

**SECOND EUROPEAN SUMMER SCHOOL on
MICROSCOPIC QUANTUM MANY-BODY THEORIES
and their APPLICATIONS**

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**HYPERSPHERICAL HARMONIC METHODS
FOR STRONGLY INTERACTING SYSTEMS:
A SUMMARY AND NEW DEVELOPMENTS**

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These are preliminary lecture notes, intended only for distribution to participants

Hyperspherical harmonic methods for strongly interacting systems: a review and some recent developments.

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There is a variety of microscopic systems which require a quantum mechanical treatment. Amongst them, we can recall:

- atoms and molecules
- muonic atoms (in particular, a, b, μ with $a, b \in p, d, t$)
- helium drops
- nuclei and hypernuclei
- quark systems.

There are today available a few techniques useful to accurately calculate on those systems.

① Variational methods (requiring calculation (mostly analytical))

② Monte Carlo methods

i) Variational approach (VMC)

ii) Green Function Monte Carlo (GFMC) technique

- ③ Faddeev and Faddeev-Yakubovsky equations, both in momentum and coordinate representation
 - ④ Hyperspherical Harmonic (HH) techniques.
-

- ① Variational methods based on gaussian-type wave functions, allow most of integrations to be done analytically. They are very useful to calculate on atomic systems and light nuclei too.
- ② Monte Carlo techniques, in particular GFM, have been successfully applied to many systems. An important example, it has been applied to calculate the spectra of nuclei up to $A=8$ using realistic nuclear interactions.
(See figures)
- ③ Faddeev-type calculations have been very successful to calculating $A=3, 4$ bound states and $n-d$ scattering states.
- ④ HH technique has been successfully applied to $A=3, 4$ bound states. The agreement with other accurate methods (in particular FY) is very good. Important applications are those ν muonic-molecules and non relativistic quark models.

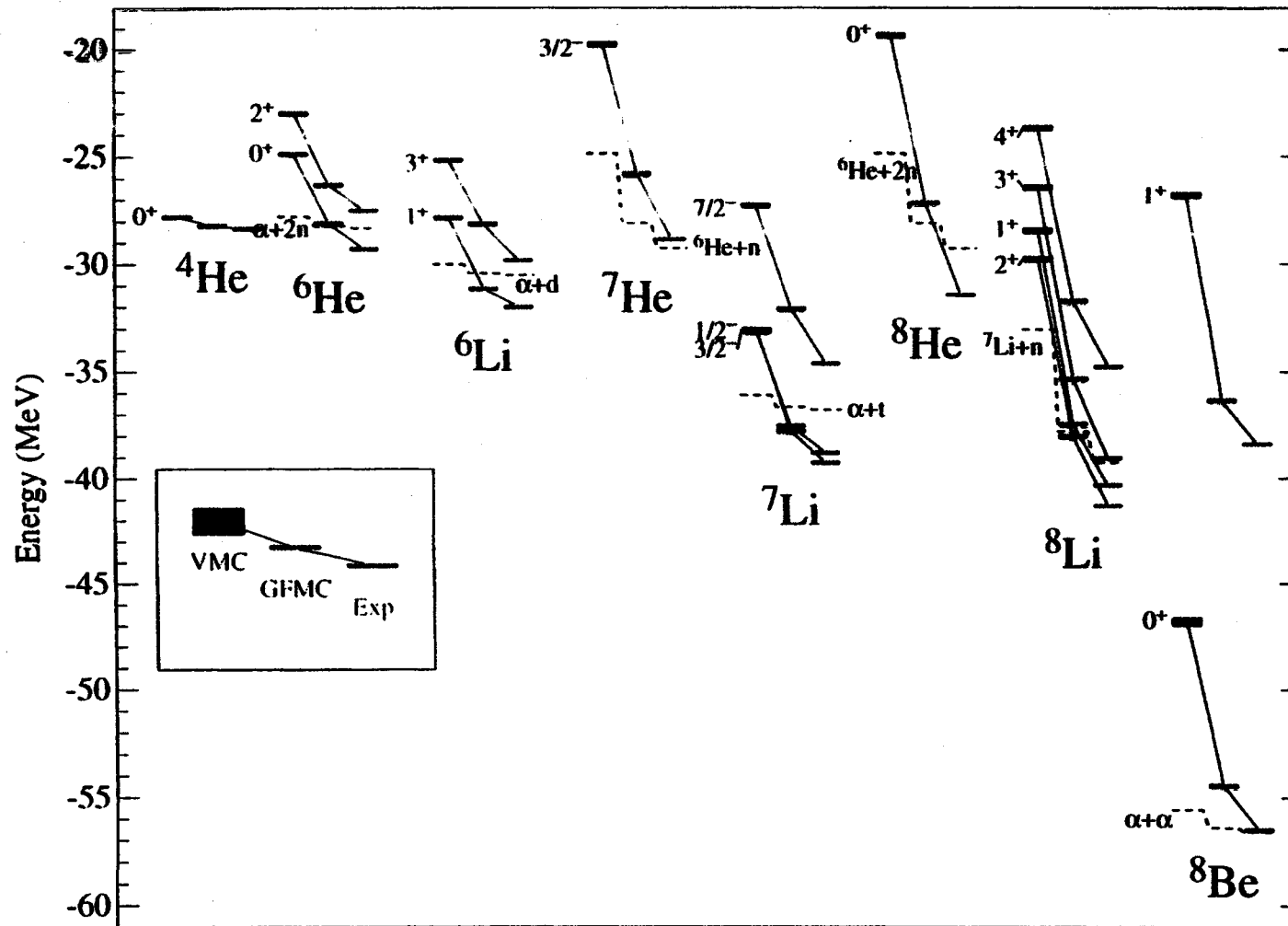


Figure 4: VMC and GFMC energies using AV18/UIX compared to experiment. Black dashed lines show the indicated breakup thresholds for each method. The Monte Carlo statistical errors are shown by the light blue and yellow bands.

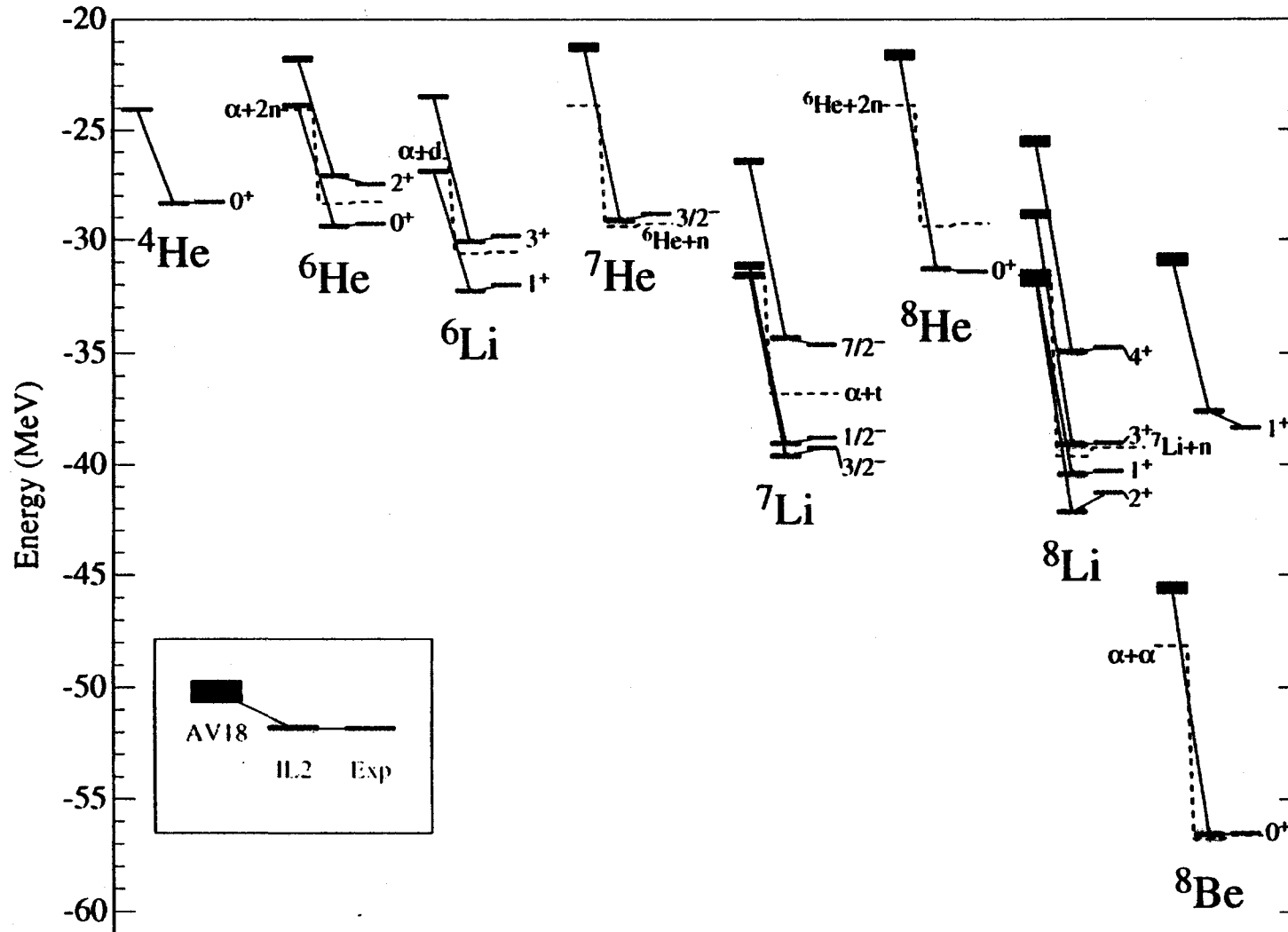


Figure 5: GFMC energies using AV18 and AV18/IL2 compared to experiment.

An useful modification is the so-called Correlated Hyperspherical Harmonic (CHH) expansion. This can be applied to systems with $A > 4$ too.

An important application has been done by the PISA group to calculating three- and four-particle states, including also, when required, the Coulomb interaction effects in an accurate way.

bound and scattering

The case here considered is the study of those systems in a non-relativistic approximation. One has to solve the Schrödinger equation. The method here discussed is the one where the w.f. is expanded in a (finite) set of suitably chosen functions F :

$$\psi_N(1, \dots, A) = \sum_I C_I F_I(1, \dots, A).$$

$\{C\}$ is a set of trial parameters and other ones can be contained in the functions F .

For bound states the variational Rayleigh principle can be used. It has to be noticed that, if only linear parameters are present in ψ_N (for example the C 's) then the variational principle is equivalent to require

$$\langle E_J | H - E | \psi_N \rangle = \sum_{I=1}^N C_I \langle F_I | H - E | F_I \rangle = 0, \quad J=1, \dots, N$$

It must be recalled that according to the Hylleraas-

(*)
Umdheim - Mac Donald's theorem there exists a one-to-one
correspondence between the approximate energy levels $E_i(N)$
and the exact levels $E_i \equiv E_i(\infty)$, the i -th approximate level
being an upper bound to the i -th exact level:

$$E_{i+1}(N+1) \geq E_i(N) \geq E_i(N+1)$$

$$\lim_{N \rightarrow \infty} E_i(N) = E_i$$

(*) E. A. Hylleraas and B. Umdheim, Z. Phys. 65 (1930) 759

J. K. L. Mac Donald, Phys. Rev. 43 (1933) 830

For calculating on scattering states, the Kohn
variational principle can be used as it will be
discussed later on.

Jacobi coordinates.

When the w.f. ψ_N is expanded in a set of basis functions the important point is the choice of the basis.

This is strongly related to the choice of the coordinate system.

For isolated systems one has first of all to introduce the C.M. coordinate so as to separate the C.M. motion from the internal one.

The internal coordinates can be selected in many ways.

Best situation \rightarrow Schrödinger equation is solved by separating the variables.

The strong interaction is highly non separable!

Useful conditions to be fulfilled:

- introduction of important collective variables;
- the corresponding basis functions should allow a rapid convergence $\psi_N \rightarrow \psi$ as possible;
- the important clustering configurations should be described by a reasonably small number of functions

|| A useful choice: C.M. coordinate and internal ones so as to separate the kinetic energy operator

$$T = \sum_{i=1}^A \frac{\vec{p}_i^2}{2m_i} = -\frac{\hbar^2}{2} \sum_{i=1}^A \nabla_{X_i}^2, \quad \vec{X}_i = \sqrt{m_i} \vec{r}_i$$

$$= -\frac{\hbar^2}{2m_{\text{tot}}} \nabla_R^2 - \frac{\hbar^2}{2M} \sum_{i=1}^N \nabla_{y_i}^2$$

where $N = A - 1$, $\vec{R} = (m_1 \vec{r}_1 + \dots + m_A \vec{r}_A) / m_{\text{tot}}$

$$m_{\text{tot}} = m_1 + \dots + m_A$$

M is a reference mass (possibly = 1)

The Jacobi coordinates \vec{y}_i , $i = 1, \dots, N$ are linear combinations of $\vec{r}_1, \dots, \vec{r}_A$.

Let $\vec{y}_A = \vec{R}$ and

$$\vec{y}_i = \sum_{j=1}^A c_{ij} \vec{r}_j, \quad (c_{ij} = m_j / m_{\text{tot}})$$

It is

$$\nabla_{X_i} = \sum_{j=1}^A c_{ji} \nabla_{y_j}$$

$$T = -\frac{\hbar^2}{2m_{\text{tot}}} \nabla_{y_A}^2 - \frac{\hbar^2}{2M} \sum_{j,k=1}^N c_{ji} c_{ki} \nabla_{y_i} \cdot \nabla_{y_k} =$$

$$= -\frac{\hbar^2}{2m_{\text{tot}}} \nabla_{y_A}^2 - \frac{\hbar^2}{2M} \sum_{i=1}^N \nabla_{y_i}^2,$$

if the following conditions are satisfied:

$$\sum_{i=1}^A c_{ji} c_{ji} = \frac{1}{M} \quad j=1, \dots, N$$

$$\sum_{i=1}^A c_{ji} c_{ki} = 0 \quad \text{if } j \neq k = 1, \dots, A$$

Jacobi coordinates : A=2

$$c_{11}^2 + c_{12}^2 = \frac{1}{M} \quad c_{11} c_{21} + c_{12} c_{22} = 0$$

with $c_{11} = \frac{1}{\sqrt{M}} \cos \alpha$, $c_{12} = \frac{1}{\sqrt{M}} \sin \alpha \Rightarrow \tan \alpha = -\sqrt{\frac{m_2}{m_1}}$

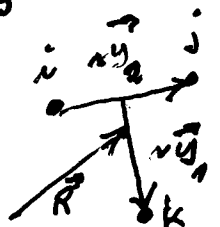
$$\left\{ \begin{aligned} \vec{y}_1 &= \sqrt{\frac{m_2}{M m_{\text{tot}}}} \vec{x}_1 - \sqrt{\frac{m_1}{M m_{\text{tot}}}} \vec{x}_2 = \sqrt{\frac{m_1 m_2}{M(m_1 + m_2)}} (\vec{r}_2 - \vec{r}_1) \\ \vec{y}_2 &= \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \end{aligned} \right.$$

A=3 let us take $\vec{y}_2 \sim (\vec{r}_2 - \vec{r}_1)$, which means $c_{23} = 0$.

One easily gets:

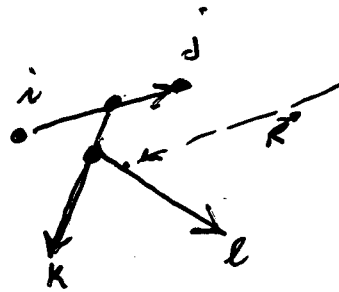
$$c_{21} = -\sqrt{\frac{m_2}{M(m_1 + m_2)}} , \quad c_{22} = \sqrt{\frac{m_1}{M(m_1 + m_2)}}$$

$$c_{11} = -\sqrt{\frac{m_1 m_2}{M m_{\text{tot}} (m_1 + m_2)}} , \quad c_{12} = -\sqrt{\frac{m_2 m_2}{M m_{\text{tot}} (m_1 + m_2)}} , \quad c_{13} = +\sqrt{\frac{m_1 m_2}{M m_{\text{tot}}}}$$



$$\underline{A=4}$$

Case a)



Let $i=1, j=2, k=3, l=4$:

$$E_{31} = -\sqrt{\frac{\mu_{12}}{M m_1}}, \quad C_{32} = \sqrt{\frac{\mu_{12}}{M m_2}}, \quad C_{33} = C_{34} = 0$$

$$\left\{ \begin{array}{l} C_{21} = -\sqrt{\frac{m_1 m_3}{m_{123} M m_{tot}}} \\ C_{23} = \sqrt{\frac{m_{12}}{M m_{123} m_{tot}}} \end{array} \right. , \quad \left\{ \begin{array}{l} C_{22} = -\sqrt{\frac{m_2 m_3}{M m_{12} m_{123}}} \\ C_{24} = 0 \end{array} \right.$$

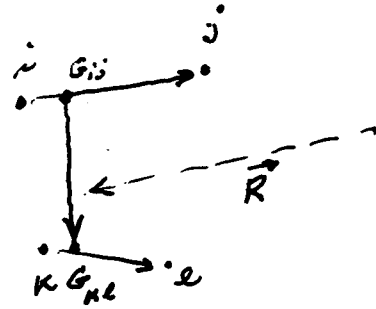
$$\left\{ \begin{array}{l} C_{11} = -\sqrt{\frac{m_1 m_3}{M m_{tot} m_{123}}} \\ C_{13} = -\sqrt{\frac{m_3}{M m_{tot} m_{123}}} \end{array} \right. , \quad \left\{ \begin{array}{l} C_{12} = -\sqrt{\frac{m_2 m_3}{M m_{tot} m_{123}}} \\ C_{14} = \sqrt{\frac{m_3 m_{123}}{M m_{tot} m_4}} \end{array} \right.$$

$$\mu_{ij} = \frac{m_i m_j}{m_i + m_j}$$

$$m_{ijk} = m_i + m_j + m_k$$

case b)

let $i=1, j=2, k=3, l=4$.



$$C_{31} = -\sqrt{\frac{\mu_{12}}{M m_1}}$$

$$C_{32} = \sqrt{\frac{\mu_{12}}{M m_2}}$$

$$C_{33} = C_{34} = 0$$

$$C_{21} = C_{22} = 0$$

$$C_{23} = -\sqrt{\frac{\mu_{34}}{M m_3}}$$

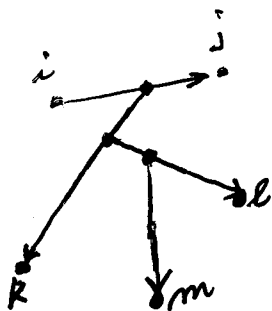
$$C_{24} = \sqrt{\frac{\mu_{34}}{M m_4}}$$

$$\left\{ \begin{aligned} C_{11} &= -\sqrt{\frac{m_1 m_{34}}{M m_{tot} m_{12}}} \\ C_{13} &= \sqrt{\frac{m_{12} m_3}{M m_{tot} m_{34}}} \end{aligned} \right.$$

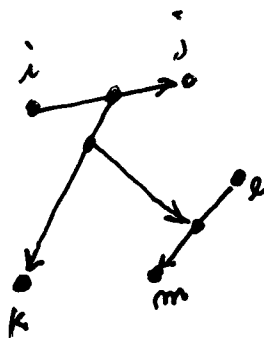
$$C_{12} = -\sqrt{\frac{m_2 m_{34}}{M m_{tot} m_{12}}}$$

$$C_{14} = \sqrt{\frac{m_4 m_{12}}{M m_{tot} m_{34}}}$$

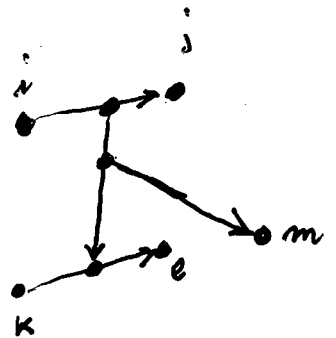
A=5



a)



b)



c)

Hyperspherical coordinates

Let (y_1^2, \dots, y_N^2) be a set of Jacobi coordinates.

$$\nabla^2 = \sum_{i=1}^N \nabla_{y_i^2}^2 = \sum_{i=1}^N \left[\frac{\partial^2}{\partial y_i^2} + \frac{2}{y_i} \frac{\partial}{\partial y_i} + \frac{\ell^2(\hat{\omega}_i)}{y_i^2} \right]$$

$$\hat{\omega}_i \equiv (\vartheta_i, \varphi_i).$$

when $N=1$ one has the polar coordinates $r \equiv y_1$, $\hat{\omega} \equiv (\vartheta, \varphi)$

$N=2$ it results to be convenient to introduce only one "length"

$$r^2 = y_1^2 + y_2^2, \quad r \equiv \text{hyperradius}$$

and a second hyperangular variable $\Phi \equiv \Phi_2$

$$y_2 = r \cos \Phi \quad y_1 = r \sin \Phi$$

together with $\hat{\omega}_1$ and $\hat{\omega}_2$.

$N > 2$ The generalization is straightforward:

N angular variables $\hat{\omega}_i$, $i = 1, \dots, N$

the hyperradius

$$r^2 = \sum_{i=1}^N y_i^2$$

$N-1$ hyperangles Φ_1, \dots, Φ_2 given by

$$\left\{ \begin{array}{l} y_N = r \cos \Phi_N \\ \vdots \\ y_i = r \sin \Phi_N \dots \sin \Phi_{i+1} \cos \Phi_i \\ \vdots \\ y_1 = r \sin \Phi_N \dots \sin \Phi_2 \end{array} \right. \quad (\Phi_N = 0)$$

One gets

$$\frac{\partial}{\partial y_i} = \frac{\partial r}{\partial y_i} \frac{\partial}{\partial r} + \frac{D_{ii}}{r} = \frac{y_i}{r} \frac{\partial}{\partial r} + D_{ii}/r$$

$$\frac{\partial^2}{\partial y_i^2} = \left(\frac{1}{r} - \frac{y_i^2}{r^3} \right) \frac{\partial}{\partial r} + \frac{y_i^2}{r^2} \frac{\partial^2}{\partial r^2} + D_{2ii}/r^2$$

D_{ii}, D_{2ii} involve derivatives with respect the hyperangles.

Then

$$\nabla^2 = \sum_{i=1}^N \nabla_{y_i}^2 = \frac{\partial^2}{\partial r^2} + \frac{3N-1}{r} \frac{\partial}{\partial r} + \frac{\Lambda(\Omega)}{r^2}$$

where $\Omega \equiv (\hat{\omega}_1, \dots, \hat{\omega}_N, \phi_1, \dots, \phi_N)$

$\Lambda(\Omega)$ is the grandangular momentum

By calculating the expression of D_{ii} and D_{2i} , with the help of the definition of z and $\{\phi_i\}$ one gets the following recurrence relation (Fabre de la Ripelle, Rev. Roum. Phys. 14 (1969) 1215)

$$\begin{aligned} \Lambda_i^2(\Omega_i) &= \frac{c^2}{\omega_i^2} + \left[3(i-2) \cot \phi_i + 2(\cot \phi_i - \tan \phi_i) \right] \frac{\partial}{\partial \phi_i} \\ &+ \frac{\ell^2(\hat{\omega}_i)}{\omega_i^2} + \frac{\Lambda_{i-1}^2(\Omega_{i-1})}{\sin^2 \phi_i} = \\ &= 4(1-z_i) \frac{\partial}{\partial z_i} + \left[1 - i(2+z_i) \right] \frac{\partial}{\partial z_i} + 2 \frac{\ell^2(\hat{\omega}_i)}{1+z_i} + 2 \frac{\Lambda_{i-1}^2(\Omega_{i-1})}{1-z_i} \end{aligned}$$

where $\underline{z_i = \cos 2\phi_i}$.

The volume element $d\vec{v} = dy_1^{\vec{}} \dots dy_N^{\vec{}}$ can be expressed in terms of the hyperspherical coordinates:

$$\begin{aligned} d\vec{v} &= d\vec{\omega}_1 \dots d\vec{\omega}_N y_1^2 \dots y_N^2 dy_1 \dots dy_N = \\ &= d\vec{\omega}_1 \dots d\vec{\omega}_N y_1^2 \dots y_N^2 J dr d\phi_1 \dots d\phi_N \end{aligned}$$

$J \equiv J(y_1, \dots, y_N \rightarrow r, \phi_1, \dots, \phi_N)$

One easily obtains

$$J = \begin{vmatrix} y_1/r & y_2/r & y_3/r & \dots & y_N/r \\ y_1 \cot \phi_N & -y_2 \tan \phi_N & -y_3 \tan \phi_N & \dots & -y_N \tan \phi_N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1 \cot \phi_2 & -y_2 \tan \phi_2 & 0 & \dots & 0 \end{vmatrix}$$

$$= \frac{1}{2} y_1 \dots y_N \cot \phi_1 \dots \cot \phi_N \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & -\tan^2 \phi_N \\ 1 & 1 & 1 & \dots & -\tan^2 \phi_{N-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -\tan^2 \phi_2 & 0 & \dots & \dots & 0 \end{vmatrix}$$

$$= \frac{1}{2} y_1 \dots y_N \frac{1}{\sin \phi_2 \cos \phi_2} \dots \frac{1}{\sin \phi_N \cos \phi_N}$$

and in conclusion writing $d\tilde{v} = r^{D-1} dy d\Omega$ ($D=3N$)

one has

$$d\Omega = d\hat{\omega}_1 \dots d\hat{\omega}_N \prod_{j=2}^N (\sin \phi_j)^{3j-4} \cos^2 \phi_j d\phi_j$$

Hyperspherical Functions.

Let us write the Laplace operator in the form

$$\nabla^2 = \sum_{i=1}^D \nabla_{y_i}^2 = \sum_{j=1}^D \frac{\partial^2}{\partial x_j^2}$$

x_j , $j=1, \dots, D$ being the cartesian components of the Jacobi vectors.

A generic homogeneous polynomial of order n can be written as

$$f_n = \sum_{(n)} a_{(n)} x_1^{n_1} \dots x_D^{n_D}, \quad (n) \equiv (n_1, \dots, n_D)$$

$n_1 + n_2 + \dots + n_D = n$

It can be verified that

$$\sum_{j=1}^D x_j \frac{\partial f_n}{\partial x_j} = n f_n.$$

f_n is an harmonic polynomial, denoted as h_n , if the coefficients $a_{(n)}$, in general complex, are such that

$$\nabla^2 h_n = 0.$$

Any homogeneous polynomial of order n can be expressed as ^(*)

$$f_n = h_n + r^2 h_{n-2} + \dots$$

which is called the canonical decomposition of f_n .

(*) J. Aron, "Hyperspherical Harmonics", Kluwer Academic Publishers (1989) -16-

If h_G is an harmonic polynomial, the function

$$y_G(\Omega) = r^{-G} h_G,$$

is independent on r . Since $\nabla^2 h_G = 0$, then

$$\nabla^2 h_G = \left(\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} + \frac{\Lambda^2(\Omega)}{r^2} \right) r^G y_G(\Omega) =$$

$$\left(\Lambda^2(\Omega) + G(G+D-2) \right) r^{G-2} y_G(\Omega) = 0$$



$$\left[\Lambda^2(\Omega) + G(G+D-2) \right] y_G(\Omega) = 0$$

A function satisfying the latter equation is called a hyperspherical harmonic function.

Many harmonic polynomials of order G can be constructed so the same is true for the HH functions too.

As an example, for $N=1$ i.e. $D=3$, the Λ^2 reduces to the ^{minus} square of the usual angular momentum L^2 . Introducing the index m to label a set of linearly independent eigenfunctions of L^2 to a given $l(l+1) - G(G+D-2)$ eigenvalue we have

$$L^2 Y_{l,m}(\hat{\omega}) = l(l+1) Y_{l,m}(\hat{\omega})$$

If $Y_{l,m}(\hat{\omega})$ is required to be also an eigenfunction of L^2 to the eigenvalue $l(l+1)$, then $Y_{l,m}(\hat{\omega})$ are the well known spherical functions.

Analogously, various functions corresponding to a given G value can be linearly combined to get an orthonormal set. Let $\{G\}$ stand for G and all other quantum numbers specifying the orthonormal functions:

$$\int d\Omega y_{\{G\}}^*(\Omega) y_{\{G'\}}(\Omega) = \delta_{\{G\}, \{G'\}}$$

The $y_{\{G\}}$ functions can be constructed by following a recursive relation (Zernike and Brinkman)

For $N=2$ the equation to be solved is

$$\Lambda_G^2(\Omega_2) y_G(\Omega_2) = -G(G+4) y_G(\Omega_2)$$

Let us consider

$$y_G = F(\cos 2\phi_2) (\cos \phi_2)^{l_2} (\sin \phi_2)^{l_1} Y_{l_1, m_1}(\hat{\omega}_1) Y_{l_2, m_2}(\hat{\omega}_2)$$

in terms of the variable $z = \cos 2\phi_2$ one obtains the equation

$$\begin{cases} (1-z^2) F'' + (\alpha - \beta z) F' + \gamma F = 0 \\ \alpha = l_2 - l_1 \quad \beta = l_1 + l_2 + 3 \quad \gamma = G(G+4) - (l_1 + l_2)(l_1 + l_2 + 4) \end{cases}$$

$$(*) \quad \Lambda_i^2(\Omega_i) = \frac{\partial^2}{\partial \phi_i^2} z + [3(i-2) \cot \phi_i + 2(\cot \phi_i - \tan \phi_i)] \frac{\partial}{\partial \phi_i} - \frac{l_i^2 \hat{\omega}_i}{\cos^2 \phi_i} + \frac{\Lambda_{i-1}^2(\Omega_{i-1})}{\sin^2 \phi_i}$$

and in conclusion the HH can be cast in the form (Faber)

$$y_{[G]}(\Omega_N) = \left[\prod_{j=1}^N Y_{l_j, m_j}(\hat{\omega}_j) \right] \cdot \left[\prod_{j=2}^N P_{m_j}^{G_{j-1}, l_j}(\phi_j) \right]$$

where

$$P_{m_j}^{G_{j-1}, l_j}(\phi_j) = N_{m_j}^{l_j, \nu_j}(\cos \phi_j) (\sin \phi_j)^{l_j} P_{m_j}^{\nu_{j-1}, l_j + \frac{1}{2}}(\cos 2\phi_j)$$

with

$$G_j = \sum_{i=1}^j (l_i + 2m_i), \quad m_1 = 0 \quad G \equiv G_N \quad \nu_j = G_j + \frac{3j}{2} - 1$$

$$\{G\} \equiv \{l_1, \dots, l_N, m_1, \dots, m_N, n_1, \dots, n_N\}$$

3N-1 quantum numbers

The explicit expression of the normalization constants is

$$N_{m_j}^{l_j, m_j} = \left[\frac{2\nu_j \Gamma(\nu_j - m_j) m_j!}{\Gamma(\nu_j - m_j - l_j - \frac{1}{2}) \Gamma(m_j + l_j + \frac{3}{2})} \right]^{1/2}$$

Again, it can be easily verified that $z^G y_{[G]}(\Omega)$ is an harmonic polynomial of order G .

The functions $y_{\{\bar{G}\}}(\Omega)$ can be combined to obtain
 a function of the total angular momentum.

They can be constructed by the following coupling scheme

$$H_{\{\bar{G}\}}(\Omega_N) = \sum_{m_1 \dots m_n} (l_1 m_1 l_2 m_2 | L_2 M_2) (l_3 m_3 L_2 M_2 | L_3 M_3) \dots (l_N m_N L_{N-1} M_{N-1} | LM) Y_{\{\bar{G}\}}(\Omega_N)$$

where $\{\bar{G}\}$ now stands for the $3N-1$ quantum numbers

$$\{\bar{G}\} \equiv (l_1 \dots l_N, L_2, \dots, L_N=L, L_2, m_2 \dots m_N)$$

As an example, for $N=4$ the HH function is

$$H_{\{l_1 l_2 l_3 L_2 L_2 m_2 m_3\}}(\Omega_3) = \left\{ \left[Y_{l_1 m_1}(\hat{\omega}_1) Y_{l_2 m_2}(\hat{\omega}_2) \right]_{L_2} Y_{l_3 m_3}(\hat{\omega}_3) \right\}_{L_2}$$

$$\cdot \begin{matrix} (2) P_{m_2}^{l_1 l_2}(\vartheta_2) & (3) P_{m_3}^{2m_2+l_1+l_2, l_3}(\vartheta_3) \end{matrix}$$

Of course, other coupling schemes can be used

For systems including Fermi particles, the H functions can be multiplied by spin functions and combined so as to get definite J, J_z values. In that case the corresponding functions will be denoted as

$$H_{\{\tilde{G}\}} \quad \{\tilde{G}\} \text{ includes some spin quantum numbers too.}$$

|| In general the w.f. of a system must satisfy certain symmetry relations.

Let (i, j, k, \dots) be a permutation p of the indices $1, 2, \dots, A$ and Ω^p the corresponding hyperangular coordinates. The w.f. can be written in the form

$$\Psi(1, \dots, A) = \pi^{-(D-1)/2} \sum_{\{\tilde{G}\}} \left(\sum_p \alpha_p H_{\{\tilde{G}\}}(\Omega^p) \right) u_{\{\tilde{G}\}}(r),$$

where $\alpha_p = 1$ for identical bosons and $\alpha_p = (-1)^P$

for fermions, $P = 0, 1$ being the "parity" of the permutation

The summation over p is extended to all the necessary permutations.

With the position

$$H_{\{\tilde{G}\}}^{(S)} = \sum_p \alpha_p H_{\{\tilde{G}\}}(\Omega^p)$$

we can rewrite

$$\Psi(1, \dots, A) = \pi^{-(D-1)/2} \sum_{\{\tilde{G}\}} H_{\{\tilde{G}\}}^{(S)} u_{\{\tilde{G}\}}(r)$$