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SECOND EUROPEAN SUMMER SCHOOL on MICROSCOPIC QUANTUM MANY-BODY THEORIES and their APPLICATIONS

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ELEMENTARY EXCITATIONS AND DYNAMIC STRUCTURE OF QUANTUM FLUIDS
Part I

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Elementary excitations and dynamic structure of quantum fluids

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Contents

1 Introduction: historical developments

Compared to the ground state, the basic theoretical understanding of the microscopic structure of the excitations and of the physical processes involved is relatively incomplete. On the experimental side, however, the elementary excitation spectrum of liquid ^{4}He, along with the underlying dynamic structure function, are known to a **high precision from x-ray and neutron scattering experiments**. (A review of the subject with references to earlier experimental and theoretical work is given by Glyde in his book.[1]) Thus, modern many-body methods face a demanding challenge of reproducing the correct spectrum; a solid agreement would create confidence in the applied tools and indicate that the essential physical ingredients have been understood and included into the calculations. A fully microscopic approach is needed here, because these phenomena defy a simple hydrodynamic Al description.

Before bringing the variational point of view into play, let us quickly summarize the milestones on this path. The preliminary work on the subject was done by Bijl and Landau in the 1940s.[2] **Landau** proposed that there are two separate collective excitation modes in liquid ^{4}He: **phonons**, thought of as collective density (sound) modes having linear dispersion, and **rotons**, assumed to be a collective rotation of the fluid having a separate dispersion curve. Later on, he joined these excitations into a single collective mode dispersion curve continuous in the wave vector.Phonons and rotons were then interpreted as the low- and high-k regions of the same collective excitation. (In this sense the name "roton" is a misnomer.) Between them we have what is called the **maxon** region. This seemed to be in qualitative agreement with experimental data and, at the same time, consistent with the continuous dispersion curve for excitations in a dilute Bose gas derived microscopically by Bogoliubov in his seminal paper.[3]

After these early developments **Feynman** took the first steps towards a microscopic description by suggesting a specific trial excited-state wave function.[4] Specifically, he wrote the wave function of the excited state $\Psi_{\mathbf{k}}$ of momentum $\hbar\mathbf{k}$ as a product $\Psi_{\mathbf{k}} = \rho_{\mathbf{k}}\Psi_0$ of the ground-state wave function Ψ_0 and of a density-fluctuation operator $\rho_{\mathbf{k}} = \Sigma_j \exp(i\mathbf{k}\cdot\mathbf{r}_j)$, offering thus a microscopic explanation for phonons and rotons as collective density excitations at all k. Although the spectrum calculated variationally using the proposed wave function, leading to the dispersion relation

$$\varepsilon_F(k) = \frac{\hbar^2 k^2}{2mS(k)} \tag{1}$$

seemed to contain much of the relevant physics, quantitatively the agreement was far from being satisfactory. The dispersion relation (1) does provide an upper bound for the lowest-lying excitation and is exact in the long-wavelength limit, but has severe deficiencies at shorter wavelengths.

For example, the computed roton energy is twice as large as the experimentally observed value. Owing to this discrepancy, the theory was subsequently supplemented by Feynman and Cohen[5] to include so-called **backflow corrections** which increased the flexibility of the wave function and, thus, lowered the roton energy significantly towards measured values. The term backflow is used to describe the correlated motion of neighboring particles around a given reference atom. In the initial picture put forward by Feynman and Cohen the form of this backflow was not unconstrained: instead, they assumed that the particles move in a dipolar flow field, behaving in a sense like a smoke ring.

After Feynman's original arguments the method of correlated basis functions (CBF) was developed along the same lines to further improve the agreement between theoretical predictions and experimental data, most notably by **Feenberg** and his collaborators,[6, 7, 8, 9, 10] In the **CBF approach** the excited-state wave function $\Psi_{\mathbf{k}}$ is written as $\Psi_{\mathbf{k}} = F_{\mathbf{k}}\Psi_0$, and the excitation operator $F_{\mathbf{k}}$ is further expressed as a **polynomial in the density-fluctuation operators** $\{\rho_{\mathbf{k}}\}$. Thus, in the lowest order we have the usual Feynman form for the excited states, and terms beyond the linear one introduce the backflow effects. Attempts to calculate the dynamic structure function were also made.[11]

More recently the **shadow wave function (SWF) method** has been extended to permit the investigations of excited states.[12, 13] By the provision that the momentum-carrying factor in $\Psi_{\mathbf{k}}$ is a density fluctuation in the subsidiary (shadow) variables, one has, in principle, a parameter-free wave function of the Feynman form in which the fluctuations in the subsidiary variables allow for the presence of backflow effects in the particle variables. Again, this backflow is represented by terms of all orders in the density fluctuation $\{\rho_{\mathbf{k}}\}$ of the real variables. The CBF and SWF methods, along with the application of **released-node Monte Carlo simulations**, all give results which agree reasonably well with experimental data.

Three-dimensional **vortex rings** are also candidates for rotons, and expanding rings could account for the lambda transition.[14] One additional view to the problem is that unlike phonons, rotons might involve the motion of only few ^{4}He atoms, a group forming a **quasi-particle**. In the extreme case this quasi-particle would be just one atom dressed in a superfluid backflow.[15] In the novel picture put forward by Glyde and Griffin the roton is viewed as a **renormalized single-particle mode**. This model has been used extensively to analyze neutron scattering data.[16, 1, 17]

The excitation spectrum of liquid ^{4}He is in two dimensions qualitatively very similar to the three-dimensional spectrum, as one might expect since the physics of liquid ^{4}He is dominated by short-range correlations. What makes the 2D system especially interesting, however, is that the **vortex-antivortex pair excitation** occurs there naturally as a low-lying elementary excitation mode. [18, 19, 20, 21, 22, 23] Therefore, this system forms an ideal framework within which it is possible to study the specific differences between the roton and vortex excitations.

The equation motion method for excited states and dynamics of quantum fluids and fluid mixtures which I will present here has its roots in the early works by Saarela and Suominen.[24, 25, 26] and by Krotscheck[27]. In that method one starts with an Hamiltonian which contains an infinitesimal, external, dynamic interaction which drives excitations into the system and then one looks for the response in the one- and two-particle density distributions. The time dependence of the external interaction creates currents into the system, which are solved together with the density fluctuations from the equations of motion. We limit ourselves into the linear response and thus calculate the **linear response function** and from that the **dynamic structure function**. The method has been applied to homogeneous and inhomogeneous quantum fluids and their mixtures.

To **summarize** our present understanding of the dynamic structure of liquid ^{4}He.
The nature of the collective excitations is fairly well understood.
The role of the Bose condensate in the excitations needs to be clarified.
Quantitative understanding of the behavior of the dynamic structure function $S(k,\omega)$ is still missing.

2 Optimized ground state

In the **microscopic variational theory** we start from the empirical Hamiltonian for the system of N particles with mass m

$$H_0 = -\sum_{i=1}^{N} \frac{\hbar^2}{2m}\nabla_i^2 + \frac{1}{2}\sum_{i\neq j}^{N} V(|\mathbf{r}_i - \mathbf{r}_j|). \tag{2}$$

We assume that the two-particle interaction $V(|\mathbf{r}_i - \mathbf{r}_j|$ is known.
For the **variational wave function** we take the Jastrow-Feenberg ansatz [6, 28, 7]

$$\begin{aligned} \Psi_0(\mathbf{r}_1,\ldots,\mathbf{r}_N) &= e^{\frac{1}{2}U(\mathbf{r}_1,\ldots,\mathbf{r}_N)}\Phi_0(\mathbf{r}_1,\ldots,\mathbf{r}_N) \\ U(\mathbf{r}_1,\ldots,\mathbf{r}_N) &= \frac{1}{2}\sum_{i\neq j}^{N} u_2(\mathbf{r}_i,\mathbf{r}_j) \\ &+\frac{1}{3!}\sum_{i\neq j\neq k}^{N} u_3(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}_k) \end{aligned} \tag{3}$$

The **variational problem** is to minimize the total energy

$$E_0 = \frac{\langle\Psi_0|H_0|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{4}$$

with respect to the correlation functions $u_2(\mathbf{r}_i,\mathbf{r}_j)$ and $u_2(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}_k)$. This leads to the Euler equations

$$\frac{\delta E_0}{\delta u_2(\mathbf{r}_1,\mathbf{r}_2)} = 0 \tag{5}$$

$$\frac{\delta E_0}{\delta u_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)} = 0 \tag{6}$$

2.1 Feenberg's prime-derivative technique

Let's derive the Euler equation by using Feenberg's prime-derivative technique [6]. By definition we have

$$u_2(r;\lambda) \equiv u_2(r) + \lambda \left[V(r) - \frac{\hbar^2}{4m}\nabla^2 u_2(r)\right] \tag{7}$$

where $V(r)$ is the two-particle interaction. In the following we use the notation

$$u_2'(r) = \left.\frac{\partial u_2(r;\lambda)}{\partial \lambda}\right|_{\lambda=0} \tag{8}$$

and ignore λ from the list of arguments of the functions. Then

$$u_2'(r) = V(r) - \frac{\hbar^2}{4m}\nabla^2 u_2(r) \tag{9}$$

Arturo Polls showed to you how to write the energy/particle in the form

$$\frac{E}{N} = \frac{1}{2}\rho_0 \int d^3r\; g(r) \left[V(r) - \frac{\hbar^2}{2m}\nabla^2 u_2(r)\right] \tag{10}$$

where $g(r)$ is the **radial distribution function** and the derived the Euler equation (5) both in coordinate space

$$g'(r) = \frac{\hbar^2}{4m}\nabla^2 g(r) \tag{11}$$

and in momentum space

$$S'(k) = -\frac{\hbar^2 k^2}{4m}\left(S(k) - 1\right) \tag{12}$$

using the structure function $S(k)$.

$$S(k) = 1 + \rho_0 \int d^3r\; (g(r) - 1)\; e^{i\mathbf{k}\cdot\mathbf{r}} \tag{13}$$

Let us define two new quantities the **induced potential** in terms of the sum of nodal diagrams $N(r)$ as

$$w_{\text{ind}}(r) = N'(r) + \frac{\hbar^2}{4m}\nabla^2 N(r)\,, \tag{14}$$

and the **particle-hole effective interaction** in terms of the direct correlation function $X(r)$

$$V_{p-h}(r) = X'(r) + \frac{\hbar^2}{4m}\nabla^2 X(r) \tag{15}$$

The **HNC-equation** connects these quantities

$$\begin{aligned} & g(r) = e^{u_2(r)+N(r)} && (16)\\ \Rightarrow\quad & u_2(r) = \log g(r) - N(r) && (17)\\ & g'(r) = g(r)\left(u_2'(r) + N'(r)\right) && (18)\end{aligned}$$

Inserting $u_2(r)$ into the definitions (9) and (18) we get

$$\begin{aligned} g'(r) &= g(r)V(r) + g(r)N'(r) \\ &+ \frac{\hbar^2}{4m}\left[4\left(\nabla\sqrt{g(r)}\right)^2 - \nabla^2 g(r) + g(r)\nabla^2 N(r)\right] \end{aligned} \tag{19}$$

Subtract $N'(r)$ from both sides of Eq. (19) and use the **definitions**

$$\begin{aligned} X(r) &= g(r) - 1 - N(r) \\ X'(r) &= g'(r) - N'(r)\,. \end{aligned} \tag{20}$$

Then

$$\begin{aligned} X'(r) &= g(r)V(r) \\ &+ (g(r)-1)\left[N'(r) + \frac{\hbar^2}{4m}\nabla^2 N(r)\right] \\ &+ \frac{\hbar^2}{4m}\left[4\left(\nabla\sqrt{g(r)}\right)^2 - \nabla^2 X(r)\right] \end{aligned} \tag{21}$$

Inserting the definitions (14) and (15) we get the expression for the particle-hole potential

$$\begin{aligned} V_{p-h}(r) &= g(r)V(r) + (g(r)-1)\,w_{\text{ind}}(r) \\ &+ \frac{\hbar^2}{m}\left(\nabla\sqrt{g(r)}\right)^2 \end{aligned} \tag{22}$$

On the other hand from the **Ornstein-Zernike relation**

$$N(|\mathbf{r}_1 - \mathbf{r}_2|) = \int d^3r_3 (g(|\mathbf{r}_1 - \mathbf{r}_3|) - 1) X(|\mathbf{r}_3 - \mathbf{r}_2|) \tag{23}$$

together with the definition (20) we get

$$\tilde{X}(k) = 1 - \frac{1}{S(k)} \tag{24}$$

The priming operation and the use of Eq. (12) gives

$$\tilde{X}'(k) = \frac{S'(k)}{S^2(k)} = -\frac{\hbar^2 k^2}{4m}\frac{(S(k)-1)}{S^2(k)} \tag{25}$$

Inserting these into the definition (15) we get

$$\begin{aligned} \tilde{V}_{p-h}(k) &= -\frac{\hbar^2 k^2}{4m}\left(\frac{S(k)-1}{S^2(k)} + \frac{S(k)-1}{S(k)}\right) \\ &= -\frac{\hbar^2 k^2}{4m}\left(1 - \frac{1}{S^2(k)}\right) \end{aligned} \tag{26}$$

which is the **Euler equation** for the homogeneous one-component quantum fluid,

$$S(k) = \frac{k}{\sqrt{k^2 + \frac{4m}{\hbar^2}\tilde{V}_{p-h}(k)}} \tag{27}$$

We still need to calculate the **induced potential** $w_{ind}(r)$ defined in Eq. (14). From the Ornstein-Zernike we get

$$\tilde{N}(k) = \frac{(S(k)-1)^2}{S(k)} \tag{28}$$

The priming operation and the use of Eqs. (12) and (25) gives

$$\tilde{N}'(k) = S'(k) - \tilde{X}'(k) = -\frac{\hbar^2 k^2}{4m}\frac{(S(k)-1)^2 (S(k)+1)}{S^2(k)} \tag{29}$$

Inserting these into the definition (14) we get

$$\begin{aligned} \tilde{w}_{ind}(k) &= -\frac{\hbar^2 k^2}{4m}(S(k)-1)^2\left(\frac{S(k)+1}{S^2(k)} + \frac{1}{S(k)}\right) \\ &= -\frac{\hbar^2 k^2}{4m}(2S(k)+1)\left(1 - \frac{1}{S(k)}\right)^2 \end{aligned} \tag{30}$$

3 Equation of motion method

We begin with an assumption that the correlation functions in the ground state wave function are optimized. It means that the system is stable against small perturbations (linear) around that solution. Let us assume that we disturb the system with an infinitesimal, external interaction. The system responds to that by changing its density, but because it is in the optimized ground state all terms linear in small changes in correlation functions disappear.

$$\frac{E}{N} = \frac{E_0}{N} + \frac{\delta E/N}{\delta U}\bigg|_{\min}\delta U + \frac{1}{2}\frac{\delta^2 E/N}{\delta U^2}\bigg|_{\min}[\delta U]^2 + O[\delta U^3] \tag{31}$$

and one is left with the quadratic terms. Using the least action principle we can derive the continuity equations which optimize the fluctuations in the correlation functions and give the change in the density caused by the infinitesimal external disturbance. These ideas lead to the linear response theory.

3.1 Linear response theory

Let us disturb the system using an external interaction $U_{\rm ext}(k,\omega)$ with a given frequency ω and wave number k. The change in the density of a homogeneous system $\delta\rho_1(k,\omega)$ will have the same frequency and wave number and the information of the dynamic properties of the system is contained in the **linear-response function** defined as

$$\chi(k,\omega) = \frac{\delta\tilde{\rho}_1(\mathbf{k};\omega)}{\rho_0 U_{\rm ext}(k,\omega)}, \tag{32}$$

The imaginary part of the linear-response function defines the **dynamic structure function**

$$S(k,\omega) = -\frac{1}{2\pi}\Im m\,[\chi(k,\omega)], \tag{33}$$

which is the measured quantity in the scattering experiments.

At low temperatures $S(k,\omega)$ consists of a sharp peak and of a broad contribution. It is therefore customary to write $S(k,\omega)$ as

$$S(k,\omega) = Z(k)\delta(\omega-\omega_0(k)) + S_{\rm mp}(k,\omega). \tag{34}$$

This suggests that the linear response function can be written in the form

$$\chi(k,\omega) = \frac{2S(k)}{\hbar\omega - \hbar\omega' - \Sigma(k,\omega')} \tag{35}$$

The quantity $Z(k)$, the residue of the response function at the pole $\omega = \omega_0(k)$, can be evaluated from the derivative of the **self energy** $\Sigma(k,\omega)$

$$Z(k) = \left[1 - \left.\frac{d\Sigma(k,\omega)}{d(\hbar\omega)}\right|_{\omega=\omega_0}\right]^{-1}, \tag{36}$$

and gives the strength of the sharp peak, whereas $S_{\rm mp}(k,\omega)$ gives what is called the **multi-phonon background**, *i.e.* the contribution in which the neutron probing the system exchanges energy with two or more excitations.

In addition, the relative weight $Z(k)/S(k)$ gives the efficiency of the single collective excitation scattering processes, as seen from the (zeroth-moment) **sum rule**

$$\int_0^\infty S(k,\omega)d\omega = S(k) = \langle\rho_{-\mathbf{k}}\rho_{\mathbf{k}}\rangle/N \tag{37}$$

In other words it gives the fraction of the available scattering processes at a given wave number, which go through a single collective mode. If the excitation were a simple density wave, as assumed in the **Feynman theory**, this ratio would be

$$\frac{Z(k)}{S(k)} = 1 \tag{38}$$

3.2 Time-dependent correlation functions

If a weak, time dependent interaction perturbs the system then the ground-state wave function, $\Psi_0(\mathbf{r}_1, \ldots, \mathbf{r}_N)$, is modified accordingly and the correlation functions become time dependent,

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N; t) = e^{-iE_0 t/\hbar}\Phi(\mathbf{r}_1, \ldots, \mathbf{r}_N; t) \tag{39}$$

with

$$\begin{aligned} \Phi(\mathbf{r}_1, \ldots, \mathbf{r}_N; t) &= \frac{1}{\sqrt{\mathcal{N}(t)}}\,\phi(\mathbf{r}_1, \ldots, \mathbf{r}_N; t) \\ \phi(\mathbf{r}_1, \ldots, \mathbf{r}_N; t) &= e^{\frac{1}{2}\delta U(\mathbf{r}_1, \ldots, \mathbf{r}_N; t)}\Psi_0(\mathbf{r}_1, \ldots, \mathbf{r}_N). \end{aligned} \tag{40}$$

The **excitation operator**

$$\delta U(\mathbf{r}_1, ..., \mathbf{r}_N; t) = \sum_i \delta u_1(\mathbf{r}_i; t) + \sum_{i<j} \delta u_2(\mathbf{r}_i, \mathbf{r}_j; t) \tag{41}$$

is a complex function and represents fluctuations in the correlation functions due to this external perturbance.

The time-dependent **one-body function** $\delta u_1(\mathbf{r}_i; t)$ must be included into the description since the dynamics will normally break the translational invariance of the system, but restricting the time dependence to the one-body component only would lead directly to the Feynman theory of excitations.
The time-dependent **two-body component** is significant in situations where the external field excites fluctuations of wavelengths comparable to the inter-particle distance, as explicitly demonstrated in Refs. [29, 25, 26, 24, 10] for liquid ^{4}He and in Ref. [30] for the bosonic Coulomb system. With these terms included the excitation operator then has a two-phonon basis,

$$\sum_{i,j} \delta u_2(\mathbf{r}_i, \mathbf{r}_j; t) = \int \frac{d^3k_1 d^3k_2}{(2\pi)^6 \rho_0} \delta u_2(\mathbf{k}_1, \mathbf{k}_2; t)\rho_{\mathbf{k}_1}\rho_{\mathbf{k}_2}\,. \tag{42}$$

In the wave function (40) the **optimized** ground state $\Psi_0(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ satisfies the Schrödinger equation

$$H_0\Psi_0 = E_0\Psi_0 \tag{43}$$

where H_0 is the **ground state Hamiltonian** given in Eq. (2) and E_0 appearing in the phase factor of definition Eq. (39) is the **ground state energy**. The

normalization factor contains the ratio between the ground state and excited state normalizations.

$$\mathcal{N}(t) = \frac{\int d^3r_1 \ldots d^3r_N |\Psi_0(\mathbf{r}_1, \ldots \mathbf{r}_N)|^2 e^{\Re e[\delta U(\mathbf{r}_1, \ldots \mathbf{r}_N; t)]}}{\langle \Psi_0 | \Psi_0 \rangle} \tag{44}$$

3.3 Action integral

The new Hamiltonian

$$H(t) = H_0 + \sum_i U_{\text{ext}}(\mathbf{r}_i; t) \,. \tag{45}$$

which contains the infinitesimal external potential $U_{\text{ext}}(\mathbf{r}; t)$ is now time dependent and must satisfy the least-action principle [31, 32]

$$\begin{aligned} \delta \mathcal{S} &= \delta \int_{t_0}^{t} dt' \mathcal{L}(t') \qquad (46) \\ &= \delta \int_{t_0}^{t} dt' \left\langle \Psi(t') \middle| H(t) - i\hbar \frac{\partial}{\partial t'} \middle| \Psi(t') \right\rangle = 0, \end{aligned}$$

We make two **assumptions** in the evaluation of the action integral. Firstly, we require that the ground-state correlation functions are **optimized**. This is important because it eliminates all contributions to the action integral that are linear in the time-dependent correlation functions. Secondly, we assume that the **perturbation is weak** which allows us to keep only the quadratic terms – and warrants the use of the **linear-response theory**.

Using the ground state Schrödinger equation (43) we can write the **integrand** in the form

$$\begin{aligned} \mathcal{L}(t) &= \left\langle \Phi(t) \middle| H - E_0 - \frac{\hbar}{2}\left(i\frac{\partial}{\partial t} + h.c.\right) \middle| \Phi(t) \right\rangle \\ &= \frac{1}{\mathcal{N}(t)} \left\langle \Psi_0 \middle| e^{\frac{1}{2}\delta U^*(t)} [H_0, e^{\frac{1}{2}\delta U(t)}] \middle| \Psi_0 \right\rangle \\ &+ \left\langle \Phi \middle| -\frac{\hbar}{2}\left(i\frac{\partial}{\partial t} + h.c.\right) + U_{ext}(t) \middle| \Phi \right\rangle \qquad (47) \end{aligned}$$

The potential energy term commutes with $\delta U(t)$ and thus only the **kinetic energy** gives contribution to the commutator. This can be evaluated with a little bit of algebra,

$$\begin{aligned} & \int d\tau \; \Psi_0 \; e^{\frac{1}{2}\delta U^*} \nabla^2 \left(e^{\frac{1}{2}\delta U} \; \Psi_0 \right) \\ = & \int d\tau \Psi_0 e^{\frac{1}{2}\delta U^*} \nabla \cdot \left(\frac{1}{2} e^{\frac{1}{2}\delta U} \Psi_0 \nabla \delta U + e^{\frac{1}{2}\delta U} \nabla \Psi_0 \right) \end{aligned}$$

$$\begin{aligned}
&= \int d\tau\Big[-e^{\frac{1}{2}\delta U^*}\left(\frac{1}{2}\Psi_0\nabla\delta U^* + \nabla\Psi_0\right)\frac{1}{2}e^{\frac{1}{2}\delta U}\Psi_0\nabla\delta U \\
&+ \Psi_0 e^{\Re e[\delta U]}\left(\frac{1}{2}\nabla\Psi_0\cdot\nabla\delta U + \nabla^2\Psi_0\right)\Big] \\
&= \int d\tau\left[-\frac{1}{4}|\Psi_0|^2 e^{\Re e[\delta U]}\left|\nabla\delta U\right|^2 + \Psi_0 e^{\Re e[\delta U]}\nabla^2\Psi_0\right],
\end{aligned}$$

giving the result

$$\begin{aligned}
&\frac{1}{\mathcal{N}(t)}\left\langle\Psi_0\left|e^{\frac{1}{2}\delta U^*}[H_0, e^{\frac{1}{2}\delta U}]\right|\Psi_0\right\rangle \\
&= \frac{\hbar^2}{8m}\left\langle\Phi\left|\sum_{j=1}^{N}\left|\nabla_j\delta U\right|^2\right|\Phi\right\rangle
\end{aligned} \tag{48}$$

The evaluation of the **time derivative** gives

$$\begin{aligned}
&- \frac{\hbar}{2}\left\langle\Psi_0\left|\frac{e^{\frac{1}{2}\delta U^*(t)}}{\sqrt{\mathcal{N}(t)}}\left(i\frac{\partial}{\partial t} + h.c.\right)\frac{e^{\frac{1}{2}\delta U(t)}}{\sqrt{\mathcal{N}(t)}}\right|\Psi_0\right\rangle \\
&= \frac{1}{2}\hbar\left\langle\Phi\left|\Im m[\delta\dot{U}(t)]\right|\Phi\right\rangle
\end{aligned} \tag{49}$$

where we have used the dot-notation $\dot{f}(t) = \frac{\partial f(t)}{\partial t}$.
Collecting all together we have the integrand

$$\mathcal{L}(t) = \left\langle\Phi\left|\frac{\hbar^2}{8m}\sum_{j=1}^{N}\left|\nabla_j\delta U\right|^2 + \frac{1}{2}\hbar\ \Im m[\delta\dot{U}] + U_{ext}\right|\Phi\right\rangle \tag{50}$$

3.4 Least action principle

In the **least action principle** we search for the correlation function which minimizes the action integral (46). Let's assume that our excitation operator $\delta U = \delta U(\mathbf{r}_1, \ldots, \mathbf{r}_n; t)$ depends on n coordinates and the time. Then the variation of the action integral

$$\delta\mathcal{S} = \delta\int_{t_0}^{t} dt'\mathcal{L}(t') = 0 \tag{51}$$

with respect to δU^* gives

$$\begin{aligned}
&\int dr_{n+1}\ldots dr_N\Big[-\frac{\hbar^2}{8m}\sum_{j=1}^{n}\nabla_j\cdot\left(|\Phi|^2\nabla_j\delta U\right) \\
&- \frac{i\hbar}{4}\frac{\partial}{\partial t}|\Phi|^2 + \left(\frac{\hbar}{2}\Im m[\delta\dot{U}] + U_{ext}\right)\frac{1}{\mathcal{N}(t)}\frac{\partial|\phi|^2}{\partial\delta U^*}\Big] \\
&- \left\langle\Phi\left|\frac{\hbar}{2}\ \Im m[\delta\dot{U}] + U_{ext}\right|\Phi\right\rangle\frac{1}{\mathcal{N}(t)}\frac{\partial\mathcal{N}(t)}{\partial\delta U^*}
\end{aligned} \tag{52}$$

The derivatives can be calculated from the definitions (40)

$$
\begin{aligned}
\frac{\partial|\phi|^2}{\partial\delta U^*} &= \frac{1}{2}|\phi|^2 \\
\frac{1}{\mathcal{N}}\frac{\partial\mathcal{N}}{\partial\delta U^*} &= \frac{1}{2}\int dr_{n+1}\ldots dr_N|\Phi|^2 \\
\frac{\partial}{\partial t}|\Phi|^2 &= |\Phi|^2\left[\Re e\delta\dot{U} - \int dr_1\ldots dr_N|\Phi|^2\Re e\delta\dot{U}\right]
\end{aligned}
$$

and the **least action principle** can be written in the form

$$
\begin{aligned}
&\int dr_{n+1}\ldots dr_N\Big[-\frac{\hbar^2}{4m}\sum_{j=1}^{n}\nabla_j\cdot\left(|\Psi|^2\nabla_j\delta U\right) \\
&+|\Psi|^2\left(-\frac{i\hbar}{2}\delta\dot{U}+U_{ext}\right) \\
&-|\Psi|^2\left\langle\Psi\left|-\frac{i\hbar}{2}\delta\dot{U}+U_{ext}\right|\Psi\right\rangle\Big]=0
\end{aligned}
\tag{53}
$$

3.5 Many-particle densities

In order to simplify Eq. (53) we define the n**-particle density**,

$$
\begin{aligned}
&\rho_n(\mathbf{r}_1,\ldots\mathbf{r}_n;t) \\
=&\ \frac{N!}{(N-n)!}\int d^3r_{n+1}\ldots d^3r_N|\Psi(\mathbf{r}_1,\ldots\mathbf{r}_N;t)|^2
\end{aligned}
\tag{54}
$$

In the linear response theory one assumes that the time dependent perturbance of the system is infinitesimal and hence we can separate the time dependent and independent parts in the density,

$$
\rho_n(\mathbf{r}_1,\ldots\mathbf{r}_n;t)=\bar{\rho}_n(\mathbf{r}_1,\ldots\mathbf{r}_n)+\delta\rho_n(\mathbf{r}_1,\ldots\mathbf{r}_n;t) \tag{55}
$$

Expanding to the first order in δU we get

$$
\begin{aligned}
\delta\rho_n(\mathbf{r}_1,\ldots\mathbf{r}_n;t) &= \frac{N!}{(N-n)!}\int dr_{n+1}\ldots dr_N|\Psi_0|^2 \\
&\times\ \left[\delta U-\langle\Psi_0\,|\delta U|\,\Psi_0\rangle\right]
\end{aligned}
\tag{56}
$$

The physical density is a real quantity and δU should be replaced with its real part. Here we have generalized the definition to complex density fluctuations.

3.6 Many-particle currents

Similarly we define the **n-particle current**

$$\begin{aligned} & \mathbf{j}_n(\mathbf{r}_1,\ldots\mathbf{r}_n;t) \\ = & \frac{\hbar}{2mi}\frac{N!}{(N-n)!}\int d^3r_{n+1}\ldots d^3r_N \\ & \Psi^*(\mathbf{r}_1,\ldots\mathbf{r}_N;t)\sum_{j=1}^{n}\nabla_j\Psi(\mathbf{r}_1,\ldots\mathbf{r}_N;t) \end{aligned} \tag{57}$$

$$\begin{aligned} \approx & \frac{\hbar}{2mi}\frac{N!}{(N-n)!}\int d^3r_{n+1}\ldots d^3r_N \\ & |\Psi_0(\mathbf{r}_1,\ldots\mathbf{r}_N)|^2\sum_{j=1}^{n}\nabla_j\delta U \end{aligned} \tag{58}$$

3.7 Continuity equations in homogeneous fluids

Let us assume that our **system is homogeneous** and the single-particle density of the ground state is constant.

$$\bar{\rho}_1(\mathbf{r}) = \rho_0 = \text{constant} \tag{59}$$

It is convenient to introduce the **n-particle distribution function**

$$g_n(\mathbf{r}_1,\ldots\mathbf{r}_n) = \frac{1}{(\rho_0)^n}\bar{\rho}_n(\mathbf{r}_1,\ldots\mathbf{r}_n) \tag{60}$$

With these assumptions we derive the **one- and two-particle continuity equations** from the general Euler equation (53). These are the **equations of motion** of the system [29, 25, 26]

$$\nabla_1\cdot\mathbf{j}_1(\mathbf{r}_1;t) + i\delta\dot{\rho}_1(\mathbf{r}_1;t) = D_1(\mathbf{r}_1;t) \tag{61}$$

$$\begin{aligned} & [\nabla_1\cdot\mathbf{j}_2(\mathbf{r}_1,\mathbf{r}_2;t) + \text{same for } (1\leftrightarrow 2)] \\ & + i\delta\dot{\rho}_2(\mathbf{r}_1,\mathbf{r}_2;t) = D_2(\mathbf{r}_1,\mathbf{r}_2;t)\,. \end{aligned} \tag{62}$$

The **terms with time derivatives** in Eq. (53) are exactly time derivatives of the density. Inserting the definition of the excitation operator (41) into the definition of the density (56) we get for the one-particle density

$$\begin{aligned} & \delta\rho_1(\mathbf{r}_1;t) = \rho_0\;\delta u_1(\mathbf{r}_1;t) \\ + & \rho_0^2\int d^3r_2(g_2(\mathbf{r}_1,\mathbf{r}_2)-1)\delta u_2(\mathbf{r}_2;t) \\ + & \rho_0^2\int d^3r_2\Big[(g_2(\mathbf{r}_1,\mathbf{r}_2)\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t) \\ + & \frac{\rho_0}{2}\int d^3r_3\,(g_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)-g_2(\mathbf{r}_2,\mathbf{r}_3))\,\delta u_2(\mathbf{r}_2,\mathbf{r}_3;t)\Big] \end{aligned} \tag{63}$$

From the definition (56) one sees directly that the **particle number is conserved** in the fluctuations

$$\int d^3r\delta\rho_1(\mathbf{r}) = 0 \tag{64}$$

and that the **sequential relation is satisfied,**

$$\int d^3r_2\delta\rho_2(\mathbf{r}_1,\mathbf{r}_2;t) = (N-1)\delta\rho_1(\mathbf{r}_1;t) \tag{65}$$

The **one- and two-particle currents** are

$$\begin{aligned}
\mathbf{j}_1(\mathbf{r}_1;t) &= \frac{\hbar\rho_0}{2mi}\Big\{\nabla_1\delta u_1(\mathbf{r}_1;t) \qquad (66)\\
&+ \rho_0\int d^3r_2 g_2(\mathbf{r}_1,\mathbf{r}_2)\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t)\Big\}\\
\mathbf{j}_2(\mathbf{r}_1,\mathbf{r}_2;t) &= \frac{\hbar\rho_0^2}{2mi}\Big\{g_2(\mathbf{r}_1,\mathbf{r}_2)[\nabla_1\delta u_1(\mathbf{r}_1;t)\\
&+ \nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t)] \qquad (67)\\
&+ \rho_0\int d^3r_3 g_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_3;t)\Big\}.
\end{aligned}$$

The currents also satisfy the **sequential condition**

$$\int d^3r_2\; \mathbf{j}_2(\mathbf{r}_1,\mathbf{r}_2;t) = (N-1)\; \mathbf{j}_1(\mathbf{r}_1;t)\,. \tag{68}$$

The terms which depend on the **external potential** are collected into the functions $D_1(\mathbf{r}_1;t)$ and $D_2(\mathbf{r}_1,\mathbf{r}_2;t)$ and they drive excitations into the system

$$\begin{aligned}
D_1(\mathbf{r}_1;t) = \frac{2\rho_0}{\hbar}\Big\{&U_{\text{ext}}(\mathbf{r}_1;t) \qquad (69)\\
+\rho_0\int d^3r_2\Big[&g_2(\mathbf{r}_1,\mathbf{r}_2)-1\Big]U_{\text{ext}}(\mathbf{r}_2;t)\Big\}
\end{aligned}$$

$$\begin{aligned}
D_2(\mathbf{r}_1,\mathbf{r}_2;t) &= \frac{2\rho_0^2}{\hbar}\Big\{g_2(\mathbf{r}_1,\mathbf{r}_2)\,[U_{\text{ext}}(\mathbf{r}_1;t)+U_{\text{ext}}(\mathbf{r}_2;t)]\\
&+\rho_0\int d^3r_3\Big[g_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)-g_2(\mathbf{r}_1,\mathbf{r}_2)\Big]U_{\text{ext}}(\mathbf{r}_3;t)\Big\}\,. \qquad (70)
\end{aligned}$$

4 Solving the continuity equations

Up to now, we have formulated the problem in terms of a Hamiltonian, a trial wave function, and the action principle. What we still need to do is to find a way to actually solve the continuity equations. They still contain four unknown quantities, namely $\delta u_1(\mathbf{r};t)$ and $\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t)$ and time derivatives of $\delta\rho_1(\mathbf{r};t)$ and $\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t)$. Assuming that all ground state quantities are known. Clearly they are not independent, but connected by the definition (56). In the following we introduce various approximation schemas to put that definition into solving the continuity equations.

In the homogeneous system fluctuations are weak and it is more convenient to work in the **Fourier space**. We define the one-particle Fourier transform as

$$\delta\tilde{u}_1(\mathbf{k};\omega) = \rho_0 \int d^3r\ dt\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\delta u_1(\mathbf{r};t)$$
$$\delta u_1(\mathbf{r};t) = \int \frac{d^3k\ d\omega}{(2\pi)^4\rho_0} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\delta\tilde{u}_1(\mathbf{k};\omega)$$

and similarly for the two-particle Fourier transforms

$$\mathcal{F}[f(\mathbf{r}_1,\mathbf{r}_2;t)] = \rho_0^2 \int d^3r_1 d^3r_2\ dt\ e^{i(\mathbf{k}\cdot\mathbf{R}+\mathbf{p}\cdot\mathbf{r}-\omega t)} f(\mathbf{r}_1,\mathbf{r}_2;t)$$
$$\mathcal{F}^{-1}[\tilde{f}(\mathbf{k},\mathbf{p};t)] = \int \frac{d^3k d^3p\ d\omega}{(2\pi)^7\rho_0^2} e^{-i(\mathbf{k}\cdot\mathbf{R}+\mathbf{p}\cdot\mathbf{r}-\omega t)}\tilde{f}(\mathbf{k},\mathbf{p};\omega)$$

where $\mathbf{R} = (\mathbf{r}_1+\mathbf{r}_2)/2$ is the center-of-mass vector, and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ the relative position vector.

4.1 Feynman approximation

Let us first calculate the simplest approximation where we let only the one-particle correlation function vary with time. This leads to the Feynman result for excitations.

We need to solve the first continuity equation (61) in momentum space with the assumption $\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t) = 0$ Then the current is simply

$$\mathbf{j}_1(\mathbf{r}_1;t) = \frac{\hbar\rho_0}{2mi}\nabla_1\delta u_1(\mathbf{r}_1;t) \tag{71}$$

and the time dependent part of the density

$$\begin{aligned}\delta\rho(\mathbf{r}_1;t) &= \rho_0\delta u_1(\mathbf{r}_1;t) \\ &+ \rho_0^2\int d^3r_2 h(\mathbf{r}_1,\mathbf{r}_2)\delta u_1(\mathbf{r}_2;t)\end{aligned} \tag{72}$$

with $h(\mathbf{r}_1,\mathbf{r}_2) \equiv g(\mathbf{r}_1,\mathbf{r}_2) - 1$. The Fourier transforms can be readily calculated giving

$$\delta\tilde{\rho}_1(\mathbf{k};\omega) = S(k)\rho_0\delta\tilde{u}_1(\mathbf{k};\omega) \tag{73}$$

and then

$$\delta\tilde{u}_1(\mathbf{k};\omega) = \frac{1}{\rho_0 S(k)}\delta\tilde{\rho}_1(\mathbf{k};\omega) \tag{74}$$

Similarly we can calculate the Fourier transform of the contribution from the external potential (69). Inserting these results into the continuity equation (61) we get

$$-\frac{\hbar k^2}{2mS(k)}\delta\tilde{\rho}_1(\mathbf{k};\omega)+\omega\delta\tilde{\rho}_1(\mathbf{k};\omega)=\frac{2\rho_0}{\hbar}S(\mathbf{k})U_{\rm ext}(k,\omega) \tag{75}$$

and we can solve the linear response function

$$\chi(k,\omega)=\frac{\delta\tilde{\rho}_1(\mathbf{k};\omega)}{\rho_0 U_{\rm ext}(k,\omega)}=\frac{2S(k)}{\hbar\omega-\frac{\hbar^2k^2}{2mS(k)}} \tag{76}$$

The poles of $\chi(k,\omega)$ give the elementary excitations of the system

$$\varepsilon_F(k)=\hbar\omega=\frac{\hbar^2k^2}{2mS(k)} \tag{77}$$

and the limit $\omega=0$ the static response function

$$\chi(k,0)=-\frac{4mS^2(k)}{\hbar^2k^2} \tag{78}$$

The imaginary part of the linear response function determines the dynamic structure function (33)

$$S(k,\omega)=-\frac{1}{2\pi}\Im m\left[\chi(k,\omega)\right]=S(k)\delta(\hbar\omega-\varepsilon_F(k)) \tag{79}$$

which means that the Feynman approximation is a single pole approximation and the strength of the pole is the structure function.
In ^{4}He the excitation mode is linear in the long wave length limit and proportional to the speed of sound c,

$$\varepsilon_F\to\hbar kc\,. \tag{80}$$

The structure function is also linear at small k

$$S(k)\to\frac{\hbar k}{2mc} \tag{81}$$

and the inverse of the static response function determines the incompressibility

$$-\chi^{-1}(k,0)\to mc^2 \tag{82}$$

The elementary excitation modes of the system are obtained also directly by setting the external potential $U_{ext}=0$ in the continuity equations. Using the Feynman approximation and the results (71) and (72) we get a differential equation

$$\begin{aligned}&\frac{\hbar\rho_0}{2mi}\nabla_1^2\delta u_1(\mathbf{r}_1;\omega)-i\omega\Big[\rho_0\delta u_1(\mathbf{r}_1;\omega) \\ +\;&\rho_0^2\int d^3r_2h(\mathbf{r}_1,\mathbf{r}_2)\delta u_1(\mathbf{r}_2;\omega)\Big]=0\end{aligned} \tag{83}$$

which has the solution (77) and

$$\delta u_1(\mathbf{r}_1;\omega) = e^{i\mathbf{k}\cdot\mathbf{r}_1} \tag{84}$$

The excitation operator has then the Feynman form

$$\delta U = \sum_j \delta u_1(\mathbf{r}_j;\omega) = \sum_j e^{i\mathbf{k}\cdot\mathbf{r}_j} \tag{85}$$

5 CBF-approximation

When the two-particle correlation function is allowed to vary with time then the first continuity equations (61) can be written in momentum space in a general form

$$\hbar\omega - \varepsilon_F(k) - \Sigma(k,\omega) = 2S(k)\chi^{-1}(k,\omega). \tag{86}$$

where $\Sigma(k,\omega)$ is the **self energy** and the linear response function as

$$\chi(k,\omega) = \frac{2S(k)}{\hbar\omega - \varepsilon_F(k) - \Sigma(k,\omega)} \tag{87}$$

For real values of the self-energy the response function can have poles which define the collective, elementary excitations. When the decay of the excited modes becomes possible then the self-energy acquires imaginary part and the sharp δ-function in the imaginary part of the response function spreads into a broader peak. The notation $\varepsilon_F(k)$ stands for the energy of the Feynman collective mode.

5.1 Convolution approximation

The derivation of the self-energy starts [29, 33] with a convolution approximation of the **three-particle distribution function**, but including also a special set of diagrams with the triplet correlation function $u_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$. The terms included are shown in Fig. 1.

In the algebraic form it becomes

$$\begin{aligned}
&g_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = 1 + h(\mathbf{r}_1,\mathbf{r}_2) + h(\mathbf{r}_1,\mathbf{r}_3) + h(\mathbf{r}_2,\mathbf{r}_3)\\
&+h(\mathbf{r}_1,\mathbf{r}_2)h(\mathbf{r}_1,\mathbf{r}_3) + h(\mathbf{r}_1,\mathbf{r}_2)h(\mathbf{r}_2,\mathbf{r}_3)\\
&+h(\mathbf{r}_1,\mathbf{r}_3)h(\mathbf{r}_2,\mathbf{r}_3) + \int d^3r_4 h(\mathbf{r}_1,\mathbf{r}_4)h(\mathbf{r}_2,\mathbf{r}_4)h(\mathbf{r}_3,\mathbf{r}_4)\\
&+ \text{terms with triplet correlations}
\end{aligned} \tag{88}$$

We ignore triplet correlations for a moment and return to them at the end of this section.

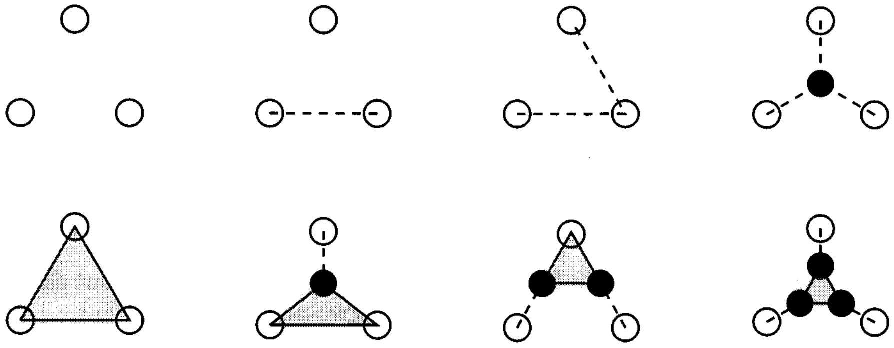

Figure 1: Convolution approximation of $g_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$. Circles are particle positions, black circles are integrated and open circles not. Dashed lines are functions $h(\mathbf{r}_1,\mathbf{r}_2)$ and triangles are triplet correlation functions $u_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$. The second, third fifth and sixth diagrams have three of the same kind, but with different particle coordinates.

5.2 Two-particle equation

Our aim is to get an approximation for $\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t)$ using Eq.(62). The simplest term to approximate is $D_2(\mathbf{r}_1,\mathbf{r}_2;t)$ in Eq. (70)

$$\begin{aligned} D_2(\mathbf{r}_1,\mathbf{r}_2;t) &= \frac{2\rho_0^2}{\hbar}\Big\{ g_2(\mathbf{r}_1,\mathbf{r}_2)\left(D_1(\mathbf{r}_1;t)+D_1(\mathbf{r}_2;t)\right) \\ + \quad & \rho_0 \int d^3r_3 U_{\rm ext}(\mathbf{r}_3;t) \qquad (89) \\ \times \quad & \left[h(\mathbf{r}_1,\mathbf{r}_3)h(\mathbf{r}_2,\mathbf{r}_3) + \int h(\mathbf{r}_1,\mathbf{r}_4)h(\mathbf{r}_2,\mathbf{r}_4)h(\mathbf{r}_3,\mathbf{r}_4)\right]\Big\}. \end{aligned}$$

The last two lines can be written in the form

$$\rho_0 \int d^3r_3 Y(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3) D_1(\mathbf{r}_3;t) \qquad (90)$$

with $Y(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3) = h(\mathbf{r}_1,\mathbf{r}_3)h(\mathbf{r}_2,\mathbf{r}_3)$. The triplet correlation will have additional contribution to that. Thus we can express the **two particle driving** term entirely in terms of $D_1(\mathbf{r};t)$.

$$\begin{aligned} D_2(\mathbf{r}_1,\mathbf{r}_2;t) &= \frac{2\rho_0^2}{\hbar}\Big\{ g_2(\mathbf{r}_1,\mathbf{r}_2)\left(D_1(\mathbf{r}_1;t)+D_1(\mathbf{r}_2;t)\right) \\ + \quad & \rho_0 \int d^3r_3 Y(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3) D_1(\mathbf{r}_3;t)\Big\} \qquad (91) \end{aligned}$$

Similarly we can write an expression for the time dependent **two-particle density**

$$\delta\rho_2(\mathbf{r}_1,\mathbf{r}_2;t) = \frac{2\rho_0^2}{\hbar}\Big\{ g_2(\mathbf{r}_1,\mathbf{r}_2)\left(\delta\rho_1(\mathbf{r}_1;t)+\delta\rho_1(\mathbf{r}_2;t)\right)$$

$$\begin{aligned}
+ & \quad \rho_0 \int d^3r_3 Y(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3)\delta\rho_1(\mathbf{r}_3;t)\Big\} \\
+ & \quad \rho_0^2 g_2(\mathbf{r}_1,\mathbf{r}_2)\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t) + \mathcal{F}[\delta u_2] \qquad (92)
\end{aligned}$$

We have removed the dependence on $\delta u_1(\mathbf{r},t)$ in favor of $\delta\rho_1(\mathbf{r},t)$. The functional $\mathcal{F}[\delta u_2]$ contains all the rest of the terms with $\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t)$. They can be written explicitly using the definition (56), but they are not included in the CBF-approximation.

The **two particle current** has a term with one-particle current, but also structure which comes from the time-dependent two-particle correlations.

$$\begin{aligned}
& \quad \mathbf{j}_2(\mathbf{r}_1,\mathbf{r}_2;t) = \rho_0 g_2(\mathbf{r}_1,\mathbf{r}_2)\, \mathbf{j}_1(\mathbf{r}_1) \qquad (93) \\
+ & \quad \frac{\hbar\rho_0^2}{2mi}\Big\{ g_2(\mathbf{r}_1,\mathbf{r}_2)\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t) + \rho_0 \int d^3r_3 \\
\times & \quad [g_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) - g_2(\mathbf{r}_1,\mathbf{r}_2)g_2(\mathbf{r}_1,\mathbf{r}_3)]\,\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_3)\Big\}
\end{aligned}$$

The final steps of the derivation are the approximations necessary to bring the two-body equation in a numerically tractable form. Our scheme follows the general strategy of the **uniform limit approximation** [6] which has been quite successful for the calculation of the optimal static three-body correlations [34, 35, 36]. The essence of the approximation is to consider **all products of two or more two-body functions small in coordinate space.**

In our specific case, the uniform limit approximation amounts to taking $g_2(\mathbf{r}_1,\mathbf{r}_2)\delta u_2(\mathbf{r}_1,\mathbf{r}_2) \approx \delta u_2(\mathbf{r}_1,\mathbf{r}_2)$ and a similar expression for $\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_2)$. While this approximation places more emphasis on the structure of $\delta u_2(\mathbf{r}_1,\mathbf{r}_2)$ it is physically appealing since it simply removes the *redundant* relevant short-range structure shared by $g_2(\mathbf{r}_1,\mathbf{r}_2)$ and $\delta u_2(\mathbf{r}_1,\mathbf{r}_2)$. Invoking the equivalent uniform limit for the three-body distribution function, the terms in Eq. (93) which depend on $\delta u_2(\mathbf{r}_1,\mathbf{r}_2)$ become

$$\begin{aligned}
& \quad \frac{\hbar\rho_0^2}{2mi}\Big[g_2(\mathbf{r}_1,\mathbf{r}_2)\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_2) + \rho_0 \int d^3r_3 \\
\times & \quad [g_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) - g_2(\mathbf{r}_1,\mathbf{r}_3)g_2(\mathbf{r}_1,\mathbf{r}_2)]\,\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_3)\Big] \\
\approx & \quad \frac{\hbar\rho_0}{2mi}\int d^3r_3\, [\delta(\mathbf{r}_3-\mathbf{r}_2) + h(\mathbf{r}_3,\mathbf{r}_2)]\,\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_3)\,.
\end{aligned}$$

We can now put together the approximate **two-particle continuity equation**

$$\begin{aligned}
& \quad \nabla_1 \cdot \Big[g_2(\mathbf{r}_1,\mathbf{r}_2)\, \mathbf{j}_1(\mathbf{r}_1) \\
+ & \quad \frac{\hbar\rho_0^2}{2mi}\int d^3r_3\, [\delta(\mathbf{r}_3-\mathbf{r}_2) + h(\mathbf{r}_3,\mathbf{r}_2)]\,\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_3)\Big]
\end{aligned}$$

$$\begin{aligned}
&+ \quad \text{same with } (1 \leftrightarrow 2) \\
&= \quad g_2(\mathbf{r}_1,\mathbf{r}_2)\Big(\frac{2\rho_0}{\hbar}D_1(\mathbf{r}_1;t) - i\delta\dot{\rho}_1(\mathbf{r}_1;t) \\
&+ \quad \frac{2\rho_0}{\hbar}D_1(\mathbf{r}_2;t) - i\delta\dot{\rho}_1(\mathbf{r}_2;t)\Big) \\
&+ \quad \rho_0 \int d^3r_3 Y(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3)\left(\frac{2\rho_0}{\hbar}D_1(\mathbf{r}_3;t) - i\delta\dot{\rho}_1(\mathbf{r}_3;t)\right) \\
&+ \quad \rho_0 \delta\dot{u}_2(\mathbf{r}_1,\mathbf{r}_2;t) \qquad (94)
\end{aligned}$$

From the terms containing the time derivative $\delta\dot{u}_2(\mathbf{r}_1,\mathbf{r}_2;t)$ we have kept only the leading term in accordance with the uniform limit approximation and left out the term $\mathcal{F}[\delta\dot{u}_2]$.

If we further more use the one-particle continuity equation to replace the one-particle quantities with one-particle currents we arrive at our final approximate form

$$\begin{aligned}
&\left[\frac{\hbar\rho_0}{2mi}\int d^3r_3\,[\delta(\mathbf{r}_3-\mathbf{r}_2)+h(\mathbf{r}_3,\mathbf{r}_2)]\,\nabla_1\delta u_2(\mathbf{r}_1,\mathbf{r}_3)\right. \\
&\left. +\text{same with } (1 \leftrightarrow 2)\right] - \delta\dot{u}_2(\mathbf{r}_1,\mathbf{r}_2;t) \qquad (95) \\
= \quad & \mathbf{j}_1(\mathbf{r}_1;t)\cdot\nabla_1\cdot g_2(\mathbf{r}_1,\mathbf{r}_2) + \mathbf{j}_1(\mathbf{r}_2;t)\cdot\nabla_2\cdot g_2(\mathbf{r}_1,\mathbf{r}_2) \\
+ \quad & \rho_0\int d^3r_3 Y(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3)\nabla_3\cdot\mathbf{j}_1(\mathbf{r}_3;t)
\end{aligned}$$

Approximating now the one-particle current (66) by the **Feynman current**

$$\begin{aligned}
\mathbf{j}(\mathbf{r}) \quad &= \quad \frac{\hbar\rho_0}{2mi}\nabla_1\delta u_1(\mathbf{r};\mathbf{t}) \qquad (96) \\
&= \quad \frac{\hbar\rho_0}{2mi}\nabla_1\left[\delta\rho_1(\mathbf{r}_1;t) - \rho_0\int d^3r_2 X(\mathbf{r}_1,\mathbf{r}_2)\delta\rho_1(\mathbf{r}_2;t)\right]
\end{aligned}$$

allows us to decouple the equations of motion. In other words the fluctuating two-point function can be expressed, in closed form, as a functional of one-body quantities alone.

Within this approximation the **second continuity equation** can be given in the form

$$\begin{aligned}
&\left[\hbar\omega - \varepsilon_{\mathrm{F}}\left(\left|\tfrac{\mathbf{k}}{2}+\mathbf{p}\right|\right) - \varepsilon_{\mathrm{F}}\left(\left|\tfrac{\mathbf{k}}{2}-\mathbf{p}\right|\right)\right] \\
&\times S\left(\left|\tfrac{\mathbf{k}}{2}+\mathbf{p}\right|\right) S\left(\left|\tfrac{\mathbf{k}}{2}-\mathbf{p}\right|\right)\delta u_2(\mathbf{k},\mathbf{p};\omega) \\
&+\varepsilon_{\mathrm{F}}(k)\sigma_{\mathbf{k}}(\mathbf{p})\delta\rho_1(\mathbf{k};\omega) = 0. \qquad (97)
\end{aligned}$$

Here $\sigma_{\mathbf{k}}(\mathbf{p})$ corresponds to

$$\begin{aligned} \sigma_{\mathbf{k}}(\mathbf{p}) &= -\frac{1}{k^2}\left[\mathbf{k}\cdot\left(\tfrac{\mathbf{k}}{2}+\mathbf{p}\right) S\left(\left|\tfrac{\mathbf{k}}{2}-\mathbf{p}\right|\right) + (\mathbf{p}\leftrightarrow -\mathbf{p})\right] \\ +\quad & S\left(\left|\tfrac{\mathbf{k}}{2}+\mathbf{p}\right|\right) S\left(\left|\tfrac{\mathbf{k}}{2}-\mathbf{p}\right|\right)\left[1+\tilde{u}_3\left(\tfrac{\mathbf{k}}{2}+\mathbf{p},\tfrac{\mathbf{k}}{2}-\mathbf{p},-\mathbf{k}\right)\right]. \end{aligned} \tag{98}$$

This is an algebraic equation from which we can solve the fluctuating **two-particle correlation function,**

$$\frac{\delta u_2(\mathbf{k},\mathbf{p};\omega)}{\delta\rho_1(\mathbf{k};\omega)} = \frac{\varepsilon_{\mathrm{F}}(k)\sigma_{\mathbf{k}}(\mathbf{p})\left[S\left(\left|\tfrac{\mathbf{k}}{2}+\mathbf{p}\right|\right) S\left(\left|\tfrac{\mathbf{k}}{2}-\mathbf{p}\right|\right)\right]^{-1}}{\hbar\omega-\varepsilon_{\mathrm{F}}\left(\left|\tfrac{\mathbf{k}}{2}+\mathbf{p}\right|\right)-\varepsilon_{\mathrm{F}}\left(\left|\tfrac{\mathbf{k}}{2}-\mathbf{p}\right|\right)},$$

needed for the self energy.

5.3 One-particle equation

Let us now return to the **first continuity equation** (61) and to the one-particle current (66). We want again to remove $\delta u_1(\mathbf{r}_1;t)$ in favor of $\delta\rho_1(\mathbf{r}_1;t)$ within the convolution approximation. In that approximation the one-particle density (63) can be written in the form

$$\delta\rho_1(\mathbf{r}_1;t) = \rho_0\,\delta v_1(\mathbf{r}_1;t) + \rho_0^2\int d^3r_2 h(\mathbf{r}_1,\mathbf{r}_2;t)\delta v_1(\mathbf{r}_2;t) \tag{99}$$

with

$$\begin{aligned} &\delta v_1(\mathbf{r}_1;t) = \delta u_1(\mathbf{r}_1;t) \\ +\quad & \rho_0\int d^3r_2 g_2(\mathbf{r}_1,\mathbf{r}_2)\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t) \\ +\quad & \frac{1}{2}\rho_0^2\int d^3r_2 d^3r_3 Y(\mathbf{r}_2,\mathbf{r}_3;\mathbf{r}_1)\delta u_2(\mathbf{r}_2,\mathbf{r}_3;t)\Big] \end{aligned} \tag{100}$$

Eq. (99) can be readily solved for $\delta v_1(\mathbf{r}_1;t)$

$$\rho_0\delta v_1(\mathbf{r}_1;t) = \delta\rho_1(\mathbf{r}_1;t) - \rho_0\int d^3r_2 X(\mathbf{r}_1,\mathbf{r}_2)\delta\rho_1(\mathbf{r}_2;t) \tag{101}$$

where $X(\mathbf{r}_1,\mathbf{r}_2)$ is the direct correlation function.

From that we can solve the **one-particle correlation function,**

$$\begin{aligned} &\rho_0\delta u_1(\mathbf{r}_1;t) = \delta\rho_1(\mathbf{r}_1;t) - \rho_0\int d^3r_2 X(\mathbf{r}_1,\mathbf{r}_2)\delta\rho_1(\mathbf{r}_2;t) \\ &-\rho_0^2\int d^3r_2 g_2(\mathbf{r}_1,\mathbf{r}_2)\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t) \\ &-\frac{1}{2}\rho_0^3\int d^3r_2 d^3r_3 Y(\mathbf{r}_2,\mathbf{r}_3;\mathbf{r}_1)\delta u_2(\mathbf{r}_2,\mathbf{r}_3;t)\Big] \end{aligned} \tag{102}$$

and taking its gradient and inserting into equation (66) we get the **one-particle current**

$$\begin{aligned}\mathbf{j}_1(\mathbf{r}_1;t) &= \nabla_1\Big[\delta\rho_1(\mathbf{r}_1;t) - \rho_0\int d^3r_2 X(\mathbf{r}_1,\mathbf{r}_2)\delta\rho_1(\mathbf{r}_2;t)\Big] \\ &-\rho_0^2\int d^3r_2\delta u_2(\mathbf{r}_1,\mathbf{r}_2;t)\nabla_1 g_2(\mathbf{r}_1,\mathbf{r}_2) \qquad (103) \\ &-\frac{1}{2}\rho_0^3\int d^3r_2 d^3r_3\nabla_1 Y(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\delta u_2(\mathbf{r}_2,\mathbf{r}_3;t)\Big]\end{aligned}$$

5.4 Self-energy

In momentum space the one-particle equation can now be written in the form

$$\begin{aligned}&[\hbar\omega - \varepsilon_{\mathrm{F}}(k)]\delta\rho_1(\mathbf{k};\omega) \qquad (104) \\ + &\frac{\hbar^2k^2}{4m}\int\frac{d^3p}{(2\pi)^3\rho_0}\sigma_{\mathbf{k}}(\mathbf{p})\delta u_2(\mathbf{k},\mathbf{p};\omega) = 2S(k)U_{ext}(k)\end{aligned}$$

where $\sigma_{\mathbf{k}}(\mathbf{p})$ is the same as in the two-particle equation (98).
By dividing this equation with $\delta\rho_1(\mathbf{k};\omega)$ we get the self-energy and the inverse of the linear response function

$$\hbar\omega - \varepsilon_{\mathrm{F}}(k) + \Sigma(\mathbf{k},\omega) = 2S(k)\chi^{-1}(\mathbf{k},\omega) \qquad (105)$$

with

$$\Sigma(\mathbf{k},\omega) = \frac{\hbar^2k^2}{4m}\int\frac{d^3p}{(2\pi)^3\rho_0}\sigma_{\mathbf{k}}(\mathbf{p})\frac{\delta u_2(\mathbf{k},\mathbf{p};\omega)}{\delta\rho_1(\mathbf{k};\omega)} \qquad (106)$$

Let's change the variables $\frac{\mathbf{k}}{2}+\mathbf{p}\to\mathbf{p}$ and $\frac{\mathbf{k}}{2}-\mathbf{p}\to\mathbf{q}$ and then introduce the Dirac delta function to insure that $\mathbf{p}+\mathbf{q}=\mathbf{k}$. The self energy correction is written as

$$\Sigma^{\mathrm{CBF}}(k,\omega) = \frac{1}{2}\int\frac{d^3pd^3q}{(2\pi)^3\rho}\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})\frac{|V_3(\mathbf{k};\mathbf{p},\mathbf{q})|^2}{\hbar\omega - \varepsilon_{\mathrm{F}}(p) - \varepsilon_{\mathrm{F}}(q)}, \qquad (107)$$

where the three-plasmon/phonon coupling matrix element

$$\begin{aligned}V_3(\mathbf{k};\mathbf{p},\mathbf{q}) &= \frac{\hbar^2}{2m}\frac{1}{\sqrt{S(p)S(q)S(k)}} \qquad (108) \\ \times &\Big[-\mathbf{k}\cdot\mathbf{p}\tilde{S}(p) - \mathbf{k}\cdot\mathbf{q}\tilde{S}(q) + k^2S(p)S(q)\left(1+\tilde{u}_3(\mathbf{k},\mathbf{p},\mathbf{q})\right)\Big] \\ = &\frac{\hbar^2}{2m}\sqrt{\frac{S(p)S(q)}{S(k)}}[\mathbf{k}\cdot\mathbf{p}\tilde{X}(p) + \mathbf{k}\cdot\mathbf{q}\tilde{X}(q) - k^2\tilde{u}_3(\mathbf{k},\mathbf{p},\mathbf{q})].\end{aligned}$$

is given in terms of the ground-state structure function $S(k)$, the direct correlation function $\tilde{X}(k) = 1 - S(k)^{-1}$, and the three-body correlation function $\tilde{u}_3$.

The integrand (107) can have poles, which makes the self energy a complex function. Let us look next in detail how it is calculated numerically.
After integrating the δ- function and the ϕ-coordinate we are left with the double integral

$$\begin{aligned}\Sigma^{\rm CBF}(k,\omega) &= \frac{1}{2}\frac{1}{(2\pi)^2\rho}\int_0^\infty q^2 dq \int_{-1}^{1} dx \\ &\times \frac{|V_3(k;q,x)|^2}{\hbar\omega - \varepsilon_{\rm F}(|\mathbf{k}+\mathbf{q}) - \varepsilon_{\rm F}(q)},\end{aligned}$$

where we have chosen the following variables

$$\begin{aligned}\mathbf{p} &= -(\mathbf{k}+\mathbf{p}) \\ p^2 &= k^2 + 2\mathbf{k}\cdot\mathbf{q} = k^2 + q^2 + 2kpx \\ x &= \frac{p^2 - k^2 - q^2}{2kq}\end{aligned} \tag{109}$$

Replacing yet x with p we write the integral in the form

$$\begin{aligned}\Sigma^{\rm CBF}(k,\omega) &= \frac{1}{8\pi^2\rho k}\int_0^\infty q dq \int_{|k-p|}^{k+p} p dp \\ &\times \frac{|V_3(k;q,p)|^2}{\hbar\omega - \varepsilon_{\rm F}(p) - \varepsilon_{\rm F}(q)},\end{aligned} \tag{110}$$

This integral has a pole when

$$\hbar\omega = \varepsilon_{\rm F}(p) + \varepsilon_{\rm F}(q) \tag{111}$$

In other words when the energy of the excitation is equal to the energy of two elementary Feynman modes. In such a case the self energy becomes a complex function. Assuming that this pole is the only pole in the integrand and that the integrand converges fast enough at infinity we can separate the real and imaginary parts

$$\Sigma(k,\omega) = \Delta(k,\omega) + \frac{i}{2}\Gamma(k,\omega) \tag{112}$$

by remembering that

$$\frac{1}{\omega' - \omega + i\eta} = \mathcal{P}\frac{1}{\omega' - \omega} - i\pi\delta(\omega' - \omega) \tag{113}$$

The **imaginary part** can then be calculated with one integration only

$$\begin{aligned}\Gamma(k,\omega) &= \frac{1}{4\pi\rho k}\int_0^\infty q dq \int_{|k-p|}^{k+p} p dp \\ &\times |V_3(k;q,p)|^2 \delta(\hbar\omega - \varepsilon_{\rm F}(p) - \varepsilon_{\rm F}(q))\end{aligned} \tag{114}$$

The **real part** could be calculated from the above principle value integral, but it is much more convenient for numerics to calculate it from the imaginary part using **Kramers-Kronic relations** which connect the real and imaginary parts. If $f(\omega)$ is a analytic complex function

$$f(\omega) = a(\omega) + ib(\omega) \tag{115}$$

then

$$\begin{aligned} a(\omega) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{b(\omega')}{\omega' - \omega} \\ b(\omega) &= -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{a(\omega')}{\omega' - \omega} \end{aligned}$$

Provided that $a(\omega)$ and $b(\omega)$ converge fast enough at large ω.
Using the first relation we can write the real part of the self energy in the form

$$\Delta(k,\omega) = \frac{1}{2\pi}\mathcal{P}\int_{0}^{\infty} d\omega' \frac{\Gamma(k,\omega')}{\omega' - \omega} \tag{116}$$

The imaginary part is non-zero only when $\omega' > 0$. In the numerical integration of the principle value one distributes the integration mesh symmetrically around ω and leaves out the point $\omega' = \omega$.

5.5 Analytic structure of the self-energy

The collective modes of the system are found by determining the poles (note that all poles are on the real axis[37]) of the response function (87), in other words by solving the implicit equation

$$\hbar\omega_0(k) = \varepsilon(k) + \Sigma(k, \omega_0(k)) \tag{117}$$

and the strength of the collective mode is given by

$$Z(k) = 2S(k)\left[1 - \frac{\partial}{\partial\omega}\Sigma(k,\omega)\bigg|_{\omega=\omega_0(k)}\right]^{-1} . \tag{118}$$

From our definition (107) of the self-energy follows the inequality

$$\Sigma^{\mathrm{CBF}}(k,\omega) \le \Sigma^{\mathrm{CBF}}(k,0) \le 0 \tag{119}$$

from which one immediately sees that the lowest collective mode satisfies the exact inequality [38]

$$\hbar\omega_0(k) \le -\frac{2S(k)}{\chi^{\mathrm{CBF}}(k,0)} . \tag{120}$$

While it is reassuring that our microscopic approach satisfies known exact sum-rules and inequalities as a consequence of its structure, we will see momentarily that the inequality (120) is of rather limited use in determining features of either the excitation spectrum, or the static response function. The reason is that it gives *neither* information on the pole strength $Z(k)$, *nor* on the existence of stable collective modes. We shall encounter examples of both: a case where the pole strength of the lowest collective mode is infinitesimal, and a case where no real collective mode exists. The latter example is in fact a well-known consequence of anomalous dispersion.

In writing down Eq. (117) we have to assume that $\Sigma(k, \omega_0(k))$ is real. This is the case when the energy denominator in Eq. (107) does not change sign, which is true when the collective energy is *below* the critical value

$$\hbar\omega_0 < \hbar\omega_{\rm crit}(k) \equiv \min_{\mathbf{q}} \left[\varepsilon(q) + \varepsilon(|\mathbf{k}+\mathbf{q}|)\right] \tag{121}$$

determining the continuum boundary. Above that energy, the self-energy is complex. Moreover, for $\hbar\omega < \hbar\omega_{\rm crit}(k)$, it follows from Eq. (107) that

$$\frac{d\Sigma(k,\omega)}{d\omega} < 0 \qquad \text{for} \qquad \hbar\omega < \hbar\omega_{\rm crit}(k)\ . \tag{122}$$

In order to determine if Eq. (117) has a solution, we must find out whether $\Sigma(k,\omega)$ becomes singular at the branch-cut $\omega = \omega_{\rm crit}(k)$ or not. This depends, of course, on the details of the reference spectrum $\varepsilon(k)$ in the energy denominator of Eq. (117). We shall study here two relevant cases.

5.6 Example: Anomalous dispersion in liquid ^{4}He

The first case is that the reference spectrum $\varepsilon(k)$ is **convex**. This refers typically to the regime of low momentum transfer in ^{4}He where the sound mode has an **anomalous dispersion** or to high momentum transfer where the **spectrum approaches the single particle kinetic energy**. When $\hbar\omega_{\rm crit}(k) = 2\varepsilon(k/2) < \varepsilon(k)$, this critical energy is *below* the reference energy. In order to determine whether Eq. (117) has a solution, we must therefore study the analytic behavior of $\Sigma(k,\omega)$ as a function of ω near the branch-point $\omega = \omega_{\rm crit}(k)$. We shall treat only the simplest cases here, assuming a monotonically growing, convex spectrum $\varepsilon'(k/2) > 0$ and $\varepsilon''(k/2) > 0$ and we are interested in the singular behavior only.

To evaluate the imaginary part of the integral (107) for $\omega \approx \omega_{\rm crit}$ it is sufficient to consider the area where the angle θ between $\mathbf{k}$ and $\mathbf{q}$ is close to zero, i.e.

$$\Im m\Sigma(q,\omega) \approx -\frac{1}{2}\Im m \int \frac{d^2k}{(2\pi)^2} \frac{|V(k,|k-q|,q)|^2}{\varepsilon_F(k) + \varepsilon_F(|\mathbf{k}-\mathbf{q}|) - \hbar\omega + i\eta}. \tag{123}$$

Next we expand the energy denominator in the vicinity of it's minimum value, $\omega_{\rm crit}(q)$. Letting $p = |\mathbf{k}-\mathbf{q}|$, we have

$$\varepsilon_F(k) + \varepsilon_F(p) \tag{124}$$

$$
\begin{aligned}
&= 2\varepsilon_F(\frac{q}{2}) + \varepsilon'(p+k-q) + \frac{1}{2}\varepsilon''\left((k-\frac{q}{2})^2 + (p-\frac{q}{2})^2\right) \\
&= 2\varepsilon_F(\frac{q}{2}) + \frac{q}{2}\varepsilon''\left[\lambda(p+k-q) + \frac{1}{q}\left(k^2+p^2-\frac{q^2}{2}\right)\right]
\end{aligned}
$$

where

$$
\varepsilon' = \left.\frac{d\varepsilon_F(q)}{dq}\right|_{\frac{q}{2}}, \qquad \varepsilon'' = \left.\frac{d^2\varepsilon_F(q)}{dq^2}\right|_{\frac{q}{2}} \tag{125}
$$

and

$$
\lambda = \frac{2\varepsilon'}{k\varepsilon''} - 1. \tag{126}
$$

For $\cos\theta \approx 1$, we can also expand (recall that we are considering momentum transfers $k \approx \frac{q}{2}$

$$
\begin{aligned}
p &= \sqrt{k^2+q^2-2kq+2kq(1-\cos\theta)} \\
&\approx q-k+\frac{kq}{(q-k)}(1-\cos\theta) \\
&\approx q-k+(4k-q)(1-\cos\theta)
\end{aligned} \tag{127}
$$

and therefore

$$
\begin{aligned}
\omega(k)+\omega(p) &= 2\omega(\frac{q}{2}) + \frac{q^2\varepsilon''}{2}\left[\lambda\left(4\frac{k}{q}-1\right)(1-\cos\theta)\right. \\
&+ \left.\frac{2k}{q}\left(\frac{k}{q}-\cos\theta\right)+\frac{1}{2}\right].
\end{aligned} \tag{128}
$$

This form of the energy denominator is correct, to second order in the momentum, in the vicinity of it's minimum value and for small angles. We can now carry out the angle integration and find

$$
\begin{aligned}
\Im m\Sigma(q,\omega) &\approx -\frac{1}{2}\Im m\int_0^\infty \frac{kdk}{(2\pi)^2}\int_0^{2\pi} d\theta \frac{|V(k,|k-q|,q)|^2}{A+B\cos\theta} \\
&= -\frac{1}{4\pi}\Im m\int_0^\infty kdk\frac{|V(k,|k-q|,q)|^2}{\hbar\sqrt{A^2-B^2}}
\end{aligned} \tag{129}
$$

with

$$
A = 2\varepsilon_F(\frac{q}{2}) + \frac{q^2\varepsilon''}{2}\left[\lambda\left(4\frac{k}{q}-1\right)+\frac{2k^2}{q^2}+\frac{1}{2}\right] - \omega + i\eta \tag{130}
$$

and

$$
B = -\frac{q^2\varepsilon''}{2}\left[\lambda\left(4\frac{k}{q}-1\right)+\frac{2k}{q}\right]. \tag{131}
$$

For $k \approx \frac{q}{2}$ and $\omega \approx 2\varepsilon_F(\frac{q}{2})$, we can further simplify

$$
A^2 - B^2 = 2k\varepsilon'\varepsilon''(k-k_-)(k-k_+) \tag{132}
$$