, x) to Fourier space (ω, q). The three equations for the stress tensors are then written in matrix notation $\hat{\mathbf{A}}\mathbf{X}$ where $\mathbf{X} = (\sigma_{zz}, \sigma_{xx}, \sigma_{xz})$

is a vector and $\hat{\mathbf{A}}$ a 3 by 3 matrix. A non-trivial solution occurs if $det(\hat{\mathbf{A}}) = 0$. Introducing a parameter X for the relation between ω and q

$$q = X\omega \tag{24}$$

the equation for the determinant becomes the following:

$$X^4 + bX^3 + 2(1-a)X^2 + bX + 1 = 0 \tag{25}$$

where

$$a = \frac{2\lambda_4 - \lambda_3}{\lambda_1 + \lambda_4} \tag{26}$$

$$b = \frac{2\lambda_2}{\lambda_1 + \lambda_4} \tag{27}$$

the 2 parameters which characterize the solutions of the equation. They may be parametrized as follows:

$$X_{\pm\pm\pm} = \frac{1}{4}\left(-b \pm \sqrt{b^2+8a} \pm \sqrt{(b \pm \sqrt{b^2+8a})^2 - 16}\right) \tag{28}$$

The four roots actually occurring are X_{++-}, X_{+--}, X_{-++}, X_{--+}. For any solution X to Eq.(25), $1/X$ is also a solution. So the four roots are pairwise related,

$$X_{++-} = \frac{1}{X_{+--}}, \quad X_{-++} = \frac{1}{X_{--+}} \tag{29}$$

As both a and b are real, the independent parameters X_{++-} and X_{-++} may be either real, imaginary or complex, with interesting physical consequences.

In Fig. 1, a 'phase diagram' is shown for X_{++-}, X_{+--} displaying parameter regions (a, b) where the solutions for the stress tensor have a propagating (hyperbolic) behavior corresponding to real roots X, classically elastic (elliptic) hehavior corresponding to imaginary roots, and a 'mixed (hyperbolic plus elliptic) behavior corresponding to complex roots. The diagram for the pair X_{-++}, X_{--+} is obtained by the mirror image of Fig. 1 with respect to the a axis. Where in the phase diagram are the actual roots of Eq.(25) ? To answer this question the scattering cross section $\Psi(\mathbf{n'} \to \mathbf{n}_1, \mathbf{n}_2|\mathbf{N})$ needs to be specified explicitly. Work in this direction is in progress. At present, the appearance of real roots indicating a hyperbolic nature of the equations cannot be ruled out. The location of points in the phase diagram depends on the strength of the bias imposed by $\mathbf{N}$ and microscopic details of the scattering model used, such as the cone of scattering angles allowed. The roots will finally be used to construct a solution for the response function. In the hyperbolic region, the response function will have a two-peak shape, whereas in elliptic region, a bell-shaped curve is expected. In the 'mixed' region, the response function should have a two-peak structure with widened peaks. Work on this is currenly in progress.

5 Conclusion

The force chain-splitting model for an isotropic, strongly disordered granular medium introduced in [15] gives rise to a pseudo-elastic response function, both determined

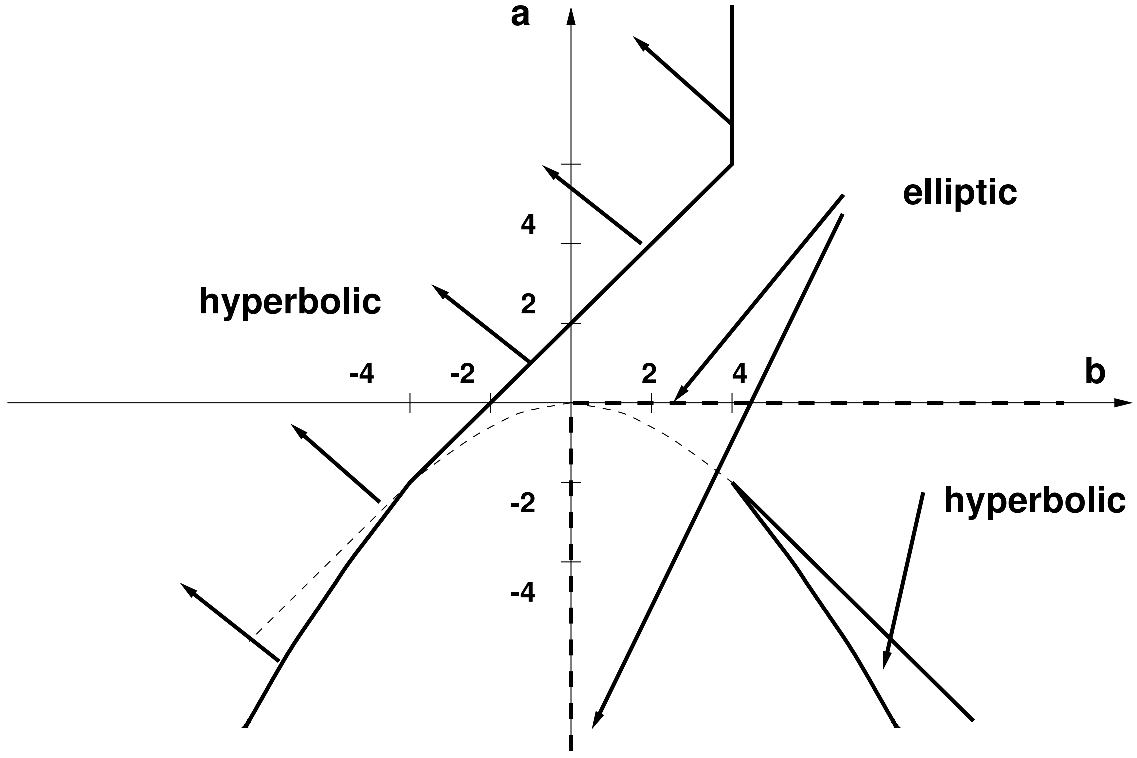

Figure 1: Sketch for the phase diagram for the solutions X_{++-}, X_{+--} of Eq.(25) in the (a, b)-plane. The hyperbolic and elliptic regions corresponding to real and imaginary roots X respectively are denoted explicitly, the remaining region gives complex roots ('mixed' behavior). The phase boundaries are formed by the linear curves $a = \pm b + 2$ and the parabola $a = -b^2/8$. The elliptic region is given by $b \geq 0$ and $a \leq 0$.

from simulational results and from the pseudo-elastic equations derived from the Boltzmann equation, which is in good agreement with experimental findings [20]. Despite the formal equivalence of the hydrodynamic equations with elasticity theory, a difference remains as far as boundary conditions are concerned. The propagative character of the Boltzmann equation is retained in the sense that boundary conditions for the stress tensor are to be specified self-consistently on both sides of the system. This point is particularly illuminated by simulationsin [15] where a point force applied on top of the layer gives a a distribution of force chains on the bottom, but also on the top by the fact that a few force chains go up to the top layer. When extended to an anisotropic medium characterized by global director $\mathbf{N}$, a whole 'phase diagram' is found which allows for hyperbolic (propagative), elliptic (elastic), and 'mixed' (hyperbolic and elliptic) behavior for stresses. It remains to be seen by studying a detailed microscopic scattering process, which points in the phase diagram actually occur, and in particular, on which parameters a possible crossover between a hyperbolic and elliptic character of the equations for the stresses depends.

Acknowledgments

We wish to thank M. Cates, J.P. Wittmer, E. Clément, J. Duran, G. Grest, A. Kamenev, E. Kolb, J.M. Luck, Y. Nahmias, G. Ovarlez and G. Reydellet for very useful discussions. D.L. acknowledges support from Israel Science Foundation grant 211/97 and from U.S. - Israel Binational Science Foundation grant 1999235. M.O. is supported by a DFG research fellowship.

References

[1] J.-P. Bouchaud, M.E. Cates, and P. Claudin, J. Phys. (France) I **5**, 639 (1995).

[2] J.P. Wittmer, P. Claudin, M.E. Cates and J.-P. Bouchaud, Nature **382**, 336 (1996); J.P. Wittmer, P. Claudin, M.E. Cates, J. Phys. (France) I **7**, 39 (1997).

[3] P.-G. de Gennes, Physica A **261**, 293 (1998).

[4] S.B. Savage, in *Physics of Dry Granular Media*, H.J. Herrmann, J.P. Hovi and S. Luding, Eds., NATO ASI, 25 (1997).

[5] J.-P. Bouchaud, P. Claudin, M.E. Cates, J.P. Wittmer, in *Physics of Dry Granular Media*, H.J. Herrmann, J.P. Hovi and S. Luding, Eds., NATO ASI, 97 (1997).

[6] M.E. Cates, J.P. Wittmer, J.-P. Bouchaud and P. Claudin, Phil. Trans. Roy. Soc. Lond. A **356**, 2535 (1998).

[7] L. Landau and E. Lifshitz, *Elasticity theory*, Pergamon, New York (1986).

[8] G. Combe and J.-N. Roux, Phys. Rev. Lett. **85**, 3628 (2000).

[9] P. Claudin, Ann. de Physique, **24**, 1 (1999).

[10] R.M. Nedderman, *Statics and Kinematics of Granular Materials* Cambridge University Press (1992).

[11] L. Vanel, P. Claudin, J.-P. Bouchaud, M.E. Cates, E. Clément, and J.P. Wittmer, Phys. Rev. Lett. **84**, 1439 (2000).

[12] M.E. Cates, J.P. Wittmer, J.-P. Bouchaud and P. Claudin, Phys. Rev. Lett. **81**, 1841 (1998).

[13] J. Šmíd, and J. Novosad, Proc. Powtech. Conference 1981, Ind. Chem. Eng. Symp. **63**, D3V 1 (1981); R. Brockbank, J.M. Huntley and R.C. Ball, J. Phys. (France) II **7**, 1521 (1997).

[14] L. Vanel, D.W. Howell, D. Clark, R.P. Behringer and E. Clément, Phys. Rev. E **60**, R5040 (1999)

[15] J.P. Bouchaud, P. Claudin, D. Levine, and M. Otto, Eur. Phys. J E **4**, 451 (2001).

[16] C.-h. Liu, S.R. Nagel, D.A. Schecter, S.N. Coppersmith, S. Majumdar, O. Narayan and T.A. Witten, Science **269**, 513 (1995); S.N. Coppersmith, C.-h. Liu, S. Majumdar, O. Narayan and T.A. Witten, Phys. Rev. E **53**, 4673 (1996). See also S. Luding, preprint.

[17] J. Geng and R.P. Behringer, G. Reydellet and E. Clément, preprint.

[18] S. Nagel et al., in preparation.

[19] V. Tkachenko and T.A. Witten, Phys. Rev. E **60**, 687 (1999); see also S.F. Edwards and D.V. Grinev, Phys. Rev. Lett **82**, 5397 (1999).

[20] G. Reydellet and E. Clément, Phys. Rev. Lett. **86**, 3308 (2001).

[21] E. Clément, G. Reydellet, L. Vanel, D.W. Howell, J. Geng and R.P. Behringer, XIIIth Int. Cong. on Rheology, Cambridge (UK), vol. 2, 426 (2000).

[22] J.-J. Moreau, in the proceedings of the *colloque 'Physique et mécanique des matériaux granulaires'*, Champs-sur-Marne (France), 199 (2000).

[23] M. da Silva and J. Rajchenbach, Nature **406**, 70 (2000).

[24] S.F. Edwards, Physica A **249**, 226 (1998).

[25] P. Dantu, Ann. des Ponts et Chaussées **4**, 144 (1967).

[26] see also the numerical simulations of F. Radjai, D.E. Wolf, M. Jean, J.J. Moreau, Phys. Rev. Lett. **80**, 61 (1998), and references therein.

[27] V.M. Kenkre, J.E. Scott, E.A. Pease and A.J. Hurd, Phys. Rev. E **57**, 5841 (1998); J.E. Scott, V.M. Kenkre and A.J. Hurd, Phys. Rev. E **57**, 5850 (1998).

[28] P. Claudin, J.-P. Bouchaud, M.E. Cates and J.P. Wittmer, Phys. Rev. E **57**, 4441 (1998).

[29] M.C.W. van Rossum and Th.M. Nieuwenhuizen, Rev. Mod. Phys. **71**, 313 (1999).

[30] see: J. Duran, *Sables, poudres et grains*, Eyrolles Sciences, Paris (1997), H. Makse, N. Gland, D. L. Johnson, L. Schwartz, cond-mat/9911110.

[31] E. Amic, J.-M. Luck and Th. Nieuwenhuizen, J. Phys **A29**, 4915 (1996).