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SMR.1348 - 20

**SECOND EUROPEAN SUMMER SCHOOL on
MICROSCOPIC QUANTUM MANY-BODY THEORIES
and their APPLICATIONS**

(3 - 14 September 2001)

HYPERSPHERICAL HARMONIC METHODS FOR STRONGLY INTERACTING SYSTEMS: A SUMMARY AND NEW DEVELOPMENTS PART III

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HH expansion in momentum space

Let $\vec{y}_1, \dots, \vec{y}_A$ be a set of Jacobi coordinates and $\hbar\vec{k}_1, \dots, \hbar\vec{k}_A$ the corresponding(*) Jacobi variables in momentum space.

From $(H-E)|\psi\rangle = 0$ we get

$$\langle \vec{k}_1 \dots \vec{k}_A | H-E | \vec{k}'_1 \dots \vec{k}'_A \rangle \langle \vec{k}'_1 \dots \vec{k}'_A | \psi \rangle = 0 ,$$

where the integration over all the $\vec{k}'$s is understood.

It holds the following relation (see Avery's book):

$$e^{i \sum_{j=1,A} \vec{k}_j \cdot \vec{y}_j} = e^{i \vec{k}_A \cdot \vec{y}_A} \frac{(2\pi)^{3/2}}{(k r)^{D/2-1}} \sum_{[G]} i^G \mathcal{Y}_{[G]}(\Omega) \mathcal{Y}_{[G]}(\Omega_k) J_{\ell+1/2}(kr)$$

where Ω and Ω_k are the sets of corresponding hyperangles, $D = 3N = 3(A-1)$, $\ell = G+(D-3)/2$, $\hbar\vec{k}_A = \vec{P}_{tot}$, $\vec{y}_A = \vec{R}$ and

$$k^2 = k_1^2 + \dots + k_N^2 .$$

Let $\psi(1, \dots, A) = \dfrac{e^{i \vec{K}_{tot} \cdot \vec{R}}}{(2\pi)^{3/2}} \sum_{[G]} \mathcal{Y}_{[G]}(\Omega)\, u_{[G]}(r)$

then $$\langle \vec{k}'_1, \dots, \vec{k}'_A | \psi \rangle = \int d\vec{y}_1 \dots d\vec{y}_A \frac{e^{-i \sum_1^A \vec{k}'_j \cdot \vec{y}_j}}{(2\pi)^{\frac{3}{2}A}} \left[\sum_{[G]} \mathcal{Y}_{[G]}(\Omega)\, u_{[G]}(r) \right] \frac{e^{i \vec{K}_{tot} \cdot \vec{R}}}{(2\pi)^{3/2}} =$$

(*) The expressions of $\hbar\vec{k}_1, \dots, \hbar\vec{k}_A$ in terms of $\vec{p}_1, \dots, \vec{p}_A$ are the same as those of $\vec{y}_1, \dots, \vec{y}_A$ in terms of $\vec{r}_1, \dots, \vec{r}_A$

$$= \delta(\vec{K}'_A - \vec{K}_{tot}) \sum_{[G]} \mathcal{Y}_{[G]}(\Omega_{k'})\, v_{[G]}(k')$$

with

$$v_{[G]}(k) = (-i)^G \int dr\, \frac{r^{D-1}}{(kr)^{D/2-1}} J_{\ell+1/2}(kr)\, u_{[G]}(r),$$

we get

$$\langle \vec{K}_1 \cdots \vec{K}_A | H - E | \vec{K}'_1 \cdots \vec{K}'_A \rangle \langle \vec{K}'_1 \cdots \vec{K}'_A | \Psi \rangle =$$

$$= \Big\{ \int d\vec{K}'_1 \cdots d\vec{K}'_A \Big[\big(T(\vec{K}'_1, \cdots, \vec{K}'_A) - E \big) \delta(\vec{K}_1 - \vec{K}'_1) \cdots \delta(\vec{K}_A - \vec{K}'_A) \Big] +$$

$$\delta(\vec{K}_A - \vec{K}'_A) \int d\vec{k}'_1 \cdots d\vec{k}'_N\, V(\vec{k}_1, \cdots \vec{k}_N; \vec{k}'_1 \cdots \vec{k}'_N) \Big\} \delta(\vec{K}'_A - \vec{K}_{tot}) \sum_{[G]} \mathcal{Y}_{[G]}(\Omega_{k'}) v_{[G]}(k')$$

$$= \delta(\vec{K}_A - \vec{K}_{tot}) \Big[\big(T(\vec{K}_1 \cdots \vec{K}_A) - E \big) \sum_{[G]} \mathcal{Y}_{[G]}(\Omega_k)\, v_{[G]}(k) +$$

$$+ \int d\vec{k}'_1 \cdots d\vec{k}'_N\, V(\vec{k}_1 \cdots \vec{k}_N; \vec{k}'_1 \cdots \vec{k}'_N) \sum_{[G]} \mathcal{Y}_{[G]}(\Omega_{k'}) v_{[G]}(k') \Big] = 0$$

By taking $\vec{K}_{tot} = 0$, we have to solve the following set of integral equations

$$\Big[T(\vec{k}_1 \cdots \vec{k}_N) - E \Big] \sum_{[G]} \mathcal{Y}_{[G]}(\Omega_k)\, v_G(k) +$$

$$+ \int d\vec{k}'_1 \cdots d\vec{k}'_N\, V(\vec{k}_1 \cdots \vec{k}_N; \vec{k}'_1 \cdots \vec{k}'_N) \sum_{[G]} \mathcal{Y}_{[G]}(\Omega_{k'}) u_{[G]}(k') = 0$$

The set of integral equations can be solved by standard numerical methods.

The relativistic kinetic energy

$$T = \sum_{i=1}^{A} \sqrt{\vec{p}_i^{\,2} c^2 + m_i^2 c^4} = \sum_{i=1}^{A} m_i c^2 + \sum_{i=1}^{A} \frac{\vec{p}_i^{\,2}}{2m_i} + \cdots$$

can be expressed in terms of $\vec{K}_1, \dots, \vec{K}_N$ since $\vec{K}_A = \vec{K}_{tot} = 0$.
As an example, in the case of a three-particle system of equal mass, it is

$$\vec{y}_3 = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3) \qquad \hbar\vec{K}_3 = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$$

$$\vec{y}_2 = \frac{1}{\sqrt{2}}(\vec{r}_2 - \vec{r}_1) \qquad \hbar\vec{K}_2 = \frac{1}{\sqrt{2}}(\vec{p}_2 - \vec{p}_1)$$

$$\vec{y}_1 = \sqrt{\frac{2}{3}}\left[\vec{r}_3 - \frac{1}{2}(\vec{r}_1 + \vec{r}_2)\right] \qquad \hbar\vec{K}_1 = \sqrt{\frac{2}{3}}\left[\vec{p}_3 - \frac{1}{2}(\vec{p}_1 + \vec{p}_2)\right]$$

so that

$$T = \hbar c \left[\sqrt{\frac{1}{6}K_1^2 + \frac{1}{2}K_2^2 + \frac{1}{\sqrt{3}}\vec{K}_1\cdot\vec{K}_2 + \frac{m^2c^2}{\hbar^2}} + \sqrt{\frac{1}{6}K_1^2 + \frac{1}{2}K_2^2 - \frac{1}{\sqrt{3}}\vec{K}_1\cdot\vec{K}_2 + \frac{m^2c^2}{\hbar^2}} \right.$$

$$\left. + \sqrt{\frac{2}{3}K_1^2 + \frac{m^2c^2}{\hbar^2}} \right] =$$

$$= 3mc^2 + \frac{\hbar^2}{2m}(K_1^2 + K_2^2) + \cdots$$

The adiabatic expansion

The so-called adiabatic approximation (*) represents an alternative efficient expansion basis for the w.f. The original idea of the adiabatic approximation was originally introduced by Born and Oppenheimer to calculating the structure of a diatomic molecule. For a fixed internuclear distance R the electronic w.f. and eigenvalue $V(R)$ are calculated. The eigenvalue $V(R)$ is then used to determine the vibrational and rotational levels of the molecule (for a discussion of the method see the paper of Kolos and Wolniewicz (°))
With

$$\psi(r,\Omega) = r^{-(D-1)/2} \; \Phi(r,\Omega)$$

the Schroedinger equation is as follows

$$\left\{ -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial r^2} + \frac{\Lambda^2(\Omega) - (D-1)(D-3)/4}{r^2} \right] + V(r,\Omega - E \right\} \Phi(r,\Omega) = 0$$

The Adiabatic Hyperspherical Harmonics (AHH) are the eigenfunctions of the following operator:

$$\left\{ -\frac{\hbar^2}{2M}\left[\frac{\Lambda^2(\Omega)}{r^2} - \frac{(D-1)(D-3)}{4r^2} \right] + V(r,\Omega) \right\} \Phi_m(r,\Omega) = U_m(r)\, \Phi_m(r,\Omega)$$

where m numbers the various eigenfunctions and the corresponding "eigenpotentials" $U_m(r)$.

(*) J.H. Macek, J. Phys. B1 (1968) 831

M. Fabre de la Ripelle, C.R. Acad. Sci. Paris, 274 (1972) 104

(°) W. Kolos and L. Wolniewicz, Rev. Mod. Phys., 35 (1963) 473

The w.f. of the system can be expanded in terms of the HH functions

$$\Psi_N = r^{-(D-1)/2} \sum_{m=1}^{M} \phi_m(r,\Omega)\, u_m(r).$$

and then from the Schroedinger equation, the following set of coupled equations is obtained

$$-\frac{\hbar^2}{2M} u''_m(r) + \sum_{n=1}^{M} \left[B_{mn}(r)\, u'_n(r) + C_{mn}(r)\, u_n(r) \right] + (U_m(r) - E)\, u_m(r) = 0$$

where

$$B_{mn}(r) = 2\int d\Omega\, \phi^*_m(r,\Omega) \frac{\partial}{\partial r} \phi_n(r,\Omega)$$

$$C_{mn}(r) = \int d\Omega\, \phi^*_m(r,\Omega) \frac{\partial^2}{\partial r^2} \phi_n(r,\Omega)$$

The important point is how to calculate the AHH functions. A possibility is to expand the $\phi_m(r,\Omega)$ in the HH basis. More recently, the spline technique has been used to this aim, with very accurate results for atomic and μ-atoms systems. The application to triton and alpha particle using realistic potentials and the AHH basis has been done by Kievsky and Viviani (Few-Body Systems Suppl. 99 (1995) 1. The AHH functions were expanded in a number K of PHH functions and the w.f. was expanded in a number M of AHH elements. The results obtained show a rapid convergence with M.

See pag. 31

M	B(A=3) (MeV)	B(A=4) (MeV)
1	7.44	20.01
5	7.61	21.05
9	7.65	21.08
13	7.66	21.09

= reduced number of channels

- AV14 potential
- K = 48 for A = 3 and K = 81 for A = 4

As it can be seen by inspection of the table, just the first term of the AHH expansion is sufficient by alone to obtain a good estimate of the upper bound energy. The reason for that lies in the fact that the lowest eigenpotential ($m=1$) is the only one with an attractive part at medium interparticle distances, whilst the other eigenpotentials contain larger repulsions.

The extended HH expansion

This technique has been just briefly discussed for the helium atom.

Let again consider the case of a three-body system. It can be noticed that

$$r^{2n} P_n^{\ell_1+1/2,\, \ell_2+1/2}(\cos 2\phi_2) = \sum_{m=0}^{n} a_{nm}\, y_2^{2m}\, r^{2(n-m)}$$

- only even powers of $y_2 \sim r_{ij}$ enter the expansion
- the dependence of ψ on r_{ij}, for small values of this distance is, in general, linear
- odd powers of $\cos\phi_2$ can be expressed in terms of the even powers but an infinite expansion is required:

e.g. $$\cos\phi_2 = \sum_{i=0}^{\infty} a_i P_{2i}(\cos\phi_2), \quad a_i = \frac{(-)^{i+1}(4i+1)\Gamma(i-1/2)}{4\sqrt{\pi}(i+1)!} \to i^{-5/2}.$$

- Note that for breakup N-d reactions, when $r \to \infty$

$$\psi \to A(\phi_2)\, e^{ikr}, \quad A(\phi_2) \simeq \cos\phi_2 \quad \text{when } \phi_2 \to \pi/2$$

EHH basis

$$\Phi^{EHH}_{\alpha, n_2 \ldots n_N, \lambda_2 \ldots \lambda_N}(i,j,k,\ldots) = (\cos\phi_2)^{\lambda_2} \cdots (\cos\phi_N)^{\lambda_N} \cdot \Phi^{HH}_{\alpha, n_2 \ldots n_N}(i,j,k\ldots),$$

where λ_j can be either 0 or 1. Let ψ^{HH} an ψ^{EHH} the corresponding antisymmetrized functions, then for $A=3$

$$\psi_N = r^{-5/2} \sum_{\alpha=1}^{N_c} \left[\sum_{n_2=0}^{N_\alpha} w_{\alpha,n_2}(r)\, \psi^{HH}_{\alpha,n_2,0} + \sum_{n_2=0}^{j} w_{\alpha n_2}(r)\, \psi^{EHH}_{\alpha,\alpha_2 1} \right]$$

AV14 Potential

	$N_c = 4$		$N_c = 8$	$N_c = 12$
G	HH	EHH_2	EHH_2	EHH_2
$\times$	–	–	7.518	7.677
0	–	2.961	7.617	7.677
2	–	4.737	7.650	7.678
4	–	6.180	7.656	7.678
6	0.750	6.747	7.658	
8	2.803	6.991	7.659	
10	4.432	7.131	7.660	
12	5.635	7.230	7.660	
14	6.152	7.278	7.660	
16	6.532	7.309		
20	6.973	7.345		
24	7.173	7.361		
28	7.262	7.367		
36	7.339	7.373		
40	7.353	7.375		
48	7.367	7.375		
52	7.370			
60	7.372			
80	7.374			
PHH	7.375		7.660	7.678

The CHH Expansion

The rate of convergence of the HH expansion results to be extremely slow when the particle interaction contains large repulsion at small distances

The binding energies of the three– and four–nucleon systems can be calculated with accuracy only including a large number of expansion terms (≈ 600 for $A = 3$ and ≈ 3000 for $A = 4$)

Problem how to extend the calculation for

- 3N potential
- Bound states of larger ($A > 4$) systems
- Scattering states

Correlated HH–spin–isospin basis

$$\Phi^{CHH}_{\alpha,n_2,\ \ ,n_N}(i,j,k,\ldots) = F_\alpha(r_{ij}, r_{ik},\ldots)\Phi^{HH}_{\alpha,n_2,\ \ ,n_N}(i,j,k,\ldots)$$

F_α= correlation factor For $A = 3$.

$$\begin{aligned} F_\alpha &= f_\alpha(r_{jk})g_\alpha(r_{ij})g_\alpha(r_{ik}) && \text{CHH} \\ F_\alpha &= f_\alpha(r_{jk}) && \text{PHH} \end{aligned}$$

For $A = 4$

$$F_\alpha = f_\alpha(r_{ij})g_\alpha(r_{ik})g_\alpha(r_{jk})g_\alpha(r_{im})g_\alpha(r_{jm})h_\alpha(r_{km}) \ ,$$

The correlation functions

- included to accelerate the convergence
- at small interparticle distance F_α takes into account the correlations induced by the strong repulsion of V_{NN}
- choice of the correlation: solution of a two–body Schroedinger equation

The relative motion of the pair j,k in the angular–spin–isospin state $\beta \equiv \ell_\beta,\, S_\beta,\, T_\beta$ is given by

$$\sum_{\beta'}\Big\{-\frac{\hbar^2}{M}\Big[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\ell_\beta(\ell_\beta+1)}{r^2}\Big]\delta_{\beta\beta'}+V_{\beta\beta'}(r)+\lambda_{\beta\beta'}(r)\Big\}f_{\beta'}(r)=0$$

The term $\lambda_{\beta\beta'}(r)$ has the role to simulate the effect of the other particles on the pair

$$\lambda_{\beta\beta'}(r)=\Lambda_\beta \exp(-\gamma r)\delta_{\beta\beta'}$$

Boundary condition $f_\beta(r)/r^{\ell_\beta} \to 1$ for $r \to \infty$

$\gamma =$ variational parameter

Λ_β determined from the boundary condition

Antisymmetrization of the w.f.

$$|\mu) \equiv \psi^{CHH}_{\alpha,n_2,\ldots,n_N} = \sum_{i,j,k,\ldots} \Phi^{CHH}_{\alpha,n_2,\ldots,n_N}(i,j,k,\ldots)$$

$$\Psi = \rho^{-\frac{3N-1}{2}} \sum_{\mu=1}^{M} u_\mu(\rho)|\mu)$$

$$\sum_{\mu'=1}^{M}\left[A_{\mu,\mu'}(\rho)\frac{d^2}{d\rho^2}+B_{\mu,\mu'}(\rho)\frac{d}{d\rho}+C_{\mu,\mu'}(\rho)-E\;N_{\mu,\mu'}(\rho)\right]u_{\mu'}(\rho)=0\;,$$

E = total energy of the system.

NUMERICAL INTEGRATION

$$N_{\mu,\mu'}(\rho)=\int d\Omega_N\;(\mu|\mu')\;,\quad A_{\mu,\mu'}(\rho)=-\frac{\hbar^2}{M}N_{\mu,\mu'}(\rho)\quad \text{etc.}$$

^{3}H binding energy

	S3		AV14	
G	HH	PHH	HH	PHH
0	0.358	4.189	–	3.233
2	–	5.317	–	5.626
4	2.130	6.652	–	7.255
6	4.606	6.698	–	7.335
8	5.208	6.699	2.803	7.373
10	5.807	6.700	4.432	7.374
12	6.238	6.700	5.635	7.375
16	6.501		6.532	
20	6.620		6.973	
30	6.689		7.283	
40	6.697		7.353	
50	6.698		7.362	

α–particle binding energy $\longrightarrow$ L. E. Marcucci

$\boxed{\text{Scattering states}}$

Decomposition of Ψ in **internal** and **asymptotic** part.

$$\Psi_{L_0S_0J} = \Psi_C + \Phi_{L_0S_0J}$$

The internal part describes the system when the particles are all close each other.

$$\Psi_C = \rho^{-\frac{3N-1}{2}} \sum_{\mu} u_{n\alpha}(\rho)\psi^{\mathrm{CHH}}_{\alpha,n_2,\ldots,n_N}$$ $\rho \equiv r$

<u>*"Asymptotic" part* $(A=3)$</u>

$$\begin{aligned}\Phi_{L_0S_0J} &= \sum_{LS}\sum_{i=1}^{3}\{Y_L(\hat{r}_i)[\phi_d(j,k)s_i]_S\}_{JJ_z}[\Xi_d(j,k)t_i]_{T,T_z} \\ &\quad\times\left(\frac{F_L(pr_i)}{pr_i}\delta_{LL_0}\delta_{SS_0} + {}^J\mathcal{R}^{L_0S_0}_{LS}\frac{\widetilde{G}_L(pr_i)}{pr_i}\right)\end{aligned}$$

- L_0S_0 quantum numbers of the incident wave
- r_i = N–d distance
- $\phi_d(j,k)\times\Xi_d(j,k)$ = deuteron bound state w.f.
- F and G regular and irregular Coulomb functions
- p = relative $N-d$ wave number

$$\frac{3\hbar^2}{4M}p^2 - B_2 = E_{c.m.}$$

- $n-d$ case $\rightarrow$ spherical Bessel functions
- $\widetilde{G}_L(qr) = G_L(qr)(1-e^{-\beta r})^{2L+1}$; β variational parameter
- $\mathcal{R}_{L_0S_0,LS}$ R–matrix elements

Kohn variational principle

$$[{}^J\mathcal{R}^{SS'}_{LL'}] = {}^J\mathcal{R}^{SS'}_{LL'} - \langle\Psi_{L'S'J}|H-E|\Psi_{LSJ}\rangle$$

Solution

$$\sum_{\mu'}\Big(A_{\mu,\mu'}(\rho)\frac{d^2}{d\rho^2}+B_{\mu,\mu'}(\rho)\frac{d}{d\rho}+C_{\mu,\mu'}(\rho)-E\ N_{\mu,\mu'}(\rho)\Big)u_{\mu'}(\rho) = D_{\mu}(\rho)$$

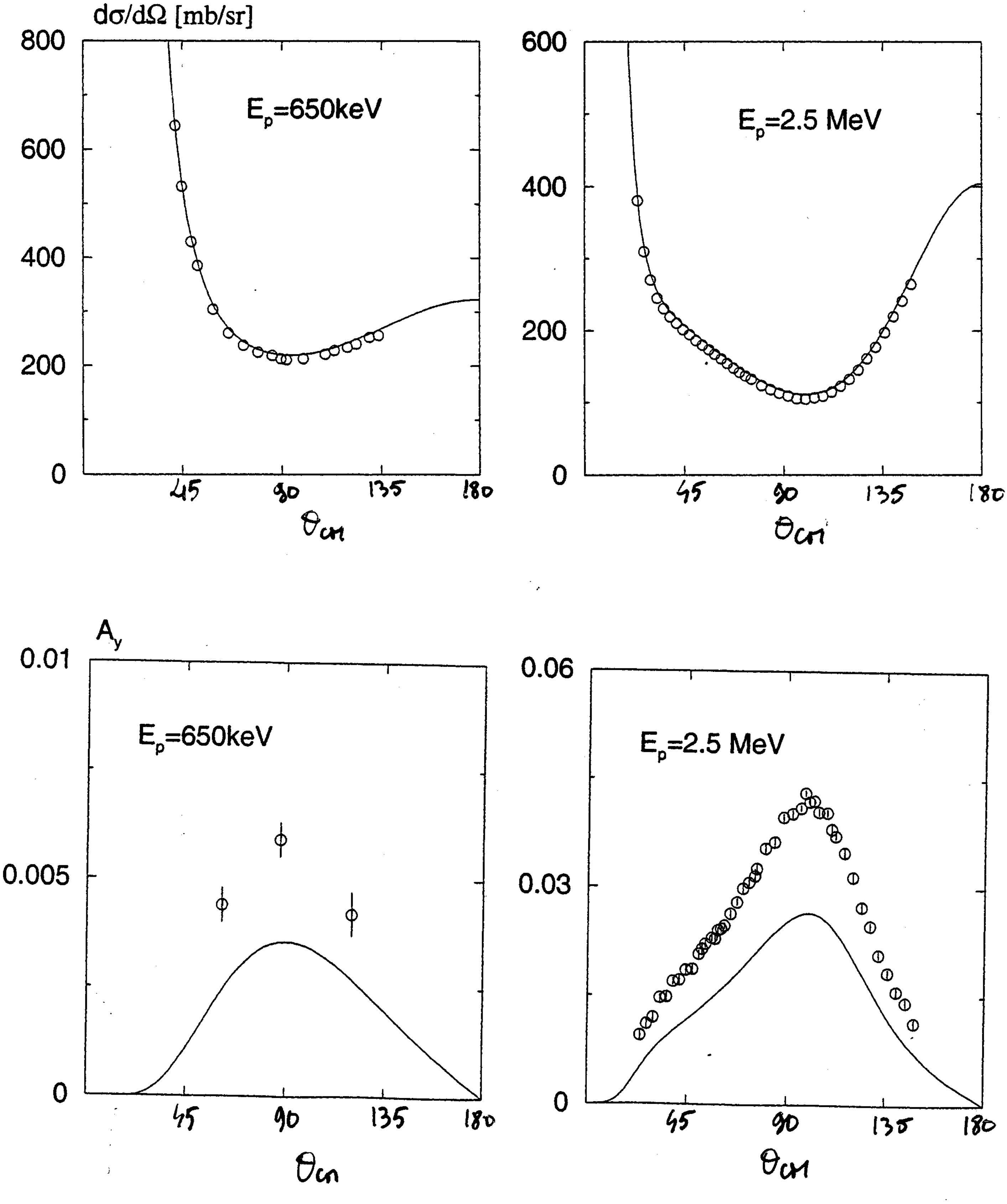
$n-d$ Reactance matrix – AV14 potential

N_c	G_α	${}^{J}\mathcal{R}^{SS'}_{LL'}$	$1^{\underline{st}}$ order	$\langle\Psi_{L'S'J}\mid\mathcal{L}\mid\Psi_{LSJ}\rangle$	$2^{\underline{nd}}$ order
8	4	${}^{1/2}\mathcal{R}^{\frac{1}{2}\frac{1}{2}}_{00}$	2.778	−0.003	2.775
		${}^{1/2}\mathcal{R}^{\frac{1}{2}\frac{3}{2}}_{02}$	0.821	+0.028	0.849
		${}^{1/2}\mathcal{R}^{\frac{3}{2}\frac{1}{2}}_{20}$	0.850	−0.001	0.849
		${}^{1/2}\mathcal{R}^{\frac{3}{2}\frac{3}{2}}_{22}$	62.07	+3.665	65.74
8	10	${}^{1/2}\mathcal{R}^{\frac{1}{2}\frac{1}{2}}_{00}$	2.753	+0.002	2.755
		${}^{1/2}\mathcal{R}^{\frac{1}{2}\frac{3}{2}}_{02}$	0.857	−0.008	0.849
		${}^{1/2}\mathcal{R}^{\frac{3}{2}\frac{1}{2}}_{20}$	0.847	+0.002	0.849
		${}^{1/2}\mathcal{R}^{\frac{3}{2}\frac{3}{2}}_{22}$	65.84	+0.316	65.52
12	10	${}^{1/2}\mathcal{R}^{\frac{1}{2}\frac{1}{2}}_{00}$	2.746	+0.001	2.747
		${}^{1/2}\mathcal{R}^{\frac{1}{2}\frac{3}{2}}_{02}$	0.854	−0.009	0.845
		${}^{1/2}\mathcal{R}^{\frac{3}{2}\frac{1}{2}}_{20}$	0.845	+0.000	0.845
		${}^{1/2}\mathcal{R}^{\frac{3}{2}\frac{3}{2}}_{22}$	65.88	−0.416	65.46

Comparison with FE results (Glöckle *et al*, 1998)

		FE	CHH		
J^{Π}	$\delta_{\Sigma\lambda}$	$j_{max}=6$	$\ell_1+\ell_2 \leq 2$	$\ell_1+\ell_2 \leq 4$	$\ell_1+\ell_2 \leq 6$
$\frac{1}{2}^+$	$\delta_{(3/2)2}$	-3.904	-3.899	-3.905	-3.905
	$\delta_{(1/2)0}$	-34.81	-35.33	-34.81	-34.81
	η	1.251	1.271	1.252	1.253
$\frac{1}{2}^-$	$\delta_{(1/2)1}$	-7.529	-7.534	-7.533	-7.533
	$\delta_{(3/2)1}$	25.06	25.04	25.05	25.05
	ϵ	7.254	7.252	7.255	7.255
$\frac{3}{2}^+$	$\delta_{(3/2)0}$	-70.48	-70.52	-70.50	-70.50
	$\delta_{(1/2)2}$	2.421	2.421	2.420	2.420
	$\delta_{(3/2)2}$	-4.215	-4.216	-4.216	-4.216
	η	$-.3881$	$-.3869$	$-.3873$	$-.3874$
	ϵ	.7785	.7747	.7801	.7800
	ξ	1.438	1.429	1.438	1.438
$\frac{3}{2}^-$	$\delta_{(3/2)3}$	.9441	.9425	.9436	.9436
	$\delta_{(1/2)1}$	-7.191	-7.201	-7.195	-7.195
	$\delta_{(3/2)1}$	26.41	26.39	26.40	26.41
	η	-3.809	-3.819	-3.806	-3.805
	ϵ	-2.765	-2.762	-2.768	-2.765
	ξ	$-.2574$	$-.2577$	$-.2573$	$-.2575$

$p-d$ cross section and A_y BELOW THE DEUTERON BREAKUP THRESHOLD



Correction from $\vec{L}\cdot\vec{S}$ term in the TNI

$$V_{3N}^{LS}(i,j,k) = W(i,j,k)\,\vec{L}_{ij}\cdot\vec{S}_{ij} + \vec{L}_{ij}\cdot\vec{S}_{ij}\,W(i,j,k)$$

$$W(i,j,k) = W(r_{ij}, r_{ik}, r_{jk})$$

Simple choice: $W(i,j,k) = W(\rho)$

then $W(i,j,k)$ and $\vec{L}_{ij}\cdot\vec{S}_{ij}$ commute

Moreover, only the channel $S=1$, $T=1$ is considered

$$V_{3N}^{LS} = W(\rho)\sum_{i<j}\vec{L}_{ij}\cdot\vec{S}_{ij}\;P_{11}(ij)$$ (A. Kievsky)

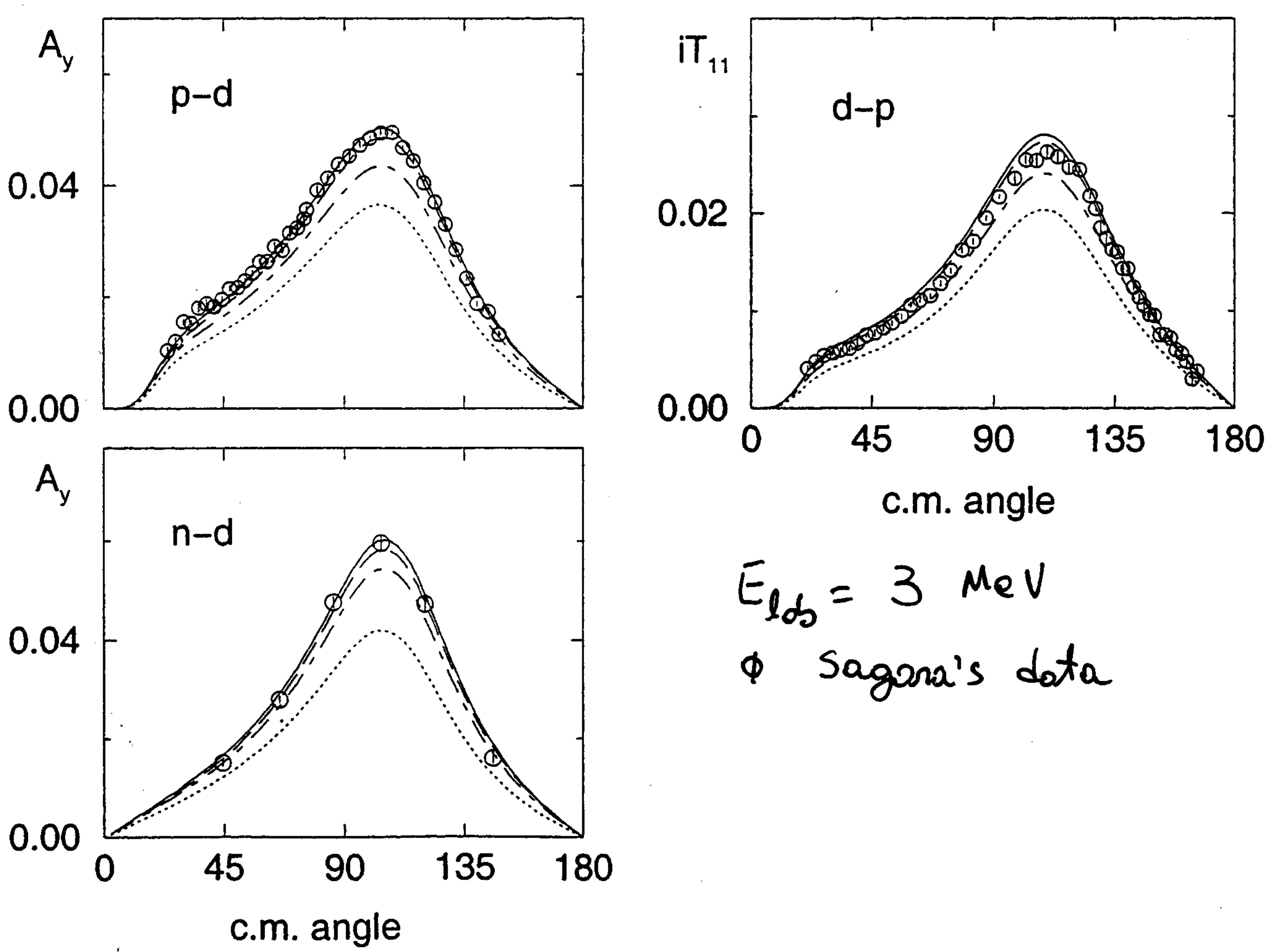
with

$$W(\rho) = W_0\, e^{-\alpha\rho}$$

W_0 (MeV)	α (fm^{-1})
-1	0.7
-10.	1.2
-20.	1.5

Fix W_0, then α is changed to reproduce A_y, iT_{11} at $E_{LAB}=3\,MeV$

		B(^3He)
·····	AV18	6.94 MeV
———	AV18 + LS1 (-1 MeV, 0.7 fm^{-1})	6.92 MeV
– – – –	AV18 + LS2 (-10 MeV, 1.2 fm^{-1})	6.90 MeV
–·–·–·	AV18 + LS3 (-20 MeV, 1.5 fm^{-1})	6.90 MeV



E_{lab} = 3 MeV

Φ Sagara's data

$n - {}^3$H and $p - {}^3$He Scattering

Benchmark calculations of the S-wave scattering lengths and effective range parameters

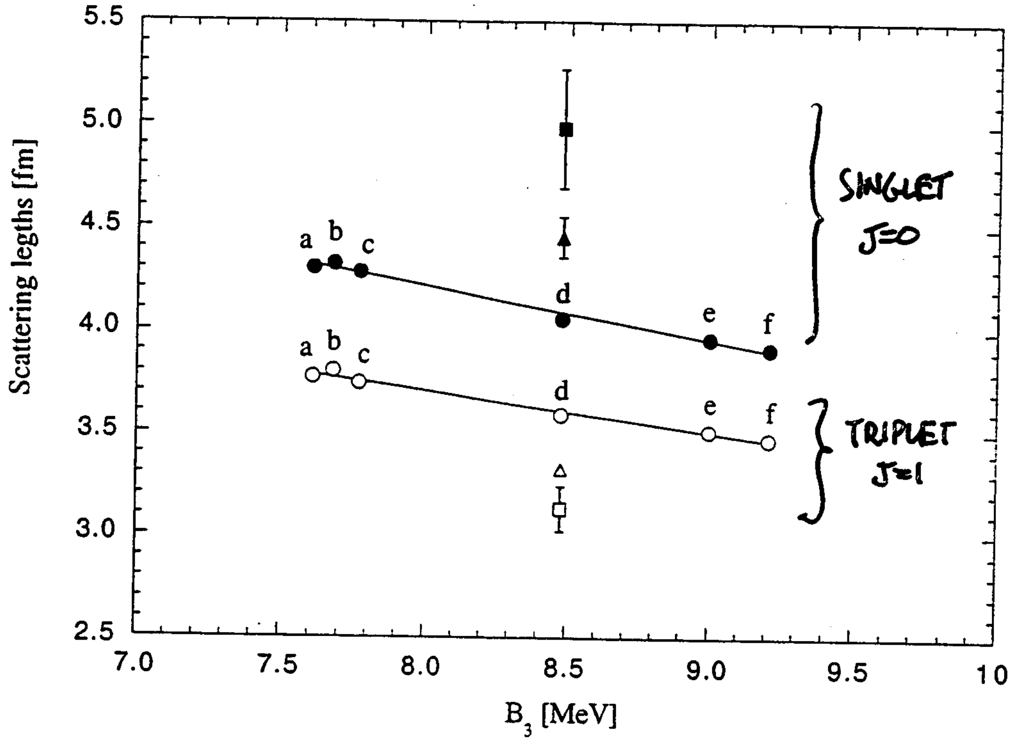
Interaction: S-wave MT I/III potential

T	J	Method	a (fm)	r_0 (fm)
0	0	CHH	14.82	6.6
		FY, Carbonell *et al.*	14.72	6.7
		FY, Yakolev *et al.*	14.7	
0	1	CHH	3.10	1.7
		FY, Carbonell *et al.*	3.08	1.8
		FY, Yakolev *et al.*	2.8	
		FY, Tjon	2.65	
1	0	CHH	4.10	2.0
		FY, Carbonell *et al.*	4.10	2.0
		FY, Yakolev *et al.*	4.0	
		FY, Tjon	4.09	
1	1	CHH	3.64	1.7
		FY, Carbonell *et al.*	3.63	1.9
		FY, Yakolev *et al.*	3.6	
		FY, Tjon	3.61	

Interaction: AV14 potential; States $T = 1$, $L = 0$, $J = S = 0, 1$

Method	$a(J = 0)$	$a(J = 1)$
CHH	4.32	3.80
FY, Carbonell *et al.*	4.31	3.79

$n - {}^3$H scattering lengths versus the ^{3}H binding energy



a) AV18
b) AV14
c) AV8
d) AV18 + TNI
e) AV14 + BR (5.6
f) AV14 + BR (5.8

Zero energy total cross section (σ_T) and coherent scattering length (a_c)

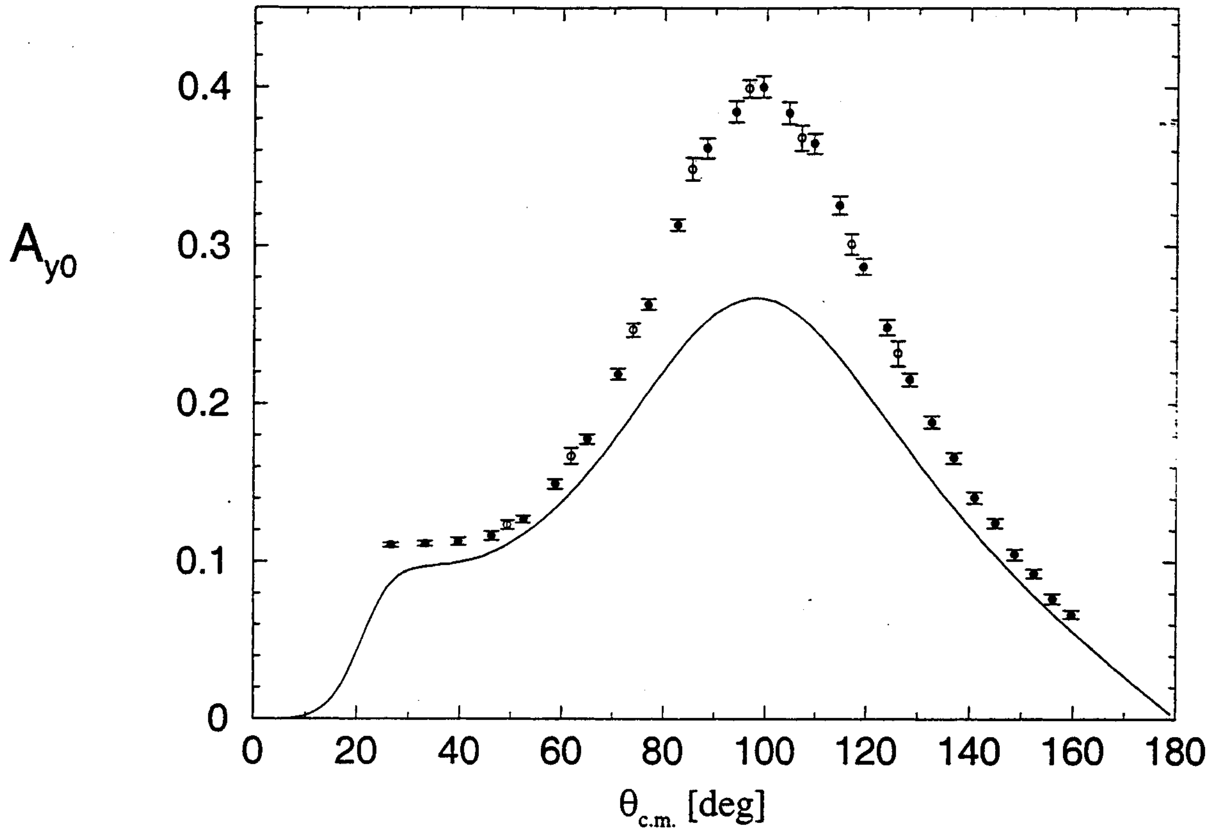
Model	σ_T (b)	a_c (fm)
AV14 + Urbana VIII	1.74	3.71
AV18 + Urbana IX	1.73	3.71
Expt.	1.70 ± 0.03 [a]	3.82 ± 0.07 [b]
		3.59 ± 0.02 [c]
		3.607 ± 0.017 [d]

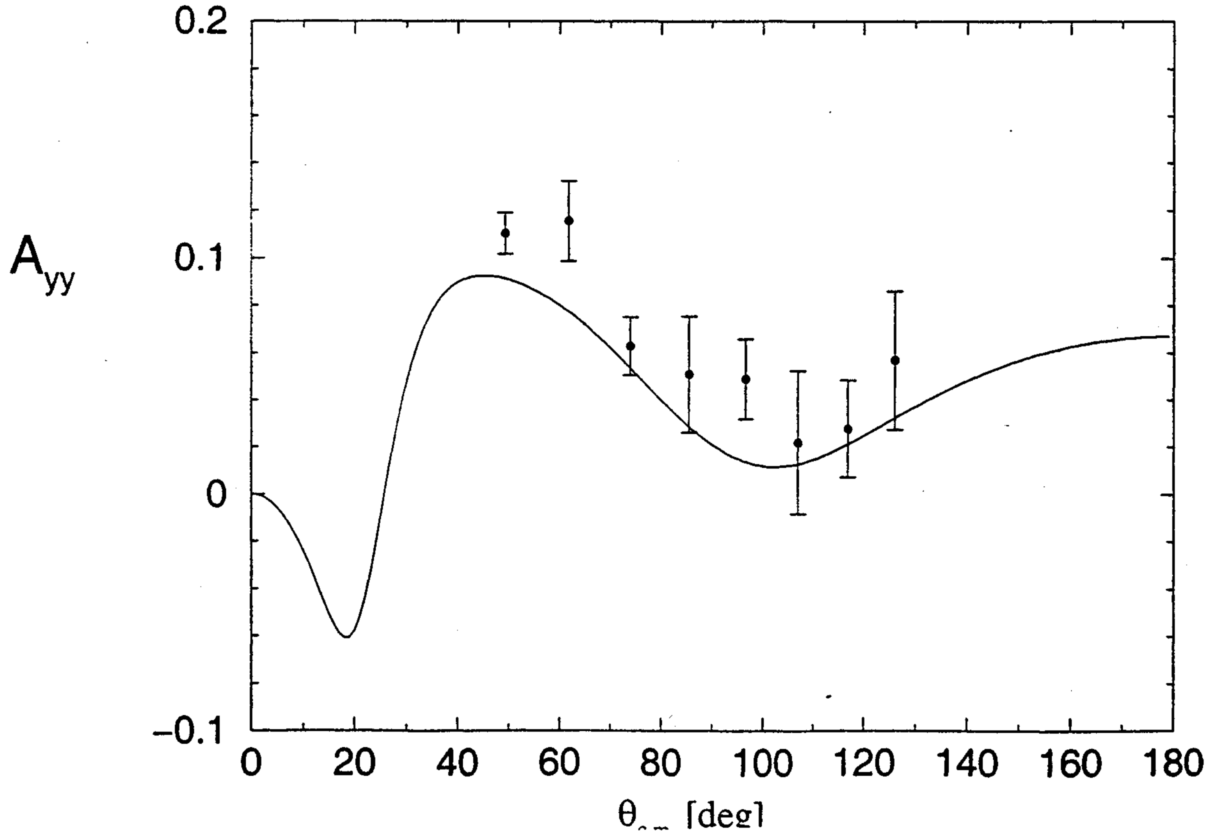
[a] T.W. Phillips *et al.* (1980)
[b] S. Hammerschmied *et al.* (1981)
[c] H.Rauch *et al.* (1985)
[d] G.M. Hale *et al.* (1990)

$$\sigma_T = \pi(|a(J=0)|^2 + 3|a(J=1)|^2) \qquad a_c = \frac{1}{4}a(J=0) + \frac{3}{4}a(J=1)$$

$p - {}^3$He scattering at $E_{c.m.} = 3$ MeV

Proton analysing power and A_{yy} at $E_{c.m.} = 3$ MeV





<u>$n-d$ and $p-d$ breakup scattering</u>

Application of the Kohn Variational Principle above deuteron breakup threshold.

The problem of the boundary conditions:

$$\sum_{\mu'=1}^{M}\left[A_{\mu,\mu'}(\rho)\frac{d^2}{d\rho^2}+B_{\mu,\mu'}(\rho)\frac{d}{d\rho}+C_{\mu,\mu'}(\rho)-E\ N_{\mu,\mu'}(\rho)\right]u_{\mu'}(\rho)=D_\mu(\rho)$$

for $\rho > 80$ fm, neglecting terms going to zero faster than ρ^{-3},

$$\sum_{\mu'=1}^{M}[n_{\mu,\mu'}(\frac{d^2}{d\rho^2}-\frac{\mathcal{L}_\mu(\mathcal{L}_\mu+1)}{\rho^2}+Q^2)-\frac{2\,Q\,\chi_{\mu,\mu'}}{\rho}+\frac{h_{\mu,\mu'}}{\rho^3}]u_{\mu'}(\rho)=0$$

where $E=\frac{\hbar^2}{M}Q^2$; M independent solutions:

$$\begin{aligned} w_\mu^{(\mu_0)}(\rho) &= \sum_{n=0,1,2,\ldots}\frac{\Gamma_\mu^{(\mu_0)}(n)}{\rho^n}e^{iQ\rho} \\ w_\mu^{(\mu_0)}(\rho) &= \sum_{\mu_0'}\sum_{n=0,1,2,\ldots}\frac{\Gamma_\mu^{(\mu_0')}(n)}{\rho^n}\left(e^{-i\chi\log 2Q\rho}\right)_{\mu_0'\mu_0}e^{iQ\rho} \end{aligned}$$

by choosing $\Gamma_\mu^{(\mu_0)}(n=0)=\delta_{\mu\mu_0}$. The $n>0$ coefficients Γ are determined by recurrence relations.

$$u_\mu(\rho)=\sum_{\mu_0=1}^{M}a_{\mu_0}w_\mu^{(\mu_0)}(\rho)\quad \text{at}\quad \rho=\rho_0>80 \text{ fm}$$

It has been numerically tested that the solutions are **insensitive** to variation of ρ_0, **even in the presence of Coulomb potential terms**.

$$u_\mu(\rho)\to\sum_{\mu_0}\left(e^{-i\chi\log 2Q\rho}\right)_{\mu,\mu_0}a_{\mu_0}e^{iQ\rho}$$

$\rightarrow$ Merkuriev's boundary conditions

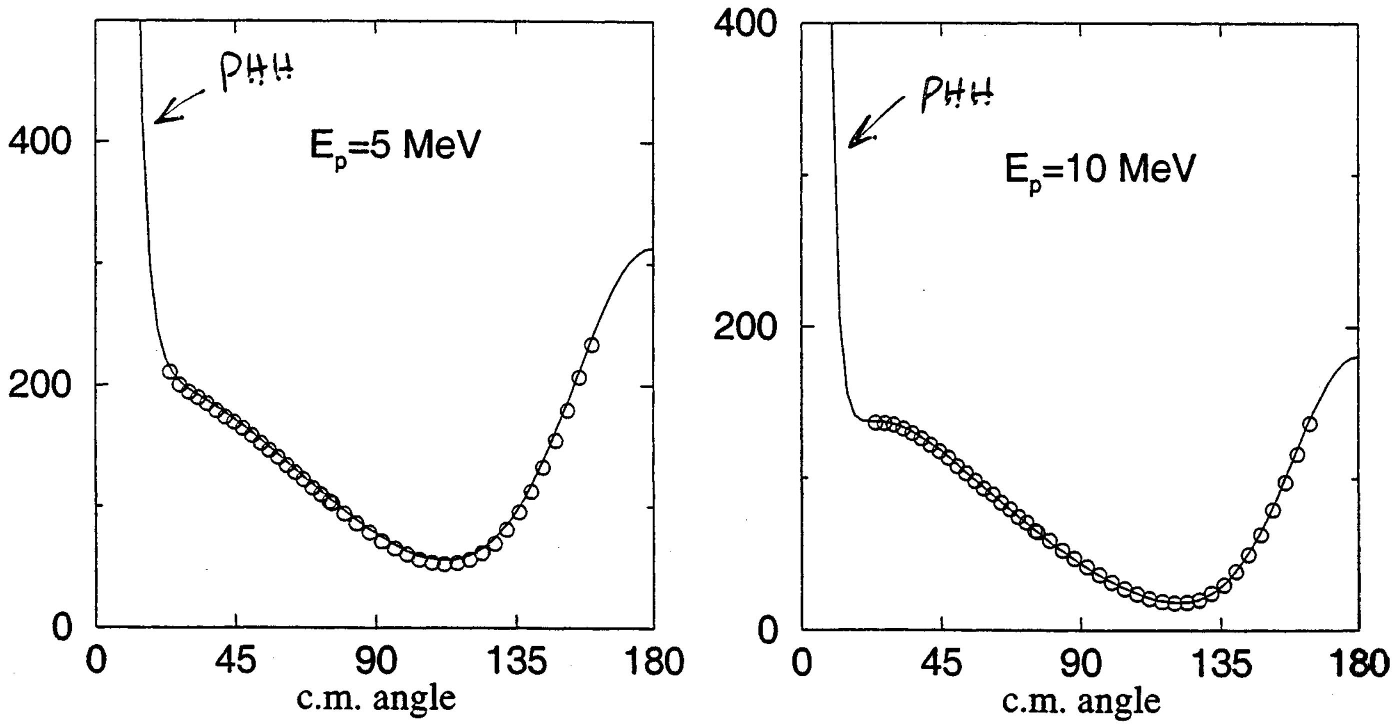
$n-d$ doublet and quartet phase shifts – MT(I–III) potential

N_α	${}^2\delta_0$	${}^2\eta_0$	${}^4\delta_0$	${}^4\eta_0$
2	97.96	0.5093	67.01	0.9933
4	105.47	0.4652	68.88	0.9788
6	105.51	0.4650	68.94	0.9784
8	105.50	0.4649	68.95	0.9782
FE/C	105.48	0.4648	68.95	0.9782
FE/Q	105.50	0.4649	68.96	0.9782

Friar et al 1996

$p-d$ elastic cross section above the deuteron breakup threshold

AV18+TN1



Electro–weak reactions → L.E. Marcucci

- $n + d \to {}^3\mathrm{H} + \gamma$ and $p + d \to {}^3\mathrm{He} + \gamma$
- $\vec{e} + {}^3\vec{\mathrm{He}} \to e' + \ldots$ inclusive reaction
- $p + {}^3\mathrm{He} \to {}^4\mathrm{He} + e^+ + \nu_e$ (*hep*) reaction

Nuclear Current Matrix Element ($A = 3$)

$$j^m_{\sigma_3\sigma_2\sigma}(\mathbf{p}, \mathbf{q}) = \langle \Psi^{(-)}_{\mathbf{p},\sigma_2\sigma} | J_m(\mathbf{q}) | \Psi_3^{\frac{1}{2}\sigma_3} \rangle$$

- $\mathbf{q}$ three–momentum transfer
- J_m nuclear electromagnetic or weak current operator
- $\Psi_3^{\frac{1}{2}\sigma_3} = {}^3\mathrm{He}$ bound state wave function
- Spherical wave expansion of the $p - d$ wave function

  $$\Psi^{(-)}_{\mathbf{p},\sigma_2\sigma} = 4\pi \sum_{SS_z} \langle \frac{1}{2}\sigma, 1\sigma_2 | SS_z \rangle \sum_{LMJJ_z} i^L \langle SS_z, LM | JJ_z \rangle Y^*_{LM}(\hat{\mathbf{p}}) \Psi^{LSJJ_z}_{2+1}$$

  Ψ_3 and $\Psi^{LSJJ_z}_{2+1}$ scattering wave function calculated as described previously

- Monte Carlo evaluation of

  $$j^{LSJ}_{J_z m\sigma} = \langle \Psi^{LSJJ_z}_{2+1} | J_m(q\hat{z}) | \Psi_3^{\frac{1}{2}\sigma_3} \rangle$$

Photodisintegration of ^{3}He

$$\mathrm{I}(E) = \int_{E_t}^{E} dE_\gamma \frac{\sigma_P(E_\gamma) - \sigma_A(E_\gamma)}{E_\gamma}$$

- $\sigma(E_\gamma)$=inelastic $\gamma - {}^3$He cross section

  P=parallel spins

  A=antiparallel spins

- $E_t = 5.495$ MeV threshold energy for inelastic processes

Gerasimov, Drell, Hearn sum rule

- GDH sum rule: $I(\infty) = 2\pi\alpha(\kappa/m_{^3\mathrm{He}})^2 = 496$ mb.
- $\kappa = -8.364$ anomalous magnetic moment of ^{3}He

Estimation of $I(5.55 MeV)$ – H.R. Weller & E. Wulf (TUNL)

$\sigma_P(E_\gamma) - \sigma_A(E_\gamma)$ can be estimated in terms of the matrix elements entering the $p+d \rightarrow {}^3$He$+\gamma$ radiative capture measured at TUNL:

$$\sigma_P(E_\gamma) - \sigma_A(E_\gamma) = \frac{32}{3}\pi^2 \frac{mp\alpha}{E_\gamma}\Big[\frac{1}{2}|M_1^{J=3/2}|^2 - |M_1^{J=1/2}|^2 + \frac{1}{2}|E_1^{J=3/2}|^2 - |E_1^{J=1/2}|^2 \Big]$$

large cancellations between the various terms

Model	I(5.55 MeV) [nb]
"Experiment"	-1.105 ± 0.219
IA	-0.15
IA+MEC	-0.35
FULL	-0.44

Theoretical calculation directly fro the $\gamma - {}^3$He reaction, including also the (small) contribution of the quadrupoles, etc.

hep reaction $\quad p + {}^3He \rightarrow {}^4He + e^+ + \nu_e$

Results obtained for astrophysical S-factor (in 10^{-23} MeV b)
(preliminary)

- Only S-waves

IA	8.24
+ MEC	8.41
+ Δ	2.60

- Only P-waves ($J = 0, 1, 2$)

$J=1$	IA	1.59
	+ MEC	2.17

$J = 0, 2$ are being calculated

present total S-factor $\quad$ 4.77