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Value Function in Optimal Control
Lecture 4

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VALUE FUNCTION OF BOLZA PROBLEM AND RICCATI EQUATIONS

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VALUE FUNCTION OF BOLZA PROBLEM

Consider the minimization problem

$$(P) \qquad \text{minimize} \int_{t_0}^{T} L(t, x(t), u(t))dt \; + \; g(x(T))$$

over solution-control pairs (x, u) of the control system

$$(1) \quad \begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U \\ x(t_0) = x_0 \end{cases}$$

where $t_0 \in [0, T]$, $x_0 \in \mathbf{R}^n$, U is a complete separable metric space,

$$g : \mathbf{R}^n \mapsto \mathbf{R}, \quad L : [0, T] \times \mathbf{R}^n \times U \mapsto \mathbf{R}$$

$$f : [0, T] \times \mathbf{R}^n \times U \mapsto \mathbf{R}^n$$

We denote by $\mathcal{U}$ the set of all measurable controls $u : [0, T] \mapsto U$ and by $x(\cdot; t_0, x_0, u)$ the solution of (1) starting at time t_0 from the initial condition x_0 and corresponding to the control $u(\cdot) \in \mathcal{U}$. Of course not to every $u \in \mathcal{U}$ corresponds a solution $x(\cdot; t_0, x_0, u)$ of (1).

For all $(t_0, x_0, u) \in [0, T] \times \mathbf{R}^n \times \mathcal{U}$ set

$$\Phi(t_0, x_0, u) =$$

$$\int_{t_0}^{T} L(t, x(t; t_0, x_0, u), u(t))dt + g(x(T; t_0, x_0, u))$$

if this expression is well defined and $\Phi(t_0, x_0, u) = +\infty$ otherwise.

The value function associated to the Bolza problem (P) is defined by

$$V(t_0, x_0) = \inf_{u \in \mathcal{U}} \Phi(t_0, x_0, u)$$

when (t_0, x_0) range over $[0, T] \times \mathbf{R}^n$.

In this lecture we address only locally Lipschitz value functions.

The **Hamiltonian** $H : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R}$ is defined by

$$H(t,x,p) = \sup_{u \in U} \left(\langle p, f(t,x,u) \rangle - L(t,x,u) \right)$$

Proposition 4.1 *Assume that $H(t,\cdot,\cdot)$ is differentiable. Then*

$$\frac{\partial H}{\partial p}(t,x,p) =$$

$$\{ f(t,x,u) \mid \langle p, f(t,x,u) \rangle - L(t,x,u) = H(t,x,p) \}$$

and

$$\frac{\partial H}{\partial x}(t,x,p) = \{ \frac{\partial f}{\partial x}(t,x,u)^{\star} p - \frac{\partial L}{\partial x}(t,x,u) \mid$$

$$\langle p, f(t,x,u) \rangle - L(t,x,u) = H(t,x,p)$$

Consider the Hamiltonian system

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(t, x(t), p(t)), \ x(T) = x_T \\ \\ -p'(t) = \dfrac{\partial H}{\partial x}(t, x(t), p(t)), \ p(T) = -\nabla g(x_T) \end{cases}$$

(2)

Definition 4.2 Hamiltonian system *(2) is called* complete *if for every x_T, the solution of (2) is defined on $[0, T]$ and depends continuously on the "initial" state in the following sense:*

Let (x_i, p_i) be solutions of (2) satisfying $x_i(t_i) \to x_0$, $p_i(t_i) \to p_0$ for some $t_i \to t_0$, $x_0 \in \mathbf{R}^n$, $p_0 \in \mathbf{R}^n$. Then (x_i, p_i) converge uniformly to the solution (x, p) of (2) such that $x(t_0) = x_0$ and $p(t_0) = p_0$.

We impose the following hypothesis:

$\mathbf{H}_1$) $f,\ L$ are continuous and
$\forall\, r > 0,\ \exists\, k_r \in L^1(0,T)$ such that $\forall\, u \in U$,

$(f(t,\cdot,u), L(t,\cdot,u))$ is $k_r(t)-$Lipschitz on $B_r(0)$

$\mathbf{H}_2$) $f(t,\cdot,u),\ L(t,\cdot,u)$ are differentiable and
$g \in \mathcal{C}^1$

$\mathbf{H}_3$) H and $\frac{\partial H}{\partial p}$ are continuous on $[0,T]\times\mathbf{R}^n\times\mathbf{R}^n$

$\mathbf{H}_4$) The Hamiltonian system (2) is complete

$\mathbf{H}_5$) For all $(t,x) \in [0,T] \times \mathbf{R}^n$, the set

$$\{(f(t,x,u), L(t,x,u)+r)\,|\,u \in U,\ r \geq 0\}$$

is closed and convex }

NECESSARY CONDITIONS

Theorem 4.3 *Assume* $H_1), H_2)$ *and let* $(\overline{x}, \overline{u})$ *be an optimal solution-control pair of* (P) *for some* $(t_0, x_0) \in [0,T] \times \mathbf{R}^n$. *If* $H(t,\cdot,\cdot)$ *is differentiable, then there exists* $p : [t_0, T] \mapsto \mathbf{R}^n$ *such that* $(\overline{x}, p)$ *solves the Hamiltonian system*

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(t, x(t), p(t)), \quad x(t_0) = x_0 \\ \\ -p'(t) = \dfrac{\partial H}{\partial x}(t, x(t), p(t)), \quad p(T) = -\nabla g(\overline{x}(T)) \\ \\ p(t_0) \in -\partial_+ V_x(t_0, x_0) \end{cases}$$

where $\partial_+ V_x(t_0, x_0)$ *denotes the superdifferential of* $V(t_0, \cdot)$ *at* x_0.

Consequently for almost all $t \in [t_0, T]$,

$$H(t, \overline{x}(t), p(t)) = \langle p(t), \overline{x}'(t) \rangle - L(t, \overline{x}(t), \overline{u}(t))$$

DIFFERENTIABILITY OF VALUE FUNCTION AND UNIQUENESS OF OPTIMAL SOLUTIONS

Theorem 4.4 *Assume $H_1) - H_5)$, that V is locally Lipschitz and for every $(t_0, x_0) \in [0,T] \times \mathbf{R}^n$ the problem (P) has an optimal solution. Then for every $\overline{p} \in$*

$$\partial_x^\star V(t_0, x_0) \ := \ \mathrm{Limsup}_{x_i \to x_0,\, t_i \to t_0} \left\{ \frac{\partial V}{\partial x}(t_i, x_i) \right\}$$

there exists a solution (x, p) of (2) satisfying

$$x(t_0) = x_0 \ \ \& \ \ p(t_0) = \overline{p}$$

and x is optimal for problem (P).

In particular if (P) has a unique optimal trajectory, then the set $\partial_x^\star V(t_0, x_0)$ is a singleton. Consequently, $V(t_0, \cdot)$ is differentiable at x_0.

CHARACTERISTIC SYSTEM OF HAMILTON-JACOBI EQUATION

Consider the Hamilton-Jacobi equation **(HJB)**

$$-\frac{\partial V}{\partial t} + H\left(t, x, -\frac{\partial V}{\partial x}\right) = 0, \quad V(T, \cdot) = g(\cdot)$$

Lemma 4.5 *Assume* $H_1) - H_5)$, *that* V *is locally Lipschitz and for every* $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$ *the problem (P) has an optimal solution. Consider* $(t_0, x_0) \in]0, T[\times \mathbf{R}^n$ *such that* V *is differentiable at* (t_0, x_0). *Then*

$$-\frac{\partial V}{\partial t}(t_0, x_0) + H\left(t_0, x_0, -\frac{\partial V}{\partial x}(t_0, x_0)\right) = 0$$

i.e., V *satisfies the Hamilton-Jacobi-Bellman equation almost everywhere in* $[0, T] \times \mathbf{R}^n$. *Consequently for all* $(p_t, p_x) \in \partial^\star V(t, x)$

$$-p_t + H(t, x, -p_x) = 0$$

Corollary 4.6 *Under all the assumptions of Lemma 4.5,* V *is a viscosity solution to* **(HJB)**. *Furthermore, for all* $0 < t < T$ *and* x

$$\forall\ (p_t, p_x) \in \partial_- V(t, x), \quad -p_t + H(t, x, -p_x) = 0$$

The characteristic system of **(HJB)** is the **Hamiltonian system**

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(t, x(t), p(t)), \quad x(T) = x_T \\ \\ -p'(t) = \dfrac{\partial H}{\partial x}(t, x(t), p(t)), \quad p(T) = -\nabla g(x_T) \end{cases}$$

(3)

By the maximum principle, if $x : [t_0, T] \mapsto \mathbf{R}^n$ is optimal, then there exists $p : [t_0, T] \mapsto \mathbf{R}^n$ such that (x, p) solves (3) with $x_T = x(T)$.

This is not a sufficient condition for optimality: it may happen that to a given $x_0 \in \mathbf{R}^n$ correspond two distinct solutions (x_i, p_i), $i = 1, 2$ of (3) satisfying

(4) $$x_i(t_0) = x_0$$

and with one of x_i being not optimal.

If the Hamiltonian system enjoys uniqueness of solutions, then

(5) $$p_1(t_0) \neq p_2(t_0)$$

Whenever (4) and (5) hold true for some solutions (x_i, p_i), $i = 1, 2$ of (3), we say that the system (3) has a **shock** at time t_0.

Shocks are the very reason why the value function is not smooth and why, in general, one should not expect smooth solutions to the Hamilton-Jacobi-Bellman equation **(HJB)**.

If we could guarantee that on some time interval $[t_0, T]$ there are no shocks, then the value function would be a continuously differentiable on $[t_0, T] \times \mathbf{R}^n$ solution of **(HJB)**. In the same time we would have the uniqueness of optimal trajectories and would derive the optimal feedback low $G : [t_0, T] \times \mathbf{R}^n \rightsquigarrow U$ by setting

$$G(t,x) = \{u \mid H(t,x,-\frac{\partial V}{\partial x}(t,x)) =$$

$$\langle -\frac{\partial V}{\partial x}(t,x), f(t,x,u)\rangle - L(t,x,u)\}$$

Then the closed loop control system

$$x' = f(t,x,u(t,x)), \quad u(t,x) \in G(t,x), \quad x(t_0) = x_0$$

would have exactly one solution which is optimal for the Bolza problem.

MATRIX RICCATI EQUATIONS AND SHOCKS

We relate the absence of shocks of the Hamilton-Jacobi-Bellman equation **(HJB)** with the existence of solutions to matrix Riccati equations

$$\begin{cases} P' + \dfrac{\partial^2 H}{\partial p \partial x}(t, x(t), p(t))P + P\dfrac{\partial^2 H}{\partial x \partial p}(t, x(t), p(t)) + \\ \\ +P\dfrac{\partial^2 H}{\partial p^2}(t, x(t), p(t))P + \dfrac{\partial^2 H}{\partial x^2}(t, x(t), p(t)) = 0 \\ \\ P(T) = -g''(x(T)) \end{cases}$$

Consider $H : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R}$ and a locally Lipschitz $\psi : \mathbf{R}^n \mapsto \mathbf{R}^n$. We assume that $H(t,\cdot,\cdot)$ is twice continuously differentiable and that for every $r > 0$, there exists $k_r \in L^1(0,T)$ satisfying

$$\frac{\partial H}{\partial(x,p)}(t,\cdot,\cdot) \text{ is } k_r(t) - \text{Lipschitz on } B_r(0)$$

We associate to these data the Hamiltonian system

$$(6)\quad \begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(t,x(t),p(t)), \ x(T) = x_T \\ \\ -p'(t) = \dfrac{\partial H}{\partial x}(t,x(t),p(t)), \ p(T) = \psi(x_T) \end{cases}$$

and assume that it is complete.

Define for every $t \in [0,T]$ the set

$$M_t = \{(x(t),p(t)) \mid (x,p) \text{ solves } (6), \ x_T \in \mathbf{R}^n\}$$

Theorem 4.7 *The following statements are equivalent:*

i) $\forall\, t,\ M_t$ *is the graph of a locally Lipschitz function from an open set* $\mathcal{D}(t)$ *into* $\mathbf{R}^n$

ii) $\forall\ (x,p)$ *solving (6) on* $[0,T]$ *and* $P_T \in \partial^\star\psi(x(T))$*, the matrix Riccati equation*

$$\begin{cases} P' + \dfrac{\partial^2 H}{\partial p \partial x}(t,x(t),p(t))P + P\dfrac{\partial^2 H}{\partial x \partial p}(t,x(t),p(t)) + \\ \\ +P\dfrac{\partial^2 H}{\partial p^2}(t,x(t),p(t))P + \dfrac{\partial^2 H}{\partial x^2}(t,x(t),p(t)) = 0 \\ \\ P(T) = P_T \end{cases}$$
(7)

has a solution on $[0,T]$.

Furthermore, if i) (or equivalently ii)) holds true, we have : if ψ *is differentiable, then*

M_t is the graph of a differentiable function

and if $\psi \in C^1$*, then*

M_t is the graph of a C^1-function

Corollary 4.8 *Under all assumptions of Theorem 4.7, suppose that for every (x, p) solving (6) on $[0, T]$ and $P_T \in \partial^\star \psi(x(T))$, the matrix Riccati equation (7) has a solution on $[0, T]$. Then the Hamiltonian system (6) has no shocks on $[0, T]$.*

Lemma 4.9 *Let $K \subset \mathbf{R}^n$ be a compact set. Consider a locally Lipschitz function $\psi : \mathbf{R}^n \mapsto \mathbf{R}^n$ and the subsets $M_t(K)$, $t \in [0, T]$ defined by*

$$M_t(K) = \{(x(t), p(t)) | (x, p) \text{ solves } (6),\ x_T \in K\}$$

Then there exists $\delta > 0$ such that for all $t \in [T - \delta, T]$, $M_t(K)$ is the graph of a Lipschitz function.

MATRIX RICCATI EQUATIONS

$$P' + A(t)^\star P + PA(t) + PE(t)P + D(t) = 0, P(T) = P_T$$

Theorem 4.10 *Let* $A, E_i, D_i : [0,T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$, $i = 1, 2$ *be integrable. Assume that* $E_i(t)$, $D_i(t)$ *are self-adjoint for almost every* t *and*

$$D_1(t) \leq D_2(t),\ E_1(t) \leq E_2(t) \text{ a.e. in } [0,T]$$

Consider self-adjoint operators $P_{iT} \in L(\mathbf{R}^n, \mathbf{R}^n)$ *such that*

$$P_{1T} \leq P_{2T}$$

and solutions $P_i(\cdot) : [t_0, T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ *to the matrix equations* $P_i(T) = P_{iT}$

$$P' + A(t)^\star P + PA(t) + PE_i(t)P + D_i(t) = 0$$

for $i = 1, 2$. *Then* $P_1 \leq P_2$ *on* $[t_0, T]$.

Theorem 4.11 *Under all assumptions of Theorem 4.10 assume that for almost every* $t \in [0,T]$, $E_1(t) \geq 0$ *Consider solutions* $P_i(\cdot) : [t_i, T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ *to the matrix equations* $P_i(T) = P_{iT}$,

$$P' + A(t)^\star P + PA(t) + PE_i(t)P + D_i(t) = 0$$

where $i = 1, 2$. *Then the solution* P_1 *is defined at least on* $[t_2, T]$ *and* $P_1 \leq P_2$.

EXISTENCE OF SOLUTIONS

Theorem 4.12 *Let* $A, E, D : [0, T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ *be integrable. We assume that* $E(t)$, $D(t)$ *are self-adjoint and* $E(t) \geq 0$ *for almost every* $t \in [0, T]$. *Consider a self-adjoint operator* $P_T \in L(\mathbf{R}^n, \mathbf{R}^n)$ *and assume that there exists an absolutely continuous* $P : [t_0, T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ *such that for every* $t \in [t_0, T]$, $P(t)$ *is self-adjoint and*

$$P' + A^{\star}P + PA + PEP + D \leq 0$$

a.e. in $[t_0, T]$ *and* $P_T \leq P(T)$. *Then the solution* $\overline{P}$ *to the equation* $P(T) = P_T$

$$(8) P' + A(t)^{\star}P + PA(t) + PE(t)P + D(t) = 0$$

is defined at least on $[t_0, T]$ *and* $\overline{P} \leq P$ *on* $[t_0, T]$.

Corollary 4.13 *Under all the assumptions of Theorem 4.12, consider a self-adjoint nonpositive* $P_T \in L(\mathbf{R}^n, \mathbf{R}^n)$. *If for almost all* $t \in [0, T]$, $D(t) \leq 0$, *then the solution* $\overline{P}$ *to the matrix Riccati equation (8) is well defined on* $[0, T]$ *and* $\overline{P} \leq 0$.

SMOOTHNESS OF THE VALUE FUNCTION

Differentiabililty of the value function is related to solutions of the Riccati Equation (7) in the following way.

Theorem 4.14 *Assume $H_1) - H_5)$, that V is locally Lipschitz and for every $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$ the problem (P) has an optimal solution.*

The following four statements are equivalent:

i) The value function V is continuously differentiable

ii) For every $t_0 \in [0, T]$, $V(t_0, \cdot) \in C^1$

iii) $\forall\ (t_0, x_0) \in [0, T] \times \mathbf{R}^n$ the optimal trajectory of (P) is unique

iv) For the Hamiltonian system (2) the set

$$M_t := \{(x(t), p(t)) \mid (x, p) \text{ solves (2) on } [t, T]\}$$

is the graph of a continuous function $\pi_t : \mathbf{R}^n \mapsto \mathbf{R}^n$.

Furthermore, iv) yields that $\pi_t(\cdot) = -\frac{\partial V}{\partial x}(t, \cdot)$ and every solution (x, p) of (2) restricted to $[t_0, T]$ satisfies: x is optimal for (P) with $x_0 = x(t_0)$ and $p(t) = -\frac{\partial V}{\partial x}(t, x(t))$ for all $t \in [0, T]$.

Corollary 4.15 *Under all assumptions of Theorem 4.14, suppose that U is a finite dimensional space, that for some $\overline{f} : [0,T] \times \mathbf{R}^n \mapsto \mathbf{R}^n$, $b : [0,T] \times \mathbf{R}^n \mapsto L(U, \mathbf{R}^n)$ we have*

$$\forall\ (t,x),\ \ f(t,x,u) = \overline{f}(t,x) + b(t,x)u$$

and $\frac{\partial L}{\partial u}(t,x,\cdot)$ is bijective. Then the (equivalent) statements $i) - iv)$ of Theorem 4.14 are equivalent to

v) For every $(t_0, x_0) \in [0,T] \times \mathbf{R}^n$ there exists a unique optimal control $\overline{u}(\cdot)$ solving the problem (P). Furthermore, if z denotes the corresponding optimal trajectory, then for all $t \in [t_0, T]$,

$$\overline{u}(t) = \left(\frac{\partial L}{\partial u}(t, z(t), \cdot)\right)^{-1} \left(-b(t, z(t))^{\star}\frac{\partial V}{\partial x}(t, z(t))\right)$$

Corollary 4.16 *Under all assumptions of Theorem 4.14, suppose that $\nabla g(\cdot)$ is locally Lipschitz, $H(t,\cdot,\cdot)$ is twice continuously differentiable and $\forall\, r > 0$, $\exists\, k_r \in L^1(0,T)$ such that*

$$\frac{\partial H}{\partial(x,p)}(t,\cdot,\cdot) \text{ is } k_r(t) - \text{Lipschitz on } B_r(0)$$

The following two statements are equivalent:

i) $\forall\, t \in [0,T]$, $\frac{\partial V}{\partial x}(t,\cdot)$ *is locally Lipschitz*

ii) $\forall\ (x,p)$ *solving (2) on* $[0,T]$ *and every* $P_T \in \partial^\star(\nabla g)(x(T))$, *the matrix Riccati equation*

$$\begin{cases} P' + \dfrac{\partial^2 H}{\partial p \partial x}(t,x(t),p(t))P + P\dfrac{\partial^2 H}{\partial x \partial p}(t,x(t),p(t)) + \\ \\ +P\dfrac{\partial^2 H}{\partial p^2}(t,x(t),p(t))P + \dfrac{\partial^2 H}{\partial x^2}(t,x(t),p(t)) = 0 \\ \\ P(T) = P_T \end{cases}$$

has a solution on $[0,T]$.

Furthermore, if i) (or equivalently ii)) holds true, then

$$\nabla g \text{ is differentiable} \Longrightarrow \frac{\partial V}{\partial x}(t, \cdot) \text{ is differentiable}$$

and for every (x, p) solving (2),

$$P(t) = -\frac{\partial^2 V}{\partial x^2}(t, x(t))$$

If moreover $g \in C^2$, then $V(t, \cdot) \in C^2$.

Proof — Let M_t be defined as in Theorem 4.14. If $i)$ holds true, then, by Theorem 4.14, M_t is the graph of a locally Lipschitz function π_t. By Theorem 4.3, $\pi_t(\cdot) = -\frac{\partial V}{\partial x}(t, \cdot)$. Applying Theorem 4.7, we deduce $ii)$. Conversely, assume that $ii)$ is verified. Thus, by Theorem 4.7, M_t is the graph of a locally Lipschitz function from an open set $\mathcal{D}(t) \subset \mathbf{R}^n$ into $\mathbf{R}^n$. By Theorem 4.3, $M_t = \text{Graph}(-\frac{\partial V}{\partial x}(t, \cdot))$. Hence $i)$.

The last statement follows from Theorem 4.7, because $P(t)$ describes the evolution of tangent space to M_t at $(x(t), p(t))$.

PROBLEMS WITH CONCAVE-CONVEX HAMILTONIANS

Observe that in general one has

$$\frac{\partial^2 H}{\partial p^2}(t, x(t), p(t)) \geq 0$$

for every solution (x, p) of the Hamiltonian system

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(t, x(t), p(t)), & x(T) = x_T \\ \\ -p'(t) = \dfrac{\partial H}{\partial x}(t, x(t), p(t)), & p(T) = -\nabla g(x_T) \end{cases} \tag{9}$$

and that whenever in addition $H(t, \cdot, p(t))$ is concave for all $t \in [0, T]$, then

$$\frac{\partial^2 H}{\partial x^2}(t, x(t), p(t)) \leq 0$$

If g is convex, then every matrix from the generalized Jacobian $\partial^\star g(x(T))$ is nonnegative.

By Corollary 4.13 for every $P_T \in \partial^\star(\nabla g)(x(T))$, the solution $P(\cdot)$ of the matrix Riccati equation

$$\begin{cases} P' + \dfrac{\partial^2 H}{\partial p \partial x}(t, x(t), p(t))P \;+\; P\,\dfrac{\partial^2 H}{\partial x \partial p}(t, x(t), p(t)) \\[2ex] +P\dfrac{\partial^2 H}{\partial p^2}(t, x(t), p(t))P + \dfrac{\partial^2 H}{\partial x^2}(t, x(t), p(t)) = 0 \\[2ex] P(T) = -P_T \end{cases}$$

(10)

exists on $[0, T]$.

By Theorem 4.7, no shocks of (9) may occur backward in time. From Theorem 4.14 and Corollary 4.16 we get

Theorem 4.17 *Assume* $H_1) - H_5)$, *that* V *is locally Lipschitz and for every* $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$ *the problem (P) has an optimal solution.*

Further assume that $\nabla g(\cdot)$ *is locally Lipschitz,* $H(t, \cdot, \cdot)$ *is twice continuously differentiable and for all* $r > 0$ *there exists* $k_r \in L^1(0, T)$ *such that*

$$\frac{\partial H}{\partial (x, p)}(t, \cdot, \cdot) \text{ is } k_r(t) - \text{Lipschitz on } B_r(0)$$

If for every solution (x, p) *of (9),* $H(t, \cdot, p(t))$ *is concave and* g *is convex, then* $V \in C^1$ *and* $\frac{\partial V}{\partial x}(t, \cdot)$ *is locally Lipschitz.*

Moreover, every solution (x, p) *of (9) is an optimal trajectory-co-state pair. If in addition* $g \in C^2$, *then* $V(t, \cdot) \in C^2$ *and, in this case,*

$$P(t) = -\frac{\partial^2 V}{\partial x^2}(t, x(t))$$

solves the matrix Riccati equation (10) with $P_T = -g''(x(T))$.