Annex to SMR1327/9

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LINEAR QUADRATIC CONTROL THEORY FOR INFINITE DIMENSIONAL SYSTEMS. HOMEWORKS.

(1) Consider the following problem:

Minimize

$$J(u) = \int_0^2 \int_0^1 [|y(t,\xi)|^2 + |u(t,\xi)|^2] dt d\xi + \int_0^1 y^2(2,\xi) d\xi, \tag{1}$$

over all controls $u \in L^2([0,2] \times [0,1])$ subject to the state equation

$$\begin{cases} D_t y(t,\xi) = D_\xi^2 y(t,\xi) + 12 y(t,\xi) + u(t,\xi), \text{ in } [0,2] \times [0,1], \\ y(t,\xi) = 0, \text{ on } [0,2] \times [0,1], \\ y(0,\xi) = x(\xi), \text{ in } [0,1]. \end{cases} \tag{2}$$

Show that there is an unique optimal pair and find an explicit solution of the Riccati equation.

Hint. Set $H = Y = U = L^2(0,1)$ and define the linear operator in H :

$$\begin{cases} A_0 y = D_\xi^2 y \\ D(A_0) = H^2(0,1) \cap H_0^1(0,1). \end{cases} \tag{3}$$

Show that A_0 has a complete orthonormal basis of eigenfunctions given by

$$e_k(\xi) = \sqrt{\frac{2}{\pi}} \sin \pi k \xi, \ k \in \mathbb{N}, \ \xi \in [0,1],$$

and that

$$A_0 e_k = -\pi^2 k^2 e_k, \ k \in \mathbb{N}.$$

Then write (2) on the form

$$\begin{cases} y'(t) = A_0 y(t) + 12 y(t) + B u(t), \ t \geq 0, \\ y(0) = x \in H, \end{cases} \tag{4}$$

and (1) as

$$J(u) = \int_0^2 \left[|y(s)|_H^2 + |u(s)|_H^2\right] ds + |y(2)|_H^2. \tag{5}$$

Look for a solution of the Riccati equation in the diagonl form with respect to the basis $\{e_k\}$.

(2) Consider the following problem:

Minimize

$$J(u) = \int_0^{+\infty} \int_0^1 [|y(t,\xi)|^2 + |u(t,\xi)|^2] dt d\xi, \tag{6}$$

over all controls $u \in L^2([0,+\infty) \times [0,1])$ subject to the state equation (2).

Show that there is an unique optimal pair and find an explicit solution of the Algebraic Riccati equation.

(3) Consider the following problem: Minimize the cost functional (1) over all controls $u \in L^2([0,+\infty) \times [0,1])$, subject to the state equation

$$\begin{cases} D_t y(t,\xi) = D_\xi^2 y(t,\xi) + 100 y(t,\xi) + \int_0^1 \sin(k\eta) u(t,\eta) d\eta, \text{ in } [0,2] \times [0,1], \\ y(t,\xi) = 0, \text{ on } [0,2] \times [0,1], \\ y(0,\xi) = x(\xi), \text{ in } [0,1], \end{cases} \tag{7}$$

where k is a fixed positive integer.

Find if and oly if conditions on k in order that there is an optimal pair, and and find an explicit solution of the Algebraic Riccati equation.

(4) Let K be a Hilbert space and $\{e_k\}$ a complete orthonormal basis on K. Consider the following problem:

Minimize the cost functional

$$J(u) = \int_0^3 \left[|y(s)|_H^2 + |u(s)|_H^2\right] ds + |y(3)|_H^2, \tag{8}$$

over all controls $u \in L^2([0,3];K)$ subject to the state equation

$$\begin{cases} y''(t) = Ay(t) + u(t) \\ \\ y(0) = x_0, \ y'(0) = x_1, \end{cases} \tag{9}$$

where A is the self–adjoint operator defined in H by

$$D(A) = \left\{ x \in H : \sum_{k=1}^{\infty} k^8 |\langle x, e_k \rangle|^2 < +\infty \right\},$$

and

$$Ae_k = -k^4 e_k, \ k \in \mathbb{N}.$$

Prove the existence of an optimal pair gived in feedback form and study the corresponding problem with infinite horizon.

Hint. Reduce (9) to a first order equation. Choose the control space $U = K$. Then set

$$|(y_1, y_2)|_{\mathcal{H}}^2 = \sum_{k=1}^{\infty} k^4 |\langle y_1, e_k \rangle|^2 + |y_2|_H^2,$$

and choose the state space $\mathcal{H}$ as

$$\mathcal{H} = \left\{ (y_1, y_2) : \ |(y_1, y_2)|_{\mathcal{H}}^2 < +\infty \right\}.$$