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Linear Quadratic Control Theory for Infinite Dimensional Systems

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LINEAR QUADRATIC CONTROL THEORY FOR INFINITE DIMENSIONAL SYSTEMS

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Preface

These notes contain a short course on the linear quadratic controls problems in Hilbert spaces.

We have essentially followed the book: A, Bensoussan, G. Da Prato, M. Delfour and S.K. Mitter, *Representation and Control of Infinite Dimensional Systems,* Birkhäuser, (1992).

See the book above for generalizations and references.

Pisa, September 11, 2001 Giuseppe Da Prato

Contents

Chapter 1

Control in finite horizon

1.1 Introduction and setting of the problem

We are concerned with a dynamical system governed by the following differential equation

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \ t \geq 0, \\ \\ y(0) = x \in H, \end{cases} \tag{1.1.1}$$

where $A : D(A) \subset H \to H$, $B : U \to H$ are linear operators defined on the Hilbert spaces H (*state space*) and U (*control space*). We shall also consider another Hilbert space Y (*observation space*). The inner product and norm in H, U, Y will be denoted by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ respectively.

Given $T > 0$, we want to minimize the cost function

$$J(u) = \int_0^T \left[|Cy(s)|^2 + |u(s)|^2 \right] ds + \langle P_0 y(T), y(T) \rangle, \tag{1.1.2}$$

where $P_0 : H \to H$, $C : H \to Y$ are linear operators defined in H and Y respectively, over all controls $u \in L^2(0, T; U)$ subject to (1.1.1).

Concerning the operators A, B, C and P_0 we shall assume that

Hypothesis 1.1 *(i) A generates a strongly continuous semigroup e^{tA} on H.*

(ii) $B \in L(U, H)$ ([1]).

(iii) $P_0 \in L(H)$ *is symmetric and nonnegative.*

(iv) $C \in L(H, Y)$.

Under Hypothesis 1.1–(i)–(ii) problem (1.1.1) has a unique *mild* solution y given by the *variation of constants* formula (see Appendix A),

$$y(t) = e^{tA}x + \int_0^t e^{(t-s)A}Bu(s)ds. \tag{1.1.3}$$

A function $u^* \in L^2(0, T; U)$ is called an *optimal control* if

$$J(u^*) \leq J(u), \ \forall\, u \in L^2(0, T; U). \tag{1.1.4}$$

In this case the corresponding solution y^* of (1.1.1) is called an *optimal state* and the pair (u^*, y^*) an *optimal pair.*

Under Hypothesis 1.1 it is easy to see that there is a unique optimal control (since the quadratic form $J(u)$ on $L^2(0, T; U)$ is coercive). However we are interested in showing that the optimal control can be obtained as a *feedback control* (*synthesis problem*). For this reason we shall describe the *Dynamic Programming* approach which consists in the following two steps:

Step 1. We solve the *Riccati operator equation*

$$\begin{cases} P' = A^*P + PA - PBB^*P + C^*C, \\ \\ P(0) = P_0, \end{cases} \tag{1.1.5}$$

where A^*, B^* and C^* are the adjoint operators of A, B and C respectively.

Step 2. We prove that the optimal control u^* is related to the optimal state y^* by the *feedback formula*

$$u^*(t) = -B^*P(T-t)y^*(t), \ \ t \in [0, T], \tag{1.1.6}$$

[1]Let X, Y be Hilbert spaces. We denote by $L(X, Y)$ the Banach space of all linear bounded operators $T : X \to Y$ endowed with the norm $\|T\| = \sup\{|Tx| : \ x \in X, \ |x| \leq 1\}$. We set $L(X, X) = L(X)$.

and moreover that y^* is the mild solution of the *closed loop equation*

$$\begin{cases} y'(t) = [A - BB^*P(T-t)]y(t), \ t \geq 0, \\ \\ y(0) = x \in H. \end{cases} \tag{1.1.7}$$

Finally the optimal cost is given by

$$J^* := \langle P(T)x, x \rangle.$$

Example 1.1.1 Let D be an open subset of $\mathbb{R}^n$ with regular boundary ∂D. Consider the equation

$$\begin{cases} D_t y(t,\xi) = (\Delta_\xi + c)y(t,\xi) + u(t,\xi), \text{ in } (0,T] \times D, \\ \\ y(t,\xi) = 0, \text{ on } (0,T] \times \partial D, \\ \\ y(0,\xi) = x(\xi), \text{ in } D. \end{cases} \tag{1.1.8}$$

We choose $H = U = Y = L^2(D)$, we set $B = C = P_0 = I$ and we denote by A the linear operator in H :

$$\begin{cases} Ay = (\Delta_\xi + c)y \\ \\ D(A) = H^2(D) \cap H_0^1(D). \end{cases} \tag{1.1.9}$$

It is well known that A generates a strongly continuous semigroup on $H = L^2(D)$.

Setting $y(t) = y(t,\cdot)$, $u(t) = u(t,\cdot)$, we can write (1.1.8) in the abstract form (1.1.1).

In this case the control problem consists in minimizing the cost

$$J(u) = \int_0^T \int_D [|y(t,\xi)|^2 + |u(t,\xi)|^2] dt d\xi + \int_D |y(T,\xi)|^2 d\xi. \tag{1.1.10}$$

Note that the control is distributed on all D.

1.2 Riccati equation

Let us introduce some notation. We set

$$\Sigma(H) = \{T \in L(H) : \ T \text{ is symmetric}\},$$

$$\Sigma^+(H) = \{t \in \Sigma(H) : \langle Tx, x\rangle \geq 0, \ \forall\, x \in H\}.$$

$\Sigma(H)$ is a closed subspace of $L(H)$, and $\Sigma^+(H)$ is a cone in $L(H)$.

For any interval $[a,b] \subset \mathbb{R}$, we shall denote by $C([a,b];\Sigma(H))$ the set of all continuous mappings from $[a,b]$ to $\Sigma(H)$.

$C([a,b];\Sigma(H))$, endowed with the norm

$$\|F\| = \sup_{t\in[a,b]} \|F(t)\|, \ F \in C([a,b];\Sigma(H)),$$

is a Banach space.

We shall also need to consider the space $C_s([a,b];\Sigma(H))$ of all strongly continuous mappings $F : [a,b] \to \Sigma(H)$, that is such that $F(\cdot)x$ is continuous on $[a,b]$ for any $x \in H$. A typical mapping belonging to $C_s([0,T];\Sigma(H))$ is $F(t) = e^{tA}$.

Let $F, \{F_n\} \subset C_s([a,b];\Sigma(H))$. We say that $\{F_n\}$ is strongly convergent to F if

$$\lim_{n\to\infty} F_n(\cdot)x = F(\cdot)x, \ \forall\, x \in H.$$

In this case we shall write

$$\lim_{n\to\infty} F_n = F, \ \text{in } C_s([a,b];\Sigma(H)).$$

If $F \in C_s([a,b];\Sigma(H))$, then the quantity

$$\|F\| = \sup_{t\in[a,b]} \|F(t)\|,$$

is finite by virtue of the Uniform Boundedness Theorem. Endowed with the norm above $C_s([a,b];\Sigma(H))$ is a Banach space that we shall denote by $C_u([a,b];\Sigma(H))$.

Let A, B, C and P_0 be given linear operators such that Hypothesis 1.1 is fulfilled. This section is devoted to solve the following Riccati equation

$$\begin{cases} P' = A^*P + PA - PBB^*P + C^*C, \\ \\ P(0) = P_0, \end{cases} \tag{1.2.1}$$

We first notice that if $A \in L(H)$ then it is easy to see that (1.2.1) is equivalent to the following integral equation

$$\begin{aligned} P(t)x &= e^{tA^*}P_0e^{tA}x + \int_0^t e^{sA^*}C^*Ce^{sA}xds \\ &- \int_0^t e^{(t-s)A^*}P(s)BB^*P(s)e^{(t-s)A}xds, \quad x \in H. \end{aligned} \tag{1.2.2}$$

Now, since the mapping

$$[0,T] \to \Sigma(H), \ t \to e^{tA^*}Te^{tA},$$

belongs to $C_s([a,b];\Sigma(H))$, equation (1.2.2) is meaningful in $C_s([a,b];\Sigma(H))$ and we will try to solve it in this space.

Definition 1.2.1 *(i) A mild solution of equation* (1.2.1) *in the interval* $[0,T]$ *is a function* $P \in C_s([a,b];\Sigma(H))$ *that verifies the integral equation* (1.2.2). *(ii) A weak solution of equation* (1.2.1) *in the interval* $[0,T]$ *is a function* $P \in C_s([a,b];\Sigma(H))$ *such that* $P(0) = P_0$ *and for any* $x, y \in D(A)$, $\langle P(\cdot)x, y\rangle$ *is differentiable in* $[0,T]$ *and verifies the equation*

$$\begin{aligned} \frac{d}{dt}\langle P(t)x,y\rangle &= \langle P(t)x, Ay\rangle + \langle P(t)Ax, y\rangle \\ &- \langle B^*P(t)x, B^*P(t)y\rangle + \langle Cx, Cy\rangle. \end{aligned} \tag{1.2.3}$$

Proposition 1.2.2 *Let* $P \in C_s([a,b];\Sigma(H))$. *Then* P *is a mild solution of equation* (1.2.1) *if and only if* P *is a weak solution of equation* (1.2.1).

Proof. If P is a mild solution of equation (1.2.1), then for any $x, y \in H$ we have

$$\begin{aligned} \langle P(t)x, y\rangle &= \langle P_0e^{tA}x, e^{tA}y\rangle + \int_0^t \langle Ce^{sA}x, Ce^{sA}y\rangle ds \\ &- \int_0^t \langle P(s)BB^*P(s)e^{(t-s)A}x, e^{(t-s)A}y\rangle ds. \end{aligned}$$

Now if $x, y \in D(A)$ it follows that $\langle P(t)x, y\rangle$ is differentiable with respect to t and, by a simple computation, that (1.2.3) holds. Conversely if P is a weak solution, then it is easy to check that for all $x, y \in D(A)$

$$\frac{d}{ds}\langle P(s)e^{(t-s)A}x, e^{(t-s)A}y\rangle = \langle Ce^{(t-s)A}x, Ce^{(t-s)A}y\rangle$$

$$-\langle B^*P(t)e^{(t-s)A}x, B^*P(t)e^{(t-s)A}y\rangle.$$

Integrating from 0 to t we see that (1.2.2) holds for any $x \in D(A)$. Since $D(A)$ is dense in H the conclusion follows. □

It is convenient to introduce the following approximating problem

$$\begin{cases} P_n' = A_n^*P_n + P_nA_n - P_nBB^*P_n + C^*C, \\ \\ P_n(0) = P_0, \end{cases} \tag{1.2.4}$$

where $A_n = n^2R(n, A) - nI$ is the Yosida approximation of A and $R(n, A)$ is the resolvent of A. Problem (1.2.4) is equivalent to the following integral equation

$$\begin{aligned} P_n(t)x &= e^{tA_n^*}P_0e^{tA_n}x + \int_0^t e^{sA_n^*}C^*Ce^{sA_n}xds \\ &- \int_0^t e^{(t-s)A_n^*}P_n(s)BB^*P_n(s)e^{(t-s)A_n}xds, \quad x \in H. \end{aligned} \tag{1.2.5}$$

We now solve problem (1.2.1). We first prove the local existence of a solution. We recall that by the Hille–Yosida Theorem (see Appendix A) for any $T > 0$ there exists $M_T > 0$ such that

$$\|e^{tA}\| \le M_T, \quad \|e^{tA_n}\| \le M_T, \; \forall\, t \in [0, T], \; n \in \mathbb{N}.$$

Lemma 1.2.3 *Assume that Hypothesis 1.1 holds, fix $T > 0$, set*

$$\rho = 2M_T^2\|P_0\| \tag{1.2.6}$$

and let τ be such that

$$\tau \in [0, T], \quad \tau\left(\|C\|^2 + \rho^2\|B\|^2\right) \le \|P_0\|, \quad 2\rho\tau M_T^2\|B\|^2 \le \frac{1}{2}. \tag{1.2.7}$$

Then problems (1.2.1) *and* (1.2.5) *have unique mild solutions* P *and* P_n *in the ball*

$$B_{\rho,\tau} = \{F \in C_u([0,\tau]; \Sigma(H)) : \ \|F\| \leq \rho\}.$$

Moreover

$$\lim_{n\to\infty} P_n = P, \text{ in } C_s([a,b]; \Sigma(H)). \tag{1.2.8}$$

Proof. Equation (1.2.2) (resp. the integral version of equation (1.2.5)) can be written in the form

$$P = \gamma(P)(\text{resp. } P_n = \gamma_n(P_n)),$$

where for $x \in H$

$$\begin{aligned}\gamma(P)(t)x &= e^{tA^*}P_0e^{tA}x \\ &+ \int_0^t e^{(t-s)A^*}[C^*C - P(s)BB^*P(s)]e^{(t-s)A}xds\end{aligned}$$

and

$$\begin{aligned}\gamma_n(P)(t)x &= e^{tA_n^*}P_0e^{tA_n}x \\ &+ \int_0^t e^{(t-s)A_n^*}[C^*C - P_n(s)BB^*P_n(s)]e^{(t-s)A_n}xds.\end{aligned}$$

Choose now ρ and τ such that (1.2.6) and (1.2.7) hold. We show that γ and γ_n are $1/2$–contractions on the ball $B_{\rho,\tau}$ of $C_u([0,\tau]; \Sigma(H))$. Let in fact $P \in B_{\rho,\tau}$. It follows that

$$|\gamma(P)(t)x| \leq M_T^2\left[\|P_0\| + \tau\|C\|^2 + \tau\rho^2\|B\|^2\right]|x| \leq 2M_T^2\|P_0\||x|,$$

and analogously

$$|\gamma_n(P)(t)x| \leq 2M_T^2\|P_0\||x|.$$

It follows that

$$\|\gamma(P)(t)\| \leq \rho, \ \ \|\gamma_n(P)(t)\| \leq \rho, \ \ \forall\, t \in [0,\tau], \ n \in \mathbb{N}, \ P \in B_{\rho,\tau},$$

so that γ and γ_n map $B_{\rho,\tau}$ into $B_{\rho,\tau}$.

For $P, Q \in B_{\rho,\tau}$ we have

$$\gamma(P)(t)x - \gamma(Q)(t)x$$

$$= \int_0^t e^{(t-s)A^*}[PBB^*(Q-P) + (Q-P)BB^*Q](s)e^{(t-s)A}x ds,$$

and a similar formula holds for $\gamma_n(P)(t)x - \gamma_n(Q)(t)x$. It follows that

$$\|\gamma(P)(t) - \gamma(Q)(t)\| \le 2\rho\tau M_T^2 \|B^2\| \|P-Q\| \le \frac{1}{2}\, \|P-Q\|,$$

$$\|\gamma_n(P)(t) - \gamma_n(Q)(t)\| \le 2\rho\tau M_T^2 \|B^2\| \|P-Q\| \le \frac{1}{2}\, \|P-Q\|.$$

Thus γ and γ_n are 1/2–contractions on $B_{\rho,\tau}$ and there exists unique mild solutions P and P_n in $B_{\rho,\tau}$. Finally (1.2.8) follows from a generalization of the classical Contraction Mapping Principle (see Appendix B). □

We now prove global uniqueness.

Lemma 1.2.4 *Assume that Hypothesis 1.1 holds, let $T > 0$ and let P, Q be two mild solutions of problem* (1.2.1) *in* $[0,T]$. *Then $P = Q$.*

Proof. Set

$$\alpha = \sup_{t\in[0,T]} \max\left\{\|P(t)\|, \|Q(t)\|\right\}.$$

α is finite by the Uniform Boundedness Theorem. Choose $\rho > 0$ and $\tau \in [0,T]$ such that

$$\rho = 2M_T^2\alpha, \quad \tau\left(\|C\|^2 + \rho^2\|B\|^2\right) \le \alpha, \quad 2\rho\tau M_T^2\|B\|^2 \le \frac{1}{2}.$$

By Lemma 1.2.3 it follows that $P(t) = Q(t)$ for any $t \in [0,\tau]$. It is now sufficient to repeat this argument in the interval $[\tau, 2\tau]$ and so on. □

The main result of this section is the following theorem.

Theorem 1.2.5 *Assume that Hypothesis 1.1 holds. Then problem* (1.2.1) *has a unique mild solution $P \in C_s([0,+\infty);\Sigma_+(H))$. Moreover for each $n \in \mathbb{N}$ problem* (1.2.5) *has a unique mild solution $P_n \in C([0,+\infty);\Sigma_+(H))$ and*

$$\lim_{n\to\infty} P_n = P \text{ in } C_s([0,T];\Sigma_+(H)),$$

for any $T > 0$.

Proof. Fix $T > 0$, set $\beta = M_T^2 (\|P_0\| + T\|C\|^2)$, and choose $\rho > 0$ and $\tau > 0$ such that

$$\rho = 2M_T^2\beta, \quad \tau \left(\|C\|^2 + \rho^2\|B\|^2\right) \leq \beta, \quad 2\rho\tau M_T^2\|B\|^2 \leq \frac{1}{2}.$$

By Lemma 1.2.3 there exists a unique solution P (resp. P_n) of (1.2.1) (resp. (1.2.5)) in $[0, \tau]$ and $P_n \to P$ in $C_s([0, \tau]; \Sigma(H))$. We now prove that

$$P_n(t) \geq 0, \quad \forall\, t \in [0, \tau]. \tag{1.2.9}$$

This will imply

$$P(t) \geq 0, \quad \forall\, t \in [0, \tau]. \tag{1.2.10}$$

To this end we notice that P_n is the solution of the following linear problem in $[0, \tau]$

$$P_n' = L_n^* P_n + P_n L_n + C^*C, \quad P_n(0) = P_0,$$

where $L_n = A_n - \frac{1}{2}BB^*P_n$. Denote by $U_n(t, s)$, $0 \leq s \leq t \leq \tau$, the evolution operator associated to L_n^*, that is the solution to

$$D_t U_n(t, s) = L_n^*(s) U_n(t, s), \; U_n(s, s) = I, \; , \; 0 \leq s \leq t \leq \tau.$$

Then we can write the solution $P_n(t)$ as

$$P_n(t) = U_n(t, 0) P_0 U_n^*(t, 0) + \int_0^t U_n(t, s) C^*C U_n^*(t, s) ds.$$

Thus (1.2.9) and (1.2.10) follow immediately.

Note that, arguing as in Lemma 1.2.3, we have

$$\|P(t)\| \leq \rho = 2M_T^2\beta$$

We now prove that we have a better estimate

$$P(t) \leq \beta I, \quad \forall\, t \in [0, \tau]. \tag{1.2.11}$$

This inequality will allow us to repeat the previous argument in the interval $[\tau, 2\tau]$ and so on. In this way the theorem will be proved. We have in fact

$$\langle P(t)x, x\rangle = \langle P_0 e^{tA}x, e^{tA}x\rangle + \int_0^t |Ce^{sA}x|^2 ds$$

$$- \int_0^t |B^*P(s)e^{(t-s)A}x|^2 ds \leq \beta |x|^2.$$

Since $P(t) \geq 0$ this implies (1.2.11). The proof is complete. □

We now prove continuous dependence with respect to data. Consider a sequence of Riccati equations

$$\begin{cases} (P^k)' = (A^k)^*P^k + P^kA^k - P^kB^k(B^k)^*P^k + (C^k)^*C^k, \\ \\ P^k(0) = P_0^k, \end{cases} \tag{1.2.12}$$

under the following assumption.

Hypothesis 1.2 *(i) For any $k \in \mathbb{N}$, (A^k, B^k, C^k, P_0^k) fulfil Hypothesis 1.1.*

(ii) For all $T > 0$ and all $x \in H$,

$$\lim_{k\to\infty} e^{tA^k}x = e^{tA}x, \text{ uniformly in } [0,T].$$

(iii) The sequences $\{B^k\}, \{(B^k)^*\}, \{C^k\}, \{(C^k)^*\}, \{P_0^k\}$ are strongly convergent to B, B^*, C, C^*, P_0, respectively.

Theorem 1.2.6 *Assume that Hypotheses 1.1 and 1.2 hold. Let P (resp. P^k) be the mild solution to (1.2.1) (resp. (1.2.12)). Then, for any $T > 0$ we have*

$$\lim_{n\to\infty} P^k = P \text{ in } C_s([0,T];\Sigma_+(H)).$$

Proof. Fix $T > 0$. By the Uniform Boundedness Theorem there exists positive numbers p, b and c such that

$$\|P_0^k\| \leq p, \quad \|(C^k)^*C^k\| \leq c, \quad \|B^k(B^k)^*\| \leq p, \ \forall\, k \in \mathbb{N}.$$

Set $\beta = M_T^2(p + cT)$ and choose ρ and $\tau \in [0,T]$ such that

$$\rho = 2\beta M_T^2, \quad \tau(c + \rho^2 b) \leq \beta \ \ 2\tau M_T^2\|B\|^2 \leq \frac{1}{2}.$$

Then, arguing as we did in the proof of Lemma 1.2.3, we can show that $P^k(\cdot)x \to P(\cdot)x$ for any $x \in H$. Finally, proceeding as in the proof of Theorem 1.2.5, we prove that this argument can be iterated in the interval $[\tau, 2\tau]$ and so on. □

We conclude this section by proving an important monotonicity property of the solutions of the Riccati equation (1.2.1).

Proposition 1.2.7 *Consider the Riccati equations:*

$$\begin{cases} P_i' = A^* P_i + P_i A - P_i B_i B_i^* P_i + C_i^* C_i, \\ P_i(0) = P_{i,0}, \ i = 1, 2. \end{cases} \tag{1.2.13}$$

Assume that $(A, B_i, C_i, P_{i,0})$ *verify Hypothesis 1.1, and, in addition, that*

$$P_{1,0} \le P_{2,0}, \ \ C_1^* C_1 \le C_2^* C_2, \ \ B_2 B_2^* \le B_1 B_1^*.$$

Then we have

$$P_1(t) \le P_2(t), \ \forall\, t \ge 0. \tag{1.2.14}$$

Proof. Due to Theorem 1.2.5 it is sufficient to prove (1.2.14) when A is bounded. Set $Z = P_2 - P_1$, then, as easily checked, Z is the solution to the linear problem

$$\begin{cases} Z' = X^* Z + Z X - P_2[B_2 B_2^* - B_1 B_1^*]P_2 + C_2^* C_2 - C_1^* C_1, \\ Z(0) = P_{2,0} - P_{1,0}, \end{cases} \tag{1.2.15}$$

where

$$X = A - \frac{1}{2}\, B_1 B_1^*(P_1 + P_2).$$

Let $V(t, s)$ be the evolution operator associated with X^*, that the solution to the problem is

$$D_t V(t, s) = X(t)^*(s) V(t, s), \ V(s, s) = I, \ , \ 0 \le s \le t \le \tau.$$

Then we have

$$\begin{aligned} Z(t) &= V(t,0)(P_{2,0} - P_{1,0})V^*(t,0) \\ &+ \int_0^t V(t,s)[C_2^* C_2 - C_1^* C_1]V^*(t,s)ds \\ &+ \int_0^t V(t,s)P_1(s)[B_1 B_1^* - B_2 B_2^*]P_1(s)V^*(t,s)ds, \end{aligned}$$

so that $Z(t) \ge 0$ and the conclusion follows. □

1.3 Solution of the control problem

In this section we consider the control problem (1.1.1)–(1.1.2). We assume that Hypothesis 1.1 is fulfilled and we denote by $P \in C_s([0,T];\Sigma^+(H))$ the mild solution of the Riccati equation (1.2.1). We first consider the closed loop equation

$$\begin{cases} y'(t) = Ay(t) - BB^*P(T-t)y(t), \ t \in [0,T], \\ \\ y(0) = x \in H. \end{cases} \tag{1.3.1}$$

We say that $y \in C([0,T];H)$ is a mild solution of equation (1.3.1) if it is a solution of the following integral equation

$$y(t) = e^{tA}x - \int_0^t e^{(t-s)A}BB^*P(T-s)y(s)ds.$$

Proposition 1.3.1 *Assume that Hypothesis 1.1 is fulfilled and let* $x \in H$. *Then equation* (1.3.1) *has a unique mild solution* $y \in C([0,T];H)$.

Proof. It follows by using standard successive approximations. □

We now prove a basic identity.

Proposition 1.3.2 *Assume that Hypothesis 1.1 is fulfilled and let* $u \in L^2(0,T,U)$ $x \in H$. *Let* y *be the solution of the state equation* (1.1.1) *and let* P *be the mild solution of the Riccati equation* (1.2.1). *Then the following identity holds*

$$J(u) = \int_0^T |u(s) + B^*P(T-s)y(s)|^2 ds + \langle P(T)x, x\rangle. \tag{1.3.2}$$

Proof. Let P_n be the mild solution of the approximated Riccati equation (1.2.5), and let y_n be the solution of the problem

$$\begin{cases} y_n'(t) = A_n y(t) + Bu(t), \ t \in [0,T], \\ \\ y(0) = x \in H. \end{cases}$$

Now, by computing the derivative

$$\frac{d}{ds}\langle P_n(T-s)y_n(s), y_n(s)\rangle$$

and completing the squares, we obtain the identity

$$\frac{d}{ds}\langle P_n(T-s)y_n(s), y_n(s)\rangle$$

$$= |u_n(s) + B^*P_n(T-s)y_n(s)|^2 - |Cy_n(s)|^2 - |u(s)|^2.$$

Integrating from 0 to T and letting n tend to infinity we obtain (1.3.2). □

We are now ready to prove the following result.

Theorem 1.3.3 *Assume that Hypothesis 1.1 is fulfilled and let $x \in H$. Then there exists a unique optimal pair (u^*, y^*). Moreover*

(i) $y^* \in C([0,T]; H)$ *is the mild solution to the closed loop equation* (1.3.1).

(ii) $u^* \in C([0,T]; U)$ *is given by the feedback formula*

$$u^*(t) = -B^*P(T-t)y^*(t),\ t \in [0,T]. \tag{1.3.3}$$

(iii) The optimal cost $J(u^*)$ *is given by*

$$J(u^*) = \langle P(T)x, x\rangle. \tag{1.3.4}$$

Proof. We first remark that by identity (1.3.2) it follows that

$$J(u^*) \geq \langle P(T)x, x\rangle, \tag{1.3.5}$$

for any control $u \in C([0,T]; U)$. Let now y^* be the mild solution to (1.3.1) and let u^* be given by (1.3.3). Setting in (1.3.2) $u = u^*$ and taking into account (1.3.5) it follows that (u^*, y^*) is an optimal pair and that (1.3.4) holds.

It remains to prove uniqueness. Let $(\overline{u}, \overline{y})$ be another optimal pair. Setting in (1.3.2) $u = \overline{u}$ and $y = \overline{y}$ we obtain

$$\int_0^T |\overline{u}(s) + B^*P(T-s)\overline{y}(s)|^2 ds = 0,$$

so that $\overline{u}(s) = -B^*P(T-s)\overline{y}(s)$ for almost every $s \in [0,T]$. But this implies that $\overline{y}$ is a mild solution of (1.3.1) so that $\overline{y} = y^*$ and consequently $\overline{u} = u^*$. □

Chapter 2

Control in infinite horizon

2.1 Introduction and setting of the problem

As in Chapter 1 we are concerned with a dynamical system governed by the following state equation

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \ t \geq 0, \\ \\ y(0) = x \in H. \end{cases} \tag{2.1.1}$$

We shall assume that

Hypothesis 2.1 *(i) A generates a strongly continuous semigroup e^{tA} on H.*

(ii) $B \in L(U, H)$.

(iii) $C \in L(H, Y)$.

We want to minimize the cost function

$$J_\infty(u) = \int_0^{+\infty} \left[|Cy(s)|^2 + |u(s)|^2\right] ds, \tag{2.1.2}$$

over all controls $u \in L^2(0, +\infty, U)$ subject to (1.1.1).

We say that the control $u \in L^2(0, +\infty; U)$ is *admissible* if $J_\infty(u) < +\infty$.

An admissible control $u^* \in L^2(0, +\infty; U)$ is called an *optimal control* if

$$J_\infty(u^*) \leq J_\infty(u), \ \forall \, u \in L^2(0, +\infty; U).$$

In this case the corresponding solution y^* of (2.1.1) is called an *optimal state* and the pair (u^*, y^*) an *optimal pair.*

An admissible controls can fail to exist, as the following simple example shows.

Example 2.1.1 Let $H = U = Y = \mathbb{R}$, $B = 0, A = C = 1$. Then for any $u \in L^2(0, +\infty; U)$ we have $y(t) = e^t x$ and

$$J_\infty(u) = \int_0^{+\infty} \left[|e^s y(s)|^2 + |u(s)|^2\right] ds = +\infty.$$

If for any $x \in H$ an admissible control exists, we say that (A, B) is *stabilizable with respect to the observation operator* C, or, for brevity, that (A, B) is C–stabilizable. In this case is still possible to solve problem (2.1.1)–(2.1.2) following the steps,

Step 1. We show that the *minimal nonnegative solution* $P_{min}(t)$ to the Riccati equation

$$P' = A^*P + PA - PBB^*P + C^*C,$$

that is the solution to (1.2.1) corresponding to $P_0 = 0$, converges, as $t \to \infty$ to a solution P_{min}^∞ to the *algebraic Riccati equation*:

$$A^*X + XA - XBB^*X + C^*C = 0 \tag{2.1.3}$$

Step 2. We show that the optimal control u^* is given by the feedback formula

$$u^*(t) = -B^* P_{min}^\infty y^*(t), \quad t \geq 0, \tag{2.1.4}$$

where y^* is the mild solution of the *closed loop equation*

$$\begin{cases} y'(t) = [A - BB^*P_{min}^\infty] y(t), \ t \geq 0, \\ \\ y(0) = x \in H. \end{cases} \tag{2.1.5}$$

Example 2.1.2 (i). Assume that A is of negative type. Then (A, B) is C–stabilizable since the control $u(t) = 0$ is clearly admissible.

(ii). Assume that $B = I$. Then (A, B) is C–stabilizable. In fact let M, ω be such that $\|e^{tA}\| \leq Me^{\omega r}$, $t \geq 0$. Choose $u(t) = -(\omega + 1)e^{t(A-\omega-1)}$, $t \geq 0$. Then $y(t) = e^{t(A-\omega-1)}$, $t \geq 0$ so that $J_\infty(u) < +\infty$.

(iii). Assume that there is $\alpha > 0, \beta > 0, K > 0$ such that

$$\|e^{t(A-2\alpha BB^*)}\| \leq Ke^{-\beta t}, \ t \geq 0. \tag{2.1.6}$$

Then (A, B) is C–stabilizable. In fact setting $u(t) = -2\alpha B^* e^{t(A-2\alpha BB^*)}$, $t \geq 0$, one has $y(t) = e^{t(A-2\alpha BB^*)}$, $t \geq 0$, and so $J_\infty(u) < +\infty$.

Moreover we shall show that equation (2.1.3) has a nonnegative solutions if and only if (A, B) is C–stabilizable.

2.2 The Algebraic Riccati Equation

We assume here that Hypothesis 2.1 holds and consider the system (2.1.1).

We consider the Riccati equation

$$P' = A^*P + PA - PBB^*P + C^*C, \tag{2.2.1}$$

and the corresponding stationary equation

$$A^*X + XA - XBB^*X + C^*C = 0. \tag{2.2.2}$$

In the sequel we shall consider only nonnegative solutions of (2.2.1) and (2.2.2).

Definition 2.2.1 *We say that $X \in \Sigma^+(H)$ is a weak solution of* (2.2.2) *if*

$$\langle Xx, Ay\rangle + \langle Ax, Xy\rangle - \langle B^*Xx, B^*Xy\rangle + \langle Cx, Cy\rangle = 0 \tag{2.2.3}$$

for all $x, y \in D(A)$.

Definition 2.2.2 *We say that $X \in \Sigma^+(H)$ is a stationary solution of* (2.2.1) *if it coincides with the mild solution of* (2.2.1) *with initial condition $P(0) = X$.*

Recalling Proposition 1.2.2 the following results follows immediately.

Proposition 2.2.3 *Let $X \in \Sigma^+(H)$, then the following statements are equivalent* *(i) X is a weak solution of* (2.2.2).

(ii) X is a stationary solution of (2.2.1).

We are going to study existence of a solution of the Algebraic Riccati equation. To this purpose it is useful to consider the solution of the Riccati equation (2.2.1) with initial condition 0. This solution will be denoted by P_{min}. It is the minimal nonnegative solution of (2.2.1). In fact if $P_0 \in \Sigma^+(H)$ and P is the mild solution of (2.2.1) such that $P(0) = P_0$, then by Proposition 1.2.7 we have

$$P_{min}(t) \le P(t), \quad \forall\, t \ge 0.$$

In particular if X is a solution of (2.2.2), then

$$P_{min}(t) \le X, \quad \forall\, t \ge 0.$$

We now prove the following properties of P_{min}.

Proposition 2.2.4 *(i) For any $x \in H$, $\langle P_{min}(\cdot)x, x\rangle$ is non decreasing.*

(ii) Assume that for some $R \in \Sigma^+(H)$, we have

$$P_{min}(t) \le R, \quad \forall\, t \ge 0.$$

Then for all $x \in H$ the limit

$$P_{min}^{\infty} x = \lim_{t \to +\infty} P_{min}(t)x, \tag{2.2.4}$$

exists, and P_{min}^{∞} is a solution of (2.2.2).

In other words there exists a nonnegative solution of (2.2.2) if and only if P_{min} is bounded.

Proof. Let $\varepsilon > 0$, $t \ge 0$ and let P be the solution of (2.2.1) such that $P(0) = P_{min}(\varepsilon)$. By Proposition 1.2.7 we have

$$P(t) = P_{min}(t + \varepsilon) = P(t) \ge P_{min}(t),$$

and (i) is proved. Assume now $P_{min}(t) \le R$; since $P_{min}(t)$ is nondecreasing and bounded we can set

$$\gamma(x) = \lim_{t \to +\infty} \langle P_{min}(t)x, x\rangle, \ \forall\, x \in H.$$

For $x, y \in H$ we have

$$2\langle P_{min}(t)x, y\rangle$$

$$= \langle P_{min}(t)(x+y), (x+y)\rangle - \langle P_{min}(t)x, x\rangle - \langle P_{min}(t)y, y\rangle.$$

So the limit

$$\Gamma(x,y) = \lim_{t\to+\infty} \langle P_{min}(t)x, y\rangle, \quad \forall\, x, y \in H,$$

exists and the following operator $P_{min}^{\infty} \in \Sigma^+(H)$ can be defined

$$\lim_{t\to+\infty} \langle P_{min}(t)x, y\rangle = \langle P_{min}^{\infty}x, y\rangle, \quad \forall\, x, y \in H.$$

It follows that

$$\lim_{t\to+\infty} \langle [P_{min}^{\infty} - P_{min}(t)]x, x\rangle = 0, \quad \forall\, x \in H,$$

which is equivalent to

$$\lim_{h\to+\infty} [P_{min}^{\infty} - P_{min}(t)]^{1/2}x = 0, \quad \forall\, x \in H$$

This implies that

$$\lim_{t\to+\infty} [P_{min}^{\infty} - P_{min}(t)]x = 0, \quad \forall\, x \in H$$

so that (2.2.4) holds.

It remains to show that P_{min}^{∞} is a solution of (2.2.2). For this we denote by P_h the solution of (2.2.1) for which $P_h(0) = P_{min}(h)$, i.e. $P_h(t) = P_{min}(h+t)$. Since

$$\lim_{h\to+\infty} P_{min}(h)x = P_{min}^{\infty}x, \ \forall\, x \in H,$$

by Theorem 1.2.6, we have

$$\lim_{h\to+\infty} P_h(\cdot)x = P_{min}^{\infty}x \text{ in } C([0,T]; H), \ \forall\, x \in H, \ T > 0.$$

Moreover P_{min}^{∞} is a solution of (2.2.1) (hence stationary). $\square$

Remark 2.2.5 Assume that there exists a solution $X \in \Sigma^+(H)$ of (2.2.2). Then by Proposition 2.2.4 the solution P_{min}^{∞} defined by (2.2.4) exists. By the above proposition it follows that

$$P_{min}^{\infty} \leq X,$$

for all solutions $X \in \Sigma^+(H)$ of (2.2.2). Thus P_{min}^{∞} is the minimal solution of the algebraic Riccati equation (2.2.2).

We now prove that a nonnegative solution of the algebraic Riccati equation exists if and only if (A, B) is C–stabilizable.

Proposition 2.2.6 *Assume that Hypothesis 2.1 is fulfilled and that* (A, B) *is* C*–stabilizable. Then there exists a minimal solution* P_{min}^{∞} *of equation* (2.2.2).

Proof. We first recall that by the basic identity (1.3.2) we have

$$\begin{aligned} &\langle P_{min}(t)x, x\rangle + \int_0^t |u(s) + B^* P_{min}(t-s)y(s)|^2 ds \\ &= \int_0^t [|Cy(s)|^2 + |u(s)|^2] ds, \end{aligned} \tag{2.2.5}$$

for any $x \in H$ and any $u \in L^2(0, +\infty; U)$, where y is the solution to (2.1.1). Let u be a control in $L^2(0, +\infty; U)$ such that the corresponding solution of (2.1.1) is such that $Cy \in L^2(0, +\infty; Y)$. By (2.2.5) it follows that

$$\sup_{t \geq 0} \langle P_{min}(t)x, x\rangle \leq \int_0^{+\infty} [|Cy(s)|^2 + |u(s)|^2] ds < +\infty$$

for any $x \in H$. By the Uniform Boundedness Theorem it follows that $P_{min}(t)$ is bounded, so that, by Proposition 2.2.4, there exists a solution of equation (2.2.2). □

In order to prove the converse it is useful to introduce, for any $t > 0$, the following auxiliary optimal control problem over the finite time horizon $[0, t]$: to minimize

$$J_t(u) = \int_0^t [|Cy(s)|^2 + |u(s)|^2] ds \tag{2.2.6}$$

over all controls $u \in L^2(0,t;U)$ subject to (2.1.1). By Theorem 1.3.3 we know that there exists a unique optimal pair (u_t, y_t) for problem (2.2.6), where y_t is the mild solution to the closed loop equation

$$\begin{cases} y_t'(s) = Ay_t(s) - BB^*P_{min}(t-s)y_t(s), \; s \in [0,t], \\ \\ y_t(0) = x, \end{cases}$$

and u_t is given by the feedback formula

$$u_t(s) = -B^*P_{min}(t-s)y_t(s), \; s \in [0,t].$$

Moreover the optimal cost is given by

$$\langle P_{min}(t)x, x\rangle = \int_0^t [|Cy_t(s)|^2 + |u_t(s)|^2]ds. \tag{2.2.7}$$

Lemma 2.2.7 *Assume that the minimal solution P_{min}^∞ of (2.1.2) exists. Denote by y_∞ the corresponding mild solution of the problem*

$$\begin{cases} y_\infty'(s) = Ay_\infty(s) - BB^*P_{min}^\infty y_\infty(s), \; s \geq 0, \\ \\ y_\infty(0) = x, \end{cases}$$

and set

$$u_\infty(s) = -B^*P_{min}^\infty y_\infty(s), \; s \geq 0. \tag{2.2.8}$$

Then we have

$$\lim_{t\to+\infty} y_t(s) = y_\infty(s), \; s \geq 0. \tag{2.2.9}$$

$$\lim_{t\to+\infty} u_t(s) = u_\infty(s), \; s \geq 0. \tag{2.2.10}$$

Proof. Fix $T > t$ and set $z_t = y_t - y_\infty$; then z_t is the mild solution to the problem:

$$\begin{cases} z_t'(s) &= [A - BB^*P_{min}(t-s)]z_t(s) \\ \\ &+ \; BB^*[P_{min}(t-s) - P_{min}^\infty]y_\infty(s) \\ \\ z_t(0) &= 0. \end{cases} \tag{2.2.11}$$

Denote by $U(r,s)$ the evolution operator corresponding to $A - BB^* P_{min}(t-\cdot)$ then for $x \in H$

$$\begin{cases} U(r,\sigma)x = e^{(r-\sigma)A}x - \displaystyle\int_\sigma^r e^{(r-\rho)A}BB^*P_{min}(t-\rho)U(\rho,\sigma)x d\rho, \\ \\ U(\sigma,\sigma) = I. \end{cases}$$

It follows that

$$\|U(r,\sigma)\| \leq Me^{(r-\sigma)\omega} + M\|B\|^2\|P_{min}^\infty\| \int_\sigma^r e^{(r-\rho)\omega}\|U(\rho,\sigma)\|d\rho.$$

By Gronwall's Lemma we have

$$\|U(r,\sigma)\| \leq Me^{(r-\sigma)[\omega + M\|B\|^2\|P_{min}^\infty\|]}, 0 \leq \sigma \leq r \leq T. \tag{2.2.12}$$

We now return to problem (2.2.11) which we write in the form

$$z_t(s) = \int_0^s U(s,\sigma)BB^*[P_{min}(t-\sigma) - P_{min}^\infty]y_\infty(\sigma)d\sigma.$$

By (2.2.12) and the dominate convergence theorem we obtain $z_t(s) \to 0$ as $t \to +\infty$. So (2.2.9) and then (2.2.10) follow. □

We can now prove the following proposition.

Proposition 2.2.8 *Assume that there exists a solution of* (2.2.2). *Then* (A, B) *is* C*-stabilizable.*

Proof. Let y_t and u_t be defined as in Lemma 2.2.7. By (2.2.7) we have for $t \geq T$

$$\langle P_{min}^\infty x, x\rangle \geq \int_0^T [|Cy_t(s)|^2 + |u_t(s)|^2]ds \tag{2.2.13}$$

and, as $t \to +\infty$,

$$\langle P_{min}^\infty x, x\rangle \geq \int_0^T [|Cy_\infty(s)|^2 + |u_\infty(s)|^2]ds. \tag{2.2.14}$$

But, since T is arbitrary we find

$$\langle P_{min}^\infty x, x\rangle \geq \int_0^{+\infty} [|Cy_\infty(s)|^2 + |u_\infty(s)|^2]ds, \tag{2.2.15}$$

and thus $u_\infty \in L^2(0, +\infty; U)$ is an admissible control. □

2.3 Solution of the control problem

We now consider the control problem (2.1.1)–(2.1.2) and prove the following result.

Theorem 2.3.1 *Assume that Hypothesis 2.1 is fulfilled, that (A, B) is C–stabilizable, and let $x \in H$. Then there exists a unique optimal pair (u^*, y^*). Moreover*

(i) $y^* \in C([0, +\infty); H)$ *is the mild solution to the closed loop equation* (2.1.5).

(ii) $u^* \in C([0, +\infty); U)$ *is given by the feedback formula*

$$u^*(t) = -B^* P_{min}^{\infty} y^*(t), \; t \geq 0. \tag{2.3.1}$$

(iii) The optimal cost $J_\infty(u^*)$ *is given by*

$$J_\infty(u^*) = \langle P_{min}^{\infty} x, x \rangle. \tag{2.3.2}$$

Proof. Let $u \in L^2([0, +\infty); U)$ and let y be the corresponding solution of the state equation (2.1.1). By the identity (2.2.5) we have

$$\langle P_{min}(t)x, x \rangle \leq \int_0^t [|Cy(s)|^2 + |u(s)|^2] ds \leq J_\infty(u).$$

It follows that

$$J_\infty(u) \geq \langle P_{min}^{\infty}(t)x, x \rangle, \; \forall \, u \in L^2([0, +\infty); U).$$

Let now u_∞ be defined by (2.2.8); by (2.2.15) we have

$$\langle P_{min}^{\infty}(t)x, x \rangle \geq J_\infty(u_\infty),$$

so that u_∞ is optimal. Formula (2.3.1) with $u^* = u_\infty$, $y^* = y_\infty$ follows from (2.2.9)–(2.2.10).

It remains to to show uniqueness. Let $(\hat{u}, \hat{y})$ be another optimal pair, then $J_\infty(\hat{u}) = \langle P_{min}^{\infty} x, x \rangle$. Fix $T > 0$. By applying (2.2.5) with $t \geq T$ we obtain

$$\int_0^T |\hat{u}(s) + B^* P_{min}(t-s)\hat{y}(s)|^2 ds \leq J_\infty(\hat{u}) - \langle P_{min}(t)x, x \rangle$$

$$\leq \langle [P_{min}^{\infty} - P_{min}(t)x, x \rangle.$$

As $t \to +\infty$ we have

$$\int_0^T |\hat{u}(s) + B^* P_{min}^{\infty} \hat{y}(s)|^2 ds = 0, thatyields$$

$\hat{u}(s) = -B^* P_{min}^{\infty} \hat{y}(s)$. Consequently $\hat{y} = y^*$ and $\hat{u} = u^*$. □

Chapter 3

Examples and generalizations

3.1 Parabolic equations

We consider here Example 1.1.1. Let D be an open subset of $\mathbb{R}^n$ with regular boundary ∂D. Consider the state equation

$$\begin{cases} D_t y(t,\xi) = (\Delta_\xi + c)y(t,\xi) + u(t,\xi), \text{ in } (0,T] \times D, \\ y(t,\xi) = 0, \text{ on } (0,T] \times \partial D, \\ y(0,\xi) = x(\xi), \text{ in } D. \end{cases} \tag{3.1.1}$$

Let $H = U = Y = L^2(D)$, $B = C = P_0 = I$ and define the linear operator by A in H :

$$\begin{cases} Ay = (\Delta_\xi + c)y \\ D(A) = H^2(D) \cap H_0^1(D). \end{cases} \tag{3.1.2}$$

It is well known that A is self–adjoint and consequently is the infinitesimal generator of a strongly continuous semigroup on $H = L^2(D)$. Moreover there exists a complete orthonormal system $\{e_k\}$ in $L^2(D)$ and a sequence $\{\lambda_k\}$ of positive numbers such that

$$Ae_k = -\lambda_k e_k, \quad k \in \mathbb{N}.$$

Setting $y(t) = y(t,\cdot)$, $u(t) = u(t,\cdot)$, we write (3.1.1) in the abstract form (1.1.1).

We want to minimze the cost

$$J(u) = \int_0^T \int_D [|y(t,\xi)|^2 + |u(t,\xi)|^2] dt d\xi + \int_D |y(T,\xi)|^2 d\xi. \tag{3.1.3}$$

By Theorem 1.3.3 there exists a unique optimal pair (u^*, y^*) where y^* is the solution of the closed loop equation

$$\begin{cases} D_t y(t,\xi) = (\Delta_\xi + c) y(t,\xi) - P(T-t) y(t,\cdot)(\xi), \text{ in } (0,T] \times D, \\ y(t,\xi) = 0, \text{ on } (0,T] \times \partial D, \\ y(0,\xi) = x(\xi), \text{ in } D. \end{cases} \tag{3.1.4}$$

Moreover u^* is given by

$$u^*(t,\xi) = -P(T-t) y(t,\cdot)(\xi),$$

and the Riccati equation reeds as follows

$$P' = 2AP - P^2 + I, \quad P(0) = I. \tag{3.1.5}$$

For any $t \geq 0$ we can find explicitly $P(t)$ as

$$P(t) e_k = p_k(t) e_k, \quad k \in \mathbb{N},$$

where p_k is the solution to the ordinary differential equation

$$p_k'(t) = -2\lambda_k(t) p_k(t) - p_k^2(t) + 1, \quad p_k(0) = 1.$$

Let us consider now the infinite horizon problem, $T = +\infty$. We want to minimze the cost

$$J_\infty(u) = \int_0^{+\infty} \int_D [|y(t,\xi)|^2 + |u(t,\xi)|^2] dt d\xi + \int_D |y(T,\xi)|^2 d\xi. \tag{3.1.6}$$

By Example 2.1.2–(ii) (A, I) is I–stabilizable, and consequently by Theorem 2.3.1 there exists a unique optimal pair (u^*, y^*) where y^* is the solution of the closed loop equation

$$\begin{cases} D_t y(t,\xi) = (\Delta_\xi + c) y(t,\xi) - P_\infty y(t,\cdot)(\xi), \text{ in } (0,+\infty) \times D, \\ y(t,\xi) = 0, \text{ on } (0,+\infty) \times \partial D, \\ y(0,\xi) = x(\xi), \text{ in } D. \end{cases} \tag{3.1.7}$$

Moreover u^* is given by

$$u^*(t,\xi) = -P_\infty y(t,\cdot)(\xi),$$

and the Algebraic Riccati equation reeds as follows

$$2AP - P^2 + I = 0 \tag{3.1.8}$$

Consequently

$$P_\infty = \sqrt{A^2+I} + A$$

and

$$P_\infty e_k = (\sqrt{\lambda_k^2+1} - \lambda_k)e_k,\ k \in \mathbb{N}.$$

3.2 Wave equation

Let D be an open subset of $\mathbb{R}^n$ with regular boundary ∂D. Consider the state equation

$$\begin{cases} D_t^2 y(t,\xi) = \Delta_\xi y(t,\xi) + u(t,\xi),\ \text{in } (0,T]\times D, \\ y(t,\xi) = 0,\ \text{on } (0,T]\times \partial D, \\ y(0,\xi) = x_0(\xi),\ D_t y(0,\xi) = x_1(\xi),\ \text{in } D. \end{cases} \tag{3.2.1}$$

We want to minimze the cost

$$\begin{aligned} J(u) = &\int_0^T \int_D [|\nabla y(t,\xi)|^2 + |y(t,\xi)|^2 + |u(t,\xi)|^2] dt d\xi \\ &+ \int_D [|\nabla y(T,\xi)|^2 + |y(T,\xi)|^2] d\xi. \end{aligned} \tag{3.2.2}$$

Setting $y(t) = y(t,\cdot)$, $u(t) = u(t,\cdot)$, we write (3.2.1) as

$$\begin{cases} y''(t) = Ay(t) + u(t) \\ y(0) = x_0,\ y'(0) = x_1, \end{cases} \tag{3.2.3}$$

where A is defined by (3.1.2). Now, setting $y'(t) = z(t)$, $Y(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$, and $X = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$, we reduce the problem to a first order problem

$$\begin{cases} D_t Y(t) = \mathcal{A} Y(t) + \mathcal{B} u(t) \\ Y(0) = X, \end{cases} \tag{3.2.4}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now we choose $H = Y = H_0^1(D) \oplus L^2(D)$, $U = L^2(D)$ and

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$D(\mathcal{A}) = (H^2(D) \oplus H_0^1(D)) \oplus H_0^1(D),$$

and $\mathcal{A}$ generates a strongly continuous semigroup of contractions on H given by:

$$e^{t\mathcal{A}} = \begin{pmatrix} \cos(\sqrt{-A}\, t) & \frac{1}{\sqrt{-A}} \sin(\sqrt{-A}\, t) \\ -\sqrt{-A}\, \sin(\sqrt{-A}\, t) & \cos(\sqrt{-A}\, t). \end{pmatrix} \tag{3.2.5}$$

Finally the cost can be written as

$$J(u) = \int_0^T \int_D [Y(t)|_H^2 + |u(t)|_{L^2(D)}^2] dt + |Y(T)|_H^2. \tag{3.2.6}$$

By Theorem 1.3.3 there exists a unique optimal pair (u^*, y^*).

Finally we can show that $(\mathcal{A}, \mathcal{B})$ is $\mathcal{C}$–stabilizable. For this we shall fulfill the conditions of Example 2.1.2–(iii) by proving that for all $\alpha < \sqrt{\lambda_0}$ (2.1.6) holds. We have in fact, by a direct computation

$$e^{t(A-2\alpha BB^*)} = e^{-t\alpha} \begin{pmatrix} \cos(Et) + \frac{\alpha}{E} \sin(Et) & \frac{1}{E} \sin(Et) \\ \frac{\alpha^2+E^2}{E} \sin(Et) & -\frac{\alpha}{E} \sin(Et) + \cos(Et) \end{pmatrix}, \tag{3.2.7}$$

where $E = \sqrt{-A - \alpha^2 I}$.

3.3 Boundary control problems

Let us consider the following state equation

$$\begin{cases} D_t y(t,\xi) = \Delta_\xi y(t,\xi), \text{ in } (0,T]\times[0,1], \\ y(t,0) = u_0(t),\ y(t,1) = u_1(t), \text{ on } (0,T], \\ y(0,\xi) = x(\xi), \text{ in } D. \end{cases} \tag{3.3.1}$$

Here the control is given on the boundary of D.

We want to minimize the cost

$$J(u) = \int_0^T \int_0^1 |y(t,\xi)|^2 dt\, d\xi + \int_0^T [|u_0(t)|^2 + u_1(t)|^2] dt + \int_0^1 |y(T,\xi)|^2 d\xi. \tag{3.3.2}$$

In order to reduce this problem to the standard form (1.1.1), it is convenient to introduce the *Dirichlet mapping*

$$\delta : \mathbb{R}^2 \to L^2(0,1),\ \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \to \delta(\alpha_0,\alpha_1)$$

where

$$\delta(\alpha_0,\alpha_1)(\xi) = (\alpha_1 - \alpha_0)\xi + \alpha_0.$$

Notice that $\delta(\alpha_0,\alpha_1)$ is the unique harmonic function on $[0,1]$ that holds α_0 at $\{0\}$ and α_1 at $\{1\}$.

Let us now proceed formally by setting

$$\begin{aligned} z(t,\xi) &= y(t,\xi) - \delta(u_0(t),u_1(t)) \\ &= y(t,\xi) - (u_1(t) - u_0(t))\xi - u_0(t),\ \xi \in [0,1], t \ge 0, \end{aligned}$$

so that $z(t,0) = z(t,1) = 0$. Then

$$D_t z(t,\xi) = D_t y(t,\xi) - \delta u'(t),$$

where $u(t) = (u_0(t), u_1(t))$, and we can write problem (3.3.1) as

$$\begin{cases} D_t z(t,\xi) = \Delta_\xi z(t,\xi) - \delta u'(t), \text{ in } (0,T]\times[0,1], \\ z(t,0) = z(t,1) = 0, \text{ on } (0,T], \\ y(0,\xi) = x(\xi), \text{ in } D. \end{cases} \tag{3.3.3}$$

Now this problem can be written in the abstract form

$$\begin{cases} z'(t) = Az(t) - \delta u'(t), \\ z(0) = z(t,1) = x - \delta u(0), \end{cases}$$

where A denotes the operator (3.1.2) (with $D = [0,1]$). Using the variation of constants formula we find

$$z(t) = e^{tA}(x - \delta u(0)) - \int_0^t e^{(t-s)A}\delta u'(s)ds,$$

and, integrating by parts, we find (always formally),

$$y(t) = e^{tA}x - \int_0^t Ae^{(t-s)A}\delta u(s)ds. \tag{3.3.4}$$

We show now that this formula is meaningful. For this we recall that

$$\delta \in L(\mathbb{R}^2; H^{1/2}(0,1)),$$

and consequently $\delta(t) \in D((-A)^{\varepsilon})$ for any $\varepsilon \in [0,1/4)$. This implies that, for a suitable constant $c > 0$ we have

$$|Ae^{(t-s)A}\delta u(s)| \le \frac{c}{(t-s)^{3/4}},$$

so that formula (3.3.4) is meaningful.

Equation (3.3.4) can be considered as the mild form of the state equation

$$\begin{cases} y'(t) = Ay(t) - A\delta u(t), \ t \ge 0, \\ y(0) = x \in H, \end{cases} \tag{3.3.5}$$

so that $B = -A\delta$. This is not meaningful because the intersection of the range of δ with the domain of A is $\{0\}$. However one is able to give a meaning to the Riccati equation by writing

$$B = (-A)^{1-\gamma}[(-A)^{\gamma}\delta] := (-A)^{1-\gamma}\delta_{\gamma},$$

where $\gamma \in (0,1/4)$ and consequently the operator δ_γ is bounded. In this way the term PBB^*P can be written as

$$P(-A)^{1-\gamma}\delta_\gamma\delta_\gamma^*[P(-A)^{1-\gamma}]^*.$$

Now the idea is to try to write an equation for $P(-A)^{1-\gamma}$.

Appendix A

Linear Semigroups Theory

In all this appendix X represents a complex Banach space (norm $|\cdot|$), and $L(X)$ the Banach algebra of all linear bounded operators from X into X endowed with the sup norm:

$$\|T\| = \sup\{|Tx| : \ x \in X, \ |x| \leq 1\}.$$

A.1 Some preliminaries on spectral theory

Let $A : D(A) \subset X \to X$ be a linear closed operator. We say that $\lambda \in \mathbb{C}$ belongs to the *resolvent set* $\rho(A)$ of A if $\lambda - A$ is bijective e and $(\lambda - A)^{-1} \in L(X)$; in this case the operator $R(\lambda, A) := (\lambda - A)^{-1}$ is called the *resolvent* of A at λ. The complementary set $\sigma(A)$ of $\rho(A)$ is called the *spectrum* of A.

Example A.1.1 Let $X = C([0,1])$ be the Banach space of all continuous functions on $[0,1]$ endowed with the sup norm, and let $C^1([0,1])$ be the subspace of $C([0,1])$ of all functions u that continuously differentiable. Let us consider the two following linear operators on X :

$$D(A) = C^1([0,1]), \ Au = u' \ , \forall \ u \in D(A),$$

$$D(B) = \{u \in C^1([0,1]); \ u(0) = 0\}, \ Bu = u' \ \ \forall \ u \in D(B).$$

We have

$$\rho(A) = \emptyset, \ \sigma(A) = \mathbb{C}.$$

In fact, given $\lambda \in \mathbb{C}$, the mapping $\lambda - A$ is not injective since, for all $c \in \mathbb{C}$ the function $u(\xi) = ce^{\lambda\xi}$ belongs to $D(A)$ and $(\lambda - A)u = 0$.

For as the operator B is concerned, we have

$$\rho(B) = \mathbb{C}, \ \sigma(A) = \emptyset.$$

and

$$(R(\lambda, B)f)(\xi) = -\int_0^\xi e^{\lambda(\xi-\eta)} f(\eta) d\eta, \forall \ \lambda \in \mathbb{C}, \forall \ f \in X, \forall \ \xi \in [0,1].$$

In fact $\lambda \in \rho(B)$ if and only if the problem

$$\begin{cases} \lambda u(\xi) - u'(\xi) = f(\xi) \\ \\ u(0) = 0 \end{cases}$$

has a unique solution $f \in X$.

Let us prove the important *resolvent identity.*

Proposition A.1.2 *If $\lambda, \mu \in \rho(A)$ we have*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A) \tag{A.1.1}$$

Proof. For all $x \in X$ we have

$$(\mu - \lambda) R(\lambda, A)x = (\mu - A + A - \lambda) R(\lambda, A)x = (\mu - A) R(\lambda, A)x - x$$

Applying $R(\mu, A)$ to both sides of the above identity, we find

$$(\mu - \lambda) R(\mu, A) R(\lambda, A)x = R(\lambda, A)x - R(\mu, A)x$$

and the conclusion follows. □

Proposition A.1.3 *Let A be a closed operator. Let $\lambda_0 \in \rho(A)$, and $|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|}$. Then $\lambda \in \rho(A)$ and*

$$R(\lambda, A) = R(\lambda_0, A)(1 + (\lambda - \lambda_0) R(\lambda_0, A))^{-1} \tag{A.1.2}$$

Thus $\rho(A)$ is open and $\sigma(A)$ is closed. Moreover

$$R(\lambda, A) = \sum_{k=1}^{\infty} (-1)^k (\lambda - \lambda_0)^k R^{k+1}(\lambda_0, A), \tag{A.1.3}$$

and so $R(\lambda, A)$ is analytic on $\rho(A)$.

Proof. The equation $\lambda x - Ax = y$ is equivalent to

$$(\lambda - \lambda_0)x + (\lambda_0 - A)x = y,$$

and, setting $z = (\lambda_0 - A)x$, to

$$z + (\lambda - \lambda_0)R(\lambda_0, A)z = y.$$

Since $\|(\lambda - \lambda_0)R(\lambda_0, A)\| < 1$ it follows

$$z = (1 + (\lambda - \lambda_0)R(\lambda_0, A))^{-1}y,$$

that yields the conclusion. □

A.2 Strongly continuous semigroups

A *strongly continuous semigroup* on X is a mapping $T : [0, \infty) \to L(X),\ t \to T(t)$ such that

(i) $T(t + s) = T(t)T(s), \forall\, t, s \geq 0,\ T(0) = I.$

(ii) $T(\cdot)x$ is continuous for all $x \in X$.

Remark A.2.1 $\|T(\cdot)\|$ is locally bounded by the uniform boundedness theorem.

The *infinitesimal generator* A of $T(\cdot)$ is defined by

$$\begin{cases} D(A) = \left\{ x \in X : \exists \lim_{h \to 0+} \Delta_h x \right\} \\ \\ Ax = \lim_{h \to 0+} \Delta_h x, \end{cases} \tag{A.2.1}$$

where

$$\Delta_h = \frac{T(h) - I}{h}, h > 0.$$

Proposition A.2.2 *$D(A)$ is dense in X.*

Proof. For all $x \in H$ and $a > 0$ we set

$$x_a = \frac{1}{a} \int_0^a T(s)xds.$$

Since $\lim_{a \to 0} x_a = x$, it is enough to show that $x_a \in D(A)$. We have in fact for any $h \in (0, a)$,

$$\Delta_h x_a = \frac{1}{ah} \left[\int_a^{a+h} T(s)xds - \int_0^h T(s)xds \right],$$

and, consequently $x_a \in D(A)$ since

$$\lim_{h \to 0} \Delta_h x_a = \Delta_a x.$$

□

Exercise A.2.3 *Prove that $D(A^2)$ is dense in X.*

We now study the derivability of the semigroup $T(t)$. Let us first notice that, since

$$\Delta_h T(t)x = T(t)\Delta_h x,$$

if $x \in D(A)$ then $T(t)x \in D(A), \forall\, t \geq 0$ and $AT(t)x = T(t)Ax$.

Proposition A.2.4 *Assume that $x \in D(A)$, then $T(\cdot)x$ is differentiable $\forall\, t \geq 0$ and*

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax \tag{A.2.2}$$

Proof. Let $t_0 \geq 0$ be fixed and let $h > 0$. Then we have

$$\frac{T(t_0 + h)x - T(t_0)x}{h} = \Delta_h T(t_0)x \overset{h \to 0}{\to} AT(t_0)x.$$

This shows that $T(\cdot)x$ is right differentiable at t_0. Let us show left differentiability, assuming $t_0 > 0$. For $h \in]0, t_0[$ we have

$$\frac{T(t_0 - h)x - T(t_0)x}{h} = T(t_0 - h)\Delta_h x \overset{h \to 0}{\to} T(t_0)Ax,$$

since $\|T(t)\|$ is locally bounded by Remarl A.2.1. □

Proposition A.2.5 *A is a closed operator.*

Proof. Let $(x_n) \subset D(A)$, and let $x, y \in X$ be such that

$$x_n \to x, \quad Ax_n = y_n \to y$$

Then we have

$$\Delta_h x_n = \frac{1}{h} \int_0^h T(t) y_n dt.$$

As $h \to 0$ we get $x \in D(A)$ and $y = Ax$, so that A is closed. □

We end this section by studying the asymptotic behaviour of $T(\cdot)$. We define the *type* of $T(\cdot)$ as

$$\omega_0 = \inf_{t>0} \frac{\log \|T(t)\|}{t}.$$

Clearly $\omega_0 \in [-\infty, +\infty)$.

Proposition A.2.6 *We have*

$$\omega_0 = \lim_{t \to +\infty} \frac{\log \|T(t)\|}{t}. \tag{A.2.3}$$

Proof. It is enough to show that

$$\limsup_{t \to \infty} \frac{\log \|T(t)\|}{t} \leq \omega_0.$$

Let $\varepsilon > 0$ and $t_\varepsilon > 0$ be such that

$$\frac{\log \|T(t_\varepsilon)\|}{t_\varepsilon} < \omega_0 + \varepsilon.$$

Set

$$t = n(t)t_\varepsilon + r(t), \; n(t) \in \mathbb{N}, r(t) \in [0, t_\varepsilon).$$

Since $\|T(\cdot)\|$ is locally bounded, there exists $M_\varepsilon > 0$ such that

$$\|T(t)\| \leq M_\varepsilon, \; t \in [0, t_\varepsilon].$$

We have

$$\frac{\log \|T(t)\|}{t} = \frac{\log \|T(t_\varepsilon)^{n(t)}T(r(t))\|}{t}$$

$$\leq \frac{n(t)\ \log \|T(t_\varepsilon)\| + \log \|T(r(t))\|}{n(t)t_\varepsilon + r(t)} \leq \frac{\log \|T(t_\varepsilon)\| + \frac{M_{t_\varepsilon}}{n(t)}}{t_\varepsilon + \frac{r(t)}{n(t)}}.$$

As $t \to +\infty$, we obtain

$$\limsup_{t\to\infty} \frac{\log \|T(t)\|}{t} \leq \frac{\log \|T(t_\varepsilon)\|}{t_\varepsilon} \leq \omega_0 + \varepsilon.$$

□

Corollary A.2.7 *Let T be of type ω_0. Then for all $\varepsilon > 0$ there exists $N_\varepsilon \geq 1$ such that*

$$\|T(t)\| \leq N_\varepsilon e^{(\omega_0+\varepsilon)t}, \forall\, t \geq 0 \tag{A.2.4}$$

Proof. Let $t_\varepsilon, n(t), r(t)$ as in the previous proof. Then we have

$$\|T(t)\| \leq \|T(t_\varepsilon)\|^{n(t)}\|T(r(t))\| \leq e^{t_\varepsilon n(t)(\omega_0+\varepsilon)}M_{t_\varepsilon} \leq M_{t_\varepsilon}e^{(\omega_0+\varepsilon)t}.$$

and the conclusion follows.□

In the sequel we shall denote by $\mathcal{G}(M,\omega)$ the set of all strongly continuous semigroups T such that

$$\|T(t)\| \leq Me^{\omega t}, t \geq 0$$

Example A.2.8 Let $X = L^p(\mathbb{R}), p \geq 1$, $(T(t)f)(\xi) = f(\xi - t), f \in L^p(\mathbb{R})$. Then we have $\|T(t)\| = 1$ and so $\omega_0 = 0$.

Example A.2.9 Let $X = L^p(0,T), T > 0, p \geq 1$, and let

$$(T(t)f)(\xi) = \begin{cases} f(\xi - t) \ \text{ if } \ \xi \in [t,T] \\ 0 \ \text{ if } \xi \in [0,t[\end{cases}$$

Then we have $T(t) = 0$ if $t \geq T$ and so $\omega_0 = -\infty$.

Exercise A.2.10 Let $A \in \mathcal{L}(X)$ compact and let $\{\lambda_i\}_{i\in\mathbb{N}}$ be its eigenvalues. Set $T(t) = e^{tA}$. Then we have

$$\omega_0 = \sup_{i\in\mathbb{N}}\ \text{Re}\ \lambda_i.$$

A.3 The Hille–Yosida theorem

We assume here that $T \in \mathcal{G}(M,\omega)$. We denote by A its infinitesimal generator.

Proposition A.3.1 *We have*

$$\rho(A) \supset \{\lambda \in \mathbb{C} \ \text{Re} \ \lambda > \omega\} \tag{A.3.1}$$

$$R(\lambda, A)y = \int_0^\infty e^{-\lambda t}T(t)ydt, \ y \in X, \ \ \text{Re} \ \lambda > \omega \tag{A.3.2}$$

Proof. Set

$$\Sigma = \{\lambda \in \mathbb{C}; \ \ \text{Re}\,\lambda > \omega\}$$

$$F(\lambda)y = \int_0^\infty e^{-\lambda t}T(t)ydt, \ y \in X, \ \ \text{Re} \ \lambda > \omega.$$

This is meaningful since $T \in \mathcal{G}(M,\omega)$. We have to show that, given $\lambda \in \Sigma$ and $y \in X$ the equation $\lambda x - Ax = y$ has a unique solution given by $x = F(\lambda)y$.

Existence

Let $\lambda \in \Sigma, y \in X$, $x = F(\lambda)y$. Then we have

$$\Delta_h x = \frac{1}{h}(e^{\lambda h} - 1)x - \frac{1}{h}e^{\lambda h}\int_0^h e^{-\lambda t}T(t)ydt$$

and so, as $h \to 0$,

$$\lim_{h\to 0+} \Delta_h x = \lambda x - y = Ax$$

that is x is a solution of the equation $\lambda x - Ax = y$.

Uniqueness

Let $x \in D(A)$ be a solution of the equation $\lambda x - Ax = y$. Then we have

$$\begin{aligned}\int_0^\infty e^{-\lambda t}T(t)(\lambda x - Ax)dt &= \lambda \int_0^\infty e^{-\lambda t}T(t)xdt \\ &- \int_0^\infty e^{-\lambda t}\frac{d}{dt}T(t)xdt = x,\end{aligned}$$

so that $x = F(\lambda)y$.

We are now going to prove the *Hille–Yosida* theorem.

Theorem A.3.2 *Let $A : D(A) \subset X \to X$ be a closed operator. Then A is the infinitesimal generator of a strongly continuous semigroup belonging to $\mathcal{G}(M,\omega)$ if and only if*

$$\begin{aligned}&(i)\ \ \rho(A) \supset \{\lambda \in \mathbb{R};\ \lambda > \omega\}\\ &(ii)\ \ \|R^n(\lambda, A)\| \le \tfrac{M}{(\lambda-\omega)^n},\ \forall\, n \in \mathbb{N}\ \ \forall\, \lambda > \omega \qquad (A.3.3)\\ &(iii)\ \ D(A)\ \text{is dense in}\ X.\end{aligned}$$

Given a linear operator A fulfilling $(A.3.3)$ it is convenient to introduce a sequence of linear operators (called the *Yosida approximations* of A). They are defined as

$$A_n = nAR(n, A) = n^2 R(n, A) - n \qquad (A.3.4)$$

Lemma A.3.3 *We have*

$$\lim_{n\to\infty} nR(n, A)x = x, \ \ \forall\, x \in X, \qquad (A.3.5)$$

and

$$\lim_{n\to\infty} A_n x = Ax, \ \ \forall\, x \in D(A). \qquad (A.3.6)$$

Proof. Since $D(A)$ is dense in X and $\|nR(n, A)\| \le \frac{Mn}{n-\omega}$, to prove (A.3.5) it is enough to show that.

$$\lim_{n\to\infty} nR(n, A)x = x, \ \ \forall\, x \in D(A).$$

In fact for any $x \in D(A)$ we have

$$|nR(n, A)x - x| = |R(n, A)Ax| \le \frac{M}{n-\omega}|Ax|,$$

and the conclusion follows.

Finally if $x \in D(A)$ we have

$$A_n x = nR(n, A)Ax \to Ax,$$

and $(A.3.6)$ follows.□

Proof of Theorem $A.3.2$. **Necessity**. (i) follows from Proposition A.3.1 and (iii) from Proposition A.2.2. Let us show (ii). Let $k \in \mathbb{N}$ and $\lambda > \omega$. It follows

$$\frac{d^k}{d\lambda^k} R(\lambda, A)y = \int_0^\infty (-t)^k e^{-\lambda t} T(t) y dt, \ y \in X,$$

from which

$$\left\| \frac{d^k}{d\lambda^k} R(\lambda, A) \right\| \le M \int_0^\infty t^k e^{-\lambda t + \omega t} dt$$

that yields the conclusion.

Sufficiency.

Step 1. We have

$$\|e^{tA_n}\| \le M e^{\frac{\omega n t}{n-\omega}}, \forall\, n \in \mathbb{N}. \tag{A.3.7}$$

In fact, by the identity

$$e^{tA_n} = e^{-nt} e^{tn^2 R(n,A)} = e^{-nt} \sum_{k=0}^\infty \frac{n^{2k} t^k R^k(n, A)}{k!},$$

it follows

$$\|e^{tA_n}\| \le M e^{-nt} \sum_{k=0}^\infty \frac{n^{2k} t^k}{(n-\omega)^k k!}.$$

Step 2. There exists $C > 0$ such that, for all $m, n > 2\omega$, and $x \in D(A^2)$,

$$\|e^{tA_n} x - e^{tA_m} x\| \le Ct \frac{|m-n|}{(m-\omega)(n-\omega)} \|A^2 x\|. \tag{A.3.8}$$

Setting $u_n(t) = e^{tA_n} x$, we have

$$\frac{d}{dt}(u_n(t) - u_m(t)) = A_n(u_n(t) - u_m(t)) - (A_m - A_n) u_m(t)$$

$$= A_n(u_n(t) - u_m(t)) - (n-m) A^2 R(m, A) R(n, A) u_m(t).$$

It follows

$$u_n(t) - u_m(t) = (n-m)A^2R(m,A)R(n,A)\int_0^t e^{(t-s)A_n}u_m(s)ds$$

$$= (n-m)R(m,A)R(n,A)\int_0^t e^{(t-s)A_n}e^{sA_m}A^2x.$$

Step 3. For all $x \in X$ there exists the limit

$$\lim_{n\to\infty} e^{tA_n}x =: T(t)x \tag{A.3.9}$$

and $T : [0,\infty) \to L(X), t \to T(t)$ is strongly continuous.

¿From the second step it follows that the sequence $(u_n(t)$ is Cauchy, uniformly in t on compact subsets of $[0,+\infty[$, for all $x \in D(A^2)$. Since $D(A^2)$ is dense in X (see Exercise A.2.3) this holds for all $x \in X$. Finally it is easy to check that $T(\cdot)$ is strongly continuous.

Step 4. If $x \in D(A)$, then $T(\cdot)x$ is differentiable and

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x.$$

In fact let $x \in D(A)$, and $v_n(t) = \frac{d}{dt}u_n(t)$.Then

$$v_n(t) = e^{tA_n}A_nx$$

Since $x \in D(A)$ there exists the limit

$$\lim_{n\to\infty} v_n(t) = e^{tA}Ax$$

This implies that u is differentiable and $u'(t) = v(t)$ so that $u \in C^1([0,+\infty);X)$. Moreover

$$A(nR(n,A)u_n(t)) = u_n'(t) \to v(t)$$

Since A is closed and $nR(n,A)u_n(t) \to u(t)$ it follows that $u(t) \in D(A)$ and $u'(t) = Au(t)$.

Step 5. A is the infinitesimal generator of $T(\cdot)$.

Let B be the infinitesimal generator of $T(\cdot)$. By Step 4 $B \supset A$ ([1]). It is enough to show that if $x \in D(B)$ then $x \in D(A)$. Let $x \in D(B)$, $\lambda_0 > \omega$, setting $z = \lambda_0 x - Bx$ we have

$$z = (\lambda_0 - A)R(\lambda_0, A)z$$

$$= \lambda_0 R(\lambda_0, A)z - BR(\lambda_0, A)z = (\lambda_0 - B)R(\lambda_0, A)z.$$

Thus $x = R(\lambda_0, B)z = R(\lambda_0, A)z \in D(A)$.□

Remark A.3.4 To use the Hille-Yosida theorem requires to check infinite conditions. However if $M = 1$ it is enough to ask (ii) only for $n = 1$. In such a case $T \in \mathcal{G}(1, \omega)$. If $\omega \leq 0$ we say that $T(\cdot)$ is a *contraction semigroup*.

Example A.3.5 Let $X = C_0([0, \pi])$ the Banach space of all continuous functions in $[0, \pi]$ that vanish at 0 and π. Let A be the linear operator in X defined as

$$\begin{cases} D(A) = \{y \in C^2([0, \pi]); y(0) = y''(0) = y(\pi) = y''(\pi) = 0\} \\ Ay = y'', \; \forall\, y \in D(A) \end{cases}$$

It is easy to check that $\sigma(A) = \{-n^2;\; n \in \mathbb{N}\}$. Moreover any element of $\sigma(A)$ is a simple eigenvalue whose corresponding eigenvector is given by

$$\varphi_n(\xi) = \sin\, n\xi, \quad \forall\, n \in \mathbb{N}.$$

We have

$$A\varphi_n = -n^2 \varphi_n$$

Moreover if $\lambda \in \rho(A)$ and $f \in C_0([0, \pi])$, $u = R(\lambda, A)f$ is the solution of the problem

$$\begin{cases} \lambda u(\xi) - u''(\xi) = f(\xi) \\ u(0) = u(\pi) = 0. \end{cases}$$

[1]That is $D(B) \supset D(A)$ and $Ax = Bx \; \forall\, x \in D(A)$

By a direct verification we find

$$u(\xi) = \frac{\sinh(\sqrt{\lambda}\xi)}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)} \int_\xi^\pi \sinh[\sqrt{\lambda}(\pi - \eta)]f(\eta)d\eta$$
$$+\frac{\sinh[\sqrt{\lambda}(\pi - \xi)]}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)} \int_0^\xi \sinh[\sqrt{\lambda}\eta]f(\eta)d\eta. \tag{A.3.10}$$

¿From $(A.3.10)$ it follows that

$$\|R(\lambda, A)\| \le \frac{1}{\lambda}, \ \forall\, \lambda > 0. \tag{A.3.11}$$

Therefore the assumptions of the Hille-Yosida theorem are fulfilled.

A.4 Cauchy problem

Let $A \in \mathcal{G}(M, \omega)$, and let $T(\cdot)$ be the semigroup generated by A.

We are here concerned with the following problem

$$\begin{cases} u'(t) = Au(t) + g(t), \ t \in [0,T] \\ u(0) = x, \end{cases} \tag{A.4.1}$$

where $x \in H$ and $g \in C([0,T]; X)$.

We say that $u : [0,T] \to X$ is a *strict solution* of problem $(A.4.1)$ if

(i) $u \in C^1([0,T]; X)$.

(ii) $u(t) \in D(A), \forall\, t \in [0,T]$.

(iii) $u'(t) = Au(t) + g(t), \ \forall\, t \in [0,T], \ u(0) = x$

We first cosider the homogeneous problem

$$\begin{cases} u'(t) = Au(t), \ t \in [0,T] \\ u(0) = x, \end{cases} \tag{A.4.2}$$

Theorem A.4.1 *Let $x \in D(A)$. Then problem $(A.4.2)$ has a unique strict solution given by $u(t) = T(t)x$.*

Proof. Existence follows from the Hille–Yosida theorem. Let us prove uniqueness. Let v be a strict solution of $(A.4.2)$. Let us fix $t > 0$ and set

$$f(s) = T(t-s)v(s), \quad s \in [0,t].$$

$f(s)$ is differentiable for $s \in [0,t)$, since

$$\begin{aligned} \frac{1}{h}(f(s+h) - f(s)) &= \frac{1}{h}(T(t-s+h)v(s+h) - T(t-s)v(s)) \\ &+ T(t-s-h)\frac{v(s+h)-v(s)}{h} \\ &+ \frac{T(t-s-h)v(s) - T(t-s)v(s)}{h}. \end{aligned} \tag{A.4.3}$$

As $h \to 0$ we find

$$\begin{aligned} f'(s) &= T(t-s)v'(s) - T'(t-s)v(s) \\ &= T(t-s)Av(s) - AT(t-s)v(s) = 0. \end{aligned}$$

Therefore f is costant and $T(t)x = v(t)$. □

We now consider problem $(A.4.1)$.

Theorem A.4.2 *Let $x \in D(A)$ and $g \in C^1([0,T];X)$. Then there is a unique strict solution $u(\cdot)$ di $(A.4.1)$ given by*

$$u(t) = T(t)x + \int_0^t T(t-s)g(s)ds. \tag{A.4.4}$$

Proof. Uniqueness can be proved as in Theorem $A.4.1$. Let us prove existence. We shall prove that the function $u(\cdot)$, defined by $(A.4.4)$ is a solution of $(A.4.1)$ and

$$u \in C^1([0,T];X) \cap C([0,T];D(A)).$$

First it is easy to check that $u \in C^1([0,T];X)$ and

$$u'(t) = T(t)g(0) + \int_0^t T(t-s)g'(s)ds. \tag{A.4.5}$$

Let us prove now that $v(t) \in D(A)$. We have in fact

$$\frac{1}{h}(T(h)u(t) - u(t))$$

$$= \frac{1}{h}\left[\int_0^t T(s+h)g(t-s)ds - \int_0^t T(s)g(t-s)ds\right]$$

$$+\frac{1}{h}\int_t^{t+h} T(s)g(t-s+h))ds - \frac{1}{h}\int_0^h T(s)g(t-s))ds.$$

As $h \to 0$ we have

$$\lim_{h\to 0}\frac{1}{h}(T(h)u(t) - u(t)) = \int_0^t T(s)g'(t-s)ds + T(t)g(0) - g(t). \tag{A.4.6}$$

¿From $(A.4.5)$ and $(A.4.6)$ it follows that $u \in C([0,T]; D(A))$ and the conclusion follows. □

Let $x \in H$ and $f \in C([0,T]; H)$. The the function u defined by

$$u(t) = T(t)x + \int_0^t T(t-s)g(s)ds \tag{A.4.7}$$

clearly belongs to $C([0,T]; H)$. We say that u is a *mild solution* of $(A.4.1)$.

Appendix A

Linear Semigroups Theory

In all this appendix X represents a complex Banach space (norm $|\cdot|$), and $L(X)$ the Banach algebra of all linear bounded operators from X into X endowed with the sup norm:

$$\|T\| = \sup\{|Tx| : \ x \in X, \ |x| \leq 1\}.$$

A.1 Some preliminaries on spectral theory

Let $A : D(A) \subset X \to X$ be a linear closed operator. We say that $\lambda \in \mathbb{C}$ belongs to the *resolvent set* $\rho(A)$ of A if $\lambda - A$ is bijective e and $(\lambda - A)^{-1} \in L(X)$; in this case the operator $R(\lambda, A) := (\lambda - A)^{-1}$ is called the *resolvent* of A at λ. The complementary set $\sigma(A)$ of $\rho(A)$ is called the *spectrum* of A.

Example A.1.1 Let $X = C([0,1])$ be the Banach space of all continuous functions on $[0,1]$ endowed with the sup norm, and let $C^1([0,1])$ be the subspace of $C([0,1])$ of all functions u that continuously differentiable. Let us consider the two following linear operators on X :

$$D(A) = C^1([0,1]), \ Au = u' \ , \forall \ u \in D(A),$$

$$D(B) = \{u \in C^1([0,1]); \ u(0) = 0\}, \ Bu = u' \ \ \forall \ u \in D(B).$$

We have

$$\rho(A) = \emptyset, \ \sigma(A) = \mathbb{C}.$$

In fact, given $\lambda \in \mathbb{C}$, the mapping $\lambda - A$ is not injective since, for all $c \in \mathbb{C}$ the function $u(\xi) = ce^{\lambda\xi}$ belongs to $D(A)$ and $(\lambda - A)u = 0$.

For as the operator B is concerned, we have

$$\rho(B) = \mathbb{C}, \ \sigma(A) = \emptyset.$$

and

$$(R(\lambda, B)f)(\xi) = -\int_0^\xi e^{\lambda(\xi-\eta)} f(\eta) d\eta, \forall\, \lambda \in \mathbb{C}, \forall\, f \in X, \forall\, \xi \in [0,1].$$

In fact $\lambda \in \rho(B)$ if and only if the problem

$$\begin{cases} \lambda u(\xi) - u'(\xi) = f(\xi) \\ \\ u(0) = 0 \end{cases}$$

has a unique solution $f \in X$.

Let us prove the important *resolvent identity.*

Proposition A.1.2 *If* $\lambda, \mu \in \rho(A)$ *we have*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A) \tag{A.1.1}$$

Proof. For all $x \in X$ we have

$$(\mu - \lambda) R(\lambda, A)x = (\mu - A + A - \lambda) R(\lambda, A)x = (\mu - A) R(\lambda, A)x - x$$

Applying $R(\mu, A)$ to both sides of the above identity, we find

$$(\mu - \lambda) R(\mu, A) R(\lambda, A)x = R(\lambda, A)x - R(\mu, A)x$$

and the conclusion follows. □

Proposition A.1.3 *Let* A *be a closed operator. Let* $\lambda_0 \in \rho(A)$, *and* $|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|}$. *Then* $\lambda \in \rho(A)$ *and*

$$R(\lambda, A) = R(\lambda_0, A)(1 + (\lambda - \lambda_0) R(\lambda_0, A))^{-1} \tag{A.1.2}$$

Thus $\rho(A)$ *is open and* $\sigma(A)$ *is closed. Moreover*

$$R(\lambda, A) = \sum_{k=1}^{\infty} (-1)^k (\lambda - \lambda_0)^k R^{k+1}(\lambda_0, A), \tag{A.1.3}$$

and so $R(\lambda, A)$ *is analytic on* $\rho(A)$.

Proof. The equation $\lambda x - Ax = y$ is equivalent to

$$(\lambda - \lambda_0)x + (\lambda_0 - A)x = y,$$

and, setting $z = (\lambda_0 - A)x$, to

$$z + (\lambda - \lambda_0)R(\lambda_0, A)z = y.$$

Since $\|(\lambda - \lambda_0)R(\lambda_0, A)\| < 1$ it follows

$$z = (1 + (\lambda - \lambda_0)R(\lambda_0, A))^{-1}y,$$

that yields the conclusion. □

A.2 Strongly continuous semigroups

A *strongly continuous semigroup* on X is a mapping $T : [0, \infty) \to L(X)$, $t \to T(t)$ such that

(i) $T(t+s) = T(t)T(s), \forall\, t, s \geq 0,\ T(0) = I.$

(ii) $T(\cdot)x$ is continuous for all $x \in X$.

Remark A.2.1 $\|T(\cdot)\|$ is locally bounded by the uniform boundedness theorem.

The *infinitesimal generator* A of $T(\cdot)$ is defined by

$$\begin{cases} D(A) = \left\{x \in X : \exists \lim\limits_{h \to 0+} \Delta_h x\right\} \\ \\ Ax = \lim\limits_{h \to 0+} \Delta_h x, \end{cases} \tag{A.2.1}$$

where

$$\Delta_h = \frac{T(h) - I}{h}, h > 0.$$

Proposition A.2.2 *$D(A)$ is dense in X.*

Proof. For all $x \in H$ and $a > 0$ we set

$$x_a = \frac{1}{a} \int_0^a T(s)xds.$$

Since $\lim_{a \to 0} x_a = x$, it is enough to show that $x_a \in D(A)$. We have in fact for any $h \in (0, a)$,

$$\Delta_h x_a = \frac{1}{ah} \left[\int_a^{a+h} T(s)xds - \int_0^h T(s)xds \right],$$

and, consequently $x_a \in D(A)$ since

$$\lim_{h \to 0} \Delta_h x_a = \Delta_a x.$$

□

Exercise A.2.3 *Prove that $D(A^2)$ is dense in X.*

We now study the derivability of the semigroup $T(t)$. Let us first notice that, since

$$\Delta_h T(t)x = T(t)\Delta_h x,$$

if $x \in D(A)$ then $T(t)x \in D(A), \forall\, t \geq 0$ and $AT(t)x = T(t)Ax$.

Proposition A.2.4 *Assume that $x \in D(A)$, then $T(\cdot)x$ is differentiable $\forall\, t \geq 0$ and*

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax \tag{A.2.2}$$

Proof. Let $t_0 \geq 0$ be fixed and let $h > 0$. Then we have

$$\frac{T(t_0 + h)x - T(t_0)x}{h} = \Delta_h T(t_0)x \overset{h \to 0}{\to} AT(t_0)x.$$

This shows that $T(\cdot)x$ is right differentiable at t_0. Let us show left differentiability, assuming $t_0 > 0$. For $h \in]0, t_0[$ we have

$$\frac{T(t_0 - h)x - T(t_0)x}{h} = T(t_0 - h)\Delta_h x \overset{h \to 0}{\to} T(t_0)Ax,$$

since $\|T(t)\|$ is locally bounded by Remarl A.2.1. □

Proposition A.2.5 *A is a closed operator.*

Proof. Let $(x_n) \subset D(A)$, and let $x, y \in X$ be such that

$$x_n \to x, \quad Ax_n = y_n \to y$$

Then we have

$$\Delta_h x_n = \frac{1}{h} \int_0^h T(t) y_n dt.$$

As $h \to 0$ we get $x \in D(A)$ and $y = Ax$, so that A is closed. □

We end this section by studying the asymptotic behaviour of $T(\cdot)$. We define the *type* of $T(\cdot)$ as

$$\omega_0 = \inf_{t>0} \frac{\log \|T(t)\|}{t}.$$

Clearly $\omega_0 \in [-\infty, +\infty)$.

Proposition A.2.6 *We have*

$$\omega_0 = \lim_{t \to +\infty} \frac{\log \|T(t)\|}{t}. \tag{A.2.3}$$

Proof. It is enough to show that

$$\limsup_{t \to \infty} \frac{\log \|T(t)\|}{t} \leq \omega_0.$$

Let $\varepsilon > 0$ and $t_\varepsilon > 0$ be such that

$$\frac{\log \|T(t_\varepsilon)\|}{t_\varepsilon} < \omega_0 + \varepsilon.$$

Set

$$t = n(t)t_\varepsilon + r(t), \ n(t) \in \mathbb{N}, r(t) \in [0, t_\varepsilon).$$

Since $\|T(\cdot)\|$ is locally bounded, there exists $M_\varepsilon > 0$ such that

$$\|T(t)\| \leq M_\varepsilon, \ t \in [0, t_\varepsilon].$$

We have

$$\frac{\log \|T(t)\|}{t} = \frac{\log \|T(t_\varepsilon)^{n(t)}T(r(t))\|}{t}$$

$$\leq \frac{n(t)\ \log \|T(t_\varepsilon)\| + \log \|T(r(t))\|}{n(t)t_\varepsilon + r(t)} \leq \frac{\log \|T(t_\varepsilon)\| + \frac{M_{t_\varepsilon}}{n(t)}}{t_\varepsilon + \frac{r(t)}{n(t)}}.$$

As $t \to +\infty$, we obtain

$$\limsup_{t\to\infty} \frac{\log \|T(t)\|}{t} \leq \frac{\log \|T(t_\varepsilon)\|}{t_\varepsilon} \leq \omega_0 + \varepsilon.$$

□

Corollary A.2.7 *Let T be of type ω_0. Then for all $\varepsilon > 0$ there exists $N_\varepsilon \geq 1$ such that*

$$\|T(t)\| \leq N_\varepsilon e^{(\omega_0+\varepsilon)t}, \forall\, t \geq 0 \tag{A.2.4}$$

Proof. Let $t_\varepsilon, n(t), r(t)$ as in the previous proof. Then we have

$$\|T(t)\| \leq \|T(t_\varepsilon)\|^{n(t)}\|T(r(t))\| \leq e^{t_\varepsilon n(t)(\omega_0+\varepsilon)}M_{t_\varepsilon} \leq M_{t_\varepsilon}e^{(\omega_0+\varepsilon)t}.$$

and the conclusion follows.□

In the sequel we shall denote by $\mathcal{G}(M,\omega)$ the set of all strongly continuous semigroups T such that

$$\|T(t)\| \leq Me^{\omega t}, t \geq 0$$

Example A.2.8 Let $X = L^p(\mathbb{R}), p \geq 1$, $(T(t)f)(\xi) = f(\xi - t), f \in L^p(\mathbb{R})$. Then we have $\|T(t)\| = 1$ and so $\omega_0 = 0$.

Example A.2.9 Let $X = L^p(0,T), T > 0, p \geq 1$, and let

$$(T(t)f)(\xi) = \begin{cases} f(\xi - t) \text{ if } \xi \in [t,T] \\ 0 \text{ if } \xi \in [0,t[\end{cases}$$

Then we have $T(t) = 0$ if $t \geq T$ and so $\omega_0 = -\infty$.

Exercise A.2.10 Let $A \in \mathcal{L}(X)$ compact and let $\{\lambda_i\}_{i\in\mathbb{N}}$ be its eigenvalues. Set $T(t) = e^{tA}$. Then we have

$$\omega_0 = \sup_{i\in\mathbb{N}} \text{ Re } \lambda_i.$$

A.3 The Hille–Yosida theorem

We assume here that $T \in \mathcal{G}(M,\omega)$. We denote by A its infinitesimal generator.

Proposition A.3.1 *We have*

$$\rho(A) \supset \{\lambda \in \mathbb{C} \ \text{Re} \ \lambda > \omega\} \tag{A.3.1}$$

$$R(\lambda, A)y = \int_0^\infty e^{-\lambda t}T(t)ydt, \ y \in X, \ \ \text{Re} \ \lambda > \omega \tag{A.3.2}$$

Proof. Set

$$\Sigma = \{\lambda \in \mathbb{C}; \ \ \text{Re}\, \lambda > \omega\}$$

$$F(\lambda)y = \int_0^\infty e^{-\lambda t}T(t)ydt, \ y \in X, \ \ \text{Re} \ \lambda > \omega.$$

This is meaningful since $T \in \mathcal{G}(M,\omega)$. We have to show that, given $\lambda \in \Sigma$ and $y \in X$ the equation $\lambda x - Ax = y$ has a unique solution given by $x = F(\lambda)y$.

Existence

Let $\lambda \in \Sigma, y \in X$, $x = F(\lambda)y$. Then we have

$$\Delta_h x = \frac{1}{h}(e^{\lambda h} - 1)x - \frac{1}{h}e^{\lambda h}\int_0^h e^{-\lambda t}T(t)ydt$$

and so, as $h \to 0$,

$$\lim_{h\to 0+} \Delta_h x = \lambda x - y = Ax$$

that is x is a solution of the equation $\lambda x - Ax = y$.

Uniqueness

Let $x \in D(A)$ be a solution of the equation $\lambda x - Ax = y$. Then we have

$$\begin{aligned}\int_0^\infty e^{-\lambda t}T(t)(\lambda x - Ax)dt \ &= \ \lambda\int_0^\infty e^{-\lambda t}T(t)xdt \\ &- \ \int_0^\infty e^{-\lambda t}\frac{d}{dt}T(t)xdt = x,\end{aligned}$$

so that $x = F(\lambda)y$.

We are now going to prove the *Hille–Yosida* theorem.

Theorem A.3.2 *Let $A : D(A) \subset X \to X$ be a closed operator. Then A is the infinitesimal generator of a strongly continuous semigroup belonging to $\mathcal{G}(M, \omega)$ if and only if*

$$\begin{aligned} &(i)\;\; \rho(A) \supset \{\lambda \in \mathbb{R};\; \lambda > \omega\} \\ &(ii)\;\; \|R^n(\lambda, A)\| \le \tfrac{M}{(\lambda-\omega)^n},\; \forall\, n \in \mathbb{N}\;\; \forall\, \lambda > \omega \\ &(iii)\;\; D(A) \text{ is dense in } X. \end{aligned} \tag{A.3.3}$$

Given a linear operator A fulfilling $(A.3.3)$ it is convenient to introduce a sequence of linear operators (called the *Yosida approximations* of A). They are defined as

$$A_n = nAR(n, A) = n^2R(n, A) - n \tag{A.3.4}$$

Lemma A.3.3 *We have*

$$\lim_{n\to\infty} nR(n, A)x = x, \;\; \forall\, x \in X, \tag{A.3.5}$$

and

$$\lim_{n\to\infty} A_n x = Ax, \;\; \forall\, x \in D(A). \tag{A.3.6}$$

Proof. Since $D(A)$ is dense in X and $\|nR(n, A)\| \le \frac{Mn}{n-\omega}$, to prove (A.3.5) it is enough to show that.

$$\lim_{n\to\infty} nR(n, A)x = x, \;\; \forall\, x \in D(A).$$

In fact for any $x \in D(A)$ we have

$$|nR(n, A)x - x| = |R(n, A)Ax| \le \frac{M}{n-\omega}|Ax|,$$

and the conclusion follows.

Finally if $x \in D(A)$ we have

$$A_n x = nR(n, A)Ax \to Ax,$$

and $(A.3.6)$ follows.□

Proof of Theorem $A.3.2$. **Necessity**. (i) follows from Proposition A.3.1 and (iii) from Proposition A.2.2. Let us show (ii). Let $k \in \mathbb{N}$ and $\lambda > \omega$. It follows

$$\frac{d^k}{d\lambda^k}R(\lambda, A)y = \int_0^\infty (-t)^k e^{-\lambda t}T(t)ydt,\ y \in X,$$

from which

$$\left\|\frac{d^k}{d\lambda^k}R(\lambda, A)\right\| \leq M\int_0^\infty t^k e^{-\lambda t+\omega t}dt$$

that yields the conclusion.

Sufficiency.

Step 1. We have

$$\|e^{tA_n}\| \leq Me^{\frac{\omega nt}{n-\omega}}, \forall\, n \in \mathbb{N}. \tag{A.3.7}$$

In fact, by the identity

$$e^{tA_n} = e^{-nt}e^{tn^2R(n,A)} = e^{-nt}\sum_{k=0}^\infty \frac{n^{2k}t^kR^k(n, A)}{k!},$$

it follows

$$\|e^{tA_n}\| \leq Me^{-nt}\sum_{k=0}^\infty \frac{n^{2k}t^k}{(n-\omega)^kk!}.$$

Step 2. There exists $C > 0$ such that, for all $m, n > 2\omega$, and $x \in D(A^2)$,

$$\|e^{tA_n}x - e^{tA_m}x\| \leq Ct\frac{|m-n|}{(m-\omega)(n-\omega)}\ \|A^2x\|. \tag{A.3.8}$$

Setting $u_n(t) = e^{tA_n}x$, we have

$$\frac{d}{dt}(u_n(t) - u_m(t)) = A_n(u_n(t) - u_m(t)) - (A_m - A_n)u_m(t)$$

$$= A_n(u_n(t) - u_m(t)) - (n-m)A^2R(m, A)R(n, A)u_m(t).$$

It follows

$$u_n(t) - u_m(t) = (n-m)A^2R(m,A)R(n,A)\int_0^t e^{(t-s)A_n}u_m(s)ds$$

$$= (n-m)R(m,A)R(n,A)\int_0^t e^{(t-s)A_n}e^{sA_m}A^2x.$$

Step 3. For all $x \in X$ there exists the limit

$$\lim_{n\to\infty} e^{tA_n}x =: T(t)x \tag{A.3.9}$$

and $T : [0,\infty) \to L(X), t \to T(t)$ is strongly continuous.

¿From the second step it follows that the sequence $(u_n(t)$ is Cauchy, uniformly in t on compact subsets of $[0,+\infty[$, for all $x \in D(A^2)$. Since $D(A^2)$ is dense in X (see Exercise A.2.3) this holds for all $x \in X$. Finally it is easy to check that $T(\cdot)$ is strongly continuous.

Step 4. If $x \in D(A)$, then $T(\cdot)x$ is differentiable and

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x.$$

In fact let $x \in D(A)$, and $v_n(t) = \frac{d}{dt}u_n(t)$.Then

$$v_n(t) = e^{tA_n}A_nx$$

Since $x \in D(A)$ there exists the limit

$$\lim_{n\to\infty} v_n(t) = e^{tA}Ax$$

This implies that u is differentiable and $u'(t) = v(t)$ so that $u \in C^1([0,+\infty);X)$. Moreover

$$A(nR(n,A)u_n(t)) = u_n'(t) \to v(t)$$

Since A is closed and $nR(n,A)u_n(t) \to u(t)$ it follows that $u(t) \in D(A)$ and $u'(t) = Au(t)$.

Step 5. A is the infinitesimal generator of $T(\cdot)$.

Let B be the infinitesimal generator of $T(\cdot)$. By Step 4 $B \supset A$ ([1]). It is enough to show that if $x \in D(B)$ then $x \in D(A)$. Let $x \in D(B)$, $\lambda_0 > \omega$, setting $z = \lambda_0 x - Bx$ we have

$$z = (\lambda_0 - A)R(\lambda_0, A)z$$

$$= \lambda_0 R(\lambda_0, A)z - BR(\lambda_0, A)z = (\lambda_0 - B)R(\lambda_0, A)z.$$

Thus $x = R(\lambda_0, B)z = R(\lambda_0, A)z \in D(A)$.□

Remark A.3.4 To use the Hille-Yosida theorem requires to check infinite conditions. However if $M = 1$ it is enough to ask (ii) only for $n = 1$. In such a case $T \in \mathcal{G}(1, \omega)$. If $\omega \leq 0$ we say that $T(\cdot)$ is a *contraction semigroup.*

Example A.3.5 Let $X = C_0([0, \pi])$ the Banach space of all continuous functions in $[0, \pi]$ that vanish at 0 and π. Let A be the linear operator in X defined as

$$\begin{cases} D(A) = \{y \in C^2([0, \pi]); y(0) = y''(0) = y(\pi) = y''(\pi) = 0\} \\ Ay = y'', \ \forall\, y \in D(A) \end{cases}$$

It is easy to check that $\sigma(A) = \{-n^2;\ n \in \mathbb{N}\}$. Moreover any element of $\sigma(A)$ is a simple eigenvalue whose corresponding eigenvector is given by

$$\varphi_n(\xi) = \sin\ n\xi, \quad \forall\, n \in \mathbb{N}.$$

We have

$$A\varphi_n = -n^2 \varphi_n$$

Moreover if $\lambda \in \rho(A)$ and $f \in C_0([0, \pi])$, $u = R(\lambda, A)f$ is the solution of the problem

$$\begin{cases} \lambda u(\xi) - u''(\xi) = f(\xi) \\ u(0) = u(\pi) = 0. \end{cases}$$

[1]That is $D(B) \supset D(A)$ and $Ax = Bx\ \forall\, x \in D(A)$

By a direct verification we find

$$u(\xi) = \frac{\sinh(\sqrt{\lambda}\xi)}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)} \int_\xi^\pi \sinh[\sqrt{\lambda}(\pi-\eta)]f(\eta)d\eta$$
$$+\frac{\sinh[\sqrt{\lambda}(\pi-\xi)]}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)} \int_0^\xi \sinh[\sqrt{\lambda}\eta]f(\eta)d\eta. \tag{A.3.10}$$

¿From $(A.3.10)$ it follows that

$$\|R(\lambda, A)\| \le \frac{1}{\lambda}, \ \forall\, \lambda > 0. \tag{A.3.11}$$

Therefore the assumptions of the Hille-Yosida theorem are fulfilled.

A.4 Cauchy problem

Let $A \in \mathcal{G}(M, \omega)$, and let $T(\cdot)$ be the semigroup generated by A.

We are here concerned with the following problem

$$\begin{cases} u'(t) = Au(t) + g(t), \ t \in [0, T] \\ u(0) = x, \end{cases} \tag{A.4.1}$$

where $x \in H$ and $g \in C([0, T]; X)$.

We say that $u : [0, T] \to X$ is a *strict solution* of problem $(A.4.1)$ if

(i) $u \in C^1([0, T]; X)$.

(ii) $u(t) \in D(A), \forall\, t \in [0, T]$.

(iii) $u'(t) = Au(t) + g(t), \ \forall\, t \in [0, T], \ u(0) = x$

We first cosider the homogeneous problem

$$\begin{cases} u'(t) = Au(t), \ t \in [0, T] \\ u(0) = x, \end{cases} \tag{A.4.2}$$

Theorem A.4.1 *Let $x \in D(A)$. Then problem $(A.4.2)$ has a unique strict solution given by $u(t) = T(t)x$.*

Proof. Existence follows from the Hille–Yosida theorem. Let us prove u-niqueness. Let v be a strict solution of $(A.4.2)$. Let us fix $t > 0$ and set

$$f(s) = T(t-s)v(s), \quad s \in [0,t].$$

$f(s)$ is differentiable for $s \in [0,t)$, since

$$\begin{aligned} \frac{1}{h}(f(s+h) - f(s)) &= \frac{1}{h}(T(t-s+h)v(s+h) - T(t-s)v(s)) \\ &+ T(t-s-h)\frac{v(s+h)-v(s)}{h} \\ &+ \frac{T(t-s-h)v(s) - T(t-s)v(s)}{h}. \end{aligned} \tag{A.4.3}$$

As $h \to 0$ we find

$$\begin{aligned} f'(s) &= T(t-s)v'(s) - T'(t-s)v(s) \\ &= T(t-s)Av(s) - AT(t-s)v(s) = 0. \end{aligned}$$

Therefore f is costant and $T(t)x = v(t)$. □

We now consider problem $(A.4.1)$.

Theorem A.4.2 *Let $x \in D(A)$ and $g \in C^1([0,T];X)$. Then there is a u-nique strict solution $u(\cdot)$ di $(A.4.1)$ given by*

$$u(t) = T(t)x + \int_0^t T(t-s)g(s)ds. \tag{A.4.4}$$

Proof. Uniqueness can be proved as in Theorem $A.4.1$. Let us prove exis-tence. We shall prove that the function $u(\cdot)$, defined by $(A.4.4)$ is a solution of $(A.4.1)$ and

$$u \in C^1([0,T];X) \cap C([0,T];D(A)).$$

First it is easy to check that $u \in C^1([0,T];X)$ and

$$u'(t) = T(t)g(0) + \int_0^t T(t-s)g'(s)ds. \tag{A.4.5}$$

Let us prove now that $v(t) \in D(A)$. We have in fact

$$\frac{1}{h}(T(h)u(t) - u(t))$$
$$= \frac{1}{h}\left[\int_0^t T(s+h)g(t-s)ds - \int_0^t T(s)g(t-s)ds\right]$$
$$+\frac{1}{h}\int_t^{t+h} T(s)g(t-s+h))ds - \frac{1}{h}\int_0^h T(s)g(t-s))ds.$$

As $h \to 0$ we have

$$\lim_{h\to 0}\frac{1}{h}(T(h)u(t) - u(t))$$
$$= \int_0^t T(s)g'(t-s)ds + T(t)g(0) - g(t). \tag{A.4.6}$$

¿From $(A.4.5)$ and $(A.4.6)$ it follows that $u \in C([0,T]; D(A))$ and the conclusion follows. □

Let $x \in H$ and $f \in C([0,T]; H)$. The the function u defined by

$$u(t) = T(t)x + \int_0^t T(t-s)g(s)ds \tag{A.4.7}$$

clearly belongs to $C([0,T]; H)$. We say that u is a *mild solution* of $(A.4.1)$.

Appendix B

Contraction Principle

Let $T > 0$, and let $\{\gamma_n\}$ be a sequence of mappings from $C_u([0,T];\Sigma(H))$ into itself such that

$$\|\gamma_n(P) - \gamma_n(Q)\| \leq \alpha\|P - Q\|, \ \forall\ P, Q \in C_u([0,T];\Sigma(H)),\ n \in \mathbb{N},$$

where $\alpha \in [0, 1)$.

Moreover assume that there exists a mapping γ from the space $C_u([0,T];\Sigma(H))$ into itself such that

$$\lim_{n\to\infty} \gamma_n^m(P) = \gamma^m(P) \text{ in } C_s([0,T];\Sigma(H)), \tag{B.0.1}$$

for all $P \in C_u([0,T];\Sigma(H))$ and all $m \in \mathbb{N}$, where γ^m and γ_n^m are defined by recurrence as

$$\gamma^1 = \gamma,\ \gamma^{m+1}(P) = \gamma(\gamma^m(P)),$$

$$\gamma_n^1 = \gamma_n,\ \gamma_n^{m+1}(P) = \gamma_n(\gamma_n^m(P)),$$

for $m = 2, 3, \ldots$ and $P \in C_s([0,T];\Sigma(H))$. It is easy to check that

$$\|\gamma(P) - \gamma(Q)\| \leq \alpha\|P - Q\|, \ \forall\ P, Q \in C_u([0,T];\Sigma(H)).$$

Then, by the classical Contraction Mapping Principle, there exists unique P_n and P in $C_u([0,T];\Sigma(H))$ such that

$$\gamma_n(P_n) = P_n \text{ rm and } \gamma(P) = P.$$

However, since we do not assume that

$$\gamma_n(P) \to \gamma(P) \text{ in } C_u([0,T];\Sigma(H)$$

we cannot conclude that $P_n \to P$ in $C_u([0,T];\Sigma(H)$, but a weaker result holds.

Lemma B.0.3 *Under the previous hypotheses on the sequence of mappings* $\{\gamma_n\}$,

$$P_n \to P \text{ in } C_s([0,T];\Sigma(H)).$$

Proof. Set

$$P^0 = 0, \quad P_n^0 = 0,$$

and define

$$P^m = \gamma^m(P^0), \quad P_m^m = \gamma_n^m(P^0), \ m \in \mathbb{N}.$$

By the classical Contraction Mapping Principle, we have

$$\lim_{m\to\infty} P^m = P, \quad \lim_{m\to\infty} P_n^m = P_n \text{ in } C_u([0,T];\Sigma(H)), \ n \in \mathbb{N}.$$

Moreover

$$\|P - P^m\| \le \sum_{k=m}^{\infty} \alpha^k \|\gamma(P^0)\|, \quad \|P_n - P_n^m\| \le \sum_{k=m}^{\infty} \alpha^k \|\gamma_n(P^0)\|.$$

Now fix $x \in H$, then for all $t \in [0,T]$

$$\begin{aligned} |P(t)x - P_n(t)x| &\le |P(t)x - P^m(t)x| + |P^m(t)x - P_n^m(t)x| \\ &+ |P_n^m(t)x - P_n(t)x|. \end{aligned} \tag{B.0.2}$$

Given $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=m}^{\infty} \alpha^k \left[\|\gamma(P^0)\| + \|\gamma_n(P^0)\|\right] \le \frac{\varepsilon}{2}, \tag{B.0.3}$$

for all $m \ge m_\varepsilon$ and all $n \in \mathbb{N}$. By (B.0.2) and (B.0.3) it follows that

$$|P(t)x - P_n(t)x| \le \frac{\varepsilon}{2} + |P^{m_\varepsilon}(t)x - P_n^{m_\varepsilon}(t)x|, \ \forall\, t \in [0,T].$$

Now (B.0.1) yields the conclusion. □