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SECOND EUROPEAN SUMMER SCHOOL on MICROSCOPIC QUANTUM MANY-BODY THEORIES and their APPLICATIONS

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HYPERSPHERICAL HARMONIC METHODS FOR STRONGLY INTERACTING SYSTEMS: A SUMMARY AND NEW DEVELOPMENTS PART II

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The coupled equations.

The Schrodinger equation in the center of mass frame in terms of the Jacobi coordinates $\vec{y}_1, \dots, \vec{y}_\nu$ can be written as

$$(T+V-E)\psi(1,\dots,A) = \left\{-\frac{\hbar^2}{2m}\sum_i \nabla^2_{y_i} + V(\vec{y}_1,\dots,\vec{y}_\nu) - E\right\}\psi(\vec{y}_1,\dots,\vec{y}_\nu) = 0$$

In the hyperspherical formalism the W.F. ψ is expanded in HH functions satisfying the appropriate symmetry relations, namely $\mathcal{H}^{(S)}_{\{\tilde{G}\}}$.

The problem to be solved is the calculation of all the hyperradial function $u_{\{\tilde{G}\}}(r)$.

For a bound state problem one can adopt a variational procedure and use the Rayleigh-Ritz principle. First of all, the W.F. is expanded in a finite number of functions $\mathcal{H}^{(S)}_{\{\tilde{G}\}}$ by limiting the choice of the quantum numbers $\{\tilde{G}\}$:

$$\psi_\nu(1,\dots,A) = r^{-(D-1)/2} \sum_{\{\tilde{G}\}}^{\{MAX\}} \mathcal{H}^{(S)}_{\{\tilde{G}\}} \, u_{\{\tilde{G}\}}(r)$$

(to be determined)

Then from the Rayleigh-Ritz principle

$$\delta_u \langle \psi_\nu | H - E | \psi_\nu \rangle = 0,$$

one gets a set of differential equations for the u-functions.

Let us introduce

$$N_{\{\tilde{G}\},\{\tilde{G}'\}} = \sum_{spin} \int d\Omega \; \mathcal{H}^{(s)*}_{\{\tilde{G}\}} \mathcal{H}^{(s)}_{\{\tilde{G}'\}} ,$$

numerical quantities

$$T_{\{\tilde{G}\},\{\tilde{G}'\}} = \sum_{spin} \int d\Omega \; \mathcal{H}^{(s)*}_{\{\tilde{G}\}} T \mathcal{H}^{(s)}_{\{\tilde{G}'\}} ,$$

$$V_{\{\tilde{G}\},\{\tilde{G}'\}} = \sum_{spin} \int d\Omega \; \mathcal{H}^{(s)*}_{\{\tilde{G}\}} V \mathcal{H}^{(s)}_{\{\tilde{G}'\}}$$

operators functions of r

A set of second order coupled differential equations is then obtained

$$\sum_{\{\tilde{G}'\}} \left[T_{\{\tilde{G}\},\{\tilde{G}'\}} - E N_{\{\tilde{G}\},\{\tilde{G}'\}} + V_{\{\tilde{G}\},\{\tilde{G}'\}} \right] u_{\{\tilde{G}'\}}(s) = 0$$

Without loss of generality, the functions $\mathcal{H}^{(s)}_{\{\tilde{G}\}}$ can be taken to correspond to a given grand angular momentum G and the equations can be written in the form

$$\sum_{\{\tilde{G}'\}} \left[-\frac{\hbar^2}{2m} N_{\{\tilde{G}\},\{\tilde{G}'\}} \left(\frac{d^2}{dr^2} + \frac{\delta(\delta+1)}{r^2} - E \right) + \bar{V}_{\{\tilde{G}\},\{\tilde{G}'\}} \right] u_{\{\tilde{G}'\}}(s) = 0$$

where $\delta = G + (D-3)/2$

i) Calculation of N, T and V elements

ii) Solution of the eigenvalue problem $u(s) \to 0$ for $r=0$ or $r \to \infty$

iii) Convergence when the basis is enlarged

i) Note that in general $N_{\{\tilde{G}\},\{\tilde{G}'\}} \neq \delta_{\{\tilde{G}\},\{\tilde{G}'\}}$ since the $\mathcal{H}^S$ functions are sum of terms in the various coordinate sets Ω!
As an example the overlap of $\mathcal{H}_{\{G\}}(\Omega^p)$ and $\mathcal{H}_{\{G\}}(\Omega^{p'})$ is not zero and has to be calculated.
One can follows one of the following possibilities.

- The required hyperangular integrations are done by a numerical technique (Gauss, M.C, QRN, ...)

- $H^p_{\{G\}}$ for the various p can be expressed in terms of the HH functions corresponding to a given permutation, say $p=1 \Rightarrow 1,2,\cdots,A$.
  For $A=3$ the coefficients of the transformation have been obtained by Reynal and Revai

  For $A>3$ the problem is more involved and has been studied by several people (Efros, Viviani, Barnea, ...)

The main difficulty when using the HH expansion is that, due to the enormous degeneracy of the basis function as G increases, one has to manage as many as possible functions (to day, a few thousand).

ii) The solution of the eigenvalue problem does not present severe difficulties since there are available numerical codes which require acceptable computing time.

iii) The convergence problem is quite serious. The HH expansion is successful for $A=3$ and 4 while for larger A values appropriate modifications of the method are required.

The helium atom

The problem of calculating the energy levels of helium atom was just afforded by Hylleraas in 1930. A noticeable accuracy was reached later on by Pekeris. After that a great attention has been devoted to increase the accuracy of theoretical calculations due to the improvements in the experimental data.

Experimental results (B in a.u.)

state	B
$1\,^1S_0$	2.903 694 2839 (33)
$2\,^3S_1$	2.175 236 5117 (33)
$2\,^1S_0$	2.145 974 147 (12)
$2\,^3P_2$	2.133 172 1598 (62)
⋮	

In the work of Pekeris the w.f. was constructed using the "perimetric coordinates"

$$\begin{cases} u = \varepsilon(r_{32} - r_{31} + r_{12}) \\ v = \varepsilon(r_{31} - r_{32} + r_{12}) \\ z = 2\varepsilon(r_{31} + r_{32} + r_{12}) \end{cases}$$

where 3 ≡ nucleus and the w.f. (the radial part of) has the form

$$\psi_N = e^{-\frac{1}{2}(u+v+z)} \, G(u, v, z)$$

with

$$G(u, v, z) = \sum_{i,j,k}^{I,J,K} A_{ijk} \, L_i(u) \, L_j(v) \, L_k(z).$$

$L_m(\xi)$ is the Laguerre polynomial of order m and A_{ijk} are trial parameters to be determined by means of the variational principle.

Such a form of trial function has been generalized by other authors, in particular by Drake et al.:

$$\psi_N = \sum_{ijk} \left[a^{(1)}_{ijk} \chi_{ijk}(\alpha_1,\beta_1) + a^{(2)}_{ijk} \chi_{ijk}(\alpha_2,\beta_2) \right] \pm \text{exchanged term}$$

$a^{(1)}$ and $a^{(2)}$ are trial parameters,

$$\chi_{ijk}(\alpha,\beta) = r_{31}^{i}\, r_{32}^{j}\, r_{12}^{k} \exp\left(-\alpha r_{31} - \beta r_{32}\right)$$

and (α_1,β_1), (α_2,β_2) are two sets of non linear parameters. The exchanged term has the same functional form of the first one but with $r_{31} \leftrightarrow r_{32}$.

Pekeris (Phys. Rev. 126 (1962) 1470

N	B (a.m.u)
95	2.903723389
125	2.903723878
203	2.903724228
210	2.903724311
extr.	2.90372435[illegible]

Drake et al. (Phys. Rev. lett. 77 (1996) 171

N	B (a.m.u)
44	2.903724 131[illegible]
67	2.903724351
135	2.903724376 75
182	2.903724376 96
236	2.903724377 02

N is the number of linear trial parameters, B the ground state energy with $m_a/m_e \to \infty$.

Few a corrections must be added

- finite m_a value
- relativistic corrections
- QED corrections.

These have been treated up to the order $\alpha^2 m_e c^2 \ln \alpha$. One important aim is to use theoretical results and exp. data to improve our knowledge on α and Rydberg constants

The HH method has been applied to the helium atom problem by many people (see, for example, R. Krivec, Few Body Systems 25 (1998) 199). The best estimate for the ground state energy is given in the paper of Fabre de la Ripelle et al., Phys. Rev. A44 (1991) 7084:

$$B = 2.9030674 \qquad \text{(Drake et al. } B = 2.9037243770\text{)}$$

Only the first four digits are correct.

The difficulty is that the HH expansion (due to the singularity at $r = 0$ of the Coulomb potential) is extremely slow.
To remedy to this Hafstad and Mandelzweig, Ann. of Phys. 189 (1989) 29) have modified the expansion basis by multiplying the HH functions by an appropriate correlation factor:

$$\psi = \theta\phi ,$$

where

$$\theta = e^{-\beta r_{12} - \delta(r_{31} + r_{32})} , \qquad \beta = -\frac{1}{2} \qquad \delta = \frac{2m_d}{m_d + 1} .$$

With such a choice of β and δ the behavior at $r \to 0$ of the Coulomb interaction is correctly taken into account.
The function ϕ is expanded in the HH basis. Moreover, the equation to be solved is written as

$$\theta^{-1}\left[H - E\right]\theta\phi = \left(\theta^{-1} T \theta + V - E\right)\phi = 0 .$$

The details of the calculations can be found in the original paper. The conclusion is that the convergence is strongly fastened and with a reasonably small number of HH components the value

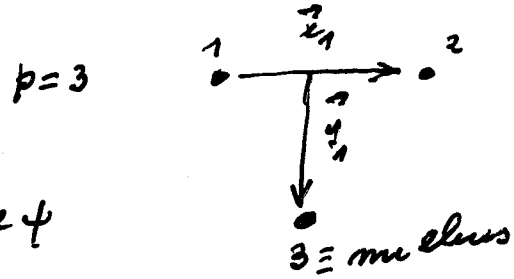
$$B = 2.9037243 68$$

has been obtained.

More recently, the HH expansion has been revisited in the thesis of Paterni (Dep. Phys. PISA). The w.f. has the form

$$\Psi = \psi \chi_{SS_z}(1,2) \qquad \psi = \phi(\vec{x}_1, \vec{y}_1) + \phi(\vec{x}_2, \vec{y}_2) + \phi(\vec{x}_3, \vec{y}_3)$$

where $\vec{x}_i, \vec{y}_i$ are the Jacobi vectors corresponding to the three even permutation of (1, 2, 3).
The amplitude ϕ are then expanded on the HH basis.



It must be noticed that:

- $\phi_3(\vec{x}_3, \vec{y}_3)$ alone is in principle sufficient to produce ψ (but convergence is slow)

- the three ϕ amplitudes are not orthogonal each other so care is necessary to avoid duplication of elements.

With a reasonable number of basis function the estimate B = 2.9028 (B(DRAKE) = 2.903724377) has been obtained, so the situation requires improvements.
To this end, the EHH expansion has been used:

$$\Psi_N = \chi_{SS_z}(1,2) \sum_{p=1}^{3} \left[{}^{(p)}\phi^{HH}(\vec{x}_p, \vec{y}_p) + {}^{(p)}\phi_E^{HH}(\vec{x}_p, \vec{y}_p) \right], \quad p = 1, 2, 3.$$

where both ϕ and ϕ_E are constructed in terms of HH functions. The hyperradial functions in ϕ and ϕ_E are taken as combinations of Laguerre functions ($\exp(-\gamma r) L_m^{\alpha}(\gamma r)$).

Results for the ground state

N_{HH}	CFHHM (Mandelzweig et al.)	EHH (+13 states in $\phi_E(\vec{x}_p, \vec{y}_p)$)
1	2.855504862	2.90347103
4	2.902870977	2.90369776
9	2.903701425	2.90371936
16	2.903718577	2.90372322
25	2.903723654	2.90372388
36	2.90372398 7	2.90372415 ← seven digits are correct
(Drake et al.)		2.90372437470

For the first excited state an accuracy of six digits has been attained.

Note that:

$$e^{-\gamma r_{12} - \delta(r_{13}+r_{23})} = 1 - \gamma s \cos\phi_3 - \delta s(\cos\phi_2 + \cos\phi_3) + \cdots$$

Further calculations are in progress.

The HH- and modified HH-type expansion.

The main difficulty when applying the HH approach to systems with particle interactions strongly state-dependent and containing large repulsion, is the very slow convergence (thousands of components can be necessary).
Also for simple central potentials, the convergence is rapid only for soft core potential, as evident from the paper of Ehrens et al. (see table).

For realistic interactions is therefore important
i) to have some rule for identifying those HH components which give the largest contributions, or

ii) to study appropriate modifications of the expansion to foster the convergence.

i) As regards this point the most interesting approach is the use of the so-called potential basis (PB) firstly introduced by Fabre de la Ripelle.
The underlying idea is to treat in an accurate way the contribution from the pair ~~central~~ correlations which, as has been discussed by J. Navarro in his contribution, are those to be firstly taken into account.
To this end, the PB prescriptions are hereafter discussed.
Let us again write the w.f. in the form

TABLE V

Eigenvalue $|E|$ For the Potentials From Table I as a Function of k_{max}

	$\|E\|$ (MeV)							
k_{max}	*B*	*V*	*EH*	*AT*	*H*	*Y*	*M VI*	*M V*
0	9.2062	7.7080	3.6660	0.3548	8.2849	2.5382	8.3960	0.5660
2	9.6134	8.0786	4.7390	2.1291	9.1954	3.1359	9.5219	1.8386
3	9.7462	8.3296	5.8895	4.6059	9.7496	3.5378	10.2383	4.2234
4	9.7646	8.3759	6.1083	5.2081	9.8681	3.6283	10.3952	4.9824
5	9.7738	8.4162	6.4934	5.8066	9.9733	3.7125	10.5375	5.8854
6	9.7779	8.4428	6.7385	6.2376	10.0536	3.7792	10.6481	6.6316
9	9.7794	8.4609	6.9592	6.5848	10.1306	3.8466	10.7566	7.3762
12	9.7795	8.4641	7.0201	6.6675	10.1577	3.8717	10.7958	7.6217
15	9.7795	8.4646	7.0379	6.6894	10.1695	3.8830	10.8131	7.7136
18	9.7795	8.4647	7.0446	6.6953	10.1754	3.8887	10.8218	7.7521
21						3.8919		7.7697
24						3.8939		7.7785
27						3.8951		7.7831

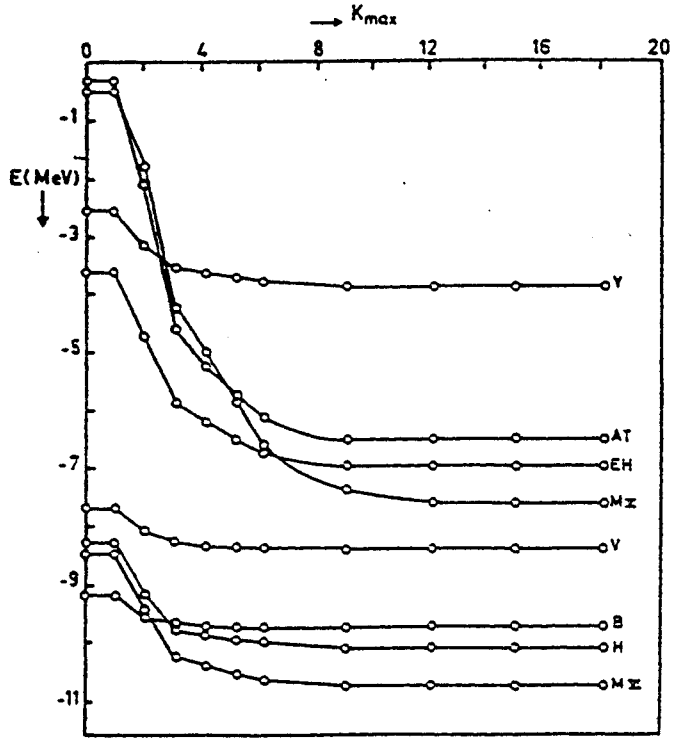


FIG. 1. Illustration of the convergence of the binding energy as a function of k_{max} for the potentials listed in Table I.

$$\psi(1,\dots,A) = \sum_{(p)} a_p \hat{\Phi}^{(p)}(i,j,k,\dots).$$

$p \equiv (i,j,k\dots)$ is a generic permutation of the indices $1,2,\dots,A$

a_p are coefficients which guarantee the correct simmetry properties.

PB basis

In the HH expansion of $\hat{\Phi}^{(p)}$

- only the minimum angular momentum value of the pair i,j (for all the important spin-isospin channels) is considered (all the others being taken $= 0$);
- only the dependence on the hyperangle $\phi_N^{(p)}$ is allowed.

Since $r_{ij}^{(p)} \sim r \cos\phi_N$, by this limited HH expansion with increasing G values, the best function of the type

$$\psi_{PB} = \sum_{(p)} a_p \psi(r_{ij}^{(p)}, r)$$

can be constructed.

For realistic NN interactions also if the PB components give the largest contributions, higher order contributions must be taken into account for an accurate description of the system structure.

iii) As regards this point the modifications more widely investigated are the adiabatic approximation, the correlated HH expansion and the EHH expansion recently proposed.

Nuclear systems ($A=3,4$) with realistic interactions

Goal: find accurate solutions of $H\Psi = E\Psi$.

H nuclear hamiltonian:

$$H = -\frac{\hbar^2}{M}\sum_{i=1}^{A}\nabla^2 + \sum_{i<j}^{A} V(i,j) + \sum_{i<j<k}^{A} W(i,j,k)$$

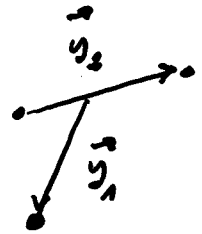
- non relativistic kinetic energy
- $V(i,j)$ nucleon–nucleon (NN) potential
  strong repulsion at short distances
  spin–isospin dependence
  non central (strong tensor component)
- $W(i,j,k)$ three–nucleon (3N) potential

Use of **variational principles**
Expansion of the spatial part of Ψ in the **HH basis**
Due to the strong repulsion at short interparticle distances, the HH basis **converges very slowly**
Modifications of the method $\rightarrow$ CHH & EHH

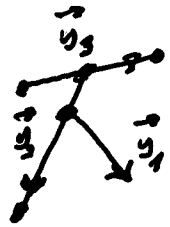
Alternative techniques

- Variational methods (Expansion in a basis of gaussians)
- Faddeev–Yakubovsky equations
- Green Function Monte Carlo method

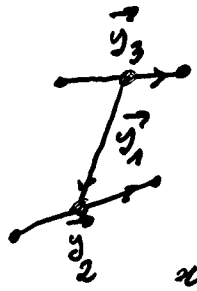
HH calculation of the binding energy.



$A = 3$



set (A)



set (B)

$A = 4$

HH coordinates $(N = A-1)$

- hyperradius $x = \left[\sum_{i=1}^{N} y_i^2\right]^{1/2}$
- $2N$ polar angles $\hat{\omega}_i \equiv (\theta_i, \varphi_i)$
- $N-1$ hyperangles $(i = 2, \dots, N)$ $\cos\phi_i = \dfrac{y_i}{\sqrt{y_1^2 + \dots + y_i^2}}$

$(0 \le \phi_i \le \frac{\pi}{2})$

HH functions

$$\mathcal{Y}_G(\Omega_N) = \left[\prod_{j=1}^{N} Y_{l_j m_j}(\hat{\omega}_j)\right]\left[\prod_{j=2}^{N} N_{n_j}^{l_j \nu_j} (\cos\phi_j)^{l_j} (\sin\phi_j)^{G_{j-1}} \cdot P_{n_j}^{\nu_{j-1}, l_j + \frac{1}{2}}(\cos 2\phi_j)\right]$$

$$G_j = \sum_{i=1}^{j} (l_i + 2n_i) \qquad n_1 = 0 \qquad G \equiv G_N$$

$$\nu_j = G_j + \frac{3j}{2} - 1$$

$$\mathcal{H}_{\{\tilde{G}\}}(\Omega_N) = \sum_{n_1 \dots n_N} (l_1 m_1 l_2 m_2 | L_2 M_2) \cdots (L_{N-1} M_{N-1} l_N m_N | L M)\, \mathcal{Y}_G(\Omega_N)$$

HH spin-isospin basis

$$\mathcal{H}_{\{\tilde{G}\}}(i,j,\dots) = \left\{\mathcal{H}_{\{\tilde{G}\}}(\Omega_N)\left[\left[\left[\sigma_i \sigma_j\right]_{S_2} \sigma_k\right]_{S_3} \cdots\right]_S\right\}_{JJ_2} \left[\left[\left[\tau_i \tau_j\right]_{T_2} \tau_k\right]_{T_3} \cdots\right]_{TT_2}$$

Channels:

$$\alpha \equiv (l_1, l_2 \cdots, l_N, L_2, \cdots, L_{N-1}, L, S_2, \cdots, S_{N-1}, S, T_2, \cdots T_{N-1}, T)$$

(LS coupling)

Antisymmetrization

$$\Phi_{\alpha, n_2, \cdots, n_N} = \sum_{\substack{p(\text{even}) \\ (i,j,\cdots)}} H_{\xi\xi_j}(i,j,k,\cdots).$$

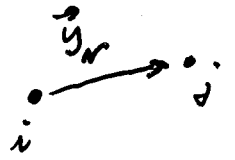
- the sum is limited to even permutations since, by construction, $H_{\xi\xi_j}(i,j,k\cdots)$ is antisymmetric with respect to $i \leftrightarrow j$.
- many functions $\Phi_{\alpha, n_2 \cdots n_N}$ are linearly dependent from each other
- Problem: choice of a small as possible set of functions

For a given channel α the states are ordered as:

1) two-body states (PB basis): only $n_N \neq 0$
2) three-body states (triplet basis): only $n_N, n_{N-1} \neq 0$

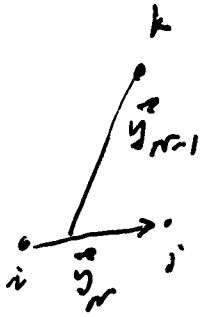
etc. etc.

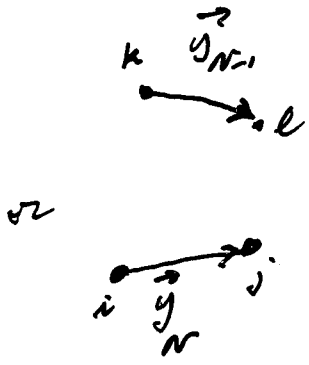
Examples:



$$\cos\phi_N = \frac{y_N}{x}$$

PB basis





$$\cos\phi_{N-1} = \frac{y_{N-1}}{x \sin\phi_N}$$

Triplet basis

Details of the calculation.

- A set of M basis functions $\Phi_{\alpha, n_1, \ldots, n_N} \equiv |\mu\rangle$ is chosen

- Orthonormalisation of the basis

$$|\mu\rangle = \sum_{\mu'=1}^{M} U_{\mu\mu'} |\mu'\rangle ,$$

so that $\langle \mu | \mu' \rangle_\Omega = \delta_{\mu\mu'}$. The linearly dependent functions are then removed.

Total w.f.

$$\psi = r^{-\frac{D-1}{2}} \sum_{\mu=1}^{M} u_\mu(r) |\mu\rangle$$

From the Rayleigh-Ritz principle $\left[\delta_u \langle \psi | H - E | \psi \rangle = 0 \right]$:

$$\left\{ -\frac{\hbar^2}{2m} \left[\frac{d^2}{dr^2} - \frac{\ell_\mu(\ell_\mu+1)}{r^2} \right] - E \right\} u_\mu(r) = - \sum_{\mu'=1}^{M} V_{\mu\mu'}(r)\, u_{\mu'}(r)$$

$$V_{\mu\mu'}(r) = \langle \mu | V | \mu' \rangle$$

$$= \frac{A(A-1)}{2} \langle \mu | V(1,2) | \mu' \rangle \qquad \text{Two-nucleon interac.}$$

$$\sim \int_0^1 d(\cos\phi_N)\, V(r\cos\phi_N)\, P(\cos 2\phi_N)$$

there are transformation coefficients to be determined

A few details on the solution of the differential eqns. set:

. A grid of N points in the hyperradius is fixed

$$r_i = h\,\frac{\eta^i - 1}{\eta - 1}$$

$r_1 = h$, $r_2 = h + \eta * h$ and so on

($h = 0.1 \div 0.5$ fm , $\eta = 1.05 \div 1.15$) $\quad 0 \le r \le 50$ fm $\quad N \approx 50$.

$$\left.\frac{d^2 u(r)}{dr^2}\right|_{r=r_i} = \sum_{j=1}^{N_2} L(i,j)\, u(r_j)$$

In this way one has to solve a standard eigenvalue problem (the quantities to be determined are $u_\mu(r_i)$, $\mu = 1, M$, $i = 1, N$)

The dimensionality of the matrices involved is $N * M * M$.

^{3}H and ^{3}He bound state

α	$\ell_{1\alpha}$	$\ell_{2\alpha}$	L_α	$S_{2\alpha}$	$T_{2\alpha}$	S_α	T_α
1	0	0	0	1	0	1/2	1/2
2	0	0	0	0	1	1/2	1/2
3	0	2	2	1	0	3/2	1/2
4	2	0	2	1	0	3/2	1/2
5	2	2	0	1	0	1/2	1/2
6	2	2	2	1	0	3/2	1/2
7	2	2	1	1	0	1/2	1/2
8	2	2	1	1	0	3/2	1/2
9	1	1	0	1	1	1/2	1/2
10	1	1	1	1	1	1/2	1/2
11	1	1	1	1	1	3/2	1/2
12	1	1	2	1	1	3/2	1/2

26 CHANNELS

The final calculations includes all the channels with $\ell_1 + \ell_2 \leq 6$.

For each channel, states with $G = 2n_2 + \ell_1 + \ell_2$ are included starting from $G^0_\alpha = \ell_1 + \ell_2$ up to a given maximum G_α.

It is very important to study the convergence channel by channel, in fact, the necessary G_α values **can be rather different**.

- Channels $1 - 4$: $G_\alpha \approx 80$
- Channels $5 - 8$: $G_\alpha \approx 40$
- Channels > 8: $G_\alpha \approx 20$ or less

^{3}H binding energy (MeV)

Convergence with G – First 4 channels

G	Volkov	S3$_a$	AV14
0	7.708	0.344	–
4	8.079	2.129	–
8	8.376	5.207	2.803
12	8.443	6.237	5.635
16	8.458	6.500	6.532
20	8.463	6.620	6.973
30	8.466	6.689	7.283
40	8.466	6.697	7.353
50		6.698	7.362
70			7.373
90			7.375
exact	8.466	6.700	7.684

Channel convergence – AV14 potential

N_c	B	$< T >$	$P_{S'}$	P_D
4	7.3751	44.200	1.231	8.169
8	7.6605	45.556	1.129	8.926
12	7.6788	45.650	1.126	8.963
18	7.6822	45.676	1.126	8.967
26	7.6844	45.678	1.126	8.968

Comparison with other techniques

Central potentials

Potential	Method	B (MeV)	T (MeV)	R (fm)	$P_{S'}$
MT(V)	HH	8.2527	30.639	1.681	0
	FE	8.25273			0
	SVM	8.2527		1.682	0
	ATMS	8.26		1.682	0
S3	HH	8.765	33.879	1.673	3.490
	FE	8.765			
	SVM	8.753		1.67	
MT(I – III)	HH	8.881	31.313	1.696	2.416
	IDEA	8.86			

Non Central potentials

Potential	Method	B (MeV)	T (MeV)	R (fm)	$P_{S'}$	P_P	P_D
AV14	HH	7.6844	45.68	1.776	1.126	0.0761	8.968
	FE	7.680					
	FE	7.670			1.12	0.08	8.96
	CRC	7.6844			1.126	0.076	8.968
	GFMC	7.670(8)					
AV18	HH	7.618	46.71	1.769	1.294	0.0658	8.511
	FE	7.62					

FE=Faddeev equations, Friar *et al*, 1985; Glöckle *et al*, 1995
SVM=Stochastic variational method, Suzuki and Varga, 1995
ATMS=amalgamation of two–body correlations into multiple scattering, Akaishi, 1974
IDEA=Integro–differental equation approach, Fabre *et al* 1988
CRC=coupled–rearrangement–channel, Kamimura *et al*, 1989

GFMC=Green function Monte Carlo, Carlson, 1988

α–particle ground state ($J = T = 0$)

The first 8 channels

α	ℓ_1	ℓ_2	ℓ_3	ℓ_{12}	L	S_2	S_3	S	T_2	T_3	T
1	0	0	0	0	0	1	1/2	0	0	1/2	0
2	0	0	0	0	0	0	1/2	0	1	1/2	0
3	0	0	2	0	2	1	3/2	2	0	1/2	0
4	0	2	0	2	2	1	3/2	2	0	1/2	0
5	0	2	2	2	0	1	1/2	0	0	1/2	0
6	0	2	2	2	1	1	1/2	1	0	1/2	0
7	0	2	2	2	1	1	3/2	1	0	1/2	0
8	0	2	2	2	2	1	3/2	2	0	1/2	0

ONLY THE SET A OF JACOBI VECTORS IS USED.

NUMBER OF CHANNELS

Choice of the expansion basis

1. Class C1. In this class are included the HH states belonging to the PB= **two–body correlations**. The PB includes states of the first three channels (the only channels with $\ell_1 = \ell_2 = 0$) and $n_2 = 0$. 3

2. Class C2. This class includes states belonging to the same three channels as class C1, but with $n_2 > 0$. This states therefore, includes also part of **three–body correlations**. 3

3. Class C3. This class includes all the states belonging to the channels $4 - 8$ reported in table. 8

4. Class C4. This class includes all the HH states belonging to the channels with $\ell_1 + \ell_2 + \ell_3 = 4$ not included in classes C1–C3 80

5. Class C5. This class includes all the states belonging to the channels with $\ell_1 + \ell_2 + \ell_3 = 6$. 189

Comparison with other techniques

Central potentials

Potential	Method	B (MeV)	T (MeV)	R(fm)	$P_{S'}$
MT(V)	HH	31.357	69.71	1.409	0
	SVM	31.360		1.4087	0
	FYE	31.36			0
	CRC	31.36			0
$S3_a$	HH	27.412	69.06	1.444	0
	DMC	27.31(5)			0
S3	HH	31.608	74.341	1.413	4.087
	FYE	31.00			

Realistic potentials

Potential	Method	B (MeV)	T (MeV)	R(fm)	$P_{S'}$	P_P	P_D
AV8	HH	25.60	99.41	1.497		0.402	14.46
	SVM	25.62	98.9	1.50			
	GFMC	25.67(3)					
AV14 + C	HH	24.21	95.80	1.524	0.355	0.389	14.40
	GFMC	24.23(3)					
AV18 + C	HH	24.11					
	GFMC	24.1(1)					

SVM=Stochastic variational method, Suzuki and Varga, 1997
FYE=Faddeev–Yakubovsky equations, Glöckle *et al*, 1995
CRC=coupled–rearrangement–channel, Kamimura *et al*, 1989
DMC=Diffusion Monte Carlo, Bishop *et al*, 1990
GFMC=Green function Monte Carlo, Carlson, 1988

Binding Energy (MeV)

Convergence:

G	$C1$	$C2$	$C3$	$C4$	$C5$
8	–	17.394	21.310	23.200	
10	–	18.290	21.593	23.544	
12	–	18.694	21.770	23.858	
14	7.532	18.992	21.898	24.007	24.180
16	9.590	19.173	21.965	24.105	24.190
18	10.929	19.267	22.009	24.147	24.203
20	11.825	19.337	22.035	24.170	24.208
22	12.363	19.381	22.051	24.176	24.210
24	12.752	19.410	22.061	24.178	
26	12.997	19.430	22.066		
28	13.166	19.444	22.070		
30	13.279	19.454	22.072		
32	13.359	19.461			
34	13.413	19.466			
36	13.452	19.470			
38	13.479	19.472			
40	13.499				
50	13.541				
60	13.551				
70	13.554				

NUMBER OF EQS. TO BE SOLVED →

A=4 with AV14 potential - Results by different groups. 2.22

TABLE I The expectation values $\langle T \rangle$ and $\langle V \rangle$ of kinetic and potential energies, the binding energies E_b in MeV, and the radius in fm

Method	$\langle T \rangle$	$\langle V \rangle$	E_b	$\sqrt{\langle r^2 \rangle}$
FY	102.39(5)	−128 33(10)	−25 94(5)	1 485(3)
CRCGV	102 30	−128 20	−25 90	1 482
SVM	102 35	−128 27	−25 92	1.486
HH	102.44	−128 34	−25.90(1)	1.483
GFMC	102.3(1.0)	−128 25(1 0)	−25 93(2)	1 490(5)
NCSM	103.35	−129.45	−25 80(20)	1.485
EIHH	100 8(9)	−126.7(9)	−25.944(10)	1 486

C(r) [fm^{-3}]

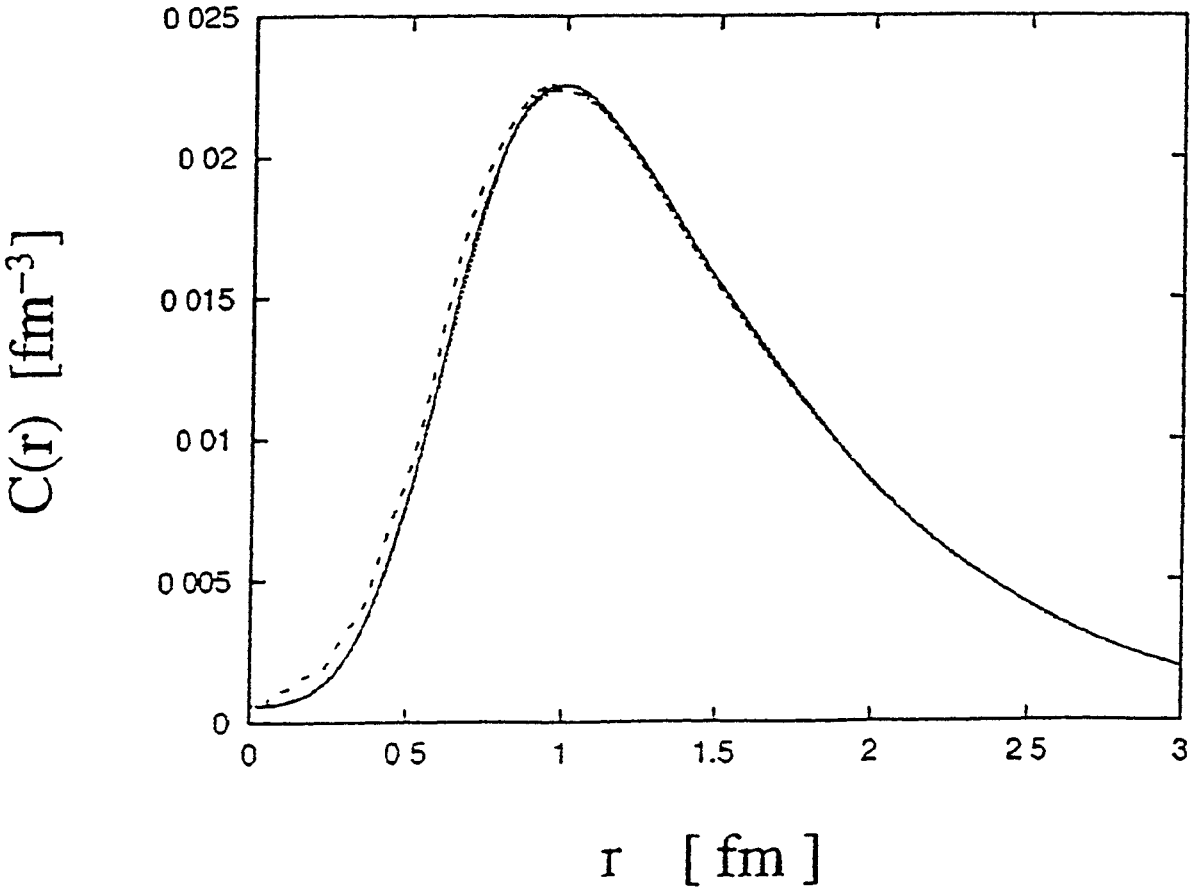


FIG 1 Correlation functions in the different calculational schemes EIHH (dashed-dotted), FY, CRCGV, SVM, HH, and NCSM (overlapping curves)