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Value Function in Optimal Control
Lecture 5

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HAMILTON-JACOBI-BELLMAN EQUATION FOR PROBLEMS UNDER STATE-CONSTRAINTS

Outline

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MAYER'S PROBLEM UNDER STATE CONSTRAINTS

Consider the optimal control problem

$$(P) \qquad \begin{cases} \text{Minimize } g(x(1)) \\ \text{over } \ x \in W^{1,1}([0,1];\mathbf{R}^n) \text{ satisfying} \\ x'(t) \in F(t,x(t)) \quad \text{a.e. } t \in [0,1], \\ x(t) \in K \quad \forall\, t \in [0,1], \\ x(0) = x_0, \end{cases}$$

the data for which comprise: a function $g : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$, a set-valued map $F : [0,1] \times \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$, a closed set $K \subset \mathbf{R}^n$ and $x_0 \in \mathbf{R}^n$.

Solutions of the above differential inclusion satisfying the constraints of (P), are called **feasible** arcs (for (P)).

Note that, since g is extended valued, (P) incorporates the endpoint constraint:

$$x(1) \in C$$

where $C := \operatorname{dom} g$.

The Hamiltonian H is defined by

$$H(t,x,p) = \sup_{v \in F(t,x)} < p, v >$$

Denote by $V : [0,1] \times K \to \mathbf{R} \cup \{+\infty\}$ the value function for (P):

for each $(t,x) \in [0,1] \times K$, $V(t,x)$ is defined to be the infimum cost for the problem

$$(P_{t,x}) \quad \begin{cases} \text{Minimize } g(y(1)) \\ \text{over } y \in W^{1,1}([t,1];\mathbf{R}^n) \text{ satisfying} \\ y'(s) \in F(s,y(s)) \quad \text{a.e. } s \in [t,1], \\ y(s) \in K \quad \forall\, s \in [t,1], \\ y(t) = x \end{cases}$$

Thus

$$V(t,x) = \inf(P_{t,x}).$$

(If $(\mathrm{P}_{t,x})$ has no feasible arcs, $V(t,x) = +\infty$.)

The Hamilton-Jacobi-Bellman Equation **(HJB)** in the constrained case is :

$$\begin{cases} -\frac{\partial V}{\partial t} + H(t,x,-\frac{\partial V}{\partial x}) = 0, \; (t,x) \in]0,1[\times \mathrm{Int}K \\ \\ V(1,x) = g(x) \text{ for } x \in K \end{cases}$$

To get uniqueness of solutions to the above PDE in the constrained case we are led to impose some kind of constraint qualification on the dynamic constraint at boundary points of the state constraint set.

We restrict attention to a special class of state constraints sets, namely a finite intersection of smooth manifolds. It is assumed that the state constraint set K is expressible as

$$K = \cap_{j=1}^{r} \{x : h_j(x) \leq 0\}$$

for a finite family of $C^{1,1}$ functions

$$\{h_j : \mathbf{R}^n \to \mathbf{R}\}_{j=1}^{r}$$

($C^{1,1}$ denotes the class of C^1 functions with locally Lipschitz continuous gradients.)

The "active set" of index values $I(x)$, at a point $x \in \operatorname{bdy} K$, is

$$I(x) := \{j \in (1, \ldots, r) : h_j(x) = 0\}.$$

Recall the notations $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for all real numbers a, b. We write

$$h^+(x) := \left(\max_{j=1,2,\ldots,r} h_j(x) \right) \vee 0.$$

$W^{1,1}([a, b]; \mathbf{R}^n)$ denotes the space of absolutely continuous n-vector valued functions on $[a, b]$, with norm

$$||x||_{W^{1,1}} = \|x(a)\| + \int_a^b \|x'(t)\| dt.$$

NEIGHBOURING FEASIBLE TRAJECTORIES THEOREM

Theorem 6.1 *Fix $r_0 > 0$. Assume that for some $c > 0$, $\alpha > 0$ and $k(\cdot) \in L^1$:*

(i) F has nonempty closed images and $F(\cdot, x)$ is measurable for all x

(ii) $F(t,x) \subset c(1+\|x\|)B \quad \forall\, (t,x) \in [0,1] \times \mathbf{R}^n$

(iii) $F(t,\cdot)$ is $k(t)-$Lipschitz for a.e. $t \in [0,1]$

Assume furthermore $\exists\, \alpha > 0$ such that

$$CQ' \quad \min_{v \in F(t,x)} \max_{j \in I(x)} \nabla h_j(x) \cdot v < -\alpha \quad , \quad \forall\, x \in B(0, e^c(r_0 + c)) \cap \mathrm{bdy}\, K,\ t \in [0,1].$$

Then there exists a constant ϑ (which depends on r_0, c, α and $k \in L^1$) with the following property: given any $t_0 \in [0,1]$ and any $\hat{x} \in \mathcal{S}_{[t_0,1]}$ such that $\hat{x}(t_0) \in B(0, r_0) \cap K$, an $x \in \mathcal{S}_{[t_0,1]}(\hat{x}(t_0))$ can be found such that

$$x(t) \in K \quad \forall\, t \in [t_0, 1]$$

and

$$\|x - \hat{x}\|_{W^{1,1}([t_0,1];\mathbf{R}^n)} \le \vartheta \max_{t \in [t_0,1]} h^+(\hat{x}(t)).$$

In the case F is continuous, condition (CQ) $'$ is implied by the condition

$$\min_{v \in F(t,x)} \max_{j \in I(x)} \nabla h_j(x) \cdot v < 0$$

$$\forall\, x \in B(0, e^c(r_0 + c)) \cap \operatorname{bdy} K,\ t \in [0,1].$$

CONSTRAINED HAMILTON-JACOBI-BELLMAN EQUATION

To investigate uniqueness of solutions in the constrained case we assume :

(H1) F is a continuous set-valued map, with non-empty, closed, convex images

(H2) There exists $c > 0$ such that

$F(t,x) \subset c(1+\|x\|)B \quad \forall\,(t,x) \in [0,1] \times \mathbf{R}^n$

(H3) $\exists\; k \in L^1$ such that $F(t,\cdot)$ is $k(t)-$Lipschitz for almost all $t \in [0,1]$

(H4) g is lower semicontinuous.

(CQ) $\forall x \in K$ and $t \in [0,1]$ there exists $v \in F(t,x)$:

$$\forall j \in I(x), \quad \nabla h_j(x) \cdot v > 0.$$

Theorem 6.2 *Let* $V : [0,1]\times K \to \mathbf{R}\cup\{+\infty\}$.
The assertions (a)-(c) below are equivalent:

(a) V *is the value function for (P).*

(b) V *is lower semicontinuous and*

(i) $\forall\ (t,x) \in ([0,1[\times K) \cap \mathrm{dom}\, V$

$$\inf_{v\in F(t,x)} D_{\uparrow}V(t,x)(1,v) \ \leq\ 0$$

(ii) $\forall\ (t,x) \in]0,1] \times \mathrm{int}\, K) \cap \mathrm{dom}\, V$

$$\sup_{v\in F(t,x)} D_{\uparrow}V(t,x)(-1,-v) \ \leq\ 0$$

(iii) $\forall\ x \in K,\ V(1,x) = g(x)$

$$\liminf_{\{(t',x')\to(1,x):t'<1,\ x'\in\mathrm{int}\, K\}} V(t',x') = V(1,x)$$

(c) V *is lower semicontinuous and*

(i) $\forall\ (t,x) \in (]0,1[\times\mathrm{int}\, K) \cap \mathrm{dom}\, V$,
$\forall\ (p_t,p_x) \in \partial_{-}V(t,x),\quad -p_t+H(t,x,-p_x) = 0$

(ii) $\forall\ (t,x) \in (]0,1[\times\mathrm{bdy}\, K) \cap \mathrm{dom}\, V$,
$\forall\ (p_t,p_x) \in \partial_{-}V(t,x),\quad -p_t+H(t,x,-p_x) \geq 0$

(iii) $\liminf_{\{(t',x')\to(0,x):t'>0\}} V(t',x') = V(0,x)$,
$\liminf_{\{(t',x')\to(1,x):t'<1,\ x'\in\mathrm{int}\, K\}} V(t',x') =$
$V(1,x) = g(x)$ *for all* $x \in K$.

Example Consider the constrained problem

$$\begin{cases} \text{Minimize } g(x(1)) \\ x'(t) \in F(t, x(t)) \\ x(t) \in K \\ x(0) = x_0, \end{cases}$$

in which $n = 1$, $g(x) = x$, $F(t, x) = \{1\}$, $K = \{x : x \leq 0\}$, $x_0 = 0$.

By inspection

$$V(t, x) = \begin{cases} +\infty & \text{if } x > -(1-t) \\ x + (1-t) & \text{if } x \leq -(1-t) \end{cases}$$

The hypotheses for application of Theorem 6.2 are satisfied, including the outward-pointing constraint qualification (CQ). Theorem 6.2 therefore tells us that V is the unique solution of (HJB) (in the sense specified).

Notice that $V(t, x) = +\infty$ at some points in $[0, 1] \times K$, despite the fact that g is everywhere finite valued (no endpoint constraints).

Lemma 6.3

(i) Take any point $x_1 \in K$*. Then there exists* $\delta \in]0,1[$ *and a solution* $y : [1-\delta, 1] \to \mathbf{R}^n$ *such that* $y(1) = x_1$ *and*

$$y(t) \in \operatorname{int} K \quad \forall\, t \in [1-\delta, 1[.$$

(ii) Take any $t_0 \in [0,1[$ *and any solution* $x : [t_0, 1] \to K$*. Take also a sequence of points* $\{(\tau_i, \xi_i)\}$ *in* $[t_0, 1[\times \operatorname{int} K$ *such that* $(\tau_i, \xi_i) \to (1, x(1))$*. Then there exists a sequence of solutions* $\{x_i : [t_0, \tau_i] \to \mathbf{R}^n\}$ *such that* $x_i(\tau_i) = \xi_i$

$$x_i(t) \in \operatorname{int} K \quad \forall\, t \in [t_0, \tau_i], \quad i = 1, 2, \ldots$$

and

$$||x_i - x||_{L^\infty([t_0,\tau_i];\mathbf{R}^n)} \to 0 \text{ as } i \to \infty.$$

Proof. According to (CQ), there exists $v \in F(1, x_1)$ and $\alpha > 0$ such that

$$\nabla h_j(x_1) \cdot v > \alpha \quad \forall\, j \in I(x_1).$$

For some $\delta \in]0, 1-t_0]$, whose magnitude will be set presently, define

$$z(t) = x_1 - (1-t)v \quad \text{for } t \in [1-\delta, 1].$$

By Filippov's Theorem, there exists a solution $x : [1-\delta, 1] \to \mathbf{R}^n$ such that $x(1) = x_1$ and

$$\|x(t) - z(t)\| \leq \exp\{\int_0^1 k(t)dt\} \int_t^1 d_{F(s,z(s))}(v)ds$$

for all $t \in [1-\delta, 1]$. We deduce from the continuity of $(t, x) \rightsquigarrow F(t, x)$ and the continuous differentiability of the h_j's that there exists a function $\eta : R^+ \to R^+$ such that $\eta(\theta) \downarrow 0$ as $\theta \downarrow 0$,

$$\|x(1-s) - (x_1 - sv)\| \leq \eta(s)s \quad \text{for } s \in [0, \delta]$$

and

$$h_j(x(1-s)) \leq h_j(x_1) + \nabla h_j(x_1)(x(1-s) - x_1) + \eta(s)s$$

for all $s \in [0, \delta]$. But then, since $h_j(x_1) = 0$ for all $j \in I(x_1)$, there exists M (M does not depend on s) such that

$$h_j(x(1-s)) \leq -s\nabla h_j(x_1) \cdot v + M\eta(s)s$$

for all $j \in I(x_1)$. Hence $\forall\ s \in [0, \delta[,\ j \in I(x_1)$

$$\frac{1}{s} h_j(x(1-s)) \leq -\alpha + M\eta(s)$$

It follows that, if we now choose δ such that $M\eta(\delta) < \alpha$, then $h_j(x(t)) < 0$ for all $j \in I(x_1)$. Since $h_j(x_1) < 0$ for all $j \notin I(x_1)$, we can arrange, by a further reduction in the size of δ, that

$$\max_{j\in\{1,\ldots,r\}} h_j(x(t)) < 0 \quad \forall\, t \in [1-\delta, 1[.$$

(ii) Define the sequence of positive numbers

$$\gamma_i := \left(-\max_{j=1,\ldots,r} h_j(\xi_i)\right) \wedge (i^{-1}) \quad \text{for } i = 1, 2, \ldots$$

Since $\{\xi_i\} \subset \operatorname{int} K$ and (CQ) holds true, it follows that $\gamma_i > 0$ for all i. Clearly $\gamma_i \downarrow 0$. For each i define

$$h^i_j(x) := h_j(x) + \gamma_i.$$

Apply the time dependent version of Filippov's Theorem to $x' \in F(t, x)$, taking as reference trajectory x restricted to $[t_0, \tau_i]$. This yields a solution $y_i : [t_0, \tau_i] \to \mathbf{R}^n$ satisfying $y_i(\tau_i) = \xi_i$ and

$$||y_i - x||_{L^\infty([t_0,\tau_i];\mathbf{R}^n)} \leq \exp\{\textstyle\int_0^1 k(t)dt\}||x(\tau_i) - \xi_i||$$

Since $(x(\tau_i) - \xi) \to 0$ as $i \to \infty$, we conclude that

$$(1) \quad ||y_i - x||_{L^\infty([t_0,\tau_i])} \to 0 \quad \text{as } i \to \infty.$$

By the comments following the statement of Theorem 6.1, (CQ) yields (CQ)$'$. So we deduce from Theorem 6.1 applied to the set-valued map $-F$ that there exists $\vartheta > 0$ and a sequence of solutions $\{x_i : [t_0, \tau_i] \to \mathbf{R}^n\}$ such that $x_i(\tau_i) = \xi_i$ and for , $i = 1, 2, \ldots$

$$||y_i - x_i||_{L^\infty([t_0,\tau_i];\mathbf{R}^n)} \leq$$

$$\vartheta \left[\max_{t \in [t_0,\tau_i]} \max_j h_j(y_i(t)) + \gamma_i \right]^+$$

$$h_j(x_i(t)) + \gamma_i \leq 0 \quad \forall\, t \in [t_0, \tau_i],\, j \in I(x_i(t))$$

This means that

$$x_i(t) \in \operatorname{int} K \quad \forall\, t \in [t_0, \tau_i],\ i = 1, 2, \ldots$$

Since $h_j(x(t)) \leq 0$ for all $t \in [0, 1]$, we deduce from (1) that

$$||x_i - x||_{L^\infty([t_0,\tau_i])} \to 0 \quad \text{as } i \to \infty.$$

In the next lemma, reference is made to the δ-tube about $\bar{x} : [t_0, t_1] \to \mathbf{R}^n$:

$$T_\delta(\bar{x}) := \{(t, x) \in [t_0, t_1] \times \mathbf{R}^n : \|x - \bar{x}(t)\| < \delta\}$$

Lemma 6.4 *Take* $[t_0, t_1] \subset [0, 1]$, *a solution* $\bar{x} : [t_0, t_1] \to \mathbf{R}^n$, $\delta > 0$ *and a lower semicontinuous function* $V : [t_0, t_1] \times \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *such that* $\forall\, (t, x) \in T_\delta(\bar{x})$ *with* $t < t_1$

$$\forall\, (p_t, p_x) \in \partial_- V(t, x), \;\; -p_t + H(t, x, -p_x) \leq 0$$

Then, for any $t_0 \leq t' \leq t'' < t_1$,

$$V(t', \bar{x}(t')) \leq V(t'', \bar{x}(t''))$$

Proof. We deduce in the same way as for the unconstrained case, that

$$V(t', \bar{x}(t')) \leq V(t'', \bar{x}(t''))$$

The fact that $t'' < t_1$ (strict inequality) is important here, since no regularity hypotheses have been imposed on $t \to V(t, \cdot)$ at $t = t_1$.

Proof of the uniqueness theorem 6.2

(a) $\Rightarrow$ (b). The value function V is lower semicontinuous by the same arguments as before.

Under the hypotheses, $(t, x) \in \operatorname{dom} V$ implies that $(\mathrm{P}_{t,x})$ has a solution. It is a straightforward matter to show that, if y is a minimizer for $(\mathrm{P}_{t,x})$, then $s \to V(s, y(s))$ is constant on $[t, 1]$; b(i) can be deduced from this property.

It can also be shown that, if $y : [t, 1] \to \mathbf{R}^n$ is a solution satisfying the constraints of $(\mathrm{P}_{t,x})$, then $s \to V(s, y(s))$ is non-decreasing on $[t, 1]$; b(ii) can be deduced from this latter property.

Since V is lower semicontinuous, it remains only to verify that for all $x \in K$

$$\liminf_{\{(t',x') \to (1,x) : t'<1, x' \in \operatorname{int} K\}} V(t', x') \leq V(1, x)$$

Lemma 6.3 tells us that there exists $\delta \in]0, 1[$ and a solution $y : [1 - \delta, 1] \to \mathbf{R}^n$ such that $y(1) = x$ and

$$y(t) \in \operatorname{int} K \quad \forall\, t \in [1 - \delta, 1[$$

But $V(t, y(t)) \leq V(1, x)$, a basic monotonicity property of the value function. Since y is continuous,

$$\liminf_{\{(t',x')\to(1,x):t'<1,\ x'\in\operatorname{int}K\}} V(t',x') \leq$$

$$\limsup_{t\uparrow 1} V(t,y(t)) \leq V(1,x).$$

as required.

(b) $\Rightarrow$ (c). This implication is a consequence duality relationships between $\partial_- V$ and $D_\uparrow V$.

(c) $\Rightarrow$ (a). Assume that V satisfies (c). Take any $x_0 \in K$ and $t_0 \in [0,1]$.

Step 1: We show that

$$V(t_0,x_0) \ \geq \ \inf(P_{t_0,x_0}). \tag{2}$$

This inequality holds true if $V(t_0,x_0) = +\infty$. So we assume that $V(t_0,x_0) < +\infty$.

Notice that, since $\operatorname{dom} V \subset K$, conditions c(i) and c(ii) imply

$$\forall\ (t,x) \in]0,1[\times K,\ \forall\ (p_t,p_x) \in \partial_- V(t,x)$$

$$-p_t + H(t,x,-p_x) \leq 0$$

and

$$\liminf_{\{(t',x')\to(0,x):t'>0\}} V(t',x') = V(0,x) \quad \forall\ x \in \mathbf{R}^n$$

(We here regard V as a function on $[0,1] \times \mathbf{R}^n$ which takes value $+\infty$ at points $(t,x) \notin [0,1] \times K$.) But then we deduce by applying the same arguments as in the unconstrained case the existence of a solution $x : [t_0, 1] \to \mathbf{R}^n$ such that $x(t_0) = x_0$ and

$$V(t_0, x_0) \geq V(t, x(t)) \quad \forall\, t \in [t_0, 1].$$

This inequality implies that $V(t, x(t)) < +\infty$ for all $t \in [t_0, 1]$. Since $\operatorname{dom} V \subset K$, we conclude that $x(\cdot)$ satisfies the state constraint. It also implies that

$$V(t_0, x_0) \geq V(1, x(1)) = g(x(1)) \geq \inf(\mathrm{P}_{t,x}).$$

This is the required inequality.

Step 2: We show that

$$(3) \qquad V(t_0, x_0) \ \leq \ \inf(P_{t_0,x_0}).$$

This will complete the proof, since (3) combines with (2) to give $V(t_0, x_0) = \inf(P_{t_0,x_0})$.

Inequality (3) is automatically satisfied if $\inf(P_{t_0,x_0}) = +\infty$. So we assume that it is finite. In this case, $\inf(P_{t_0,x_0})$ is the infimum of $g(x(1))$ over all feasible arcs of (P_{t_0,x_0}). It therefore suffices to show that

$$V(t_0, x_0) \leq g(\bar{x}(1)),$$

where $\bar{x} \in W^{1,1}([t_0, 1]; \mathbf{R}^n)$ is an arbitrary feasible arc of (P_{t_0,x_0}).

By hypothesis,

$$g(\bar{x}(1)) = \liminf_{\{(\tau,\xi)\to(1,\bar{x}(1)):\tau<1,\xi\in\operatorname{int} K\}} V(\tau, \xi)$$

There exists, therefore, a sequence $\{(\tau_i, \xi_i)\}$ in $[t_0, 1) \times \operatorname{int} K$ such that $\xi_i \to \bar{x}(1)$ and

$$V(\tau_i, \xi_i) \to g(\bar{x}(1)). \tag{4}$$

Lemma 6.3(ii) asserts the existence of a sequence of solutions $x_i : [t_0, \tau_i] \to \mathbf{R}^n$ such that $x_i(\tau_i) = \xi_i$,

$$x_i(t) \in \operatorname{int} K \quad \forall\, t \in [t_0, \tau_i]$$

and

$$||x_i - \bar{x}||_{L^\infty([t_0,\tau_i];\mathbf{R}^n)} \to 0 \quad \text{as } i \to \infty. \tag{5}$$

Filippov's Theorem tells us that x_i can be extended to all of $[t_0, 1]$ (we write the extension also x_i) as a solution to our differential inclusion. Choose $\sigma_i \in]\tau_i, 1[$ and $\epsilon_i > 0$ such that

$$x_i(t) + \epsilon_i B \subset \operatorname{int} K \quad \forall\, t \in [t_0, \sigma_i].$$

Now apply Lemma 6.4 with $\sigma_i = t_1$ and $\bar{x} = x_i$ to conclude that

$$V(t_0, x_i(t_0)) \leq V(\tau_i, \xi_i).$$

It follows from (4), (5) and the lower semicontinuity of V that

$$V(t_0, x_0) = V(t_0, \bar{x}(t_0)) \leq \liminf_i V(t_0, x_i(t_0))$$
$$\leq \lim_i V(\tau_i, \xi_i) = g(\bar{x}(1))$$

as required.

EXERCISES.

In all exercises we impose all the assumptions of Theorem 6.1.

1. Assume that g is continuous on K. Show that the value function of the problem

$$\begin{cases} \text{Minimize } g(y(1)) \\ \text{over } y \in W^{1,1}([0,1]; \mathbf{R}^n) \text{ satisfying} \\ y'(s) \in F(s, y(s)) \quad \text{a.e. } s \in [0,1], \\ y(s) \in K \quad \forall\, s \in [0,1], \\ y(0) = x \end{cases}$$

coincides with the value function of the relaxed problem

$$\begin{cases} \text{Minimize } g(y(1)) \\ \text{over } y \in W^{1,1}([0,1]; \mathbf{R}^n) \text{ satisfying} \\ y'(s) \in \overline{co}F(s, y(s)) \quad \text{a.e. } s \in [0,1], \\ y(s) \in K \quad \forall\, s \in [0,1], \\ y(0) = x \end{cases}$$

and that V is continuous on $[0,1] \times K$.

2. Assuming that g is locally Lipschitz on K, show that in this case V is locally Lipschitz on $[0,1] \times K$.

3. Show that if g is continuous on K, then the value function V satisfies the following properties

(i) $\forall\ (t, x) \in (]0,1[\times \operatorname{int} K) \cap \operatorname{dom} V$, $\forall\ (p_t, p_x) \in \partial_+ V(t, x)$

$$-p_t + H(t, x, -p_x) \leq 0.$$

(ii) $\forall\ (t, x) \in (]0,1[\times K) \cap \operatorname{dom} V$, $\forall\ (p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) \geq 0$$

4. Show that if W is continuous on $[0,1] \times K$, satisfies the boundary condition $W(1, \cdot) = g$ and the above properties (i), (ii), then W is the value function.

5. State and prove a relaxation theorem under state constraints.