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Value Function in Optimal Control
Lecture 3

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HAMILTON-JACOBI-BELLMAN EQUATION

Outline

3.1 Tangent and Normal cones.

3.2 Viability theorem.

3.3 Hamilton-Jacobi Equation

3.4 Lower Semicontinuous Solutions

3.5 Viscosity Solutions

Tangent and Normal Cones to a Subset

Definition 3.1 (Contingent Cone) *Let K be a subset of a normed vector space X and $x \in \overline{K}$ belong to the closure of K. The* contingent cone $T_K(x)$ *is defined by*

$$T_K(x) \ := \ \text{Limsup}_{h \to 0+} \frac{K - x}{h}$$

That is

$$\begin{cases} v \ \in \ T_K(x) \ \text{ if and only if } \ \exists \ h_n \to 0+ \ , v_n \to v \\ \text{such that } \ \forall \ n, \ \ x + h_n v_n \in K \end{cases}$$

It implies that when K is convex,

$$T_K(x) = \overline{\bigcup_{\lambda \geq 0} \lambda(K - x)}$$

We also observe that

$$\text{if} \quad x \in \ \text{Int}(K), \quad \text{then} \ \ T_K(x) \ = \ X$$

This situation may also happen when x does not belong to the interior of K.

Theorem 3.2 *Let X be a finite dimensional vector-space and K be a closed subset of X. Then for every $x \in K$*

$$\text{Liminf}_{y \to_K x} T_K(y) = \text{Liminf}_{y \to_K x} \overline{co} T_K(y) \subset T_K(x)$$

SUBNORMAL CONES

Definition 3.3 (Subnormal Cones) *The* **subnormal cone** $N_K^0(x)$ *is defined by*

$N_K^0(x) \ := \ \{p \in X^\star \mid \ < p, v > \leq 0 \ \ \forall\, v \in T_K(x)\}$

From the very definitions it follows that

$$\mathcal{E}p(D_\uparrow \varphi(x_0)) \ = \ T_{\mathcal{E}p(\varphi)}\,(x_0, \varphi(x_0))$$

where $\mathcal{E}p$ denotes the epigraph.

Proposition 3.4 *Let* $\varphi \ : \ R^n \ \mapsto \ R \cup \{\pm\infty\}$ *and* $x_0 \in \mathrm{Dom}(\varphi)$. *Then the following statements are equivalent*

$i) \quad p \ \in \ \partial_- \varphi(x_0)$

$ii) \ \ \forall\, u \ \in \ R^n, \ \ \langle p, u\rangle \ \leq \ D_\uparrow \varphi(x_0)(u)$

$iii) \ (p, -1) \ \in \ N^0_{\mathcal{E}p(\varphi)}(x_0, \varphi(x_0))$

Lemma 3.5 (Rockafellar) *Consider a lower semicontinuous* $\varphi \ : \ R^n \ \mapsto \ R \cup \{+\infty\}$ *and* $x_0 \in \mathrm{Dom}(\varphi)$. *Let* $p \in R^n$ *be such that*

$$(p, 0) \ \in \ N^0_{\mathcal{E}p(\varphi)}(x_0, \varphi(x_0)), \quad p \ \neq \ 0$$

Then for every $\varepsilon > 0$, *there exist* x_ε, p_ε *in* R^n *and* $q_\varepsilon < 0$ *satisfying*

$$\|x_\varepsilon - x_0\| + \|p_\varepsilon - p\| \leq \varepsilon, \ \ (p_\varepsilon, q_\varepsilon) \in N^0_{\mathcal{E}p(\varphi)}(x_\varepsilon, \varphi(x_\varepsilon))$$

VIABILITY THEOREM

Let $F : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ be a set-valued map and $K \subset \mathrm{Dom}(F)$ be a nonempty subset.

The subset K enjoys the **viability property** for the differential inclusion

$$(1) \qquad x' \in F(x)$$

if for any initial state $x_0 \in K$, there exists at least one solution $x(\cdot)$ to (1) starting at x_0 which is viable in K in the sense that $x(t) \in K$ for all $t \geq 0$.

The viability property is said to be **local** if for any initial state $x_0 \in K$, there exist $T(x_0) > 0$ and a solution starting at x_0 which is viable in K on the interval $[0, T(x_0)]$ in the sense that for every $t \in [0, T(x_0)]$, $x(t) \in K$.

We say that K is a **viability domain** of F if

$$\forall\, x \in K, \quad F(x) \cap T_K(x) \neq \emptyset$$

Theorem 3.6 (Viability Theorem) *If F is upper semicontinuous with nonempty compact convex images, then a locally compact set K enjoys the local viability property if and only if it is a viability domain of F. In this case, if for some $c > 0$, we have*

$$\forall\, x \in K, \quad \inf_{v \in F(x) \cap T_K(x)} \|v\| \;\leq\; c(\|x\| + 1)$$

and if K is closed, then K enjoys the viability property.

The following result provides a very useful **duality** characterization of viability domains:

Proposition 3.7 (Ushakov) *Assume that the set-valued map $F : K \rightsquigarrow \mathbf{R}^n$ is upper semicontinuous with convex compact values. Then the following three statements are equivalent:*

i) $\forall\, x \in K, \;\; F(x) \cap T_K(x) \; \neq \; \emptyset$

ii) $\forall\, x \in K, \;\; F(x) \cap \overline{co}\,(T_K(x)) \; \neq \; \emptyset$

iii) $\forall\, x \in K, \forall\, p \in N_K^0(x), \;\; H(x, -p) \geq 0$

where $H(x, -p) = \sup_{v \in F(x)} < -p, v > .$

Theorem 3.8 *Suppose that* $F : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ *is upper semicontinuous with compact convex values and for some* $c > 0$*, we have*

$$\forall\, x \in \mathbf{R}^n, \quad \sup_{v \in F(x)} \|v\| \;\leq\; c(\|x\| + 1)$$

For a closed subset $K \subset \mathbf{R}^n$ *the following conditions are equivalent:*

i) $\forall\, x \in K, \quad F(x) \cap \overline{co}\,(T_K(x)) \;\neq\; \emptyset$

ii) $\forall\, x_0 \in K$ *there is a solution* $x : [0, +\infty) \mapsto K$ *to*

$$\begin{cases} x'(t) \in F(x(t)) \\ \\ x(0) = x_0 \end{cases}$$

Characterization of the Value Function

Consider a lower semicontinuous $g : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$, a Lipschitz set-valued map $F : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ with convex compact images and the differential inclusion

$$(2) \quad x'(t) \in F(x(t)) \text{ almost everywhere}$$

The value function is given by:
For all $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$,

$$(3)\, V(t_0, x_0) = \inf\{g(x(T)) \mid x \in \mathcal{S}_{[t_0,T]}(x_0)\}$$

It is **lower semicontinuous**.

The value function is nondecreasing along solutions to (2): $\forall\, x \in \mathcal{S}_{[t_0,T]}(x_0)$,

$$\forall\, t_0 \le t_1 \le t_2 \le T,\ V(t_1, x(t_1)) \le V(t_2, x(t_2))$$

Furthermore $x \in \mathcal{S}_{[t_0,T]}(x_0)$ is optimal for problem (3) if and only if $V(t, x(t)) \equiv g(x(T))$.

These two properties characterize the value function.

Consider **any** $W : [0,T] \times \mathbf{R}^n \mapsto R \cup \{+\infty\}$ satisfying the boundary condition $W(T,\cdot) = g$. If W is nondecreasing along solutions to (2) and for all $(t_0, x_0) \in [0,T] \times \mathbf{R}^n$ there exists $\bar{x} \in \mathcal{S}_{[t_0,T]}(x_0)$ such that $W(t_0,x_0) \geq W(T,\bar{x}(T))$, then $W = V$.

Indeed if $\bar{x}$ is as above, then

$$W(t_0,x_0) \geq W(T,\bar{x}(T)) = g(\bar{x}(T)) \geq V(t_0,x_0)$$

So $W \geq V$. Next if $x \in \mathcal{S}_{[t_0,T]}(x_0)$ is optimal, then

$$V(t_0,x_0) = g(x(T)) \geq W(t_0,x_0)$$

Hence $V \geq W$.

Define the Hamiltonian $H : R^n \times R^n \mapsto R$ by

$$H(x,p) = \sup_{v \in F(x)} < p, v >$$

Notice that $H(x,\cdot)$ is convex, positively homogeneous.

HAMILTON-JACOBI-BELLMAN EQUATION

Consider the Hamilton-Jacobi equation **(HJB):**

$$-\frac{\partial V}{\partial t}(t,x)+H\left(x,\ -\frac{\partial V}{\partial x}(t,x)\right)=0, \quad V(T,\cdot)=g(\cdot)$$

Definition 3.9 *An extended lower semicontinuous function* $V:[0,T]\times R^n \mapsto R\cup\{+\infty\}$ *is called a* **lower semicontinuous solution** *to the Hamilton-Jacobi-Bellman equation* **(HJB)** *if it satisfies the following conditions:*

$$\begin{cases} V(T,\cdot) \;=\; g(\cdot) \ \text{ and for all } \ (t,x)\in]0,T[\times R^n, \\ \forall\ (p_t,p_x)\in\partial_- V(t,x), \quad -p_t+H(x,-p_x)=0 \\ \forall\ (p_t,p_x)\in\partial_- V(0,x), \quad -p_t+H(x,-p_x)\geq 0 \\ \forall\ (p_t,p_x)\in\partial_- V(T,x), \quad -p_t+H(x,-p_x)\leq 0 \end{cases}$$

VISCOSITY SOLUTIONS

Definition 3.10 *An extended lower semicontinuous function* $V : [0,T] \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ *is called a* viscosity supersolution *to* **(HJB)** *if for all* $t \in]0,T[$ *and* $x \in \mathbf{R}^n$

$$\forall\ (p_t, p_x) \in \partial_- V(t,x), \quad -p_t + H(t,x,-p_x) \geq 0$$

An extended upper semicontinuous function $V : [0,T] \times \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ *is called a* viscosity subsolution *to* **(HJB)** *if for all* $t \in]0,T[$ *and* $x \in \mathbf{R}^n$

$$\forall\ (p_t, p_x) \in \partial_+ V(t,x), \quad -p_t + H(t,x,-p_x) \leq 0$$

Let $V : [0,T] \times \mathbf{R}^n \mapsto \mathbf{R}$ *be a continuous function. It is called a* **viscosity solution** *to* **(HJB)** *if for all* $t \in]0,T[$ *and* $x \in \mathbf{R}^n$

$$\forall\ (p_t, p_x) \in \partial_- V(t,x), \quad -p_t + H(t,x,-p_x) \geq 0$$
$$\forall\ (p_t, p_x) \in \partial_+ V(t,x), \quad -p_t + H(t,x,-p_x) \leq 0$$

UNIQUENESS OF SOLUTIONS

Theorem 3.11 *Let* $V : [0,T] \times R^n \mapsto R \cup \{+\infty\}$ *be an extended lsc function.*
The following statements are equivalent:

$i)$ V is the value function

$ii)$ V is a lsc solution to **(HJB)**

$iii)$ V is a contingent solution to **(HJB)** :
$V(T,\cdot) = g(\cdot)$ and for all $(t,x) \in \mathrm{Dom}(V)$,
$0 \le t < T \Longrightarrow \inf_{v \in F(x)} D_{\uparrow}V(t,x)(1,v) \le 0$
$0 < t \le T \Longrightarrow \sup_{v \in F(x)} D_{\uparrow}V(t,x)(-1,-v) \le$

$iv)$ $V(T,\cdot) = g(\cdot)$ and for all $(t,x) \in]0,T[\times R^n$,
$\forall\, (p_t, p_x) \in \partial_- V(t,x), \quad -p_t + H(x, -p_x) = 0$
$\forall\, \overline{x} \in R^n, \quad V(0,\overline{x}) = \liminf_{t \to 0+,\, x \to \overline{x}} V(t,x)$
$\forall\, \overline{x} \in R^n, \quad g(\overline{x}) = \liminf_{t \to T-,\, x \to \overline{x}} V(t,x)$

Finally, if V *is continuous on* $[0,T] \times R^n$ *then the above statements are equivalent to:*

$v)$ V *is a viscosity solution to* **(HJB)**

FROM (HJB) EQUATION TO VIABILITY CONDITIONS

Consider a set-valued map $F : R^n \rightsquigarrow R^n$ with nonempty compact images.

Theorem 3.12 *Assume that F is upper semicontinuous and has convex images. Consider an extended lower semicontinuous function V : $[0,T] \times R^n \mapsto R \cup \{+\infty\}$.*

The following statements are equivalent :

i) For all $(t,x) \in \mathrm{Dom}(V)$ such that $t < T$ and for every $(p_t, p_x, q) \in N^0_{\mathcal{E}p(V)}(t,x,V(t,x))$

$$-p_t + H(x, -p_x) \;\geq\; 0$$

ii) For all $(t,x) \in \mathrm{Dom}(V)$, $t < T$, $y \geq V(t,x)$

$$(\{1\} \times F(x) \times \{0\}) \cap T_{\mathcal{E}p(V)}(t,x,y) \;\neq\; \emptyset$$

iii) For all $(t,x) \in \mathrm{Dom}(V)$ such that $t < T$

$$\inf_{v \in F(x)} D_{\uparrow}V(t,x)(1,v) \leq 0$$

iv) For all $(t,x) \in \mathrm{Dom}(V)$ such that $t < T$

$$\forall\, (p_t, p_x) \;\in\; \partial_{-}V(t,x), \;\; -p_t + H(x, -p_x) \geq 0$$

Theorem 3.13 *Consider an extended lower semi-continuous function* $V : [0,T] \times R^n \mapsto R \cup \{+\infty\}$ *and assume that* F *is lower semicontinuous.*

The following statements are equivalent :

i) For all $(t,x) \in \mathrm{Dom}(V)$ *such that* $t > 0$ *and for every* $(p_t, p_x, q) \in N^0_{\mathcal{E}p(V)}(t,x,V(t,x))$

$$-p_t + H(x, -p_x) \ \leq \ 0$$

ii) For all $(t,x) \in \mathrm{Dom}(V)$ *such that* $t > 0$ *and all* $y \geq V(t,x)$

$$\{-1\} \times (-F(x)) \times \{0\} \ \subset \ T_{\mathcal{E}p(V)}(t,x,y)$$

iii) For all $(t,x) \in \mathrm{Dom}(V)$ *such that* $t > 0$

$$\sup_{v \in F(x)} D_{\uparrow}V(t,x)(-1,-v) \leq 0$$

iv) For all $(t,x) \in \mathrm{Dom}(V)$ *such that* $t > 0$

$$\forall\ (p_t, p_x) \in \partial_{-}V(t,x), \ \ -p_t + H(x,-p_x) \ \leq \ 0$$

MONOTONE BEHAVIOR OF CONTINGENT SOLUTIONS

Consider a set-valued map $F : R^n \rightsquigarrow R^n$ with nonempty compact images and the differential inclusion

(4) $\quad x'(t) \in F(x(t))$ almost everywhere

We investigate a relationship between monotone behavior of a function V along solutions to (4) and contingent inequalities.

Theorem 3.14 *Let* $V : [0,T] \times R^n \mapsto R \cup \{+\infty\}$ *be an extended lower semicontinuous function. Assume that* F *is upper semicontinuous with convex images and linear growth* $\exists\, c > 0$

$$\forall\,(t,x) \;\in\; \sup_{v \in F(x)} \|v\| \le c(1 + \|x\|)$$

Then the following statements are equivalent:

i) $\forall\, t < T,\ \inf_{v \in F(x)} D_{\uparrow}V(t,x)(1,v) \le 0$

ii) For every $(t_0, x_0) \in [0,T] \times R^n$*, there exists* $\overline{x} \in \mathcal{S}_{[t_0,T]}(x_0)$ *such that* $V(t,\overline{x}(t)) \le V(t_0,x_0)$ *for all* $t \in [t_0,T]$.

Proof — $i) \Longrightarrow ii)$. Define $\widehat{F}$ by

$$\widehat{F}(t,x,z) = \begin{cases} \{1\} \times F(x) \times \{0\} & \text{if} t < T \\ \\ [0,1] \times \overline{co}(F(x) \cup \{0\}) \times \{0\} & \text{if} t \geq T \end{cases}$$

and consider the viability problem

$$(5) \quad \begin{cases} (t,x,z)' & \in & \widehat{F}(t,x,z) \\ (t,x,z)(t_0) & = & (t_0, x_0, V(t_0,x_0)) \\ (t,x,z) & \in & \mathcal{E}p(V) \end{cases}$$

For all $(t,x,z) \in \mathcal{E}p(V)$

$$\widehat{F}(t,x,z) \cap T_{\mathcal{E}p(V)}(t,x,z) \neq \emptyset$$

By the Viability Theorem, (5) has a solution

$$[t_0, T] \ni t \mapsto (t, \overline{x}(t), z(t)) \in \mathcal{E}p(V)$$

Thus $V(t, \overline{x}(t)) \leq z(t) = V(t_0, x_0)$ for all $t \in [t_0, T]$ and $ii)$ follows. Conversely, assume that $ii)$ is satisfied. Fix $(t_0, x_0) \in \mathrm{Dom}(V)$ with $t_0 < T$ and let $\overline{x}$ be as in $ii)$. Let $h_n \to 0+$ be such that $[x(t_0+h_n) - x(t_0)]/h_n$ converge to some $v \in F(x_0)$. On the other hand $D_{\uparrow}V(t_0,x_0)(1,v) \leq$

$$\liminf_{n \to \infty} \frac{V(t_0+h_n, x(t_0+h_n)) - V(t_0,x_0)}{h_n} \leq 0$$

Theorem 3.15 *Let* $V : [0,T] \times R^n \mapsto R \cup \{+\infty\}$ *be an extended lower semicontinuous function. If* F *is locally Lipschitz, then the following two statements are equivalent:*

$i)\forall (t,x) \in \text{Dom}\,(V),\ \sup_{v\in F(x)} D_{\uparrow}V(t,x)(-1,-v) \leq 0$

$ii)\ \forall\ x \in \mathcal{S}_{[t_0,T]}(x_0),\ t \in [t_0,T], V(t_0,x_0) \leq V(t,x(t))$.

Proof— Assume $i)$. Since $i)$ does not involve T, it is enough to prove the inequality in $ii)$ for $t = T$. We know that for all $0 \leq t < T$ and $x \in R^n$ such that $(T-t,x) \in \text{Dom}(V)$ and for all $z \geq V(T-t,x)$,

$$\{-1\} \times (-F(x)) \times \{0\} \subset T_{\mathcal{E}p(V)}(T-t,x,z)$$

Let B denote the closed unit ball in R^n and c_R the Lipschitz constant of F on $B_R(0)$. There exists a continuous $\widehat{f} : R^n \times B \mapsto R^n$ and $\vartheta > 0$

$$\begin{cases} \forall\, x \in R^n, \;\; F(x) \;=\; \widehat{f}(x,B) \\ \forall\, u \in B, \;\; \widehat{f}(\cdot,u) \;\text{ is }\; \vartheta c_R - \text{Lipschitz on }\; B_R(0) \\ \forall\, x \in R^n \;\text{ and for all }\; u,v \,\in\, B, \\ \left\|\widehat{f}(x,u) - \widehat{f}(x,v)\right\| \leq \vartheta(\sup_{y\in F(x)} \|y\|)\,\|u-v\| \end{cases}$$

Fix $x \in \mathcal{S}_{[t_0,T]}(x_0)$. It is enough to consider the case $V(T, x(T)) < \infty$. Consider a measurable map $u : [t_0, T] \mapsto B$ such that

$$x'(t) \;=\; \widehat{f}(x(t), u(t)) \;\text{ a.e.}$$

and continuous maps $u_k : [t_0, T] \mapsto B$ converging to u in $L^1(t_0, T; B)$. Let x_k denote the solution to

$$x'_k(t) = \widehat{f}(x_k(t), u_k(t)),\; t \in [t_0, T],\; x_k(T) = x(T)$$

Then x_k converge uniformly to x. The map

$$t \mapsto (T - t, x_k(T - t), V(T, x(T)))$$

is the only solution to

$$(6)\begin{cases} \gamma'(t) = -1 \\ y'(t) = -\widehat{f}(y(t), u_k(T - t)) \\ z'(t) = 0 \\ \gamma(0) = T, y(0) = x(T), z(0) = V(T, x(T)) \end{cases}$$

We know that for all $(\gamma, x, z) \in \mathcal{E}p(V)$,

$$(-1, -\widehat{f}(x, u_k(\gamma)), 0) \in T_{\mathcal{E}p(V)}(\gamma, x, z)$$

The map $(t, x) \rightsquigarrow \{-\widehat{f}(x, u_k(T-t))\}$ being continuous, by the Viability Theorem problem (6) has at least one solution satisfying

$$[0, T - t_0] \ni t \mapsto (\gamma(t), y(t), z(t)) \in \mathcal{E}p(V)$$

Consequently, $\forall\, 0 \leq t \leq T - t_0$,

$$(T - t, x_k(T - t), V(T, x(T))) \in \mathcal{E}p(V)$$

In particular, $V(t_0, x_k(t_0)) \leq V(T, x(T))$. Taking the limit when $k \to \infty$ and using that V is lower semicontinuous, we deduce $ii)$ for $t = T$.

VALUE FUNCTION & CONTINGENT SOLUTIONS

Proposition 3.16 *Let V be the value function. Then for all $(t_0, x_0) \in \mathrm{Dom}(V)$,*

$$\begin{cases} t_0 < T \implies \inf_{v \in F(x_0)} D_\uparrow V(t_0, x_0)(1, v) \leq 0 \\ t_0 > 0 \implies \sup_{v \in F(x_0)} D_\uparrow V(t_0, x_0)(-1, -v) \leq 0 \end{cases}$$

Proof — Fix (t_0, x_0) as above. Then $\exists\ x \in \mathcal{S}_{[t_0,T]}(x_0)$ such that $V(t, x(t)) \equiv g(x(T))$. Theorem 3.14 ends the proof of the first statement. The second one follows from Theorem 3.15.

Theorem 3.17 *The value function is the only lower semicontinuous function from $[0, T] \times R^n$ into $R \cup \{+\infty\}$ satisfying*

$$\begin{cases} V(T, \cdot) = g(\cdot) \ \text{ and for all } \ (t, x) \in \mathrm{Dom}(V), \\ 0 \leq t < T \implies \inf_{v \in F(x)} D_\uparrow V(t, x)(1, v) \ \leq \ 0 \\ 0 < t \leq T \implies \sup_{v \in F(x)} D_\uparrow V(t, x)(-1, -v) \leq 0 \end{cases}$$

REGULARITY OF VALUE FUNCTION AT BOUNDARY POINTS

Theorem 3.18 *If an extended lower semicontinuous function* $V : [0,T] \times R^n \mapsto R \cup \{+\infty\}$ *satisfies*

$$\begin{cases} \forall\, (t,x) \in]0,T[\times R^n, \\ \forall\, (p_t, p_x) \in \partial_- V(t,x), -p_t + H(x,-p_x) = 0 \\ \\ \forall\, \overline{x} \in R^n, \quad V(0,\overline{x}) \;=\; \liminf_{t\to 0+,\; x\to\overline{x}} V(t,x) \\ \\ \forall\, \overline{x} \in R^n, \quad V(T,\overline{x}) \;=\; \liminf_{t\to T-,\; x\to\overline{x}} V(t,x) \end{cases}$$

then for all $(t,x) \in \mathrm{Dom}(V)$,

$$\begin{cases} 0 < t \leq T \Longrightarrow \sup_{v\in F(x)} D_\uparrow V(t,x)(-1,-v) \leq 0 \\ 0 \leq t < T \Longrightarrow \inf_{v\in F(x)} D_\uparrow V(t,x)(1,v) \leq 0 \end{cases}$$

UNIQUENESS OF VISCOSITY SOLUTIONS

Theorem 3.19 *Let* $V : [0,T] \times \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ *be an extended lower semicontinuous function. Assume that* F *is upper semicontinuous and has convex compact nonempty images.*

Then the following two statements are equivalent:

i) V *is a viscosity supersolution of* **(HJB)**

ii) For all $0 < t < T$ *and* $x \in \mathbf{R}^n$ *such that* $V(t,x) \neq +\infty$, *we have*

$$\inf_{v \in F(x)} D_{\uparrow} V(t,x)(1,v) \le 0$$

Notice next that

$$T_{\mathcal{H}yp(\varphi)}(x_0, \varphi(x_0)) \;=\; \mathcal{H}yp\,\left(D_{\downarrow}\varphi(x_0)\right)$$

where $\mathcal{H}yp$ denotes for the hypograph.

Thus $p \in \partial_+\varphi(x_0)$ if and only if

$$(7) \qquad (-p,\ +1) \in N^0_{\mathcal{H}yp\,(\varphi)}(x_0, \varphi(x_0))$$

Theorem 3.20 *Let* $V : [0,T] \times \mathbf{R}^n \to \mathbf{R}$ *be continuous. Assume that* F *has compact nonempty images and is Lipschitz.*

Then the following two statements are equivalent

i) V is a viscosity subsolution of **(HJB)**

ii) $\forall t \in]0,T[, x, \sup_{v \in F(x)} D_{\uparrow}V(t,x)(-1,-v) \leq 0$

Proof — Assume that *ii)* holds true. Fix $0 < t_0 < T$. We already know that for every $t_0 \leq t_1 < T$ and every $x \in \mathcal{S}_{[t_0,t_1]}(x_0)$:

$$\forall\, t \in [t_0, t_1], \quad V(t_0, x_0) \;\leq\; V(t, x(t))$$

Fix $v \in F(t_0, x_0)$. Then there exist $t_0 < t_1 < T$ and $x \in \mathcal{S}_{[t_0,t_1]}(x_0)$ such that $x'(t_0) = v$. The above inequality yields $0 \leq D_{\downarrow}V(t_0, x_0)(1, v)$. Consequently,

$$\forall\, (p_t, p_x) \;\in\; \partial_+ V(t_0, x_0), \quad 0 \leq p_t + \langle p_x, v \rangle$$

Since $v \in F(t_0, x_0)$ is arbitrary, V is a viscosity subsolution.

Assume i). We claim that for all (t, x) such that $0 < t < T$ and all $z \leq V(t, x)$ we have

$$\forall\ (q_t, q_x, q) \in N^0_{\mathcal{H}yp\ (V)}(t, x, z), q_t + H(x, q_x) \leq 0 \quad (8)$$

Indeed it is enough to consider the case $z = V(t, x)$. Fix such (q_t, q_x, q). Clearly $q \geq 0$. If $q > 0$ then

$$\left(\frac{q_t}{q}, \frac{q_x}{q}, +1\right) \in N^0_{\mathcal{H}yp\ (V)}(t, x, V(t, x))$$

Hence, by (7) and i),

$$\frac{q_t}{q} + H\left(x, \frac{q_x}{q}\right) \leq 0$$

and therefore $q_t + H(x, q_x) \leq 0$. If $q = 0$, applying Rockafellar's Lemma 3.5 to the extended lower semicontinuous function $(s, y) \mapsto -V(s, y)$, we can find a sequence $(t_i, x_i) \to (t, x)$ and a sequence

$$\left(q_t^i,\ q_x^i,\ q^i\right) \in N^0_{\mathcal{H}yp\ (V)}(t, x, V(t, x))$$

such that $q^i > 0$ and (q_t^i, q_x^i) converge to (q_t, q_x). This and continuity of H yield (8).

We next deduce from (8) and the separation theorem that for all (t,x) such that $0 < t < T$ and all $z \leq V(t,x)$

$$\{1\} \times F(x) \times \{0\} \ \subset \ \overline{co}\left(T_{\mathcal{H}yp\,(V)}(t,x,z)\right)$$

This and lower semicontinuity of F imply that for all (t,x) satisfying $0 < t < T$

$$\{1\} \times F(x) \times \{0\} \subset$$
$$\operatorname{Liminf}_{\substack{(t',x',z') \to (t,x,V(t,x)) \\ (t',x',z') \in \mathcal{H}yp\,(V)}} \overline{co} T_{\mathcal{H}yp\,(V)}(t',x',z$$
$$\subset \ T_{\mathcal{H}yp\,(V)}(t,x,V(t,x)) \ = \ \mathcal{H}yp\,(D_{\downarrow}V(t,x))$$

Thus for all (t,x) satisfying $0 < t < T$,

$$\inf_{v \in F(x)} D_{\downarrow}V(t,x)(1,v) \ \geq \ 0$$

Define $W(t,x) = -V(T-t,x)$. Then for all (t,x) such that $0 < t < T$ and for all $v \in F(x)$, we have

$$D_{\uparrow}W(t,x)(-1,v) = -D_{\downarrow}V(T-t,x)(1,v) \ \leq \ 0$$

Applying Theorem 3.15 to W and the set-valued map

$$\widehat{F}(x) = -F(x)$$

we deduce that for every solution y to the inclusion

$$y'(t) \in \widehat{F}(y(t)) \text{ a.e. in } [t_0, t_1]$$

where $0 < t_0 \leq t_1 < T$ we have

$$\forall\, t \in [t_0, t_1], \quad W(t_0, x_0) \leq W(t, y(t))$$

Fix any $v \in F(x_0)$ and consider a solution $y(\cdot)$ to the differential inclusion

$$\begin{cases} y' \in \widehat{F}(y) \\ y(T - t_0) = x_0, \;\; y'(T - t_0) = -v \end{cases}$$

Then for all small $s > 0$,

$$W(T - t_0, x_0) \leq W(T - t_0 + s, y(T - t_0 + s))$$

and therefore for a sequence $v_s \to v$ we have

$$V(t_0 - s, x_0 - sv_s) \leq V(t_0, x_0)$$

This yields that $D_{\uparrow}V(t_0, x_0)(-1, -v) \leq 0$. Since $v \in F(x_0)$ is arbitrary, *ii*) follows.