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Value Function in Optimal Control
Lecture 2

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2. VALUE FUNCTION & OPTIMALITY

Outline

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Generalized Differentials of Nonsmooth Functions

Definition 2.1 *Let X be a normed vector space, $\varphi : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be an extended function and $x_0 \in X$ be such that $\varphi(x_0) \neq \pm\infty$.*

The superdifferential *of φ at x_0 is the closed convex set $\partial_+\varphi(x_0)$ equal to:*

$$\{p \in \mathbf{R}^n | \limsup_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - < p, x - x_0 >}{\|x - x_0\|} \leq 0$$

where $< \cdot, \cdot >$ denotes the scalar product.

The subdifferential $\partial_-\varphi(x_0)$ *is defined as the set:*

$$\{p \in \mathbf{R}^n | \liminf_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - < p, x - x_0 >}{\|x - x_0\|} \geq 0\}$$

We always have $\partial_+\varphi(x_0) = -\partial_-(-\varphi)(x_0)$.

The super and subdifferentials may also be characterized using *contingent epiderivatives*:

Definition 2.2 *Let X be a normed vector space, $\varphi : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be an extended function, $v \in X$ and $x_0 \in X$ be such that $\varphi(x_0) \neq \pm\infty$.*

The contingent epiderivative *of φ at x_0 in the direction v is given by*

$$D_{\uparrow}\varphi(x_0)(v) \;=\; \liminf_{h\to 0+,\; v'\to v} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h}$$

and the contingent hypoderivative *of φ at x_0 in the direction v by*

$$D_{\downarrow}\varphi(x_0)(v) \;=\; \limsup_{h\to 0+,\; v'\to v} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h}$$

Clearly

$$D_{\uparrow}\varphi(x_0) \;=\; -D_{\downarrow}(-\varphi)(x_0)$$

By a direct verification $D_{\uparrow}\varphi(x_0)$ is a lower semicontinuous map taking its values in $\mathbf{R} \cup \{\pm\infty\}$.

When $\varphi : \mathbf{R}^n \mapsto \mathbf{R}$ is Lipschitz at x_0, then the contingent epi and hypoderivatives are reduced to the *Dini lower and upper derivatives*:

$$D_{\uparrow}\varphi(x_0)(v) \;=\; \liminf_{h \to 0+} \frac{\varphi(x_0 + hv) - \varphi(x_0)}{h}$$

and

$$D_{\downarrow}\varphi(x_0)(v) \;=\; \limsup_{h \to 0+} \frac{\varphi(x_0 + hv) - \varphi(x_0)}{h}$$

Proposition 2.3 *Let $\varphi : \mathbf{R}^n \mapsto \mathbf{R} \cup \{\pm\infty\}$ be an extended function. Then $\partial_-\varphi(x_0) =$*

$$\{p \in \mathbf{R}^n \mid \forall\, v \in \mathbf{R}^n,\; D_{\uparrow}\varphi(x_0)(v) \geq < p, v >\}$$

and $\partial_+\varphi(x_0) =$

$$\{p \in \mathbf{R}^n \mid \;\forall\, v \in \mathbf{R}^n,\; D_{\downarrow}\varphi(x_0)(v) \leq < p, v >\}$$

Notice that φ is Fréchet differentiable at x_0 if and only if both super and subdifferentials of φ at x_0 are nonempty. Moreover in this case

$$\partial_+\varphi(x_0) \;=\; \partial_-\varphi(x_0) \;=\; \{\; \nabla\varphi(x_0)\; \}$$

Definition 2.4 *Let* $\varphi : \mathbf{R}^n \mapsto \mathbf{R}$ *be Lipschitz at* x_0. *We denote by* $\partial^\star \varphi(x_0)$ *the set of all cluster points of gradients* $\nabla \varphi(x_n)$, *when* x_n *converge to* x_0 *and* φ *is differentiable at* x_n, *i.e.,*

$$\partial^\star \varphi(x_0) \; = \; \mathrm{Limsup}_{x \to x_0} \{ \nabla \varphi(x) \}$$

Proposition 2.5 (Clarke) *If* $\partial^\star \varphi(x_0)$ *is a singleton, then* φ *is differentiable at* x_0.

We recall that the directional derivative of a function $\varphi : \mathbf{R}^m \mapsto \mathbf{R}$ at $x_0 \in \mathbf{R}^m$ in the direction $v \in \mathbf{R}^m$ (when it exists) is defined by

$$\frac{\partial \varphi}{\partial v}(x_0) \; = \; \lim_{h \to 0+} \frac{\varphi(x_0 + hv) - \varphi(x_0)}{h}$$

Semiconcave Functions

Definition 2.6 *Consider a convex subset K of R^n. A function $\varphi : K \mapsto R$ is called* **semiconcave** *if $\exists\ \omega : R_+ \times R_+ \mapsto R_+$ such that* $\lim_{s \to 0+} \omega(R, s) = 0$,

$$\forall\ r \leq R,\ \forall\ s \leq S, \omega(r, s) \leq \omega(R, S)$$

and $\forall\ R > 0,\ \lambda \in [0, 1]$ *and all* $x, y \in K \cap RB$

$$\lambda\varphi(x) + (1 - \lambda)\varphi(y) \leq$$
$$\varphi(\lambda x + (1 - \lambda)y) + \lambda(1 - \lambda)\|x - y\|\omega(R, \|x - y\|)$$

ω is called a **modulus of semiconcavity** *of φ.*

Proposition 2.7 *Let $\varphi : R^n \mapsto R$ be Lipschitz and semiconcave at x_0. If $\partial_+\varphi(x_0)$ is a singleton, then φ is differentiable at x_0 and*

$$\partial^\star\varphi(x_0) = \{ \nabla\varphi(x_0) \}$$

In particular, if $\partial_+\varphi(x)$ is a singleton for all x near x_0, then φ is continuously differentiable at x_0.

Theorem 2.8 *Let $K \subset R^n$ be a convex set, $x_0 \in K$ and let $\varphi : K \mapsto R$ be Lipschitz and semiconcave at x_0. Then for every v*

$$\liminf_{\substack{v' \to v,\, h \to 0+ \\ x' \to_K x_0,\ x' + hv' \in K}} \frac{\varphi(x' + hv') - \varphi(x')}{h}$$

$$= \lim_{\substack{v' \to v,\, h \to 0+ \\ x_0 + hv' \in K}} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h}$$

In particular, if $x_0 \in \mathrm{Int}(K)$, then

$$\partial_+ \varphi(x_0) = co\,(\partial^\star \varphi(x_0))$$

Proposition 2.9 *Let $K \subset R^n$ and $\varphi : K \mapsto R$ be locally Lipschitz. Define the set-valued map $Q : K \rightsquigarrow R^n$ by:*
for all $x \in K$, $Q(x)$ is equal to

$$\{v| \liminf_{\substack{v' \to v, h \to 0+ \\ x' \to_K x, x' + hv' \in K}} \frac{\varphi(x' + hv') - \varphi(x')}{h} \leq 0$$

Then Q has nonempty images and Graph(Q) *is closed in $K \times R^n$.*

Corollary 2.10 *Let $V : [0, T[\times R^n \mapsto R$ be locally Lipschitz and semiconcave. Define*

$$\Psi(t, x) := \{v \mid \frac{\partial V}{\partial(1, v)}(t, x) \leq 0\}$$

Then Ψ has nonempty images and Graph(Ψ) *is closed in $[0, T[\times R^n$.*

VALUE FUNCTION FOR MAYER PROBLEM

Consider $T > 0$, a complete separable metric space U and a map $f : R^n \times U \mapsto R^n$. We associate with it the control system

$$(1) \qquad x'(t) = f(x(t), u(t)), \quad u(t) \in U$$

Let an extended function $g : R^n \mapsto R \cup \{+\infty\}$ and $\xi_0 \in R^n$ be given. Consider the minimization problem, called **Mayer's problem**:

$$\min \{g(x(T)) \mid x \text{ is a solution to } (1),\ x(0) = \xi_0\}$$

The value function associated with this problem is defined by: for all $(t_0, x_0) \in [0, T] \times R^n$

$$V(t_0, x_0) = \inf\{g(x(T)) \mid x \text{ solves } (1),\ x(t_0) = x_0\}$$

Define the Hamiltonian $H : R^n \times R^n \mapsto R$ by

$$H(x,p) = \sup_{v \in f(x,U)} < p, v > = \sup_{u \in U} < p, f(x,u) >$$

In all the results of today lecture it is assumed that $f(x, U)$ are compact and for some $k \geq 0$, $f(\cdot, u)$ is $k-$Lipschitz for all $u \in U$.

Lipschitz Continuity of the Value Function

More generally consider $g : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$, a set-valued map $F : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$, $\xi_0 \in \mathbf{R}^n$ and the differential inclusion

(2) $\quad x'(t) \in F(x(t))$ almost everywhere

and the minimization problem

$$\min \{g(x(T)) \mid x \text{ solves } (2),\ x(0) = \xi_0)\}$$

The corresponding value function is given by: For all $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$,

(3) $V(t_0, x_0) = \inf\{g(x(T)) \mid x \in \mathcal{S}_{[t_0,T]}(x_0)\}$

The value function is nondecreasing along solutions to (2): $\forall\, x \in \mathcal{S}_{[t_0,T]}(x_0)$,

$$\forall\, t_0 \leq t_1 \leq t_2 \leq T,\ V(t_1, x(t_1)) \leq V(t_2, x(t_2))$$

and satisfies the following *dynamic programming principle*: $\forall\, t \in [t_0, T]$,

$$V(t_0, x_0) = \inf\left\{V(t, x(t)) \mid x \in \mathcal{S}_{[t_0,T]}(x_0)\right\}$$

Furthermore $x \in \mathcal{S}_{[t_0,T]}(x_0)$ is optimal for problem (3) if and only if $V(t, x(t)) \equiv g(x(T))$.

Theorem 2.11 *Assume that F is Lipschitz with compact nonempty images and g is locally Lipschitz. Then for every $R > 0$, there exists $L_R > 0$ such that*

i) For all $(t_0, x_0) \in [0, T] \times B_R(0)$ and every solution $x \in \mathcal{S}_{[t_0,T]}(x_0)$

$$\forall\, t \in [t_0, T], \quad \|x(t)\| \;\leq\; L_R$$

and the map $[t_0, T] \ni t \mapsto V(t, x(t))$ is absolutely continuous.

Furthermore for almost every $t \in [t_0, T]$, the directional derivative

$$\frac{\partial V}{\partial(1, x'(t))}(t, x(t))$$

does exist.

ii) For all $R > 0$, V is L_R-Lipschitz on $[0, T] \times B_R(0)$

SUFFICIENT CONDITIONS FOR OPTIMALITY

Theorem 2.12 *Assume that g is locally Lipschitz and let $(t_0, x_0) \in [0, T] \times R^n$. Consider a solution $z \in \mathcal{S}_{[t_0,T]}(x_0)$. If for a.e. $t \in [t_0, T]$, $\exists\, p(t) \in R^n$*

$$(4) \quad \left(\left\langle p(t), z'(t)\right\rangle,\ -p(t)\right) \in \partial_+ V(t, z(t))$$

then z is optimal.

Proof — The map $\psi(t) := V(t, z(t))$ is absolutely continuous. Let $t \in [t_0, T]$ be such that the derivatives $\psi'(t)$ and $z'(t)$ do exist and (4) holds true. Then

$$0 = \left\langle (< p(t), z'(t) >, -p(t)), (1, z'(t)) \right\rangle$$

$$\geq D_{\downarrow} V(t, z(t))(1, z'(t))$$

$$\geq \ \limsup_{h \to 0+} \frac{V(t+h, z(t+h)) - V(t, z(t))}{h}$$

This yields that ψ is nonincreasing. Since the value function is also nondecreasing along solutions, the map $t \mapsto V(t, z(t))$ is constant. So z is optimal.

NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

Theorem 2.13 *Assume that f is differentiable with respect to x, and g is differentiable and locally Lipschitz. A trajectory-control solution $(z, \overline{u})$ with $z(t_0) = x_0$ is optimal* **if and only if** *the solution $p : [t_0, T] \mapsto R^n$ to the adjoint system*

$$-p'(t) = \left(\frac{\partial f}{\partial x}(z(t), \overline{u}(t))\right)^{\star} p(t), \; p(T) = -\nabla g(z(T))$$

satisfies the maximum principle

$$\langle p(t), f(z(t), \overline{u}(t))\rangle \; = \; H(z(t), p(t)) \; \text{ a.e.}$$

and the generalized transversality conditions

$$(5)\; (H(z(t), p(t)), -p(t)) \in \partial_+ V(t, z(t)) \; \text{ a.e.}$$

$$-p(t) \; \in \; \partial_+ V_x(t, z(t)) \; \text{ for every } \; t \in [t_0, T]$$

where $\partial_+ V_x(t, z(t))$ denotes the superdifferential of $V(t, \cdot)$ at $z(t)$.

Furthermore, if V is semiconcave, then (5) holds true everywhere in $[t_0, T]$.

The map $p(\cdot)$ given by the above theorem is called **the co-state or the adjoint variable** corresponding to the optimal control $\overline{u}$.

EXPRESSING NECESSARY CONDITIONS USING HAMILTONIANS

If H is differentiable, then $z,\ p$ satisfy the **Hamiltonian system**

$$\begin{cases} z'(t) = \frac{\partial H}{\partial p}(z(t), p(t)) \\ \\ p'(t) = -\frac{\partial H}{\partial x}(z(t), p(t)) \ \text{a.e. in} \ [t_0, T] \end{cases}$$

Proposition 2.14 *Let* $(z, \overline{p}) \in R^n \times R^n$ *and* $\overline{u} \in U$ *be such that*

$$\langle \overline{p}, f(z, \overline{u}) \rangle = H(z, \overline{p})$$

i) If $H(\cdot, \overline{p})$ *is differentiable at* z*, then*

$$\frac{\partial H}{\partial x}(z, \overline{p}) = \left(\frac{\partial f}{\partial x}(z, \overline{u}) \right)^{\star} \overline{p}$$

ii) If $H(z, \cdot)$ *is differentiable at* $\overline{p}$*, then*

$$\frac{\partial H}{\partial p}(z, \overline{p}) = f(z, \overline{u})$$

In particular $H(z, \cdot)$ **is not differentiable at zero, when** $f(z, U)$ **is not a singleton**.

CO-STATE & SUPERDIFFERENTIALS OF VALUE

Theorem 2.15 *Assume that f is differentiable with respect to x and g is differentiable and locally Lipschitz. Suppose further that $V(t_0, \cdot)$ is differentiable at x_0 and let $(z, \overline{u})$ be an optimal state-control pair. Then the co-state $p : [t_0, T] \mapsto R^n$ corresponding to $(z, \overline{u})$ verifies*

$$\{-p(t)\} = \partial_+ V_x(t, z(t)) \text{ for all } t \in [t_0, T]$$

In particular if $V(t, \cdot)$ is differentiable at $z(t)$, then

$$\frac{\partial V}{\partial x}(t, z(t)) = -p(t)$$

HAMILTONIAN SYSTEM AND OPTIMALITY

Define $W(\cdot) = V(t_0, \cdot)$, $\partial^\star V_x(t_0, x_0) = \partial^\star W(x_0)$.

Theorem 2.16 *Assume f is differentiable with respect to x, g is differentiable and locally Lipschitz, and H is differentiable on $R^n \times (R^n \backslash \{0\})$.*

Further assume that the sets $f(x, U)$ are convex and for every $R > 0$, $\exists\ l_R > 0$ such that for all $x, y \in RB$ and $p, q \in RB \backslash \frac{1}{R}B$

$$\|\nabla H(x,p) - \nabla H(y,q)\| \leq l_R(\|x-y\| + \|p-q\|)$$

Let $(t_0, x_0) \in [t_0, T] \times R^n$ and $p_0 \neq 0$ be such that $-p_0 \in \partial^\star V_x(t_0, x_0)$. Then the Hamiltonian system

$$\begin{cases} x'(t) = \frac{\partial H}{\partial p}(x(t), p(t)), & x(t_0) = x_0 \\ p'(t) = -\frac{\partial H}{\partial x}(x(t), p(t)), & p(t_0) = p_0 \\ p(t) \neq 0 \text{ for all } t \in [t_0, T] \end{cases}$$

has a unique solution $(z(\cdot), \overline{p}(\cdot))$ defined on $[t_0, T]$. Moreover $z(\cdot)$ is optimal.

Furthermore, if $\nabla g(\cdot)$ is continuous at $z(T)$, then $\overline{p}(\cdot)$ is the co-state corresponding to $z(\cdot)$.

UNIQUENESS OF OPTIMA AND DIFFERENTIABILITY OF VALUE

Theorem 2.16 yields that if $\partial^\star V_x(t_0, x_0)\backslash\{0\}$ is **not a singleton**, then optimal solution is **not unique**. We prove a similar statement under less restrictive regularity assumptions on $H(x, \cdot)$.

Theorem 2.17 *Assume that $g \in C^1$, f is differentiable with respect to x, $f(x, U)$ are convex and $\frac{\partial H}{\partial x}$ is continuous.*

Further assume that for every $R > 0$, there exists $l_R > 0$ such that

$$\forall\ x, y, p \in RB,\ \left\|\frac{\partial H}{\partial x}(x,p) - \frac{\partial H}{\partial x}(y,p)\right\| \leq l_R \|x - y\|$$

If the Mayer problem has a **unique optimal solution** *z, then for all t, $\partial^\star V_x(t, z(t))$ is a* **singleton** *and $V(t, \cdot)$ is differentiable at $z(t)$.*

Theorem 2.18 *We posit all hypothezis of Theorem 2.16 and we assume that $g \in C^1$. Then $V(t_0, \cdot)$ is* **differentiable at x_0 with the derivative different from zero if and only if** *there exists a* **unique optimal solution** *z satisfying $\nabla g(z(T)) \neq 0$.*

SEMICONCAVITY OF VALUE

We assume the following

$$\begin{cases} \exists\, \omega : R_+ \times R_+ \mapsto R_+,\ \forall\, r \le R, s \le S, \\ \omega(r,s) \le \omega(R,S),\ \lim_{s\to 0+} \omega(R,s) = 0 \\ \\ \forall\, R > 0, x_1, x_2 \in B_R(0), u \in U \\ ||\frac{\partial f}{\partial x}(x_1,u) - \frac{\partial f}{\partial x}(x_2,u)|| \le \omega(R, ||x_1 - x_2||) \\ \\ g : R^n \mapsto R \ \text{ is semiconcave, loc. Lipschitz} \end{cases}$$

(6)

Theorem 2.19 *Assume (6). Then the value function is semi-concave on* $[0,T] \times R^n$.

Example Consider the control system

$$x' = f_0(x) + \sum_{i=1}^{k} f_i(x)u_i, \quad u_i \in [a_i, b_i]$$

If $f_i \in C^1$ for all $i \ge 0$, then (6) holds true. Furthermore $f(x,U)$ is convex and compact, where $U = [a_1,b_1] \times ... \times [a_k,b_k]$ and

$$f(x,u_1,...,u_k) = f_0(x) + \sum_{i=1}^{k} f_i(x)u_i$$

DIFFERENTIABILITY ALONG OPTIMAL SOLUTIONS

Theorem 2.20 *Under assumptions of Theorem 2.17, suppose that the Mayer problem has a unique optimal solution z and V is semiconcave. Then V is differentiable at $(t, z(t))$ for all $t \in [t_0, T]$.*

Corollary 2.21 *Under hypothezis of Theorems 2.16, assume that g is continuously differentiable and V is semiconcave. Then $V(\cdot,\cdot)$ is* **differentiable** *at (t_0, x_0) with the partial derivative $\frac{\partial V}{\partial x}(t_0, x_0)$ different from zero if and only if* **there exists a unique optimal solution** *z satisfying $\nabla g(z(T)) \neq 0$.*

Theorem 2.22 *Assume that V is semiconcave, g is convex and*

$$\text{Graph}(f(\cdot, U)) \quad \text{is convex}$$

Then $V(t,\cdot)$ is convex and V is continuously differentiable on $[0, T] \times R^n$.

OPTIMAL SYNTHESIS

The optimal synthesis is a mapping $u : [0,T] \times R^n \mapsto U$, i.e. $u(t,x) \in U$ such that **for every** $(t_0, x_0) \in [0,T] \times R^n$ the solution $x(\cdot)$ to

$$\begin{cases} x' &= f(x, u(t,x)) \\ x(t_0) &= x_0 \end{cases}$$

is optimal, i.e. $V(t_0, x_0) = g(x(T))$.

BUT

- The optimal feedback may be discontinuous
- It may be not unique

Example For instance for $g(y) = -|y|$ and the control system

$$x' = u, \quad u \in [-1, 1]$$

Set

$$G(x) = \begin{cases} +1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \\ \{-1, +1\} \text{ if } x = 0 \end{cases}$$

Then $x(\cdot; t_0, x_0, u)$ is optimal if and only if

$$x'(t) \in G(x(t)) \text{ almost everywhere}$$

OPTIMAL FEEDBACK

We introduce the following **feedback map** $G : [0,T] \times R^n \rightsquigarrow R^n$: $\forall\ (t,x) \in [0,T] \times R^n$

$$G(t,x) = \left\{ v \in F(x) \mid \frac{\partial V}{\partial(1,v)}(t,x) = 0 \right\}$$

(notice that the sets $G(t,x)$ **may be empty.**)

Theorem 2.23 *Assume that F is Lipschitz and g is locally Lipschitz and let $t_0 \in [0,T]$. The following two statements are equivalent:*

i) x is a solution to the differential inclusion

$$(7) \qquad x'(t) \in G(t, x(t)) \text{ a.e. in } [t_0, T]$$

ii) $x \in \mathcal{S}_{[t_0,T]}(x_0)$ and $V(t, x(t)) \equiv g(x(T))$.

Proof— Fix $x \in \mathcal{S}_{[t_0,T]}$ and set $\varphi(t) = V(t, x(t))$. Then φ is absolutely continuous and for a.e. $t \in [t_0, T]$

$$\varphi'(t) = \frac{\partial V}{\partial(1, x'(t))}(t, x(t))$$

Assume that $i)$ holds true. Hence, for almost every $t \in [t_0, T]$, the set $G(t, x(t))$ is nonempty and $\varphi'(t) = 0$ almost everywhere in $[t_0, T]$.

Corollary 2.24 *A solution* $x \in \mathcal{S}_{[t_0,T]}(x_0)$ *is optimal for the problem*

$$\inf\{g(x(T)) \mid x \in \mathcal{S}_{[t_0,T]}(x_0)\}$$

if and only if *it is a solution to the differential inclusion*

$$x'(t) \in G(t, x(t)) \text{ a.e. in } [t_0, T]$$

satisfying the initial condition $x(t_0) = x_0$*.*

The set-valued map G may be **very-very irregular.** In the true (and most) nonlinear cases

- G is not single-valued
- the sets $G(t, x)$ are not convex
- $G(t, \cdot)$ is not locally Lipschitz and even not upper semicontinuous

However if f, g are sufficiently smooth (C^1), G is upper semicontinuous.

REGULARITY OF OPTIMAL FEEDBACK

The **feedback map** is defined by

$$G(t,x) = \left\{ v \in F(x) \mid \frac{\partial V}{\partial(1,v)}(t,x) = 0 \right\}$$

Theorem 2.25 *If V is semiconcave, then G has compact nonempty images and its graph is closed in $[0,T[\times R^n$.*

Corollary 2.26 *If in addition G is single-valued on a subset $K \subset [0,T[\times R^n$, then the map $K \ni (t,x) \mapsto G(t,x)$ is continuous.*

Theorem 2.27 *If $V \in C^1$ on $[0,T]\times R^n$, then G has convex compact images and is upper semicontinuous. Furthermore, if for every x the set $f(x,U)$ is strictly convex, then G is single valued and continuous on the set*

$$\left\{ (t,x) \in [0,T[\times R^n \mid \frac{\partial V}{\partial x}(t,x) \neq 0 \right\}$$