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On the stabilization of some nonlinear control systems: results, tools, and applications

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On the stabilization of some nonlinear control systems: results, tools, and applications

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Abstract

It has been proved by Brockett that, contrary to the case of linear control systems, many controllable nonlinear control systems cannot be stabilized by means of stationary continuous feedback laws. In this paper we give results showing that many controllable nonlinear control systems can be stabilized by means of time-varying continuous feedback laws and that many controllable and observable nonlinear control systems can be stabilized by means of time-varying dynamic continuous feedback laws. We show the interest of time-varying feedback laws for robustness with respect to measurement disturbances. We also present methods to design stabilizing feedback laws and we give applications to satellites and fluid mechanics.

Contents

1 Introduction

A control system is controllable if, for any given states x_0 and x_1, there exists an *open* loop control $t \in [0,T] \to u_{x_0,x_1}(t)$ which, when applied to the control system, allows to go from x_0 to x_1. One does not know any interesting necessary and sufficient condition for controllability, even when x_0 and x_1 are close together and the control system is analytic. But one knows powerful necessary conditions and powerful sufficient conditions. In section 2.2 we recall two well-known conditions.

Unfortunately, open loop controls are usually very sensitive to disturbances. So in many practical situations one prefers closed loop control, i.e. controls which do not depend on the initial x_0 but on the state x which (asymptotically) stabilize the point one wants to reach. Usually such closed loop controls (or feedback laws) have the advantage to be more robust to disturbances.

It is a classical result, see e.g. [123] Theorem 13, p. 186, that any linear control system which is controllable can be asymptotically stabilized by means of continuous feedback laws. A natural question is whether this result still holds for nonlinear control systems. In 1979 Sussmann showed that the global version of this result does not hold for nonlinear control systems: in [125] he has given an example of a nonlinear analytic control system which is globally controllable but cannot be globally asymptotically stabilized by means of continuous feedback laws. In 1983 Brockett has shown that the local version also does not hold even for analytic control systems: in [9] he has given a necessary condition (Theorem 2.16 below) for local asymptotic stabilizability by means of continuous feedback laws which is not implied by local controllability even for analytic control systems; for example, as pointed in [9], the analytic control system

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = x_1 u_2 - x_2 u_1, \tag{1.1}$$

where the state is $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the control $u = (u_1, u_2) \in \mathbb{R}^2$, is locally (and even globally) controllable but does not satisfy the Brockett necessary condition (and therefore cannot be asymptotic stabilized by means of continuous feedback laws). To get around the problem of impossibility to stabilize many controllable systems by means of continuous feedback laws two main strategies have been proposed:

(i) Asymptotic stabilization by means of discontinuous feedback laws,

(ii) Asymptotic stabilization by means of continuous time-varying feedback laws.

In this paper we shall consider mainly continuous time-varying feedback laws. Let us just briefly describe some results on discontinuous feedback laws. The pioneer work on this type of control is [125] by H. Sussmann. It is proved in [125] that any controllable analytic system can be asymptotically stabilized by means of piecewise analytic feedback laws. One of the key questions for discontinuous feedback laws is what is the relevant definition of a solution of the closed loop system. In [125], this question is solved by specifying an "exit rule" on the singular set. However, it is not completely clear how to implement this exit rule (but see Remark 1.1 below for this problem), which is important in order to analyze the robustness. If, following Hermes [60] (see also [38, 89]), one considers that the solutions of the closed loop systems are the solutions in the sense of Filippov [48], then it is proved in [38] that a control system which can be stabilized by means of a discontinuous feedback law can be stabilized by means of continuous periodic time-varying feedback laws and, moreover, if the system is affine in the control, it can be stabilized by means of continuous feedback laws. In particular the control system (1.1) cannot be stabilized by means of discontinous feedback laws if one considers Filippov solutions of the closed loop system –see also [111]. Another interesting poosibility is to consider "Euler" solutions; see [16] for a definition. This is a quite natural notion for control systems since it corresponds to the idea that one uses during small intervals of time the same control. With this type of solution, Clarke, Ledyaev, Sontag and Subbotin have obtained a very strong result. They have proved in [16] that controllability (or even assymptotic controllability) implies the existence of stabilizing discontinuous feedback laws. Their feedback laws are robust to (small) actuator disturbance. But, using a result due to Clarke, Ledyaev and Stern [17], Ledyaev and Sontag have proved in [89] that these feedback laws are in general (e.g. for the control system (1.1)) not robust to arbitrarily small measurement disturbances. In [88] Ledyaev and Sontag have introduced a new class of "dynamic and hybrid" discontinuous feedback laws and have shown that controllability (or even asymptotic controllability) implies the existence of stabilizing discontinuous feedback laws in this class which are robust to (small) actuators and measurement disturbances.

Remark 1.1 It would be interesting to know if one can in some sense "implement" (a good enough approximation of) Sussmann's exit rule (see [125]) by means of Sontag-Ledyaev's "dynamic-hybrid" strategy.

For *continuous* time-varying feedback laws, let us first mention that, due to an inverse of Lyapunov's second theorem proved by Kurzweil in [87] (see also [17]), periodic time-varying feedback laws are robust to (small) actuator and measurement disturbances. From now on, all the feedback laws considered are continuous. The pioneer works concerning time-varying feedback laws are due to Sontag-Sussmann [124] and Samson [112]. In [124], it is proved that, if the dimension of the state is 1, controllability implies asymptotic stabilizability by means of time-varying feedback laws. In [112], it is proved that the control system (1.1) can be asymptotically stabilized by means of time-varying feedback laws. In Sections 2.4 and 2.5, we present results showing that, in many cases, (local) controllability implies stabilizability by means of time-varying static feedback laws.

In many practical situations only part of the state called the output– is measured and therefore state feedback cannot be implemented; only output feedback is allowed. It is well-known, see e.g. [123, Theorem 32, p. 324], that any linear control system which is controllable and observable can be asymptotically stabilized by means of dynamic feedback laws. Again it is natural to ask if this result can be extended to the nonlinear case. In the nonlinear case, there are many possible definitions for observability. The weakest requirement for observability is that, given two different states, there exists a control $t \to u(t)$ which leads to two outputs which are not identical. With this definition of observability, the nonlinear control system

$$\dot{x} = u \in \mathbb{R}, y = x^2 \in \mathbb{R}, \tag{1.2}$$

where the state is x, the control u, and the output y, is observable. This system is also clearly controllable and asymptotically stabilizable by means of (stationary) static feedback laws (e.g. $u(x) = -x$). But, see [24], this system cannot be asymptotically stabilized by means of stationary dynamic feedback laws. Again, the introduction of time-varying feedback laws improves the situation; indeed the control system (1.2) can be asymptotically stabilized by means of time-varying dynamic feedback laws. In section 2.6 we present a result contained in [24] showing that many locally controllable and observable nonlinear control systems can be locally asymptotically stabilized by means of time-varying output feedback laws. In section 2.7 we show the interest of time-varying feedback for robustness with respect to measurement disturbances.

Let us also mention that the usefulness of time-varying controls for different goals has been pointed out by many authors. For example by

- V. Polotski [103] for observers to avoid peaking;
- S.H. Wang [133] for decentralized linear systems;
- Aeyels and Willems [1] for the pole assignment problem for linear time-invariant systems;
- Khargonekar et al. [81], Ho-Mock-Qai and Dayawansa [67, 68] for simultaneous stabilization of a family of control systems.

See also the references in these papers.

In chapter 3 we present some tools (namely, control Lyapunov function, damping, homogeneity, averaging and backstepping) to design asymptotically stabilizing feedback laws and present applications to the control of the attitude of a rigid space spacecraft with control torques provided by two thruster jets and to satellite transfer by means of electric propulsion.

In chapter 4, we show how the methods of chapters 2 and 3 can be applied to the control of some nonlinear partial differential equations. We present two applications:

1. Stabilization of a rotating body-beam without damping;
2. Controllability and stabilization of incompressible fluids.

2 Time-varying feedback laws

2.1 Notation and definitions

Throughout this paper, we denote by (C) the nonlinear control system

$$(C) \qquad \dot{x} = f(x, u),$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control. We assume that

$$f(0, 0) = 0 \tag{2.1}$$

and that, unless otherwise specified, $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$.

Let us first recall the definition of asymptotically stable for a time-varying dynamic system -we should in fact say uniformly asymptotically stable.

Definition 2.1 Let X be in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$. One says that 0 is *locally asymptotically stable* for $\dot{x} = X(x,t)$ if

(i) for all $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $\tau \in \mathbb{R}$ and for all $t \geq \tau$,

$$(\dot{x} = X(x,t), |x(\tau)| < \eta) \Rightarrow |x(t)| < \varepsilon \tag{2.2}$$

(ii) there exists $\delta > 0$ such that, for all $\varepsilon > 0$, there exists $M > 0$ such that, for all s in $\mathbb{R}$,

$$\dot{x} = X(x,t) \text{ and } |x(s)| < \delta \tag{2.3}$$

imply

$$|x(\tau)| < \varepsilon, \forall \tau > s + M. \tag{2.4}$$

If, moreover, for all $\delta > 0$, there exists $M > 0$ such that (2.3) implies (2.4) for all s in $\mathbb{R}$, one says that 0 is *globally asymptotically stable* for $\dot{x} = X(x,t)$.

Throughout this paper, and in particular in (2.2) and (2.3), we denote by $\dot{x} = X(x,t)$ any *maximal* solution of this differential equation. Let us emphasize that, since the vector field X is only continuous, the Cauchy problem $\dot{x} = X(x,t), x(t_0) = x_0$, where t_0 and x_0 are given, may have many maximal solutions. Let us recall that Kurzweil in [87] has shown that, even for vector fields which are only continuous, asymptotic stability is equivalent to the existence of a Lyapunov function of class C^∞; see also [17].

Let us now define "asymptotically stabilizable by means of a stationary feedback law" and "asymptotically stabilizable by means of a time-varying feedback law".

Definition 2.2 The control system (C) is *locally* (resp. *globally*) *asymptotically stabilizable by means of a stationary feedback law* if there exists $u \in C^0(\mathbb{R}^n; \mathbb{R}^m)$ satisfying

$$u(0) = 0,$$

such that, for the system $\dot{x} = f(x, u(x))$, 0 is a locally (resp. globally) asymptotically stable point.

Definition 2.3 The control system (C) is *locally* (resp. *globally*) *asymptotically stabilizable by means of a time-varying feedback law* if there exists $u \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ satisfying

$$u(0,t) = 0, \quad \forall t \in \mathbb{R},$$

such that, for the system $\dot{x} = f(x, u(x,t))$, 0 is a locally (resp. globally) asymptotically stable point.

2.2 Small time local controllability

Let us first give the definition we use in these notes for small time locally controllable –we should in fact say small time locally controllable with small controls.

Definition 2.4 The control system (C) is *small time locally controllable* if, for all real numbers $\varepsilon > 0$, there exists a real number $\eta > 0$ such that, for all $x_0 \in B_\eta := \{x \in \mathbb{R}^n; |x| < \eta\}$, there exists a measurable function $u : [0, \varepsilon] \to \mathbb{R}^m$ such that

$$|u(t)| < \varepsilon, \forall t \in [0, T],$$
$$(\dot{x} = f(x, u(t)), x(0) = x_0) \Rightarrow (x(\varepsilon) = 0).$$

One does not know any interesting necessary and sufficient condition for small time local controllability, even for analytic control systems. But one knows powerful necessary conditions and powerful sufficient conditions. Let us recall two well-known conditions.

In order to give these conditions, let us give some new definitions.

Definition 2.5 ([22]) The *strong jet accessibility subspace* of (C) at $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is the subspace of $\mathbb{R}^n$, denoted by $a(\bar{x}, \bar{u})$, spanned by

$$\{g(\bar{x}); g \in \{\partial^{|\alpha|} f / \partial u^\alpha(\cdot, \bar{u}), \alpha \in \mathcal{N}^m, |\alpha| \geq 1\} \cup \mathrm{Br}_2(f, \bar{u}) \tag{2.5}$$

where $\mathrm{Br}_2(f, \bar{u})$ denotes the set of iterated Lie brackets of length at least 2 of vector fields in $\{\partial^{|\alpha|} f / \partial u^\alpha(\cdot, \bar{u}); \alpha \in \mathcal{N}^m\}$.

Remark 2.6 One easily checks that the usual strong accessibility subspace of (C) at $\bar{x}$ (i.e. the space denoted $\mathcal{F}_0 D(\bar{x})$ in [129, p. 109]) contains $a(\bar{x}, \bar{u})$ for all $\bar{u}$ in $\mathbb{R}^m$ and that, if f is analytic with respect to x and u or is a polynomial with respect to u, these inclusions are all equalities.

Our last definition before giving a necessary condition for small time local controllability is

Definition 2.7 The control system (C) satisfies the *strong Lie algebra rank condition* at $(\bar{x}, \bar{u})$ if

$$a(\bar{x}, \bar{u}) = \mathbb{R}^n. \tag{2.6}$$

Remark 2.8 It follows from Remark 2.6 that, if (C) satisfies the strong Lie algebra rank condition at $(x, \bar{u})$ (2.6), then it satisfies the usual strong accessibility rank condition at $\bar{x}$ (i.e. dim $\mathcal{F}_0 D(\bar{x}) = n$ with the notation of [129]) and the converse holds if f is analytic with respect to x and u or is a polynomial with respect to u.

With these definitions one has the following well-known necessary condition for small time local controllability of analytic control system due to Sussmann-Jurdjevic [129].

Theorem 2.9 *Assume that the control system (C) is locally controllable and that f is analytic. Then the control system (C) satisfies the strong Lie algebra rank condition at* $(0, 0)$.

This necessary condition is sufficient for important control systems as, for example,

- Linear control systems $\dot{x} = Ax + Bu$. This follows from the Kalman condition [123, Thm. 2 p. 88] and the fact that

$$a(0,0) = \text{ Span } \{A^i Bu;\ i \in [0, n-1],\ u \in \mathbb{R}^m\}.$$

- Driftless control systems $\dot{x} = \sum_{i=1}^m u_i f_i(x)$. This is the classical Chow theorem [14].

But, in general, this necessary condition is not sufficient, as the two following simple control systems show:

$$(n = 1, m = 1),\ \dot{x} = u^2,$$
$$(n = 2, m = 1),\ \dot{x}_1 = x_2^2, \dot{x}_2 = u.$$

One can find other necessary conditions in [61, 66, 76, 127] and the references therein.

Let us now give sufficient conditions for small time local controllability. Let us assume, for the time being, that

$$f(x, u) = f_0(x) + \sum_{i=1}^m u_i f_i(x).$$

Let $L(f_0, \dots, f_m)$ be the free Lie algebra generated by $f_0, \dots, f_m$ and let us denote by $\mathrm{Br}(f) \subset L(f_0, \dots, f_m)$ the set of *formal iterated* Lie brackets of $\{f_0, f_1, ..., f_m\}$; see [128] for more details and precise definitions. For example

$$h = [[[f_0, [f_1, f_0]], f_1], f_0] \in \mathrm{Br}(f). \tag{2.7}$$

For $h \in L(f_0, \dots, f_m)$, let $h(0) \in \mathbb{R}^n$ be the "value" of h at 0. For h in $\mathrm{Br}(f)$ and $i \in [0, m]$, let $\delta_i(h)$ be the number of times that f_i appears in h. For example, with h given by (2.7), one has $\delta_0(h) = 3$, $\delta_1(h) = 2$ and $\delta_i(h) = 0$, for any $i \in [2, m]$. Let S_m be the group of permutations of $\{1, \dots, m\}$. For π in S_m, let $\tilde{\pi}$ be the automorphism of $L(f_0, \dots, f_m)$ which sends f_0 to f_0 and f_i to $f_{\pi(i)}$ for $i \in [1, m]$. For $h \in \mathrm{Br}(f)$, we let

$$\sigma(h) = \sum_{\pi \in S_m} \tilde{\pi}(h) \in L(f_0, \dots, f_m).$$

For example, if h is given by (2.7) and $m = 2$, one has

$$h = [[[f_0, [f_1, f_0]], f_1], f_0] + [[[f_0, [f_2, f_0]], f_2], f_0].$$

For $\theta \in [0, +\infty]$, $\dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ satisfies the Sussmann [128, Section 7] condition $S(\theta)$ if it satisfies the strong Lie algebra rank condition (2.6) at $(0, 0)$ and, if, for every $h \in \mathrm{Br}(f)$ with $\delta_0(h)$ odd and $\delta_i(h)$ even for all i in $[1, m]$, $\sigma(h)(0)$ is in the span of the $g(0)$'s, where the g's are in $\mathrm{Br}(f)$ and satisfy

$$\theta \delta_0(g) + \sum_{i=1}^m \delta_i(g) < \theta \delta_0(h) + \sum_{i=1}^m \delta_i(h) \tag{2.8}$$

with the convention that when $\theta = +\infty$, (2.8) is replaced by $\delta_0(g) < \delta_0(h)$. H. Sussmann has proved in [128]:

Theorem 2.10 **([128, Thm. 7.3])** *If, for some θ in $[0,1]$, $\dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ satisfies $S(\theta)$, then the control system (C) is small time locally controllable.*

Let us notice that one can easily check:

Proposition 2.11 *Let θ be in $[0,1]$. Then $\dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ satisfies $S(\theta)$ if and only if $\dot{x} = f_0(x) + \sum_{i=1}^m y_i f_i(x)$, $\dot{y} = u$, where the state is $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ and the control is $u \in \mathbb{R}^m$, satisfies $S(\theta/(1-\theta))$ (with the convention $1/0 = +\infty$).*

This proposition allows us to extend $S(\theta)$ to $\dot{x} = f(x,u)$ in the following way.

Definition 2.12 Let $\theta \in [0,1]$. The control system $\dot{x} = f(x,u)$ satisfies $S(\theta)$ if the control system $\dot{x} = f(x,y)$, $\dot{y} = u$ satisfies $S(\theta/(1-\theta))$.

What is called the Hermes condition is $S(0)$ ([63] and [127]). It follows from [128] that:

Theorem 2.13 *If, for some θ in $[0,1]$, the control system $\dot{x} = f(x,u)$ satisfies $S(\theta)$ then it is small time locally controllable.*

Proof Apply [128] to $\dot{x} = f(x,y)$, $\dot{y} = u$ with the constraint $\int_0^t |u(s)|\, ds \leqslant r$ (instead of $|u| \leqslant 1$).

One can find other sufficient conditions for small time local controllability in Agrachev [2], Bianchin-Stefani [7], Kawski [76], Tret'yak [130] and the references therein.

Example 2.14 If $f(x,u) = \sum_{i=1}^m u_i f_i(x)$ or if $f(x,u) = Ax + Bu$ (i.e. for driftless control systems and linear control systems), the control system (C) satisfies the Hermes condition if and only if it satisfies the strong Lie algebra rank condition at $(0,0)$. Hence, Sussmann's Theorem 2.10 allows to recover Chow's theorem (i.e. that for driftless control systems, small time local controllability is implied by the strong Lie algebra rank condition at $(0,0)$) and that the Kalman condition

$$\text{Span}\,\{A^i Bu;\ i \in [0, n-1],\ u \in \mathbb{R}^m\} = \mathbb{R}^n$$

implies the controllability of the linear control system $\dot{x} = Ax + Bu$).

Example 2.15 Let us consider the following classical model for a rigid spacecraft with control torques provided by thruster jets. Let $\eta = (\phi, \theta, \psi)$ be the Euler angles of a frame attached to the spacecraft representing rotations about a reference frame. Let $\omega = (\omega_1, \omega_2, \omega_3)$ be the angular velocity of the frame attached to the spacecraft with respect to the reference frame, expressed in the frame attached to the spacecraft and let J be the inertia matrix of the satellite. The evolution of the spacecraft is governed by the equations

$$J\dot{\omega} = S(\omega)J\omega + \sum_{i=1}^m u_i b_i, \quad \dot{\eta} = A(\eta)\omega, \tag{2.9}$$

where the $u_i \in \mathbb{R}$, $1 \leqslant i \leqslant m$, are the controls ($u_i b_i \in \mathbb{R}^3$, $1 \leqslant i \leqslant m$ are the torques applied to the spacecraft), $S(\omega)$ is the matrix representation of the wedge-product, i.e.

$$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{2.10}$$

and

$$(2.11) \qquad A(\eta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta\tan\phi & 1 & -\cos\theta\tan\phi \\ -\sin\theta/\cos\phi & 0 & \cos\theta/\cos\phi \end{pmatrix}.$$

Without loss of generality, we assume that the vectors $b_1, \ldots, b_m$ are linearly independent. Then one has the following results

- If $m = 3$, control system (2.9) is small time locally controllable and globally controllable in large time (that is given two states, there exists a time $T > 0$ and an open loop control $u \in L^\infty(0, T)$ which allows to go from the first state to the second one). This result is due to Bonnard [8] (see also [40]).

- If $m = 2$, control system (2.9) satisfies the strong Lie algebra rank condition at $(0, 0) \in \mathbb{R}^6 \times \mathbb{R}^2$ if and only if (see [8, 40])

$$(2.12) \qquad \text{Span } \{b_1, b_2, S(\omega)J^{-1}\omega;\, \omega \in \text{Span } \{b_1, b_2\}\} = \mathbb{R}^3.$$

  Moreover, if (2.12) holds,

  - The control system (2.9) satisfies Sussmann's condition $S(1)$, and so is small time locally controllable; see Keraï [79];
  - The control system (2.9) is globally controllable in large time; this result is due to Bonnard [8], see also [40].

- If $m = 1$, the control system (2.9) satisfies the strong Lie algebra rank condition at $(0, 0) \in \mathbb{R}^6 \times \mathbb{R}$ if and only if (see [8, 40])

$$(2.13) \qquad \text{Span } \{b_1, S(b_1)J^{-1}b_1 S(\omega)J^{-1}\omega;\, \omega \in \text{ Span } \{b_1, S(b_1)J^{-1}b_1\}\} = \mathbb{R}^3.$$

  Moreover

  - The control system (2.9) does not satisfy a necessary condition for small time local controllability due to Hermes [61] and Sussmann [127] and so is not small time locally controllable; see [79].
  - If (2.13) holds, the control system (2.9) is globally controllable in large time; this result is due to Bonnard [8], see also [40].

2.3 Obstructions to stationary feedback stabilization

In this section all the feedback laws considered are stationary. Let us recall that they are also assumed to be continuous. Let us start by recalling the following necessary condition for stabilizability due to Brockett [9].

Theorem 2.16 *If the control system (C) can be locally asymptotically stabilized by means of feedback laws, then the image of any neighborhood of* $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ *is a neighborhood of* $0 \in \mathbb{R}^n$.

Example 2.17 Let us go back to the control system of the attitude of a rigid spacecraft, already considered in Example 2.15. One easily sees that

- If $m = 3$, the control system (2.9) satisfies Brockett's condition. In fact in that case, the control system (2.9) is indeed asymptotically stabilizable by means of feedback laws; see [40] and [12].

- If $m \in \{1, 2\}$, the control system (2.9) does not satisfy Brockett's condition (and so is not asymptotically stabilizable by means of feedback laws). Indeed if $b \in \mathbb{R}^3 \setminus (\text{Span } \{b_1, b_2\})$ there exits no $((\omega, \eta), u)$ such that

$$S(\omega)\omega + u_1 b_1 + u_2 b_2 = b, \tag{2.14}$$

$$A(\eta)\omega = 0. \tag{2.15}$$

(Note that(2.15) gives $\omega = 0$, which, with (2.14), implies that $b = u_1 b_1 + u_2 b_2$.) See also [12].

In [136] Zabczyk has observed that, from a theorem due to Krasnosel'skiĭ [83, 84], one can deduce the following stronger necessary condition, that we shall call the *index condition*.

Theorem 2.18 *If the control system (C) can be locally asymptotically stabilized by means of feedback laws, then there exists $u \in C^0(\mathbb{R}^n; \mathbb{R}^m)$ vanishing at 0 such that $f(x, u(x)) \neq 0$ for x small enough but not 0 and the index of $x \to f(x, u(x))$ at 0 is equal to $(-1)^n$.*

For a definition of the index, see, for example, [84, p. 9].

In turns out that the index condition is, in a sense, too strong, In order to explain why, let us introduce a definition

Definition 2.19 The control system (C) is *locally asymptotically stabilizable by means of dynamic feedback laws* if, for some integer $p \in \mathbb{N}$, the control system

$$\dot{x} = f(x, u), \ \dot{y} = v \in \mathbb{R}^p, \tag{2.16}$$

where the control is $(u, v) \in \mathbb{R}^m \times \mathbb{R}^p$ and the state is $(x, y) \in \mathbb{R}^n \times \mathbb{R}^p$, is locally asymptotically stabilizable by means of feedback laws. By convention, when $p = 0$, the control system (2.16) is just the control system (C).

Clearly, if the control system (C) is locally asymptotically stabilizable by means of feedback laws, it is locally asymptotically stabilizable by means of dynamic feedback laws. But it is proved in [31] that the converse does not hold. Moreover, the example given in [31] shows that the index condition is not necessary for local asymptotic stabilizability by means of dynamic feedback laws. Clearly the Brockett necessary condition is still necessary for local asymptotic stabilizability by means of dynamic feedback laws. But this condition turns out to be not sufficient for local asymptotic stabilizability by means of dynamic feedback laws even if one assumes that 0 is small time locally controllable and that the system is analytic. In [19] we have proposed a slightly stronger necessary condition; we have:

Theorem 2.20 *Assume that the control system (C) can be locally asymptotically stabilized by means of dynamic feedback laws. Then, for any positive and small enough ϵ,*

$$f_* (\sigma_{n-1} (\{(x, u); \mid x \mid + \mid u \mid < \epsilon, f(x, u) \neq 0\})) = \sigma_{n-1}(\mathbb{R}^n \setminus \{0\}) \ (= \mathbb{Z}), \tag{2.17}$$

where $\sigma_{n-1}(A)$ denotes the stable homotopy group of order $(n - 1)$ (for a definition of stable homotopy groups, see e.g. [134]).

Let us point out that the index condition implies (2.17). Moreover (2.17) implies that a "dynamic extension" of (C) satisfies the index condition if the system is analytic. More precisely, one has

Theorem 2.21 ([25, Section 2]) *Assume that f is analytic (or continuous and subanalytic). Assume that* (2.17) *is satisfied. Then, if $p \geq 2n+1$, the control system* (2.16) *satisfies the index condition.*

Let us end this section by an open problem:

Open Problem 2.22 Let us assume that f is analytic, satisfies (2.17) and that 0 is small time locally controllable (or even continuously locally reachable in small time –see Definition 2.28 below–). Is the control system (C) locally asymptotically stabilizable by means of "dynamic" stationary feedback laws?

A natural guess is that, unfortunately, a positive answer is unlikely to be true. A possible candidate for a negative answer is the control system, with $n = 3$ and $m = 1$,

$$\dot{x}_1 = x_3^2(x_1 - x_2),\ \dot{x}_2 = x_3^2(x_2 - x_3),\ \dot{x}_3 = u.$$

This system satisfies the Hermes condition $S(0)$ and so, by Sussmann's Theorem 2.10 is small time locally controllable. Moreover, it satisfies the index condition (take $u = x_3 - (x_1^2 + x_2^2)$).

2.4 Stabilization of driftless systems

In this section we assume that

$$f(x, u) = \sum_{i=1}^{m} u_i f_i(x).$$

Let us first remark that in this case, as pointed out by Pomet in [104], the control system (C) does not satisfy Brockett's necessary condition (Theorem 2.16) for asymptotic stabilizability by means of stationary feedback laws if the vectors $f_1(0), \ldots, f_m(0)$ are linearly independent, which is a generic situation. But we are going to see that most of the driftless control systems can be globally asymptotically by means of time-varying feedback laws.

Let us denote by $\text{Lie}\{f_1, \ldots, f_m\} \subset C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ the Lie sub-algebra generated by the vector fields $f_1, \ldots, f_m$. Then one has:

Theorem 2.23 *Assume that, for all $x \in \mathbb{R}^n \backslash \{0\}$,*

$$\{g(x);\, g \in \text{Lie}\{f_1, \ldots, f_m\}\} = \mathbb{R}^n. \tag{2.18}$$

Then, for all $T > 0$, there exists u in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that

$$u(0, t) = 0,\ \forall t \in \mathbb{R}, \tag{2.19}$$

$$u(x, t+T) = u(x, t),\ \forall x \in \mathbb{R}^n,\ \forall t \in \mathbb{R}, \tag{2.20}$$

and 0 is globally asymptotically stable for

$$\dot{x} = f(x, u(x, t)) = \sum_{i=1}^{m} u_i(x, t) f_i(x). \tag{2.21}$$

Remark 2.24 By Chow's theorem [14], property (2.18) implies (and by Theorem 2.9 is equivalent if the f_i, $1 \leqslant i \leqslant m$, are analytic) to the global controllability of the driftless control system (C) in $\mathbb{R}^n \setminus \{0\}$, i.e., for any $x_0 \in \mathbb{R}^n \setminus \{0\}$, any $x_1 \in \mathbb{R}^n \setminus \{0\}$, and any $T > 0$, there exists $u \in L^\infty((0,T);\mathbb{R}^m)$ such that, if $\dot{x} = \sum_{i=1}^m u_i(t) f_i(x)$ and $x(0) = x_0$, then $x(T) = 0$.

This theorem is proved in [20]. Let us just briefly describe the idea of the proof: assume that, for any positive real number T, there exists $\bar{u}$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ satisfying (2.19) and (2.20) such that, if $\dot{x} = f(x, \bar{u}(x,t))$, then

$$x(T) = x(0). \tag{2.22}$$

(2.23) If $x(0) \neq 0$, the linearized control system around $(x, \bar{u})$ is controllable on $[0, T]$.

Using (2.22) and (2.23), one easily sees that one can construct a "small" feedback v in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ satisfying (2.19) and (2.20) such that, if

$$\dot{x} = f(x, (\bar{u} + v)(x,t)) \tag{2.24}$$

and $x(0) \neq 0$, then

$$|x(T)| < |x(0)|, \tag{2.25}$$

which implies that 0 is globally asymptotically stable for (2.21) with $u = u + v$.

So it remains only to construct $\bar{u}$. In order to get (2.22) just impose on $\bar{u}$ the condition that

$$\bar{u}(x,t) = -\bar{u}(x, T-t), \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{2.26}$$

which implies that $x(t) = x(T-t)$, $\forall t \in [0,T]$ and therefore gives (2.22). Finally, one proves that (2.23) holds for "many" $\bar{u}$.

Remark 2.25 The above method, which we have called "return method", can be used also to get controllability results. The idea is the following: assume that, for some positive real number T, there exists a measurable bounded function $u : [0,T] \to \mathbb{R}^m$ such that, if we denote by x the (maximal) solution of $\dot{\bar{x}} = f(\bar{x}, u(t))$, $\bar{x}(0) = 0$, then $x(T) = 0$ and the linearized control system around $(\bar{x}, u)$ is controllable on $[0,T]$. Then it follows easily from the inverse mapping theorem - see e.g. [123], Theorem 7 p. 126 - that $\dot{x} = f(x,u)$ is locally controllable around 0 and at time T, (i.e., for any $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|x_0| + |x_1|$ small enough, there exists $u \in L^\infty([0,T];\mathbb{R}^m)$ such that $\dot{x} = f(x, u(t))$ and $x(0) = x_0$ imply that $x(T) = x_1$). So one can in some cases reduce the problem of the controllability of a nonlinear system to the problem of the controllability of a *linear* (time-varying) control system. This is specially useful for studying the controllability of partial differential equations. Indeed one has powerful methods to study the controllability of *linear* partial differential equations, for example the HUM method [93] due to J. L. Lions, but one has very few tools to study the controllability of *nonlinear* partial differential equations. In particular, the use of Lie brackets, which is very powerful for nonlinear control systems of finite dimension (see section 2.2 above), does not seem to give any interesting results for the controllability of nonlinear partial differential equations. In section 4.2.1, we shall see that the return method allows us to prove boundary controllability of the Euler equations of incompressible inviscid fluids. Using the return method, Sontag has also found in [118] numerical techniques for the steering of systems without drift.

Remark 2.26 The fact (2.23) holds for "many" $\bar{u}$ is related to the prior works [115] and [58]. In [115] Sontag has shown that if a system is completely controllable then any two points can be joined by means of a control law such that the linearized control system around the associated trajectory is controllable. In [58, Thm p. 156] M. Gromov has shown that generic under-determined linear (partial) differential equations are algebraically solvable, which is related to controllability for time-varying linear control system (and in fact equivalent if the system is analytic; see [58, 2.3.8.(B)] and [123, Cor.3.5.18]). In our situation the linear differential equations are not generic; only the controls are generic, but this is be sufficient to get the result. Moreover, as pointed out by Sontag in [122], for analytic systems, one can get (2.23) by using a result due to Sussmann on observability [126]. Note that the proof we give for (2.23) in [20] see also [21] can be used to get a C^∞-version of [126]; see [22].

Remark 2.27 Using a method due to Pomet [104], we have given in [36] a method to deduce a suitable v from $\bar{u}$; see sub-section 3.2.2 below.

2.5 Stabilization of general systems

Let us first point out that in [124] Sontag and Sussmann have proved that any one dimensional state nonlinear control system which is locally (resp. globally) controllable can be locally (resp. globally) asymptotically stabilized by means of time-varying feedback laws. Let us also point out that it follows from Sussmann [125] that a result similar to Theorem 2.23 does not hold for systems with a drift term: more precisely, there are analytic control systems (C) the controls of which are globally controllable, for which there is no u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ for which 0 is globally asymptotically stable for $\dot{x} = f(x, u(x,t))$. In fact the proof of [125] requires uniqueness of the trajectories of $\dot{x} = f(x, u(x,t))$. But this can always been assumed; indeed it follows easily from Kurzweil's result [87] that, if there exists u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that 0 is globally asymptotically stable for $\dot{x} = f(x, u(x,t))$, then there exists $\bar{u}$ in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m) \cap C^\infty((\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}; \mathbb{R}^m)$ such that 0 is globally asymptotically stable for $\dot{x} = f(x, u(x,t))$; for such a $\bar{u}$ one has uniqueness of the trajectories of $\dot{x} = f(x, \bar{u}(x,t))$. But we are going to see in this subsection that a local version of Theorem 2.23 holds for many control systems which are small time locally controllable.

Let us again introduce some definitions.

Definition 2.28 The origin (of $\mathbb{R}^n$) is *locally continuously reachable* (for the control system (C))*in small time* if, for all positive real number T, there exist a positive real number ε and u in $C^0\left(\mathbb{R}^n; L^1\left((0,T); \mathbb{R}^m\right)\right)$ such that

$$\operatorname{Sup}\{|u(a)(t)|; t \in (0,T)\} \to 0 \text{ as } a \to 0,$$

$$(\dot{x} = f(x, u(x(0))(t)), |x(0)| < \varepsilon) \Rightarrow x(T) = 0.$$

Let us notice that, following a method due to M. Kawski [76] (see also [62]), we have proved in [21, Lemma 3.1 and Section 5] that "many" sufficient conditions for small time local controllability imply that the origin is locally continuously reachable in small time. This is in particular the case for the Sussmann condition (Theorems 2.10 and 2.13); this is in fact also the case for the Bianchini and Stefani condition [7, Corollary p. 970], which extends Theorem 2.10.

Our next definition is

Definition 2.29 The control system (*C*) is *locally stabilizable in small time by means of almost smooth periodic time-varying feedback laws* if, for any positive real number T, there exist ε in $(0,+\infty)$ and u in $C^0(\mathbb{R}^n\times\mathbb{R};\mathbb{R}^m)$ of class C^∞ on $(\mathbb{R}^n\setminus\{0\})\times\mathbb{R}$ such that

$$u(0,t)=0,\ \forall t\in\mathbb{R};\tag{2.27}$$

$$u(x,t+T)=u(x,t),\ \forall t\in\mathbb{R};\tag{2.28}$$

$$((\dot x=f(x,u(x,t))\text{ and }x(s)=0)\Rightarrow(x(t)=0\,\forall t\geq s)),\ \forall s\in\mathbb{R};\tag{2.29}$$

$$((\dot x=f(x,u(x,t))\text{ and }|x(s)|\leq\varepsilon)\Rightarrow(x(t)=0,\ \forall t\geq s+T))\ \forall s\in\mathbb{R}.\tag{2.30}$$

Note that (2.28), (2.29), and (2.30) imply that 0 is locally asymptotically stable for $\dot x=f(x,u(x,t))$; see [23, Lemma 2.15] for a proof. Note that, if (*C*) is locally stabilizable in small time by means of almost smooth periodic time-varying feedback laws, then $0\in\mathbb{R}^n$ is locally continuously reachable for (*C*). The main result of this section is that the converse holds if $n\notin\{2,3\}$ and if (*C*) satisfies the strong Lie algebra rank condition at $(0,0)$. That is, we have:

Theorem 2.30 *Assume that 0 is locally continuously reachable in small time, that (C) satisfies the strong Lie algebra rank condition at $(0,0)$, and that*

$$n\notin\{2,3\}.\tag{2.31}$$

Then (C) is locally stabilizable in small time by means of almost smooth periodic time-varying feedback laws.

This theorem is proved in [23] when $n\geq 4$ and in [26] when $n=1$. Let us just give a sketch of the main steps of the proof of [23].

Let I be an interval of $\mathbb{R}$. By a trajectory of the control system (*C*) on I we mean $(\gamma,u)\in C^\infty(I;\mathbb{R}^n\times\mathbb{R}^m)$ satisfying $\dot\gamma(t)=f(\gamma(t),u(t))$ for all t in I. The linearized control system around (γ,u) is $\dot\xi=A(t)\xi+B(t)w$ where the state is $\xi\in\mathbb{R}^n$, the control is $w\in\mathbb{R}^m$, and $A(t)=\partial f/\partial x(\gamma(t),u(t))\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, $B(t)=\partial f/\partial u(\gamma(t),u(t))\in\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$, for all t in I. We first introduce the following definition.

Definition 2.31 The trajectory (γ,u) is *supple* on $S\subset I$ if, for all s in S,

$$\operatorname{Span}\{((d/dt)-A(t))^i\,B(t)|_{t=s}w\ ;\ w\in\mathbb{R}^m, i\geq 0\}=\mathbb{R}^n.\tag{2.32}$$

In (2.32) we use the classical convention $(d/dt-A(t))^0\,B(t)=B(t)$. Let us recall that Silverman and Meadows have shown in [114] that (2.1) implies that the linearized control system around (γ,u) is controllable with impulsive controls at time s (in the sense of [74] p. 614). Let T be a positive real number. For u in $C^0(\mathbb{R}^n\times[0,T];\mathbb{R}^m)$ and a in $\mathbb{R}^n$, let $x(a,\cdot\,;u)$ be the maximal solution of $\partial x/\partial t=f(x,u(a,t))$, $x(a,0;u)=a$. Let, also, C^* be the set of $u\in C^0(\mathbb{R}^n\times[0,T];\mathbb{R}^m)$ of class C^∞ on $(\mathbb{R}^n\setminus\{0\})\times[0,T]$ and vanishing on $\{0\}\times[0,T]$. For simplicity, in this sketch of proof, we omit some details which are important to take care of the uniqueness property (2.29) (note that without (2.29) one does not have stability).

Step 1. Using (1.8), (1.9), and [21] or [22], one proves that there exist ϵ_1 in $(0,+\infty)$ and u_1 in C^*, vanishing on $\mathbb{R}^n\times\{T\}$, such that

$$|a|\leq\epsilon_1\Rightarrow x(a,T;u_1)=0,$$
$$0<|a|\leq\epsilon_1\Rightarrow(x(a,\cdot\,;u_1),u_1(a,\cdot))\text{ is supple on }[0,T].$$

Step 2. Let Γ be a closed sub-manifold of $\mathbb{R}^n \setminus \{0\}$ of dimension 1 such that

$$\Gamma \subset \{x \in \mathbb{R}^n; 0 < |x| < \epsilon_1\}.$$

Perturbing u_1 in a suitable way, one obtains a map u_2 in C^*, vanishing on $\mathbb{R}^n \times \{T\}$, such that

$$|a| \leq \epsilon_1 \Rightarrow x(a, T; u_2) = 0,$$
$$0 < |a| \leq \epsilon_1 \Rightarrow x(a, \cdot; u_2), u_2(a, \cdot)) \text{ is supple on } [0, T],$$
$$a \in \Gamma \to x(t, a; u_2) \text{ is an embedding of } \Gamma \text{ into } \mathbb{R}^n \setminus \{0\}, \forall t \in [0, T).$$

Here one uses the assumption $n \geq 4$ and one proceeds as in the classical proof of the Whitney embedding theorem (see e.g. [57] Chapter II, Section 5). Let us emphasize that it is only in this step that this assumption is used.

Step 3. From Step 2, one deduces the existence of u_3^* in C^*, vanishing on $\mathbb{R}^n \times \{T\}$, and of an open neighborhood $\mathcal{N}^*$ of Γ in $\mathbb{R}^n \setminus \{0\}$ such that

$$a \in \mathcal{N}^* \Rightarrow x(a, T; u_3^*) = 0, \tag{2.33}$$

$$a \in \mathcal{N}^* \to x(a, t; u_3^*) \text{ is an embedding of } \mathcal{N}^* \text{ into } \mathbb{R}^n \setminus \{0\}, \forall t \in [0, T).$$

This embedding property allows to transform the open-loop control u_3^* into a feedback law u_3 on $\{(x(a, t; u_3), t); a \in \mathcal{N}, t \in [0, T)\}$. So –see in particular (2.33) and note that u_3^* vanishes on $\mathbb{R}^n \times \{T\}$– there exist u_3 in C^* and an open neighborhood $\mathcal{N}$ of Γ in $\mathbb{R}^n \setminus \{0\}$ such that

$$(x(0) \in \mathcal{N} \text{ and } \dot{x} = f(x, u_3(x, t))) \Rightarrow (x(T) = 0).$$

One can also impose that, for all τ in $[0, T]$,

$$(\dot{x} = f(x, u_3(x, t)) \text{ and } x(\tau) = 0) \Rightarrow (x(t) = 0, \quad \forall t \in [\tau, T]).$$

Step 4. In this last step one shows the existence of a closed sub-manifold of $\mathbb{R}^n \setminus \{0\}$ of dimension 1 included in the set $\{x \in \mathbb{R}^n; 0 < |x| < \epsilon_1\}$ such that, for any neighborhood $\mathcal{N}$ of Γ in $\mathbb{R}^n \setminus \{0\}$, there exists u_4 in C^* such that, for some ϵ_4 in $(0, +\infty)$,

$$(\dot{x} = f(x, u_4(x, t)) \text{ and } |x(0)| < \epsilon_4) \Rightarrow (x(T) \in \mathcal{N} \cup \{0\}),$$
$$((\dot{x} = f(x, u_4(x, t)) \text{ and } x(\tau) = 0) \Rightarrow (x(t) = 0 \quad \forall t \in [\tau, T])) \forall \tau \in [0, T].$$

Finally let $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ be equal to u_4 on $\mathbb{R}^n \times [0, T]$, $2T$-periodic with respect to time, and such that $u(x, t) = u_3(x, t - T)$ for all (x, t) in $\mathbb{R}^n \times (T, 2T)$. Then u vanishes on $\{0\} \times \mathbb{R}$, is continuous on $\mathbb{R}^n \times (\mathbb{R} \setminus \mathbb{Z}T)$, of class C^∞ on $(\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R} \setminus \mathbb{Z}T)$, and satisfies

$$(\dot{x} = f(x, u(x, t)) \text{ and } |x(0)| < \epsilon_4) \Rightarrow (x(2T) = 0),$$
$$(\dot{x} = f(x, u(x, t)) \text{ and } x(\tau) = 0) \Rightarrow (x(t) = 0, \quad \forall t \geq \tau), \forall \tau \in \mathbb{R},$$

which implies, see [23], that (2.30) holds, with $4T$ instead of T and $\epsilon > 0$ small enough, and that 0 is uniformly locally asymptotically stable for the system $\dot{x} = f(x, u(x, t))$. Since T is arbitrary, Theorem 2.30 is proved (modulo a problem of regularity of u at (x, t) in $\mathbb{R}^n \times \mathbb{Z}T$ that is fixed in [23]).

Remark 2.32 We conjecture that assumption (2.31) can be removed in Theorem 2.30.

Example 2.33 Let us go back again to control system (2.9) of the attitude of a rigid spacecraft, already considered in Examples 2.15 and 2.17. Let us recall that it is proved in [79] that, if $m = 2$ and (2.12) holds (which is generically satisfied), then the control system (2.9) satisfies Sussmann's condition S(1) which, by [21, Lemma 3.1 and Section 5] implies that $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^3$ is locally continuously reachable; hence, by Theorem 2.30, for any $T > 0$ there exist a T-periodic time-varying feedback laws which locally asymptotically stabilizes the control system (2.9) (if (2.12) holds). The construction of such feedback laws has been performed by Morin et al. [101] in the special case where the torque actions are exerted about the principal axis of the inertia matrix of the spacecraft. The general case has been treated in [39]; simpler feedback laws have been proposed Morin-Samson in [100]. In sections 3.3, 3.4 and 3.5, we explain how the the feedback laws of [100] are constructed.

2.6 Output feedback stabilization

In this section only part of the state (called the output) is measured; let us denote by $(\check{C})$ the control system

$$(\check{C}): \qquad \dot{x} = f(x,u),\ y = h(x), \tag{2.34}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $y \in \mathbb{R}^p$ is the output. Again $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ and satisfies (2.1); we also assume that $h \in C^\infty(\mathbb{R}^n; \mathbb{R}^p)$ and satisfies

$$h(0) = 0. \tag{2.35}$$

In order to state the main result of this section we first introduce some definitions.

Definition 2.34 The control system $(\check{C})$ is said to be *locally stabilizable in small time by means of static periodic time-varying output feedback laws* if, for any positive real number T, there exist ε in $(0, +\infty)$ and u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that (2.27), (2.28), (2.29), (2.30) hold and such that

$$u(x,t) = \bar{u}(h(x), t) \tag{2.36}$$

for some $\bar{u}$ in $C^0(\mathbb{R}^p \times \mathbb{R}; \mathbb{R}^n)$.

Our next definition concerns dynamic stabilizability.

Definition 2.35 The control system $(\check{C})$ is *locally stabilizable in small time by means of dynamic periodic time-varying state (resp. output) feedback laws* if, for some integer $k \geq 0$, the control system

$$\dot{x} = f(x,u),\ \dot{z} = v,\ \tilde{h}(x,z) = (h(x), z), \tag{2.37}$$

where the state is $(x,z) \in \mathbb{R}^n \times \mathbb{R}^k$, the control $(u,v) \in \mathbb{R}^m \times \mathbb{R}^k$, and the output $\tilde{h}(x,z) \in \mathbb{R}^p \times \mathbb{R}^k$, is locally stabilizable in small time by means of static periodic time-varying state (resp. output) feedback laws.

In the above definition, the control system (2.37) with $k = 0$ is, by convention, the system $(\check{C})$. Let us also point out that it is proved in [21, Section 3], that if, for system$(\check{C})$, 0 is continuously reachable in small time, then $(\check{C})$ is locally stabilizable in small time by means of dynamic periodic time-varying state feedback laws.

For our last definition, one needs to introduce some notations. For α in $\mathbb{N}^m$ and $\bar{u}$ in $\mathbb{R}^m$, let $f_{\bar{u}}^{\alpha}$ in $C^{\infty}(\mathbb{R}^n;\mathbb{R}^n)$ be defined by

$$f_{\bar{u}}^{\alpha}(x) = \frac{\partial^{|\alpha|} f}{\partial u^{\alpha}}(x, \bar{u}), \ \forall x \in \mathbb{R}^n. \tag{2.38}$$

Let $\mathcal{O}(\tilde{C})$ be the subspace of $C^{\infty}(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^p)$ spanned by the maps ω such that, for some integer $r \geq 0$ (depending on ω) and for some sequence $\alpha_1, ..., \alpha_r$ of r multi-indices in $\mathbb{N}^m$,we have, for all $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$,

$$\omega(x, u) = L_{f_u^{\alpha_1}} ... L_{f_u^{\alpha_r}} h(x), \tag{2.39}$$

where $L_{f_{\bar{u}}^{\alpha_i}}$ denotes Lie derivatives with respect to $f_{\bar{u}}^{\alpha_i}$ and where, by convention, if $r = 0$ the right hand side of (2.39) is $h(x)$. With this notation our last definition is

Definition 2.36 The control system $(\tilde{C})$ is *locally Lie null-observable* if there exists a positive real number $\bar{\varepsilon}$ such that

(i) for all a in $\mathbb{R}^n \setminus \{0\}$ such that $|a| < \bar{\varepsilon}$, there exists q in $\mathbb{N}$ such that

$$L_{f_0}^q h(a) \neq 0 \tag{2.40}$$

with $f_0(x) = f(x, 0)$ and the usual convention $L_{f_0}^0 h = h$;

(ii) for all $(a_1, a_2) \in (\mathbb{R}^n \setminus \{0\})^2$ with $a_1 \neq a_2, |a_1| < \bar{\varepsilon}$, and $|a_2| < \bar{\varepsilon}$, and for all u in $\mathbb{R}^m$ with $|u| < \bar{\varepsilon}$, there exists ω in $\mathcal{O}(\tilde{C})$ such that

$$\omega(a_1, u) \neq \omega(a_2, u). \tag{2.41}$$

Note that (i) implies the following property:

(i)* for any $a \neq 0$ in $B_{\bar{\varepsilon}} := \{x \in \mathbb{R}^m, |x| < \bar{\varepsilon}\}$ there exists a positive real number τ such that

$$x(\tau) \text{ exists and } h(x(\tau)) \neq 0, \tag{2.42}$$

where $x(t)$ is defined by $\dot{x} = f(x, 0), x(0) = a$.

Moreover, if f and g are analytic, (i)* implies (i). The reason of "null" in "null-observable" comes from condition (i) or (i)* : roughly speaking we want to be able to distinguish from 0 any a in $B_{\varepsilon} \setminus \{0\}$ by using the control law which vanishes identically.

When f is affine with respect to u, i.e. $f(x, u) = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ with f_1, ..., f_m in $C^{\infty}(\mathbb{R}^n; \mathbb{R}^n)$, then a slightly simpler version of (ii) can be given. Let $\tilde{\mathcal{O}}(\tilde{C})$ be the observation space –see e.g. [59] or Remark 6.4.2 in [123]– i.e. the set of maps $\tilde{\omega}$ in $C^{\infty}(\mathbb{R}^n; \mathbb{R}^p)$ such that for some integer $r \geq 0$ (depending on $\tilde{\omega}$) and for some sequence $i_1, ..., i_r$ of integers in $[0, m]$

$$\tilde{\omega}(x) = L_{f_{i_1}} ... L_{f_{i_r}} h(x), \ \forall x \in \mathbb{R}^n, \tag{2.43}$$

with the convention that, if $r = 0$, the right hand side of (2.43) is $h(x)$. Then (ii) is equivalent to

$$((a_1, a_2) \in B_{\bar{\varepsilon}}^2, \quad \tilde{\omega}(a_1) = \tilde{\omega}(a_2) \ \forall \tilde{\omega} \in \tilde{\mathcal{O}}(\tilde{C})) \Rightarrow (a_1 = a_2). \tag{2.44}$$

Finally let us remark that if f is a polynomial with respect to u or if f and g are analytic then (ii) is equivalent to (ii)* for all $(a_1, a_2) \in \mathbb{R}^n \setminus \{0\}$ with $a_1 \neq a_2, |a_1| < \varepsilon$ and $|a_2| < \varepsilon$ there exists u in $\mathbb{R}^m$ and ω in $\mathcal{O}(\tilde{C})$ such that (2.41) holds.

Indeed, in these cases, the subspace of $\mathbb{R}^p$ spanned by $\omega(x, u)$; $\omega \in \mathcal{O}(\tilde{C})$ does not depend on u: it is the observation space of $(\tilde{C})$ evaluated at x – as defined for example in [59].

With these definitions we have

Theorem 2.37 *Assume that the origin (of $\mathbb{R}^n$) is locally continuously reachable (for (C)) in small time (see Definition 2.28). Assume that $(\tilde{C})$ is locally Lie null-observable. Then $(\tilde{C})$ is locally stabilizable in small time by means of dynamic periodic time-varying output feedback laws.*

This theorem is proved in [24]. Let us just sketch the proof given in [24]. We assume that the assumptions of Theorem 2.37 are satisfied. Let T be a positive real number. The proof of Theorem 2.37 is divided into three steps.

Step 1. Using the assumption that the system (C) is locally Lie null-observable one proves, using [22], that there exist u^* in $C^\infty(\mathbb{R}^p \times [0, T]; \mathbb{R}^m)$ and a positive real number ε^* such that

$$u^*(y, T) = u^*(y, 0) = 0, \ \forall y \in \mathbb{R}^p, u^*(0, t) = 0, \ \forall t \in [0, T], \tag{2.45}$$

and, for all (a_1, a_2) in $B^2_{\varepsilon^*}$, for all s in $(0, T)$,

$$(h^{(i)}_{a_1}(s) = h^{(i)}_{a_2}(s), \ \forall i \in \mathbb{N}) \Rightarrow (a_1 = a_2), \tag{2.46}$$

where $h_a(s) = h(x^*(a, s))$ with x^* defined by $\partial x^*/\partial t = f(x^*, u^*(h(x^*), t))$, $x^*(a, 0) = a$. Let us note that in [98] a similar u^* was considered, but it was taken depending only on time and so (2.45), which is important to get stability, was not satisfied in general. In this step we do not use any reachability property for (C).

Step 2. Let $q = 2n + 1$. In this step, using (2.46), one proves the existence of $(q + 1)$ real numbers $0 < t_0 < t_1 ... < t_q < T$ such that the map $K : B_{\varepsilon^*} \to (\mathbb{R}^p)^q$ defined by

$$K(a) = \left(\int_{t_0}^{t_1} (s - t_0)(t_1 - s) h_a(s) ds, ..., \int_{t_0}^{t_q} (s - t_0)(t_q - s) h_a(s) ds \right) \tag{2.47}$$

is one-to-one and so, there exists a map $\theta : (\mathbb{R}^p)^q \to \mathbb{R}^n$ such that

$$\theta \circ K(a) = x^*(a, T), \ \forall a \in B_{\varepsilon^*/2}. \tag{2.48}$$

Step 3. In this step one proves the existence of $\bar{u}$ in $C^0(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$ and $\bar{\varepsilon}$ in $(0, +\infty)$ such that

$$\bar{u} = 0 \text{ on } (\mathbb{R}^n \times \{0, T\}) \cup (\{0\} \times [0, T]), \tag{2.49}$$

$$(\dot{x} = f(x, \bar{u}(x(0), t)) \text{ and } |x(0)| < \bar{\varepsilon}) \Rightarrow (x(T) = 0). \tag{2.50}$$

Property (2.50) means that $\bar{u}$ is a "dead-beat" open-loop control. In this last step, we use the reachability assumption on (C), but do not use the Lie null-observability assumption.

Using these three steps let us finish the proof of Theorem 2.37. The dynamic extension of system (C) that we consider is

$$\dot{x} = f(x, u), \ \ \dot{z} = v = (v_1, ..., v_q, v_{q+1}) \in \mathbb{R}^p \times ... \times \mathbb{R}^p \times \mathbb{R}^n \simeq \mathbb{R}^{pq+n}, \tag{2.51}$$

with $z_1 = (z_1, ..., z_q, z_{q+1}) \in \mathbb{R}^p \times ... \times \mathbb{R}^p \times \mathbb{R}^n \simeq \mathbb{R}^{pq+n}$. For this system the output is $\tilde{h}(x,z) = (h(x), z) \in \mathbb{R}^p \times \mathbb{R}^{pq+n}$. For $s \in \mathbb{R}$ let $s^+ = \max(s, 0)$ and let Sign $(s) = 1$ if $s > 0$, 0 if $s = 0$, -1 if $s < 0$. Finally, for r in $\mathbb{N} \setminus \{0\}$ and $b = (b_1, ..., b_r)$ in $\mathbb{R}^r$, let

$$b^{1/3} = (|b_1|^{1/3}\text{Sign }(b_1), ..., |b_r|^{1/3}\text{Sign }(b_r)). \tag{2.52}$$

We now define $u : \mathbb{R}^p \times \mathbb{R}^{pq+n} \times \mathbb{R} \to \mathbb{R}^m$ and $v : \mathbb{R}^p \times \mathbb{R}^{pq+n} \times \mathbb{R} \to \mathbb{R}^{pq+n}$ by requiring, for (y, z) in $\mathbb{R}^p \times \mathbb{R}^{pq+n}$ and for all i in $[1, q]$,

$$u(y, z, t) = u^*(y, t), \ \forall t \in [0, T), \tag{2.53}$$

$$v_i(y, z, t) = -t(t_0 - t)^+ z_i^{1/3} + (t - t_0)^+(t_i - t)^+ y, \ \forall t \in [0, T), \tag{2.54}$$

$$v_{q+1}(y, z, t) = -t(t_q - t)^+ z_{q+1}^{1/3} + 6\frac{(T-t)^+(t-t_q)^+}{(T-t_q)^3}\theta(z_1, ..., z_q), \tag{2.55}$$

$$u(y, z, t) = \bar{u}(z_{q+1}, t - T), \ \forall t \in [T, 2T), \tag{2.56}$$

$$v(y, z, t) = 0, \ \forall t \in [T, 2T), \tag{2.57}$$

$$u(y, z, t) = u(y, z, t + 2T), \ \forall t \in \mathbb{R}, \tag{2.58}$$

$$v(y, z, t) = v(y, z, t + 2T), \ \forall t \in \mathbb{R}. \tag{2.59}$$

Roughly speaking the strategy is the following.

(i) During the time interval $[0, T]$, one "excites" system (C) by means of $u^*(y, t)$ in order to be able to deduce from the observation during this interval of time what is the state at time T: at time T we have $z_{q+1} = x$.

(ii) During the time interval $[T, 2T]$, z_{q+1} does not move and one uses the dead-beat open-loop u but transforms it into an output feedback by using in its argument z_q instead of the value of x at time T (this step has been used previously in the proof of Theorem 1.7 of [21]).

Remark 2.38 This method has been previously used by Sontag in [117], Lozano [94], Mazenc and Praly [98]. A related idea is also used in Section 3 of [21], where we first recover initial data from the state. Moreover, as in [117] and [98], our proof relies on the existence of an output feedback which distinguishes every pair of distinct states (see [126] for analytic systems and [22] for C^∞ systems).

One easily sees that u and v are continuous and vanishes on $\{(0, 0)\} \times \mathbb{R}$. Let (x, z) be any maximal solution of the closed loop system

$$\dot{x} = f(x, u(\tilde{h}(x, z), t)), \quad \dot{z} = v(\tilde{h}(x, z), t); \tag{2.60}$$

then one easily checks that, if $|x(0)| + |z(0)|$ is small enough,

$$z_i(t_0) = 0, \ \forall i \in [1, q], \tag{2.61}$$

$$(z_1(t), ..., z_q(t)) = K(x(0)), \ \forall t \in [t_q, T], \tag{2.62}$$

$$z_{q+1}(t_q) = 0, \tag{2.63}$$

$$z_{q+1}(T) = \theta \circ K(x(0)) = x(T), \tag{2.64}$$

$$x(t) = 0,\ \forall t \in [2T, 3T], \tag{2.65}$$

$$z(2T + t_q) = 0. \tag{2.66}$$

Equalities (2.61) (resp. (2.63)) are proved by computing explicitly, for $i \in [1, q]$, z_i on $[0, t_0]$ (resp. z_{q+1} on $[0, t_q]$) and by seeing that this explicit solution reaches 0 before time t_0 (resp. t_q) and by pointing out that if, for some s in $[0, t_0]$ (resp. $[0, t_q]$), $z_i(s) = 0$ (resp. $z_{q+1}(s) = 0$) then $z_i = 0$ on $[s, t_0]$ (resp. $z_{q+1} = 0$ on $[s, t_q]$) –note that $z_i \dot{z}_i \leq 0$ on $[0, t_0]$ (resp. $z_{q+1}\dot{z}_{q+1} \leq 0$ on $[0, t_q]$).

Moreover one has also, for all s in $\mathbb{R}$ and all $t \geq s$,

$$((x(s), z(s)) = (0, 0)) \Rightarrow ((x(t), z(t)) = (0, 0)). \tag{2.67}$$

Indeed, first note that without loss of generality we may assume $s \in [0, 2T]$ and $t \in [0, 2T]$. If $s \in [0, T]$, then, since u^* is of class C^∞ we get, using (2.45), that $x(t) = 0$, $\forall t \in [s, T]$ and then, using (2.35) and (2.54), we get that, for all $i \in [1, q]$, $z_i\dot{z}_i \leq 0$ on $[s, T]$ and so z_i also vanishes on $[s, T]$; this, with (2.55) and $\theta(0) = 0$ (see(2.47) and (2.48)), implies that $z_{q+1} = 0$ also on $[s, T]$. Hence we may assume that $s \in [T, 2T]$. But, in this case, using (2.57), we get that $z = 0$ on $[s, 2T]$ and, from (2.49) and (2.56), we get that $x = 0$ also on $[s, 2T]$.

¿From (2.65), (2.66), and (2.67) we get –see Lemma 2.15 in [23] – the existence of ε in $(0, +\infty)$ such that, for any s in $\mathbb{R}$ and any maximal solution (x, z) of $\dot{x} = f(x, u(\tilde{h}(x, z), t))$, $\dot{z} = v(\tilde{h}(x, z), t)$, we have

$$(|x(s)| + |y(s)| \leq \varepsilon) \Rightarrow ((x(t), z(t)) = (0, 0),\ \forall t \geq s + 5T).$$

Since T is arbitrary, Theorem 2.37 is proved.

Remark 2.39 In [98] it is established that distinguishability with a universal time-varying control, global stabilizability by state feedback, and observability of blow-up are sufficient conditions for the existence of a time-varying dynamic output feedback (of infinite dimension and in a sense more general than the one considered in Definition 2.35) guaranteeing boundedness and convergence of all the solutions defined at time $t = 0$. The methods developed in [98] can be applied directly to our situation. In this case Theorem 2.37 gives two improvements: we get that 0 is asymptotically stable for the closed loop system, instead of only attractor for time 0, and our dynamic extension is of finite dimension, instead of infinite dimension.

Remark 2.40 If $(\tilde{C})$ is locally stabilizable in small time by means of dynamic periodic time-varying output feedback laws, then the origin (of $\mathbb{R}^n$) is locally continuously reachable (for $(\tilde{C})$) in small time (use Lemma 3.5 in [24]) and, if moreover f and h are analytic, then $(\tilde{C})$ is locally Lie null-observable –see [24, Proposition 4.3].

Let us remark that it follows from our proof of Theorem 2.37 that it suffices to consider dynamic extensions of dimension $n + (2n + 1)p$, i.e. under the assumption of Theorem 2.37, the control (2.37) with $k = n + (2n + 1)p$ is locally stabilizable in small time by means of static periodic time-varying output feedback laws. We conjecture that, as in the linear case, this result still holds for $k = n - 1$. Note that this conjecture is true if $n = 1$, i.e. we have the following proposition which is proved in [26].

Proposition 2.41 *Assume that $n = 1$ and that the origin (of $\mathbb{R}$) is locally continuously reachable (for $(\tilde{C})$) in small time. Assume that $(\tilde{C})$ is locally Lie null-observable. Then $(\tilde{C})$ is locally stabilizable in small time by means of periodic time-varying output feedback laws.*

Remark 2.42 There are linear control systems which are controllable and observable but which cannot be locally asymptotically stabilized by means of a time-varying static feedback law. This is for example the case for the controllable and observable linear system, with $n = 2$, $m = 1$, and $p = 1$,

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ y = x_1.$$

Assume that this system can be locally asymptotically stabilized by means of a time-varying static output feedback law $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then there exist $r > 0$ and $\tau > 0$ such that, if $\dot{x}_1 = x_2$, $\dot{x}_2 = u(x_1, t)$,

$$x_1(0)^2 + x_2(0)^2 \leq r^2 \Rightarrow x_1(\tau)^2 + x_2(\tau)^2 \leq r^2/5. \tag{2.68}$$

Let $(u^n; n \in \mathbb{N})$ be a sequence of functions from $\mathbb{R}$ into $\mathbb{R}$ of class C^∞ which converges uniformly to u on each compact subset of $\mathbb{R} \times \mathbb{R}$. Then, for n large enough, $\dot{x}_1 = x_2$, $\dot{x}_2 = u^n(x_1, t)$ implies

$$x_1(0)^2 + x_2(0)^2 \leq r^2 \Rightarrow x_1(\tau)^2 + x_2(\tau)^2 \leq r^2/4. \tag{2.69}$$

But, since the time-varying vector field X on $\mathbb{R}^2$ defined by

$$X_1(x_1, x_2, t) = x_1,\ X_2(x_1, x_2, t) = u^n(x_1, t)$$

has a divergence equal to 0, the flow associated with X preserves area, which is a contradiction to (2.69).

2.7 Time-varying feedback and ISS

Let us recall that one prefers to use feedback laws instead of open loop control since they are usually more robust to disturbances. In order to define the robustness of a feedback there is the well-established operator approach. This approach gives very useful results, but is not invariant under changes of variables. In [116], Sontag has defined a new concept, called "Input-to-State Stability" (ISS) to define a robustness which is invariant under changes of variables.

Let us first deal with actuator disturbances. We need to recall the definitions of functions of class $\mathcal{K}$, of class $\mathcal{K}_\infty$ and of class $\mathcal{KL}$.

Definition 2.43 A function $\gamma : \mathbb{R}^+ \to \mathbb{R}$ is said to be of *class $\mathcal{K}$* if it is continuous, strictly increasing and if $\gamma(0) = 0$. The function γ is said to be of *class $\mathcal{K}_\infty$* if moreover it is not bounded.

A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of *class $\mathcal{KL}$* if for each fixed t the mapping $\beta(., t)$ is of class $\mathcal{K}$ and for each fixed s the mapping $\beta(s, .)$ is decreasing to zero on t as $t \to +\infty$.

For $d \in L^\infty(I)$, where I is a subset of $\mathbb{R}$, we note $\|d\|_\infty$ it's L^∞-norm. We these definitions, one can now give Sontag's definition [116] of ISS for actuator disturbances.

Definition 2.44 A feedback law $u \in C^0(\mathbb{R};\mathbb{R}^m)$, such that $u(0) = 0$, makes the control system (C) *ISS for actuator disturbances* if there exist a function γ of class $\mathcal{K}$ and a function β of class $\mathcal{KL}$ such that any solution, defined at time 0, of the closed loop system

$$\dot{x} = f(x, u(x) + d(t))$$

where d is a continuous bounded disturbance, exists for all $t \geqslant 0$ and satisfies

$$|x(t)| < \beta(|x(0)|, t) + \gamma(\|d\|_\infty)), \ \forall t \in [0, +\infty). \tag{2.70}$$

One easily checks that any feedback law which makes the control system (C) ISS for actuator disturbances globally asymptotically stabilizes the control system (C). As pointed out by Sontag in [116], the converse does not hold, even for control systems which are affine in the controls.

Remark 2.45 But note that a local version version holds, that is, for any feedback law u which locally asymptotically stabilizes (C), there exist a function γ of class $\mathcal{K}$ and a function β of class $\mathcal{KL}$ such that (2.70) holds for $|x(0)|$ and $\|d\|_\infty$ small enough. This follows from [87].

Even if the converse does not hold, one has the following theorem proved by Sontag in [116].

Theorem 2.46 *Asumme that the control system (C) is affine in the controls and globally asymptotically stabilizable by means of stationary feedback laws. Then there exist feedback laws which make the control system (C) ISS for actuator disturbances.*

In [119], Sontag has shown that one cannot remove the assumption "(C) is affine in the controls".

A natural question is "does one have a similar result to Theorem 2.46 for ISS for measurement disturbances?". Of course the definition of ISS for measurement disturbances is:

Definition 2.47 A feedback $u \in C^0(\mathbb{R};\mathbb{R}^m)$, such that $u(0) = 0$, makes system (C) *ISS for measurement disturbances* if there exists a function γ of class $\mathcal{K}$ and a function β of class $\mathcal{KL}$ such that any solution, defined at time 0, of the closed loop system

$$\dot{x} = f(x, u(x + d(t)))$$

where d is a continuous bounded disturbance, exists for all $t \geqslant 0$ and satisfies

$$|x(t)| < \beta(|x(0)|, t) + \gamma(\|d\|_\infty), \ \forall t \in [0, +\infty).$$

Again, one easily checks that any feedback law which makes system (C) ISS for measurement disturbances globally asymptotically stabilizes the control system (C). A counterexample, given by Freeman in [45], shows that Theorem 2.46 does not hold with ISS for measurement disturbances. Nevertheless, it does hold for those systems that can be put into strict feedback form (see [47]).

Again one may wonder if the use of time-varying feedback law can help to get Theorem 2.46 with ISS for measurement disturbances. Let us first adapt Definition 2.47 to time-varying feedback laws.

Definition 2.48 A feedback $u \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$, such that, for any $t \in \mathbb{R}$, $u(0,t) = 0$, makes the system (2.71) *input-to-state stable for measurement disturbances* if and only if there exists a function γ of class $\mathcal{K}$ and a function β of class $\mathcal{KL}$ such that, for any time t_0 and any solution, defined at time t_0, of the closed loop system

$$\dot{x} = f(x) + u(x + d, t)g(x),$$

where $d \in L^\infty(t_0, +\infty)$ is a continuous bounded disturbance, exists for all $t \geqslant t_0$, and satisfies

$$|x(t)| < \beta(|x_0|, t - t_0) + \gamma(\|d\|_\infty), \forall t \in [t_0, +\infty).$$

One has the following theorem, due to Chung [15].

Theorem 2.49 *Consider the control system*

$$\dot{x} = f(x) + ug(x), \tag{2.71}$$

where $x \in \mathbb{R}$ is the state, $u \in \mathbb{R}$ the feedback, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous. Suppose that this system is globally asymptotically stabilizable by means of stationary feedback laws. Then, for any period $T > 0$, there exists a T-periodic time-varying feedback law making the closed loop system input-to-state stable with respect to measurement disturbances. Moreover, if the zeros of g are bounded, the feedback law can be taken time-invariant.

Periodic time-varying feedbacks have also been used for affine systems by R. Freeman in [46]. The problem he studies is ISS for measurement disturbances with systems that are only partially observable. More precisely he assumes that g in equation (2.71) does not vanish on $\mathbb{R} \setminus \{0\}$ but that the sign of g is unknown.

3 Feedback design tools

In this chapter we give some tools to design stabilizing feedback laws and present some applications of these tools. The tools we want to describe are

- Control Lyapunov functions,
- Damping,
- Homogeneity,
- Averaging,
- Backstepping.

There are in fact plenty of other powerful methods; e.g. zero-dynamics, center manifolds, forwarding, adaptive control, etc. See, for example, [4, 37, 71, 85, 86, 99, 101, 102, 113] and the references therein.

3.1 Control Lyapunov function

A basic tool for studying the asymptotic stability of an equilibrium point is the Lyapunov function. In the case of a control system, the control is at our disposal, so there are more "chances" that a given function, could be a Lyapunov function for a suitable choice of feedback laws. For simplicity, we restrict our attention to global asymptotic stabilization; the definitions and theorems of this section can be easily adapted to treat local asymptotic stabilization.

In the framework of control systems, the Lyapunov function approach leads to the following definition, due to Artstein [3].

Definition 3.1 A function $V \in C^1(\mathbb{R}^n;\mathbb{R})$ is a *control Lyapunov function* for the control system (C) if

$$V(x) \to +\infty, \text{ as } |x| \to +\infty,$$
$$V(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\},$$
$$\forall x \in \mathbb{R}^n \setminus \{0\}, \exists u \in \mathbb{R}^m \text{ s.t. } f(x,u)\cdot \nabla V(x) < 0.$$

Moreover, V satisfies the *small control property* if, for any strictly positive real number ε, there exits a strictly positive real number η such that, for any $x \in \mathbb{R}^n$ with $0 < |x| < \eta$, there exists $u \in \mathbb{R}^m$ such that $|u| < \varepsilon$ and $f(x,u)\cdot\nabla V(x) < 0$.

With this definition, one has the following theorem due to Artstein [3].

Theorem 3.2 *If the control system (C) is globally asymptotically stabilizable by means of a stationary feedback law, then it admits a control Lyapunov function satisfying the small control property. If the control system (C) admits a control Lyapunov function satisfying the small control property, then it can be globally asymptotically stabilizable by means of*

- *stationary feedback laws if the control system (C) is affine in the controls;*
- *relaxed controls for general f* (see [3] for a definition).

Instead of relaxed controls, one can use periodic time-varying feedback laws. Indeed one has the following theorem proved in [33].

Theorem 3.3 *The control system (C) can be globally asymptotically stabilized by means of periodic time-varying feedback laws if it admits a control Lyapunov function satisfying the small control property.*

Let us point out that, even in the case of control systems which are affine in the controls, Artstein's proof of Theorem 3.2 relies on partitions of unity and so does not give explicit stabilizing feedback laws. Explicit feedback laws are given by Sontag in [121]. He proves:

Theorem 3.4 ([121]) *Assume that V is a control Lyapunov function satisfying the small control property for the control system (C). Assume that (C) is affine in the controls, that is,*

$$f(x,u) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x),\ \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m,$$

for some $f_0, \dots, f_m$ in $C^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Then $u = (u_1, \dots, u_m) : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$u_i(x) := -\varphi\left(\cdot f_0(x)\cdot\nabla V(x), \sum_{i=1}^{m}\left(f_i(x)\cdot\nabla V(x)\cdot\right)^2\right) f_i(x)\cdot\nabla V(x), \tag{3.1}$$

with

$$\varphi(a,b) = \begin{cases} \frac{a+\sqrt{a^2+b^2}}{b} & \text{if } b \neq 0, \\ 0 & \text{if } b = 0, \end{cases} \tag{3.2}$$

is continuous and globally asymptotically stabilizes the control system (C).

Open Problem 3.5 For systems which are not affine in the controls, find some explicit formulas for globally asymptotically stabilizing periodic time-varying feedback laws when one knows a control Lyapunov function satisfying the small control property. (By Theorem 3.3, such feedback laws exist.)

3.2 Damping feedback laws

The control Lyapunov function is a very powerful tool to design stabilizing feedback laws. But one needs to guess candidates for such functions in order to apply Sontag's formula (3.1). For mechanical systems at least, a natural candidate for a control Lyapunov function is given by the total energy, i.e. the sum of potential and kinetic energy. But, in general, it does not work.

Example 3.6 Consider the classical spring-mass system. The control system is

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -kx_1 + u,$$

where m is the mass of the point attached to the spring, x_1 is the displacement of the mass (on a line), x_2 is the speed of the mass, k is the spring constant, and u is the force applied to the mass. The total energy E of the system is

$$E = \tfrac{k}{2}x_1^2 + \tfrac{m}{2}x_2^2.$$

One has, with the notations of Theorem 3.4,

$$\begin{aligned} f_0(x)\cdot\nabla E(x) &= 0, \\ f_1(x)\cdot\nabla E(x) &= x_2. \end{aligned}$$

Hence, if $x_2 = 0$, there exists no u such that $(f_0(x) + uf_1(x))\cdot\nabla E(x) < 0$. Therefore the total energy is not a control Lyapunov function. But one has

$$(f_0(x) + uf_1(x))\cdot\nabla E(x) = uf_1(x)\cdot\nabla E(x) = ux_2.$$

Hence, it is tempting to consider the feedback law

$$u(x) = -\nu\nabla E(x)\cdot f_1(x) (= -\nu x_2). \tag{3.3}$$

With this feedback law, the closed loop control system is

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{\nu}{m}x_2, \end{aligned} \tag{3.4}$$

which is the dynamic of a spring-mass-dashpot system. In other words, the feedback law adds some damping to the the spring-mass system. With this feedback law

$$\nabla E(x)\cdot(f_0(x) + u(x)f_1(x)) \leqslant 0,$$

so that $(0,0) \in \mathbb{R}^2$ is stable for the closed loop system. In fact $(0,0)$ is globally asymptotically stable for this system. Indeed, if a trajectory $x(t), t \in \mathbb{R}$, of the closed loop system is such that $E(x(t))$ does not depend on time, then

$$x_2(t) = 0, \forall t \in \mathbb{R}. \tag{3.5}$$

Differentiating (3.5) with respect to time and using (3.4), one gets

$$x_1(t) = 0, \forall t \in \mathbb{R},$$

which, with (3.5) and LaSalle's invariance principle, proves that $(0,0)$ is globally asymptotically stable for the closed loop system.

The previous example can be generalized in the following way. We assume that the control system (C) is affine in the controls, that is

$$f(x,u) = f_0(x) + \textstyle\sum_{i=1}^m u_i f_i(x), \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m,$$

for some $f_0, \ldots, f_m$ in $C^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Let $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ be such that

$$\begin{gathered} V(x) \to +\infty, \text{ as } |x| \to +\infty, \\ V(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\}, \\ f_0 \cdot \nabla V \leqslant 0 \text{ in } \mathbb{R}^n. \end{gathered}$$

Then

$$f \cdot \nabla V \leqslant \textstyle\sum_{i=1}^m u_i (f_i \cdot \nabla V).$$

Hence it is tempting to consider the feedback law $u = (u_1, \ldots, u_m)$ defined by

$$u_i = -f_i \cdot \nabla V, \forall i \in [1, m]. \tag{3.6}$$

With this feedback law

$$f(x, u(x)) \cdot \nabla V(x) = f_0(x) \cdot \nabla V(x) - \sum_{i=1}^m (f_i(x) \cdot \nabla V(x))^2 \leqslant 0.$$

Therefore, 0 is stable for the closed loop system $\dot{x} = f(x, u(x))$. By LaSalle's invariance principle it is globally asymptotically stable if the following property holds:

($\mathcal{P}$) For any $x \in C^1(\mathbb{R};\mathbb{R}^n)$ such that

$$\begin{gathered} \dot{x}(t) = f_0(x(t)), \forall t \in \mathbb{R} \\ f_i(x(t)) \cdot \nabla V(x(t)) = 0, \forall t \in \mathbb{R}, \forall i \in [0, m], \end{gathered}$$

one has

$$x(t) = 0, \forall t \in \mathbb{R}.$$

This method has been introduced by Jacobson in [72] and by Jurdjevic-Quinn [73]. There are many sufficient conditions available for property ($\mathcal{P}$). Let us give, for example the following condition, due to Jurdjevic-Quinn [73] (see also [91] for a more general condition).

Theorem 3.7 *With the above notations, assume that, for every $x \in \mathbb{R}^n \setminus \{0\}$,*

$$\mathrm{Span}\{f_0(x), \mathrm{ad}^k_{f_0} f_i(x), \; i = 1, \ldots, m, \; k \in \mathbb{N}\} = \mathbb{R}^n.$$

Then property ($\mathcal{P}$) is satisfied. In particular the feedback law defined by (3.6) *globally asymptotically stabilizes the control system* (C).

Let us recall that $\mathrm{ad}^k_{f_0} f_i \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ is defined by induction on k by

$$\mathrm{ad}^0_{f_0} f_i = f_i,$$
$$\mathrm{ad}^k_{f_0} f_i = [f_0, \mathrm{ad}^{k-1}_{f_0} f_i].$$

Let us point out that this method is also very useful when there are some constraints on the controls. Indeed if, for example, one wants that, for some $\varepsilon > 0$,

$$|u_i(x)| \leqslant \varepsilon, \forall i \in [1, m],$$

then it suffices to replace (3.6) by

$$u_i(x) = -\sigma(f_i(x) \cdot \nabla V(x)),$$

where $\sigma \in C^0(\mathbb{R}; [-\varepsilon, \varepsilon])$ is such that $s\sigma(s) > 0$ for any $s \in \mathbb{R} \setminus \{0\}$. We give an application of this possibility in the next subsection.

3.2.1 Orbit transfer with low-thrust systems

Electric propulsion systems for satellites are seriously considered for performing large amplitude transfers. Let us recall that electric propulsion is characterized by a low-thrust acceleration level but a high specific impulse. In this subsection, where we follow [32], we are interested in a large amplitude transfer of a satellite in a central gravitational field by means of an electric propulsion system.

The state of the control system is the position of the satellite (here identified with a point: we are not considering the attitude of the satellite) and the speed of the satellite. Instead of using Cartesian coordinates, we prefer to use the "orbital" coordinates. The advantage of this set of coordinates is that the first five coordinates remain unchanged if the thrust vanishes: these coordinates characterize the Keplerian elliptic orbit; when thrusts are applied they characterize the Keplerian elliptic osculating orbit of the satellite. The last component is an angle which gives the position of the satellite on the Keplerian elliptic osculating orbit of the satellite. A customary set of orbital coordinates is

$$p = a(1 - e^2),$$
$$e_x = e \cos \tilde{\omega}, \quad \text{with} \quad \tilde{\omega} = \omega + \Omega,$$
$$e_y = e \sin \tilde{\omega},$$
$$h_x = \tan \tfrac{i}{2} \cos \Omega,$$
$$h_y = \tan \tfrac{i}{2} \sin \Omega,$$
$$L = \tilde{\omega} + v,$$

where a, e, ω, Ω, i characterize the Keplerian osculating orbit:

- a is the semi-major axis,
- e is the eccentricity,
- i is the inclination with respect to the equator,
- Ω is the right ascension of the ascending node,
- ω is the angle between the ascending node and the perigee,

and where v is the true anomaly; see, e.g., [18, 109, 137].

In this set of coordinates the equations of motion are

$$(3.7)\qquad \begin{cases} \frac{dp}{dt} = 2\sqrt{\frac{p^3}{\mu}}\frac{1}{Z}S, \\ \frac{de_x}{dt} = \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left[Z(\sin L)Q + AS - e_y(h_x \sin L - h_y \cos L)W\right], \\ \frac{de_y}{dt} = \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left[-Z(\cos L)Q + BS - e_x(h_x \sin L - h_y \cos L)W\right], \\ \frac{dh_x}{dt} = \frac{1}{2}\sqrt{\frac{p}{\mu}}\frac{X}{Z}(\cos L)W, \\ \frac{dh_y}{dt} = \frac{1}{2}\sqrt{\frac{p}{\mu}}\frac{X}{Z}(\sin L)W, \\ \frac{dL}{dt} = \sqrt{\frac{\mu}{p^3}}Z^2 + \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left(h_x \sin L - h_y \cos L\right)W, \end{cases}$$

where $\mu > 0$ is a gravitational coefficient depending on the central gravitational field, Q, S, W, are the radial, orthoradial, and normal components of the thrust delivered by the electric propulsion systems and where

$$(3.8)\qquad Z = 1 + e_x \cos L + e_y \sin L,$$

$$(3.9)\qquad A = e_x + (1+Z)\cos L,$$

$$(3.10)\qquad B = e_y + (1+Z)\sin L,$$

$$(3.11)\qquad X = 1 + h_x^2 + h_y^2.$$

We study the case, useful in applications, where

$$(3.12)\qquad Q = 0,$$

and, for some $\varepsilon > 0$,

$$|S| \leqslant \varepsilon \text{ and } |W| \leqslant \varepsilon.$$

Note that ε is small, since the thrust acceleration level is low. In this subsection we give feedback laws, based on the damping approach, which (globally) asymptotically stabilized a given Keplerian elliptic orbit characterized by the coordinates $p, \bar{e}_x, e_y, \bar{h}_x, \bar{h}_y$. (We are not interested in the position at time t of the satellite on the given Keplerian elliptic orbit; if this position is important, see [13, 32], which uses forwarding techniques developed by Mazenc-Praly [99] and Sepulchre et al. [113].) In oder to simplify the notations (this is not essential for the method) we restrict our attention to the case where the desired final orbit is geostationary, that is,

$$\bar{e}_x = \bar{e}_y = \bar{h}_x = \bar{h}_y = 0.$$

Let

$$\mathcal{A} = (0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2 \times \mathbb{R},$$

where

(3.13) $$\mathcal{B}_1 := \{e = (e_x, e_y) \in \mathbb{R}^2; e_x^2 + e_y^2 < 1\}.$$

With this notation, one requires that the state $(p, e_x, e_y, h_x, h_y, L)$ belongs to $\mathcal{A}$. We are looking for two maps

$$\begin{array}{rccc} S: & \mathcal{A} & \to & [-\varepsilon, \varepsilon] \\ & (p, e_x, e_y, h_x, h_y, L) & \mapsto & S(p, e_x, e_y, h_x, h_y, L) \end{array}$$

and

$$\begin{array}{rccc} W: & \mathcal{A} & \to & [-\varepsilon, \varepsilon] \\ & (p, e_x, e_y, h_x, h_y, L) & \mapsto & W(p, e_x, e_y, h_x, h_y, L) \end{array}$$

such that $(p, 0, 0, 0, 0) \in \mathbb{R}^5$ is globally asymptotically stable for the closed loop system (see (3.7) and (3.12))

(3.14) $$\begin{cases} \frac{dp}{dt} = 2\sqrt{\frac{p^3}{\mu}}\frac{1}{Z}S, \\ \frac{de_x}{dt} = \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left[AS - e_y(h_x \sin L - h_y \cos L)W\right], \\ \frac{de_y}{dt} = \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left[BS - e_x(h_x \sin L - h_y \cos L)W\right], \\ \frac{dh_x}{dt} = \frac{1}{2}\sqrt{\frac{p}{\mu}}\frac{X}{Z}(\cos L)W, \\ \frac{dh_y}{dt} = \frac{1}{2}\sqrt{\frac{p}{\mu}}\frac{X}{Z}(\sin L)W, \\ \frac{dL}{dt} = \sqrt{\frac{\mu}{p^3}}Z^2 + \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left(h_x \sin L - h_y \cos L\right)W, \end{cases}$$

with $(p, e_x, e_y, h_x, h_y, L) \in \mathcal{A}$. Note that $\mathcal{A} \neq \mathbb{R}^6$ and that we are interested only in the first five variables. So we need to specify what we mean by "$(\bar{p}, 0, 0, 0, 0)$ is globally uniformly asymptotically stable for the closed loop system". Various natural definitions are possible. We take the one which sounds the strongest, namely we require:

- *Uniform stability*, that is, for any $\varepsilon_0 > 0$, there exists $\varepsilon_1 > 0$ such that any solution of (3.14) defined at time 0 and satisfying

  $$|p(0) - p| + |e_x(0)| + |e_y(0)| + |h_x(0)| + |h_y(0)| < \varepsilon_1,$$

  is defined for any time $t \geqslant 0$ and satisfies

  $$|p(t) - p| + |e_x(t)| + |e_y(t)| + |h_x(t)| + |h_y(t)| < \varepsilon_0$$

  for any $t \geqslant 0$.

- *Uniform global attractivity*, that is, for any $M > 0$ and for any $\eta > 0$, there exists $T > 0$ such that any solution of (3.14), defined at time 0, such that

  $$\frac{1}{p(0)} + p(0) + \frac{1}{1 - (e_x(0)^2 + e_y(0)^2)} + |h_x(0)| + |h_y(0)| < M,$$

  is defined for any time $t \geqslant 0$ and satisfies

  $$|p(t) - \bar{p}| + |e_x(t)| + |e_y(t)| + |h_x(t)| + |h_y(t)| < \eta$$

for any time $t \geq T$.

We start by a change of "time", already used in Geoffroy [54], describing the evolution of (p, e_x, e_y, h_x, h_y) as a function of L instead of t. Then system (3.14) reads

$$\text{(3.15)} \quad \begin{cases} \frac{dp}{dL} = 2KpS, \\ \frac{de_x}{dL} = K[AS - e_y(h_x \sin L - h_y \cos L)W], \\ \frac{de_y}{dL} = K[BS - e_x(h_x \sin L - h_y \cos L)W], \\ \frac{dh_x}{dL} = \frac{K}{2}X(\cos L)W, \\ \frac{dh_y}{dL} = \frac{K}{2}X(\sin L)W, \\ \frac{dt}{dL} = K\sqrt{\frac{\mu}{p}}Z, \end{cases}$$

with

$$\text{(3.16)} \quad K = \left[\frac{\mu}{p^2}Z^3 + (h_x \sin L - h_y \cos L)W\right]^{-1}.$$

Let V be a function of class C^1 from $(0, \infty) \times \mathcal{B}_1 \times \mathbb{R}^2$ into $[0, \infty)$ such that

$$\text{(3.17)} \quad V(p, e_x, e_y, h_x, h_y) = 0 \quad \Leftrightarrow \quad (p, e_x, e_y, h_x, h_y) = (\bar{p}, 0, 0, 0, 0),$$

$$\text{(3.18)} \quad V(p, e_x, e_y, h_x, h_y) \to +\infty \text{ if } (p, e_x, e_y, h_x, h_y) \to \partial((0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2).$$

In (3.18), the boundary $\partial((0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2)$ is taken in the set $[0, +\infty] \times \bar{\mathcal{B}}_1 \times [-\infty, +\infty]^2$. Therefore condition (3.18) is equivalent to the following condition: for any $M > 0$, there exists a compact set $\mathcal{K}$ included in $(0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2$ such that

$$\left((p, e_x, e_y, h_x, h_y) \in ((0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2) \setminus \mathcal{K}\right) \Rightarrow (V(p, e_x, e_y, h_x, h_y) \geq M).$$

(One can take, for example,

$$\text{(3.19)} \quad V(p, e_x, e_y, h_x, h_y) = \frac{1}{2}\left(\frac{(p - \bar{p})^2}{p} + \frac{e^2}{1 - e^2} + h^2\right),$$

with $e^2 = e_x^2 + e_y^2$ and $h^2 = h_x^2 + h_y^2$.) The time derivative of V along a trajectory of (3.15) is given by

$$\dot{V} = K(\alpha S + \beta W),$$

with

$$\text{(3.20)} \quad \alpha = 2p\frac{\partial V}{\partial p} + A\frac{\partial V}{\partial e_x} + B\frac{\partial V}{\partial e_y},$$

$$\text{(3.21)} \quad \beta = (h_y \cos L - h_x \sin L)\left(e_y\frac{\partial V}{\partial e_x} + e_x\frac{\partial V}{\partial e_y}\right) + \frac{1}{2}X\left((\cos L)\frac{\partial V}{\partial h_x} + (\sin L)\frac{\partial V}{\partial h_y}\right).$$

Following the damping method, one defines

$$\text{(3.22)} \quad S = -\sigma_1(\alpha),$$

$$\text{(3.23)} \quad W = -\sigma_2(\beta)\sigma_3(p, e, h),$$

where $\sigma_1 : \mathbb{R} \to \mathbb{R}, \sigma_2 : \mathbb{R} \to \mathbb{R}$ and $\sigma_3 : (0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2 \to (0, 1]$ are continuous functions such that

$$\sigma_1(s) \cdot s > 0, \quad \forall s \in \mathbb{R} \setminus \{0\}, \tag{3.24}$$

$$\sigma_2(s) \cdot s > 0, \quad \forall s \in \mathbb{R} \setminus \{0\}, \tag{3.25}$$

$$\| \sigma_1 \|_{L^\infty(\mathbb{R})} < \varepsilon, \tag{3.26}$$

$$\| \sigma_2 \|_{L^\infty(\mathbb{R})} < \varepsilon, \tag{3.27}$$

$$\sigma_3(p, e, h) \leq \frac{1}{1 + \varepsilon\, p^2} \frac{\mu}{p^2} \frac{(1 - |e|)^3}{|h|}, \ \forall (p, e, h) \in (0, +\infty) \times \mathcal{B}_1 \times (\mathbb{R}^2 \setminus \{0\}). \tag{3.28}$$

The reason for using σ_3 is to ensure the existence of K defined by (3.16). Indeed from (3.8), (3.15), (3.23), (3.27) and (3.28), one obtains for any $L \in \mathbb{R}$ that

$$\frac{\mu Z^3}{p^2} + (h_x \sin L - h_y \cos L) W > 0$$

on $(0, \infty) \times \mathcal{B}_1 \times \mathbb{R}^2$ and therefore K is well-defined for any $(p, e, h, L) \in (0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2 \times \mathbb{R}$ (see (3.16)). One has

$$\| S \|_{L^\infty((0,+\infty) \times \mathcal{B}_1 \times \mathbb{R}^2 \times \mathbb{R})} < \varepsilon, \tag{3.29}$$

$$\| W \|_{L^\infty((0,+\infty) \times \mathcal{B}_1 \times \mathbb{R}^2 \times \mathbb{R})} < \varepsilon, \tag{3.30}$$

$$\dot{V} \leq 0, \text{ and } ((\dot{V} = 0) \Leftrightarrow (\alpha = \beta = 0)).$$

Since the closed loop system (3.15) is L-varying but *periodic* with respect to L one may apply LaSalle's invariance principle: in order to prove that $(\bar{p}, 0, 0, 0, 0)$ is globally asymptotically stable on $(0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2$ for the closed loop system (3.15), it suffices to check that any trajectory of (3.15) such that $\alpha \equiv \beta \equiv 0$ is identically equal to $(\bar{p}, 0, 0, 0, 0)$. For such a trajectory one has – see in particular (3.15), (3.20), (3.21), (3.22), (3.23), (3.24), (3.25) and (3.28) –

$$\frac{dp}{dL} = 0\,, \ \frac{de_x}{dL} = 0\,, \ \frac{de_y}{dL} = 0\,, \ \frac{dh_x}{dL} = 0\,, \ \frac{dh_y}{dL} = 0, \tag{3.31}$$

$$2p\frac{\partial V}{\partial p} + A\frac{\partial V}{\partial e_x} + B\frac{\partial V}{\partial e_y} = 0, \tag{3.32}$$

$$(h_y \cos L - h_x \sin L) \left(e_y \frac{\partial V}{\partial e_x} + e_x \frac{\partial V}{\partial e_y} \right) + \frac{1}{2} X \left((\cos L) \frac{\partial V}{\partial h_x} + (\sin L) \frac{\partial V}{\partial h_y} \right) = 0. \tag{3.33}$$

Hence p, e_x, e_y, h_x et h_y are constant. The left hand side of (3.32) is a linear combination of the functions $\cos L, \sin L, \cos^2 L, \sin L \cos L$ and the constant functions. These functions are linearly independent, so that

$$2p\frac{\partial V}{\partial p} + e_x \frac{\partial V}{\partial e_x} + 2e_y \frac{\partial V}{\partial e_y} = 0\,, \quad \frac{\partial V}{\partial e_x} = 0\,, \quad \frac{\partial V}{\partial e_y} = 0,$$

and therefore

$$\frac{\partial V}{\partial p} = \frac{\partial V}{\partial e_x} = \frac{\partial V}{\partial e_y} = 0;$$

which, with (3.11) and (3.33), gives

$$\frac{\partial V}{\partial h_x} = \frac{\partial V}{\partial h_y} = 0.$$

Hence it suffices to impose on V that

$$(\nabla V(p, e_x, e_y, h_x, h_y) = 0) \Rightarrow ((p, e_x, e_y, h_x, h_y) = (\bar{p}, 0, 0, 0, 0)). \tag{3.34}$$

Note that, if V is given by (3.19), then V satisfies (3.34).

3.2.2 Damping feedback and driftless systems

Throughout this subsection we again assume that (C) is a driftless system, i.e.

$$f(x, u) = \sum_{i=1}^{m} u_i f_i(x).$$

We assume also that the Lie algebra rank condition (2.18) holds. Then Theorem 2.23 tells us that, for every $T > 0$, the control system (C) is globally asymptotically stabilizable by means of T-periodic time-varying feedback laws. Let us recall that the main ingredient of the proof is the existence of $\bar{u}$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ vanishing on $\{0\} \times \mathbb{R}$, T-periodic with respect to time and such that, if $\dot{x} = f(x, \bar{u}(x, t))$, then

$$x(T) = x(0). \tag{3.35}$$

(3.36) If $x(0) \neq 0$, the linearized control system around $(x, \bar{u})$ is controllable on $[0, T]$.

In this subsection we want to explain how the damping method allows to construct from this $\bar{u}$ a T-periodic time-varying feedback law u which globally asymptotically stabilizes the control system (C). With slight modifications we follow [31], which is directly inspired by Pomet [104]. Let $W \in C^\infty(\mathbb{R}^n; \mathbb{R})$ be any function such that

$$\begin{aligned} &W(x) \to +\infty \text{ as } |x| \to +\infty, \\ &W(x) > W(0), \forall x \in \mathbb{R}^n \setminus \{0\}, \\ &\nabla W(x) \neq 0, \forall x \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{3.37}$$

One can take, for example, $W(x) = |x|^2$. Let $X(x, t) = \sum_{i=1}^m \bar{u}_i(x, t) f_i(x)$ and let $\Phi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$, $(x, t, s) \mapsto \Phi(x, t, s)$, be the flow associated with the time-varying vector field X., i.e.

$$\tfrac{\partial \Phi}{\partial t} = X(\Phi, t), \tag{3.38}$$

$$\Phi(x, s, s) = x, \forall x \in \mathbb{R}^n, \forall s \in \mathbb{R}. \tag{3.39}$$

Note that by (3.35)

$$\Phi(x, 0, T) = \Phi(x, 0, 0) = x, \ \forall x \in \mathbb{R}^n. \tag{3.40}$$

Let us now define $V \in C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ by

$$V(x, t) = W(\Phi(x, 0, t)). \tag{3.41}$$

By (3.40), V is T-periodic with respect to time and one easily checks that

$$V(x,t) > V(0,t) = W(0), \forall (x,t) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R},$$
$$\lim_{|x|\to+\infty} \mathrm{Min}\{V(x,t);\, t \in \mathbb{R}\} = +\infty.$$

Moreover, from (3.38) and (3.41),

$$\frac{\partial V}{\partial t} + X \cdot \nabla V = 0, \tag{3.42}$$

so that, along the trajectories of $\dot{x} = \sum_{i=1}^{m} (\bar{u}_i + v_i) f_i(x)$, the time derivative $\dot{V}$ of V is

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + (\sum_{i=1}^{m} (\bar{u}_i + v_i) f_i) \cdot \nabla V \\ &= \sum_{i=1}^{m} (v_i (f_i \cdot \nabla V)). \end{aligned} \tag{3.43}$$

Hence, as above, one takes $v_i(x,t) = -f_i(x) \cdot \nabla V(x,t)$, which with (3.43), gives

$$\dot{V} = -\sum_{i=1}^{m} (f_i(x) \cdot \nabla V(x,t))^2.$$

By LaSalle's invariance principle, in order to prove that $u = \bar{u} + v$ globally asymptotically stabilizes the control system (C), it suffices to check that any trajectory $x : \mathbb{R} \to \mathbb{R}^n$ of X such that

$$f_i(\bar{x}(t)) \cdot \nabla V(\bar{x}(t), t) = 0,\ \forall t \in \mathbb{R}, \forall i \in [1, m], \tag{3.44}$$

satisfies

$$x(0) = 0. \tag{3.45}$$

Let us denote by $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ the set of linear maps from $\mathbb{R}^p$ into $\mathbb{R}^q$. Let $A \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$ and $B \in C^\infty(\mathbb{R}; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$ be defined by

$$A(t) = \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t)),\ \forall t \in \mathbb{R}, \tag{3.46}$$

$$B(t) = \frac{\partial f}{\partial u}(\bar{x}(t), \bar{u}(t)),\ \forall t \in \mathbb{R}. \tag{3.47}$$

Let $R : \mathbb{R} \times \mathbb{R} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $(t,s) \mapsto R(t,s)$, be the fundamental solution of the time-varying linear differential equation $\dot{y} = A(t)y$, i.e.,

$$\begin{aligned} \frac{\partial R}{\partial t} &= A(t) R \text{ on } \mathbb{R} \times \mathbb{R}, \\ R(s,s)x &= x, \forall (s,x) \in \mathbb{R} \times \mathbb{R}^n. \end{aligned}$$

We identify vectors with elements of $\mathcal{L}(\mathbb{R}, \mathbb{R}^n)$ and denote by $E^* \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ the adjoint of $E \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^p)$. By (3.41) one has

$$\begin{aligned} (\nabla V(\bar{x}(t), t))^* &= \nabla W(\bar{x}(0))^* \frac{\partial \Phi}{\partial x}(\bar{x}(t), 0, t) \\ &= \nabla W(\bar{x}(0))^* R(0,t),\ \forall x \in \mathbb{R}, \end{aligned}$$

which, with (3.44) and (3.47), gives

$$(\nabla W(x(0)))^* R(0,t)B(t) = 0, \ \forall t \in \mathbb{R}.$$

In particular

$$(\nabla W(\bar{x}(0)))^* \left(\int_0^T R(0,t)B(t)B(t)^* R(0,t)^* \mathrm{d}t\right) (\nabla W(\bar{x}(0))) = 0,$$

which, by (3.37), shows that, if $\bar{x}(0) \neq 0$, the non-negative symmetric matrix

$$\tilde{C} = \int_0^T R(0,t)B(t)B(t)^* R(0,t)^* \mathrm{d}t$$

is not invertible. But it is well-known (see, for example, [123, Theorem 5, p. 109]) that the time-varying linear control system

$$\dot{y} = A(t)y + B(t)w, \ t \in [0,T],$$

where $y \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^m$ is the control, is controllable on $[0,T]$ (if and) only if $C := R(T,0)\tilde{C}R(T,0)^*$ is invertible. Hence, using (3.36), one obtains (3.45).

3.3 Homogeneity

Let us start by recalling the following classical result:

Theorem 3.8 *Let $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$. If 0 is asymptotically stable for the linear system $\dot{y} = X'(0)y$, then 0 is locally asymptotically stable for $\dot{x} = X(x)$.*

A classical application of this theorem to feedback stabilization is the following well-known property. Consider the linearized control system of (C) around $(0,0)$, i.e. the linear control system

$$\dot{y} = \frac{\partial f}{\partial x}(0,0)y + \frac{\partial f}{\partial u}(0,0)v,$$

where $y \in \mathbb{R}^n$ is the state and $v \in \mathbb{R}^m$ is the control. Assume that this linear control system is asymptotically stabilizable by means of a feedback law. Then it is asymptotically stabilizable by means of a linear feedback law $v(x) = Kx$ with $K \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$. By Theorem 3.8, this feedback law locally asymptotically stabilizes the control system (C).

The idea of "homogeneity" is a generalization of the above procedure: one wants to deduce the asymptotic stabilizability of the control system (C) from the asymptotic stabilizability of a "simpler" system than (C).

Let us now give the definition of a homogeneous vector field. Since we are going to give an application to periodic time-varying feedback laws, the vector fields we consider depend on time and are T-periodic with respect to time. The vector fields are also assumed to be continuous. Let $r = (r_1, \ldots, r_n) \in (0,+\infty)^n$. One has the following definition (see [108, Chapitre 3] for various generalizations):

Definition 3.9 The vector field $X = (X_1, \ldots, X_n)$ is *r-homogeneous of degree* 0, if, for every $\varepsilon > 0$, every $x \in \mathbb{R}^n$, every $i \in [1,n]$ and every $t \in \mathbb{R}$,

$$X_i(\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n, t) = \varepsilon^{r_i} X_i(x_1, \ldots, x_n, t).$$

Since the degree of homogeneity will be always 0 in this paper, we shall omit "of degree 0".

Example 3.10 1. A time-varying linear system $\dot{y} = A(t)y$ is $(1, ..., 1)$-homogeneous. 2. Take $n = 2$ and $X(x_1, x_2) = (x_1 - x_2^3, x_2)$. Then X is (3,1)-homogeneous.

For applications to feedback stabilization, the key theorem is

Theorem 3.11 *Let us assume that*

$$X = Y + R, \tag{3.48}$$

where Y and R are T-periodic time-varying vector fields such that Y is r-homogeneous and, for some $\eta > 0$ and $M > 0$, one has, for every $i \in [1, n]$, every $\varepsilon \in (0, 1]$, and every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $|x| \leqslant 1$,

$$|R_i(\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n)| \leqslant M\varepsilon^{r_i+\eta}. \tag{3.49}$$

Then, if 0 is locally (=globally) asymptotically stable for $\dot{x} = Y(x, t)$, it is also locally asymptotically stable for $\dot{x} = X(x, t)$.

This theorem has been proved by Hermes in [64] when one has uniqueness of the trajectories of $\dot{x} = Y(x)$, and in the general case by Rosier in [107]. In fact [107], as well as [64], deal with the case of stationary vector fields. But the proof of [107] can be easily extended to the case of periodic time-varying vector fields. Let us briefly sketch the arguments. One first observes that Theorem 3.11 is a corollary of the following theorem, which has its own interest and goes back to Massera [97] when the vector fields is of class C^1.

Theorem 3.12 ([107, 105]) *Let Y be a T-periodic time-varying vector field which is r-homogeneous. We assume that 0 is locally (=globally) asymptotically stable for $\dot{x} = Y(x, t)$. Let p be a positive integer and let $k \in \{p \max_{1\leqslant i\leqslant n} r_i, +\infty\}$. Then there exists a function $V \in C^\infty((\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}; \mathbb{R}) \cap C^p(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ such that*

$$\begin{gathered} V(x,t) > V(0,t) = 0, \forall (x,t) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}, \\ V(x, t+T) = V(x,t), \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \lim_{|x|\to+\infty} \text{Min } \{V(x,t); t \in \mathbb{R}\} = +\infty, \end{gathered} \tag{3.50}$$

$$\frac{\partial V}{\partial t} + Y \cdot \nabla V < 0 \text{ in } (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}, \tag{3.51}$$

$$V(\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n, t) = \varepsilon^k V(x_1, \ldots, x_n, t), \forall (\varepsilon, x, t) \in (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}. \tag{3.52}$$

Let us deduce, as in [107], Theorem 3.11 from Theorem 3.12. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\varepsilon > 0$, let

$$\delta_\varepsilon^r(x) = (\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n). \tag{3.53}$$

Let V be as in Theorem 3.12 with $p = 1$. From (3.51), there exists $\nu > 0$ such that

$$\left(\frac{\partial V}{\partial t} + Y \cdot \nabla V\right)(x,t) \leqslant -\nu, \tag{3.54}$$

for any $t \in [0,T]$ and any $x \in \mathbb{R}^n$ such that $|x_1|^{1/r_1} + \ldots + |x_n|^{1/r_n} = 1$. From (3.54) and the assumption that Y is r-homogeneous, we get that

$$(3.55)\quad \left(\frac{\partial V}{\partial t} + Y \cdot \nabla V\right)(\delta_\varepsilon^r(x), t) = \varepsilon^k \left(\frac{\partial V}{\partial t} + Y \cdot \nabla V\right)(x,t),\ \forall(\varepsilon, x, t) \in (0,+\infty) \times \mathbb{R}^n \times \mathbb{R}.$$

¿From (3.52) and (3.56), we obtain

$$(3.56)\quad \left(\frac{\partial V}{\partial t} + Y \cdot \nabla V\right)(x,t) \leqslant -\nu \left(|x_1|^{1/r_1} + \ldots + |x_n|^{1/r_n}\right)^k,\ \forall (x,t) \in \mathbb{R} \times [0,T].$$

Using (3.49) and (3.52), similar computations show the existence of $\bar{C} > 0$ such that

$$(3.57)\quad (R \cdot \nabla V)(x,t) \leqslant \bar{C}\left(|x_1|^{1/r_1} + \ldots + |x_n|^{1/r_n}\right)^{\eta+k}$$

for any $t \in [0,T]$ and any $x \in \mathbb{R}^n$ with $|x| \leqslant 1$. From (3.48), (3.56) and (3.57), we get the existence of $\rho > 0$ such that

$$\left(\frac{\partial V}{\partial t} + X \cdot \nabla V\right)(x,t) < 0,\ \forall t \in [0,T],\ \forall x \in \mathbb{R}^n \text{ with } 0 < |x| \leqslant \rho,$$

which ends the proof of Theorem 3.11.

Finally we sketch the proof of Theorem 3.12, given in [107] for stationary vector fields and extended by Pomet and Samson in [105] to the case of time-varying vector fields. By Kurzweil's theorem [87], there exists $W \in C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ such that

$$\begin{gathered} W(x,t) > W(0,t) = 0,\ \forall (x,t) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}, \\ W(x, t+T) = W(x,t),\ \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \lim_{|x| \to +\infty} \operatorname{Min}\{W(x,t); t \in \mathbb{R}\} = +\infty, \\ \frac{\partial W}{\partial t} + Y \cdot \nabla W < 0 \text{ in } (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}. \end{gathered}$$

Let $a \in C^\infty(\mathbb{R};\mathbb{R})$ be such that $a' \geqslant 0$, $a = 0$ in $(-\infty, 1]$ and $a = 1$ in $[2,+\infty)$. Then one can prove that V, defined by

$$V(x,t) = \int_0^{+\infty} \frac{1}{s^{k+1}} a(V(s^{r_1}, \ldots, s^{r_n}, t))ds,\ \forall (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

satisfies all the required conditions.

Example 3.13 Following [39], let us give an application to the construction of explicit time-varying feedback laws stabilizing asymptotically the attitude of a rigid body spacecraft with two controls, a problem already considered in Examples 2.15, 2.17 and 2.33. Without loss of generality we may assume that $\{v_1 b_1 + v_2 b_2;\ (v_1, v_2) \in \mathbb{R}^2\} = \{0\} \times \mathbb{R}^2$. So, after a change of the control variables, (2.9) can be replaced by

$$(3.58)\quad \dot{\omega}_1 = Q(\omega) + \omega_1 L_1(\omega),\ \ \dot{\omega}_2 = V_1,\ \ \dot{\omega}_3 = V_2,\ \dot{\eta} = A(\eta)\omega,$$

with $L_1\omega = D_1\omega_1 + E_1\omega_2 + F_1\omega_3$, $Q(\omega) = A\omega_2^2 + B\omega_2\omega_3 + C\omega_3^2$. For system (3.58) the controls are V_1 and V_2. It is proved in [79] that Q changes sign if and only if the control system (2.9) satisfies the strong Lie algebra rank condition at $(0, 0)$ which, by Theorem 2.9, is a necessary condition for small time local controllability. From now on we assume that Q changes sign this is a generic situation. Hence, after a suitable change of coordinates of the form:

$$\omega = P\tilde{\omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_p & b_p \\ 0 & c_p & d_p \end{pmatrix} \tilde{\omega}, \tag{3.59}$$

system (3.58) can be written

$$\dot{\tilde{\omega}}_1 = \tilde{\omega}_2\tilde{\omega}_3 + \tilde{\omega}_1 L_2(\tilde{\omega}),\ \dot{\tilde{\omega}}_2 = u_1,\ \ \dot{\tilde{\omega}}_3 = u_2,\ \dot{\eta} = A(\eta)P\tilde{\omega} \tag{3.60}$$

with $L_2\tilde{\omega} = D_2\tilde{\omega}_1 + E_2\tilde{\omega}_2 + F_2\tilde{\omega}_3$. Let $c = \det P$; we can always choose P so that $c > 0$. Let

$$x_1 = \tilde{\omega}_1,\ x_5 = \tilde{\omega}_2,\ x_6 = \tilde{\omega}_3,\ x_3 = \frac{1}{c}(d_p\theta - b_p\psi);$$
$$x_4 = \frac{1}{c}(-c_p\theta + a_p\psi),\ x_2 = \phi - \frac{b_p d_p}{2}x_4^2 - \frac{a_p c_p}{2}x_3^2 - b_p c_p x_3 x_4.$$

In these coordinates, our system can be written

$$\begin{cases} \dot{x}_1 = x_5x_6 + R_1(x),\ \ \dot{x}_2 = x_1 + cx_3x_6 + R_2(x), \\ \dot{x}_3 = x_5 + R_3(x),\ \ \dot{x}_4 = x_6 + R_4(x),\ \ \dot{x}_5 = u_1,\ \ \dot{x}_6 = u_2 \end{cases} \tag{3.61}$$

where R_1, R_2, R_3, and R_4 are analytic functions on a neighborhood of 0 such that, for a suitable positive constant C, one has, for all x in $\mathbb{R}^6$ with $|x|$ small enough,

$$|R_1(x)| + |R_3(x)| + |R_4(x)| \le C\left(|x_1| + |x_2| + |x_3|^2 + |x_4|^2 + |x_5|^2 + |x_6|^2\right)^{3/2}, \tag{3.62}$$

$$|R_2(x)| \leqslant C\left(|x_1| + |x_2| + |x_3|^2 + |x_4|^2 + |x_5|^2 + |x_6|^2\right)^2. \tag{3.63}$$

Hence our control system can be written

$$\dot{x} = f(x, u) = X(x) + R(x) + uY(x) \tag{3.64}$$

where $x = (x_1, \ldots, x_6) \in \mathbb{R}^6$ is the state, $u = (u_1, u_2) \in \mathbb{R}^2$ is the control,

$$uY = u_1Y_1 + u_2Y_2 = (0, 0, 0, 0, u_1, u_2),\ \ X(x) = (x_5x_6, x_1 + cx_3x_6, x_5, x_6, 0, 0), \tag{3.65}$$

where c is a constant in $(0, +\infty)$, and R is a *perturbation* term in the following sense. Note that X is $(2, 2, 1, 1, 1, 1)$-homogeneous and that, for a suitable constant $C > 0$, the vector field R satisfies, for all ε in $(0, 1)$ and all $x = (x_1, \ldots, x_6)$ in $\mathbb{R}^6$ with $|x| \leqslant 1$,

$$|R_i(\delta_\varepsilon(x))| \le C_0\varepsilon^{1+r_i},\ \forall i \in [1, 6];$$

with

$$\delta_\varepsilon(x) = (\varepsilon^2x_1, \varepsilon^2x_2, \varepsilon x_3, \varepsilon x_4, \varepsilon x_5, \varepsilon x_6).$$

Keeping in mind this $(2, 2, 1, 1, 1, 1)$-homogeneity and Theorem 3.11 it is natural to consider time-varying feedback laws u which have the following property:

$$u(\delta_\varepsilon x, t) = \varepsilon u(x, t),\ \forall x \in \mathbb{R}^6,\ \ \forall t \in \mathbb{R}. \tag{3.66}$$

Indeed, assume that u is a periodic time-varying feedback law satisfying (3.66) which locally (=globally) asymptotically stabilizes the control system

$$\dot{x} = X(x) + uY. \tag{3.67}$$

Then, from Theorem 3.11, u locally asymptotically stabilizes the control system (3.64). In subsections 3.4 and 3.5, we shall give a method, due to Morin-Samson [100], for constructing a periodic time-varying feedback law u satisfying (3.66) which locally (=globally) asymptotically stabilizes control system (3.67); see also [39] for another method for constructing such feedback laws.

Remark 3.14 For other applications of the homogeneity techniques, see, for example, [64, 65, 77] and the references therein.

3.4 Averaging

Let us start with the following classical result (see, e.g., [80, Thm. 7.4, p. 417]).

Theorem 3.15 *Let X be a T-periodic time-varying feedback law of class C^2. Assume that the origin is locally exponentially asymptotically stable for the "averaged" system*

$$\dot{x} = \frac{1}{T}\int_0^T X(x,t)\, dt. \tag{3.68}$$

Then there exits $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the origin is locally asymptotically stable for $\dot{x} = X(x, t/\varepsilon)$.

By *locally exponentially asymptotically stable* one means the existence of $(r, C, \lambda) \in (0, +\infty)^3$ such that $|x(t)| \leqslant C|x(0)|\exp(-\lambda t)$ for any solution x of the averaged system (3.68) such that $|x(0)| \leqslant r$. This is equivalent (see, e.g., [80, Thm. 4.4, p. 179]) to the property "the origin is asymptotically for the linear system $\dot{y} = (1/T)(\int_0^T (\partial X/\partial x)(0,t)dt)y$".

In the case of homogeneous vector fields, this theorem has been improved by M'Closkey-Murray in [95]. They prove the following theorem:

Theorem 3.16 *Let X be a continuous T-periodic time-varying feedback law which is r-homogeneous (of degree 0). Assume that the origin is locally (=globally) asymptotically stable for the averaged system (3.68). Then there exits $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the origin is locally asymptotically stable for $\dot{x} = X(x, t/\varepsilon)$.*

Morin-Samson has given in [100] a proof of this theorem which provides us with a value of ε_0 if X is has the form $X(x,t) = f_0(x) + \sum_{i=1}^m g_i(t) f_i(x)$.

Example 3.17 Following [100], let us give an application of this theorem to the construction of explicit time-varying feedback laws stabilizing asymptotically the attitude of a rigid body spacecraft with two controls. In Example 3.13, we have reduced this problem to the construction of a periodic time-varying feedback law u satisfying (3.66) which locally (=globally) asymptotically stabilizes the control system (3.67). Now the strategy is to construct a periodic time-varying feedback law with good homogeneity which globally asymptotically stabilizes the control system

$$\dot{x_1} = \bar{x}_5\bar{x}_6, \;\; \dot{x_2} = x_1 + cx_3\bar{x}_6, \;\; \dot{x_3} = \bar{x}_5, \;\; \dot{x_4} = \bar{x}_6, \tag{3.69}$$

where the state is $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and the control $(\bar{x}_5, \bar{x}_6) \in \mathbb{R}^2$. By good homogeneity we mean that, for all t in $\mathbb{R}$, all (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$, all ε in $(0, +\infty)$, and i in $\{5, 6\}$,

$$\bar{x}_i(\varepsilon^2 x_1, \varepsilon^2 x_2, \varepsilon x_3, \varepsilon x_4, t) = \varepsilon \bar{x}_i(x_1, x_2, x_3, x_4, t). \tag{3.70}$$

Using the backstepping method explained in the following section, we shall see in Example 3.20 how to deduce, from such a feedback law $(\bar{x}_5, \bar{x}_6)$, a feedback law $\tilde{u} : \mathbb{R}^6 \times \mathbb{R} \to \mathbb{R}^2$, $(x_1, x_2, x_3, x_4, x_5, x_6, t) \to \tilde{u}(x_1, x_2, x_3, x_4, x_5, x_6, t)$ which is periodic in time, has a good homogeneity, and globally asymptotically stabilizes the control system obtained from the control system (3.69) by adding an integrator on x_5 and on x_6, i.e. the control system (3.67).

For $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, let $\rho = \rho(x) = (x_1^2 + x_2^2 + x_3^4 + x_4^4)^{1/4}$. Let $\bar{x}_5 \in C^0(\mathbb{R}^4; \mathbb{R})$ and $\bar{x}_6 \in C^0(\mathbb{R}^4; \mathbb{R})$ be defined by

$$x_5 = -x_3 - \rho \sin \frac{t}{\varepsilon}, \quad \bar{x}_6 = -x_4 - \frac{2}{\rho}(x_1 + x_2) \sin \frac{t}{\varepsilon}. \tag{3.71}$$

Then the closed loop system (3.69)-(3.71) is (2,2,1,1)-homogeneous and its corresponding averaged system is

$$\dot{x_1} = -x_1 - x_2, \quad \dot{x_2} = x_1 + cx_3\bar{x}_6, \quad \dot{x_3} = -x_3, \quad \dot{x_4} = -x_4. \tag{3.72}$$

The origin is locally asymptotically stable for system (3.72), since it is asymptotically stable for the linear approximation of this system. Then, by Theorem 3.16, the feedback law (3.71) locally asymptotically stabilizes the control system (3.69) if ε is small enough.

3.5 Backstepping

In backstepping, we are interested in a control system (C) having the following structure:

$$\dot{x}_1 = f_1(x_1, x_2), \tag{3.73}$$
$$\dot{x}_2 = u, \tag{3.74}$$

where the state is $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^m = \mathbb{R}^n$ and the control $u \in \mathbb{R}^m$. The key theorem for backstepping is the following one:

Theorem 3.18 *Assume that f_1 is of class C^1 and that the control system*

$$\dot{x}_1 = f_1(x_1, v), \tag{3.75}$$

where the state is $x_1 \in \mathbb{R}^{n_1}$ and the control $v \in \mathbb{R}^m$, can be globally asymptotically stabilized by means of a stationary feedback law of class C^1. Then control system (3.73)-(3.74) *can be globally asymptotically stabilized by means of a continuous stationary feedback law.*

A similar theorem holds for time-varying feedback laws and local asymptotic stabilization. Theorem 3.18 has been proved independently by Byrnes-Isidori [11], Koditschek [82] and Tsinias [131]. A local version of Theorem 3.18 has been known for a long time; see, e.g., Vidyasagar [132]. Let us give the proof of [11, 82, 131] –for a different method, see Sontag [120]. Let $v \in C^1(\mathbb{R}^{n_1}; \mathbb{R}^m)$ be a feedback law which globally asymptotically stabilizes control system (3.75). Then, by the converse of the second Lyapunov Theorem, there exists a Lyapunov function for the closed loop system $\dot{x}_1 = f(x_1, v(x_1))$, that is, there exists a function $V \in C^\infty(\mathbb{R}^{n_1}; \mathbb{R})$ such that

$$(3.76)\qquad \begin{gathered} f(x_1, v(x_1)) \cdot \nabla V(x_1) < 0,\ \forall x_1 \in \mathbb{R}^{n_1} \setminus \{0\}, \\ V(x_1) \to +\infty \text{ as } |x_1| \to +\infty, \\ V(x_1) > V(0),\ \forall x_1 \in \mathbb{R}^{n_1} \setminus \{0\}. \end{gathered}$$

A natural candidate for a control Lyapunov function for control system (3.73)-(3.74) is

$$(3.77)\qquad W(x_1, x_2) = V(x_1) + \frac{1}{2}|x_2 - v(x_1)|^2.$$

Indeed, one has

$$(3.78)\qquad W(x_1, x_2) \to +\infty \text{ as } |x_1| + |x_2| \to +\infty,$$

$$(3.79)\qquad W(x_1, x_2) > W(0),\ \forall (x_1, x_2) \in (\mathbb{R}^{n_1} \times \mathbb{R}^m) \setminus \{(0, 0)\}.$$

Moreover, if one computes the time-derivative $\dot{W}$ of W along the trajectories of (3.73)-(3.74), one gets

$$(3.80)\qquad \dot{W} = f_1(x_1, x_2) \cdot \nabla V(x_1) - (x_2 - v(x_1)) \cdot (v'(x_1) f_1(x_1, x_2) - u).$$

Since f_1 is of class C^1, there exists $G \in C^0(\mathbb{R}^{n_1} \times \mathbb{R}^m \times \mathbb{R}^m; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{n_1}))$ such that

$$(3.81)\qquad f_1(x_1, x_2) - f_1(x_1, y) = G(x_1, x_2, y)(x_2 - y),\ \forall (x_1, x_2, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^m \times \mathbb{R}^m.$$

By (3.80) and (3.81),

$$\begin{aligned} \dot{W} = {} & f(x_1, v(x_1)) \cdot \nabla V(x_1) \\ & +(u^* - (v'(x_1) f_1(x_1, x_2))^* + (\nabla V(x_1))^* G(x_1, x_2, v(x_1)))(x_2 - v(x_1)). \end{aligned}$$

Hence, taking as feedback law for the control system (3.73)-(3.74)

$$(3.82)\qquad u = v'(x_1) f_1(x_1, x_2) - G(x_1, x_2, v(x_1))^* \nabla V(x_1) - (x_2 - v(x_1)),$$

one obtains

$$\dot{W} = f(x_1, v(x_1)) \cdot \nabla V(x_1) - |x_2 - v(x_1)|^2$$

which, with (3.76), gives

$$\dot{W} < 0 \text{ on } (\mathbb{R}^{n_1} \times \mathbb{R}^m) \setminus \{(0, 0)\}.$$

Hence, the feedback law (3.82) globally asymptotically stabilizes the control system (3.73)-(3.74).

Let us point out that this proof uses the C^1 regularity of f^1 and v. In fact one knows that Theorem 3.18 does not hold in the following cases:

- f_1 is only continuous (see [33, Remark 3.2]);
- "stationary" is replaced by "periodic time-varying", and the feedback law which asymptotically stabilizes the control system (3.75) is only assumed to be continuous (see [33, Proposition 3.1]).

One does not have any counterexample to Theorem 3.18 when the feedback laws which asymptotically stabilize the control system (3.75) are only continuous. But it seems reasonable to conjecture that such counterexamples exist. It will be more interesting to know if there exists a counterexample such that the control system (3.75) satisfies the Hermes condition S(0) (see section 2.2). Let us recall (see Proposition 2.11) that, if the control system (3.75) satisfies the Hermes condition, then the control system (3.73)-(3.74) satisfies also the Hermes condition, and so, by Theorem 2.10, is small time locally controllable.

3.5.1 Desingularization

In some cases where v is not of class C^1, one can use a "desingularization" technique introduced in [106]. Instead of giving the method in its full generality (see [106] for a precise general statement), let us explain it on a simple example. We take $n = 2$, $n_1 = m = 1$, and $f_1(x_1, x_2) = x_1 - 2x_2^3$, so the control system (3.75) is

$$\dot{x}_1 = x_1 - 2v^3. \tag{3.83}$$

Clearly the feedback law $v(x_1) = x_1^{1/3}$ globally asymptotically stabilizes the control system (3.83). Kawski has given in [75] an explicit continuous stationary feedback law which asymptotically stabilizes the control system (3.73)-(3.74), which is

$$\dot{x}_1 = x_1 - 2x_2^3,\ \dot{x}_2 = u. \tag{3.84}$$

Note that the control systems (3.83) and (3.84) cannot be stabilized by means of feedback laws of class C^1 (see also [41] for less regularity). Moreover the construction of a stabilizing feedback law u given in the proof of Theorem 3.18 leads to a feedback law which is not locally bounded. Let us explain how the desingularization technique of [106] works in this example (Kawski's construction in [75] is different). Let us first point out that the reason for the term $(1/2)|x_2 - v(x_1)|^2$ in the control Lyapunov function (3.77) is to penalize $x_2 \neq v(x_1)$. But, in our case, $x_2 = v(x_1)$ is equivalent to $x_2^3 = x_1$. So a natural idea is to replace the definition of the control Lyapunov function (3.77) by

$$\begin{aligned} W(x_1, x_2) &= V(x_1) + \int_{x_1^{1/3}}^{x_2} (s^3 - x_1) ds \\ &= \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4 - x_1 x_2 + \frac{3}{4}|x_1|^{4/3}. \end{aligned}$$

With this W, one has again (3.78) and (3.79). Moreover one now obtains

$$\dot{W} = -x_1^2 + (x_2 - x_1^{1/3})[(x_2^2 + x_1^{1/3}x_2 + |x_1|^{2/3})(u - 2(x_1 - x_2 + x_1^{1/3})) + x_1].$$

Hence, if one takes for u the continuous function defined by

$$u(x_1, x_2) = \begin{cases} 2(x_1 - x_2 + x_1^{1/3}) - \frac{x_1}{x_2^2 + x_1^{1/3}x_2 + |x_1|^{2/3}} - (x_2 - x_1^{1/3}) & \text{if } (x_1, x_2) \neq (0,0), \\ 0 & \text{if } (x_1, x_2) = (0,0), \end{cases}$$

one gets $\dot{W} = -x_1^2 - (x_2 - x_1^{1/3})^2 < 0$ for $(x_1, x_2) \neq (0,0)$. Hence the feedback law u globally asymptotically stabilizes the control system (3.83).

In Section 4.1 one can find an application of this desingularization technique to the stabilization of nonlinear partial differential equations.

3.5.2 Backstepping and homogeneity

Note that, in order to construct u as in the proof of Theorem 3.5, one does not only need to know v: one also needs to know a Lyapunov function V. In many situations this is a difficult task. Rosier in [108, VI] and Morin-Samson in [100] have given interesting situations where one does not need to know V. Let us briefly describe Morin-Samson's situation. It concerns homogeneous control systems, with homogeneous feedback laws, a case already considered in [31] but the method of [31] does not lead to explicit feedback laws. Morin-Samson prove in [100]:

Theorem 3.19 *Let $T > 0$. Assume that there exists a T-periodic time-varying feedback law $v \in$ of class C^1 on $(\mathbb{R}_1^n \times \mathbb{R}) \setminus (\{0\} \times \mathbb{R})$ which globally asymptotically stabilizes the control system* (3.75)*. Assume the existence of $r = (r_1, \ldots, r_m) \in (0, +\infty)^m$ and $q > 0$ such that the closed vector field $f_1(x_1, v(x_1, t)$ is r-homogeneous (of degree 0) and that, with the notation of* (3.53),

$$v(\delta_\varepsilon^r(x_1, t)) = \varepsilon^q v(x_1, t), \forall(\varepsilon, x_1, t) \in (0, +\infty) \times \mathbb{R}^m.$$

Then, for $K > 0$ large enough, the feedback law $u = -K(x_2 - v(x_1, t))$ globally asymptotically stabilizes the control system (3.73)-(3.74).

Example 3.20 Let us go back again to the stabilization problem of the attitude of a rigid spacecraft, already considered in Examples 2.15, 2.17, 2.33, 3.13 and 3.17. It follows from these examples and Theorem 3.19 that the feedback law

$$u_1 = -K(x_5 - x_5(x_1, x_2, x_3, x_4, t)), u_1 = -K(x_6 - x_6(x_1, x_2, x_3, x_4, t)),$$

where x_5 and $\bar{x}_6$ are defined by (3.70) and $K > 0$ is large enough, locally asymptotically stabilizes the control system (3.64), i.e. the attitude of the rigid spacecraft.

4 Applications to some nonlinear partial differential equations

The goal of this chapter is to show that some methods presented in the previous chapters can also be useful for the control and stabilization of nonlinear partial differential control equations. We present two applications:

1. Stabilization of a rotating body-beam without damping;
2. Controllability and stabilization of incompressible fluids.

4.1 Stabilization of a rotating body-beam without damping

In this section we study the stabilization of a system, already considered in [5], consisting of a disk with a beam attached to its center and perpendicular to the disk's plane. The beam is confined to another plane, which is perpendicular to the disk and rotates with the disk; see Figure 1 below.

The dynamics of motion is, see [5] and [6],

$$\rho u_{tt}(x,t) + EIu_{xxxx}(x,t) + \rho B u_t(x,t) = \rho\omega^2(t)u(x,t). \tag{4.1}$$

$$u(0,t) = u_x(0,t) = u_{xx}(L,t) = u_{xxx}(L,t) = 0,$$

$$\frac{d}{dt}\left\{\omega(t)(I_d + \rho\int_0^L u^2(x,t)dx)\right\} = \Gamma(t), \tag{4.2}$$

where L is the length of the beam, ρ is the mass per unit length of the beam, EI is the flexural rigidity per unit length of the beam, $\omega(t) = \dot\theta(t)$ is the angular velocity of the disk at time t, I_d is the disk's moment of inertia, $u(x,t)$ is the beam's displacement in the rotating plane at time t with respect to the spatial variable x, Bu_t is the damping term, and $\Gamma(t)$ is the torque control variable applied to the disk at time t (see Figure 1).

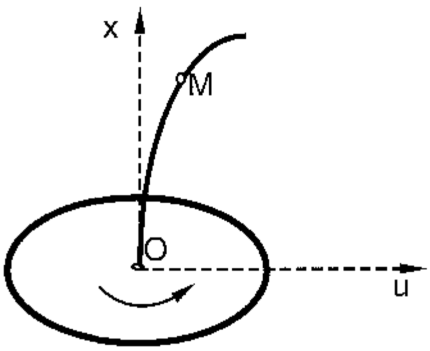

Figure 1: The body-beam structure

If there is no damping, $B = 0$ and therefore (4.1) reads

$$\rho u_{tt}(x,t) + EIu_{xxxx}(x,t) = \rho\omega^2(t)u(x,t). \tag{4.3}$$

The asymptotic behavior of the solutions of (4.3)-(4.2) when there is no control (i.e. $\Gamma = 0$), but with a damping term, has been studied by Baillieul and Levi in [5] and by Bloch and Titi in [10]. Still when there is a damping term, Xu and Baillieul have shown in [135] that the feedback torque control law $\Gamma = -\nu\omega$, where ν is any positive constant, globally asymptotically stabilizes the equilibrium point $(u,\omega) = (0,0)$. It is easy to check that such feedback laws do not asymptotically stabilize the equilibrium point when there is no damping.

In this section, we present a result given in [35] on the stabilization problem when there exists no damping. We shall see that the design tools presented in section 3.5 will allow us to construct in this case a feedback torque control law which globally asymptotically stabilizes the origin.

Of course, by suitable scaling arguments, we may assume that $L = EI = \rho = 1$. Let $H^2(0,1)$ be the usual Sobolev space

$$H^2(0,1) = \{u \in L^2(0,1)\,;\, u_{xx} \in L^2(0,1)\}.$$

Let

$$H = \{w = (u,v) \in H^2(0,1) \times L^2(0,1)\,;\, u(0) = u_x(0) = 0\}.$$

The space H with inner product

$$\langle (u_1,v_1),(u_2,v_2)\rangle_H = \int_0^1 (u_{1xx}u_{2xx} + v_1v_2)dx$$

is a Hilbert space. For $w \in H$, let

$$\mathcal{E}(w) = | w |_H^2 .$$

We consider the unbounded linear operator A in H

$$A(u,v) = (-v, u_{xxxx})$$

with domain

$$\mathrm{Dom}(A) = \{(u,v) \in H^4(0,1) \times H^2(0,1); \\ u(0) = u_x(0) = u_{xx}(1) = u_{xxx}(1) = v(0) = v_x(0) = 0\}.$$

It is well-known that A is an unbounded skew-adjoint operator and therefore generates a unitary group e^{tA} of bounded linear operators on H. With this notation, our control system (4.2)-(4.3) reads

$$\frac{dw}{dt} + Aw = \omega^2(0,u), \tag{4.4}$$

$$\frac{d\omega}{dt} = \gamma, \tag{4.5}$$

with

$$\gamma = (\Gamma - 2\omega \int_0^1 uvdx)/(I_d + \int_0^1 u^2 dx). \tag{4.6}$$

By (4.6), we may consider γ as the control. In order to explain how we have constructed our stabilizing feedback law, let us first consider, as in the usual backstepping method (see section 3.5 above), equation (4.4) as a control system where w is the state and ω is the control. Then natural candidates for a control Lyapunov function and a stabilizing feedback law are respectively $\mathcal{E}$ and $\omega = \sigma^*(\int_0^1 uvdx)$, where $\sigma^* \in C^0(\mathbb{R},\mathbb{R})$ satisfies $\sigma^* > 0$ on $(-\infty, 0)$ and $\sigma^* = 0$ on $[0, +\infty)$. One can prove (see [35, Appendix]) that such feedback laws always give *weak* asymptotic stabilization, i.e. one gets

$$w(t) \rightharpoonup 0 \text{ weakly in } H \text{ as } t \to +\infty \tag{4.7}$$

instead of

$$w(t) \to 0 \text{ in } H \text{ as } t \to +\infty. \tag{4.8}$$

But it is not clear that such feedbacks give *strong* asymptotic stabilization. It is possible to prove that one gets such stabilization for the particular case where the feedback is

$$\omega = (\max\{0, -\int_0^1 uvdx\})^{\frac{1}{2}}. \tag{4.9}$$

Let us recall that the control system (4.4)-(4.5) is obtained by adding an integrator to control system (4.4). Unfortunately, ω defined by (4.9) is not of class C^1 and so one cannot use the proof of Theorem 3.18 given in section 3.5 above. The smoothness of this ω is also not sufficient to apply the desingularization techniques introduced in [106]; see subsection 3.5.1 above. For these reasons, we use a different control Lyapunov function and a different feedback law to asymptotically stabilize control system (4.4). For the control Lyapunov function, we take

$$J(w) = \frac{1}{2}\{\mathcal{E}(w) - F(\mathcal{E}(w)) \int_0^1 u^2 dx\}, \ \forall w = (u,v) \in H,$$

where $F \in C^3([0,+\infty);[0,+\infty))$ satisfies

$$\text{Sup}_{s\geq 0} F'(s) < \frac{\mu_1}{2}, \tag{4.10}$$

where μ_1 is the first eigenvalue of the unbounded linear operator (d^4/dx^4) in $L^2(0,1)$ with domain

$$\text{Dom}(d^4/dx^4) = \{f \in H^4(0,1)\,;\, f(0) = f_x(0) = f_{xx}(1) = f_{xxx}(1) = 0\},$$

so that

$$\int_0^1 u_{xx}^2 dx \geq \mu_1 \int_0^1 u^2 dx\,,\ \forall w \in H. \tag{4.11}$$

¿From (4.10) and (4.11),

$$\frac{1}{4}\mathcal{E}(w) \leq J(w) \leq \frac{1}{2}\mathcal{E}(w)\,,\ \forall w \in H. \tag{4.12}$$

Computing the time-derivative $\dot{J}$ of J along the trajectories of (4.4) one obtains

$$\dot{J} = \left(K\omega^2 - F(\mathcal{E})\right)\left(\int_0^1 uv dx\right), \tag{4.13}$$

where, for simplicity, we write $\mathcal{E}$ for $\mathcal{E}(w)$ and where

$$K(=K(w)) := 1 - F'(\mathcal{E})\int_0^1 u^2 dx. \tag{4.14}$$

Let us impose the condition that

$$0 \leq F'(s)s < \mu_1 - F(s)\,,\ \forall s \in [0,+\infty), \tag{4.15}$$

$$\exists C_4 > 0 \text{ s. t. } \lim_{s\to 0,\, s>0} \frac{F(s)}{s} = C_4. \tag{4.16}$$

It is then natural to consider the feedback law for (4.4) vanishing at 0 and such that, on $H \setminus \{0\}$,

$$\omega = K^{-1/2}\left(F(\mathcal{E}) - \sigma\left(\int_0^1 uv dx\right)\right)^{\frac{1}{2}}, \tag{4.17}$$

where $\sigma \in C^2(\mathbb{R};\mathbb{R})$ is such that

$$s\bar{\sigma}(s) > 0\,,\ \forall s \in \mathbb{R}\setminus\{0\}, \tag{4.18}$$

$$\exists C_5 > 0 \text{ s. t. } \lim_{s\to 0,\, s\neq 0} \frac{\bar{\sigma}(s)}{s} = C_5, \tag{4.19}$$

$$\sigma(s) < F(2\sqrt{\mu_1}s)\,,\ \forall s > 0. \tag{4.20}$$

Note that, using (4.11), one gets that for every $w = (u,v) \in H$,

$$\begin{aligned}\int_0^1 uv dx &\leq \frac{1}{2}\left(\sqrt{\mu_1}\int_0^1 u^2 dx + \frac{1}{\sqrt{\mu_1}}\int_0^1 v^2 dx\right)\\ &\leq \frac{1}{2\sqrt{\mu_1}}\left(\int_0^1 u_{xx}^2 dx + \int_0^1 v^2 dx\right) = \frac{1}{2\sqrt{\mu_1}}\mathcal{E}(w),\end{aligned} \tag{4.21}$$

which, with (4.15), (4.16) and (4.20), implies that

$$F(\mathcal{E}(w)) - \bar{\sigma}\left(\int_0^1 uvdx\right) > 0\,,\ \forall w \in H \setminus \{0\}. \tag{4.22}$$

¿From (4.11), (4.14) and (4.15), one gets

$$K(w) \geq 1 - \frac{F'(\mathcal{E}(w))}{\mu_1}\int_0^1 u_{xx}^2 dx \geq 1 - \frac{F'(\mathcal{E}(w))}{\mu_1}\mathcal{E}(w) > 0\,,\ \forall w \in H. \tag{4.23}$$

¿From (4.22) and (4.23) we get that ω is well-defined by (4.17), and is of class C^2 on $H \setminus \{0\}$. This regularity is sufficient to apply the desingularization technique of [106] (see subsection 3.5.1 above): we note that (4.17) is equivalent to

$$\omega^3 = \psi(w) := K^{-3/2}\left(F(\mathcal{E}(w)) - \bar{\sigma}\left(\int_0^1 uvdx\right)\right)^{\frac{3}{2}} \tag{4.24}$$

and therefore, following [106], one considers the following control Lyapunov function for the control system (4.4)-(4.5):

$$V(w,\omega) = J + \int_{\psi^{\frac{1}{3}}}^{\omega}(s^3 - \psi)ds = J + \frac{1}{4}\omega^4 - \psi\omega + \frac{3}{4}\psi^{\frac{4}{3}},$$

where, for simplicity, we write J for $J(w)$ and ψ for $\psi(w)$. Then, by (4.12),

$$V(w,\omega) \to +\infty \text{ as } \mid w\mid_H + \mid \omega \mid \to +\infty,$$

$$V(w,\omega) > 0\,,\ \forall (w,\omega) \in H \times \mathbb{R} \setminus \{(0,0)\},$$

$$V(0,0) = 0.$$

Moreover, if one computes the time-derivative $\dot{V}$ of V along the trajectories of (4.4)-(4.5), one gets, using in particular (4.13),

$$\dot{V} = -\left(\int_0^1 uvdx\right)\bar{\sigma}\left(\int_0^1 uvdx\right) + (\omega - \psi^{\frac{1}{3}})[\gamma(\omega^2 + \psi^{\frac{1}{3}}\omega + \psi^{\frac{2}{3}}) + D], \tag{4.25}$$

where

$$D = -\dot{\psi} + K(\omega + \psi^{\frac{1}{3}})\int_0^1 uvdx, \tag{4.26}$$

with

$$\begin{aligned}\dot{\psi} \;=\; & \frac{3\psi^{\frac{1}{3}}}{2K}\left[2F'(\mathcal{E})\omega^2\int_0^1 uvdx - \sigma'\left(\int_0^1 uvdx\right)\left(\int_0^1 (v^2 - u_{xx}^2 + \omega^2 u^2)dx\right)\right] \\ & +3\frac{\psi}{K}\left(\int_0^1 uvdx\right)\left(\omega^2 F''(\mathcal{E})\int_0^1 u^2 dx + F'(\mathcal{E})\right).\end{aligned} \tag{4.27}$$

Hence it is natural to define a feedback law γ by

$$\gamma(0,0) = 0 \tag{4.28}$$

and, for every $(w, \omega) \in (H \times \mathbb{R}) \setminus \{(0,0)\}$,

$$\gamma = -(\omega - \psi^{\frac{1}{3}}) - \frac{D}{\omega^2 + \psi^{\frac{1}{3}}\omega + \psi^{\frac{2}{3}}}. \tag{4.29}$$

Note that by (4.22), (4.23), and (4.24)

$$\psi(w) > 0\,,\ \forall w \in H \setminus \{0\}. \tag{4.30}$$

Moreover, by (4.14), (4.16), (4.19), (4.20) and (4.21), there exists $\delta > 0$ such that

$$\psi(w) > \delta \mathcal{E}(w)^{3/2},\ \forall w \in H \text{ s. t. } \mathcal{E}(w) < \delta. \tag{4.31}$$

Using (4.14), (4.23), (4.24), (4.26), (4.27), (4.29), (4.30) and (4.31), one easily checks that γ is Lipschitz on bounded sets of on $H \times \mathbb{R}$. Therefore the Cauchy problem associated with (4.4)-(4.5) has, for the feedback law γ, one and only one (maximal) solution defined on an open interval containing 0. By (4.20), (4.25), (4.26), (4.28) and (4.29), one has

$$\dot V = -\left(\int_0^1 uvdx\right)\bar\sigma\left(\int_0^1 uvdx\right) - (\omega - \psi^{\frac{1}{3}})^2(\omega^2 + \psi^{\frac{1}{3}}\omega + \psi^{\frac{2}{3}}) \le 0. \tag{4.32}$$

In [35], we prove

Theorem 4.1 *The feedback law γ defined by* (4.28)-(4.29) *globally strongly asymptotically stabilizes the equilibrium point* $(0,0)$ *for the control system* (4.4)-(4.5), *i.e.*,

(i) *for every solution of* (4.4)-(4.5) *and* (4.28)

$$\lim_{t \to +\infty} \mid w(t) \mid_H + \mid \omega(t) \mid = 0, \tag{4.33}$$

(ii) *for every $\epsilon > 0$, there exists $\eta > 0$ such that for every solution of* (4.4)-(4.5) *and* (4.28),

$$(\mid w(0) \mid_H + \mid \omega(0) \mid < \eta) \Rightarrow (\mid w(t) \mid_H + \mid \omega(t) \mid < \epsilon,\ \forall t \ge 0).$$

4.2 Control and stabilization of incompressible inviscid fluids

4.2.1 Control of incompressible inviscid fluids

In this section we shall see that the "return" method we have used to stabilize driftless control systems can be used to prove the controllability of the Euler equations of incompressible fluids; for a description of the return method, see Remark 2.25 above.

Let us first describe the problem of the controllability of the Euler equations of incompressible fluids, a problem which has been raised by J.–L. Lions in [90, 92]. Let us introduce some notation. Let $l \in \{2,3\}$ and let Ω be a bounded nonempty connected open subset of $\mathbb{R}^l$ of class C^∞. Let $\Gamma^\#$ be a nonempty open subset of $\Gamma := \partial\Omega$. We denote by n the outward unit normal vector field on Γ. The set $\Gamma^\#$ is the part of the boundary on which the control acts. The fluid that we consider is incompressible so that the velocity field y satisfies

$$\operatorname{div} y = 0. \tag{4.34}$$

On the part of the boundary $\Gamma \setminus \Gamma^\#$ where there is no control the fluid does not cross the boundary: it satisfies

$$y \cdot n = 0 \text{ on } \Gamma \setminus \Gamma^\#. \tag{4.35}$$

Let us introduce the following definition

Definition 4.2 A *trajectory* of the Euler control system on the interval of time $[0, T]$ is a map $y : \overline{\Omega} \times [0, T] \to \mathbb{R}^l$ of class C^∞ such that, for some function $p : \overline{\Omega} \times [0, T] \to \mathbb{R}$ of class C^∞,

$$\frac{\partial y}{\partial t} + (y \cdot \nabla)y + \nabla p = 0 \text{ in } \overline{\Omega} \times [0, T], \tag{4.36}$$

$$\text{div } y = 0 \text{ in } \overline{\Omega} \times [0, T], \tag{4.37}$$

$$y(\cdot, t) \cdot n = 0 \text{ on } \Gamma \setminus \Gamma^\# \; \forall t \in [0, T]. \tag{4.38}$$

The J.-L. Lions problem of controllability is the following: let $T > 0$, let y_0 and y_1 in $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ be such that

$$\text{div } y_0 = 0 \text{ in } \overline{\Omega}, \tag{4.39}$$

$$\text{div } y_1 = 0 \text{ in } \overline{\Omega}, \tag{4.40}$$

$$y_0 \cdot n = 0 \text{ on } \Gamma \setminus \Gamma^\#, \tag{4.41}$$

$$y_1 \cdot n = 0 \text{ on } \Gamma \setminus \Gamma^\#, \tag{4.42}$$

does there exist a trajectory y of the Euler control system such that

$$y(\cdot, 0) = y_0 \text{ in } \overline{\Omega}, \tag{4.43}$$

$$y(\cdot, T) = y_1 \text{ in } \overline{\Omega}? \tag{4.44}$$

That is to say, starting with the initial data y_0 for the velocity field, we ask whether there are trajectories of the given Euler control system which, at a given fixed time T, are equal to the given velocity field y_1. If this problem has always a solution one says that the control system considered is *exactly controllable*.

Note that (4.36), (4.37), (4.38) and (4.43) have many solutions. In order to have uniqueness one needs to add extra conditions. These extra conditions are the controls. There are various possible choice for the controls. One can take for example (see also subsection 4.2.2 below) $y \cdot n$ on $\Gamma^\#$ with $\int_{\Gamma^\#} y \cdot n = 0$ and curl y if $l = 2$, curl y $\cdot\, n$ if $l = 3$, at the points of $\Gamma^\# \times [0, T]$ where $y \cdot n < 0$: these boundary conditions, (4.38), and the initial condition (4.43) imply the uniqueness of the solution to the Euler equations (4.36) up to an arbitrary function of t which may be added to p; see also [78] for the existence of the solution.

Let us first point out that in order to have controllability one needs that

$$\Gamma^\# \text{ intersects any connected component of } \Gamma. \tag{4.45}$$

Indeed, let Γ_i be a connected component of Γ which does not intersect $\Gamma^\#$ and assume that, for some smooth Jordan curve C_0 on Γ_i (if $l = 2$ take $C_0 = \Gamma_i$),

$$\int_{C_0} y_0 . ds \neq 0 \tag{4.46}$$

but that

$$y_1(x) = 0 \; \forall x \in \Gamma_i. \tag{4.47}$$

Then there is no solution to our problem, that is, there is no $y \in C^\infty(\overline{\Omega} \times [0,T]; \mathbb{R}^2)$ and no $p \in C^\infty(\overline{\Omega} \times [0,T]; \mathbb{R})$ such that (4.37), (4.36), (4.43), (4.44), and (4.38) hold. Indeed, if such a solution (y,p) exists, then by Kelvin's law,

$$\int_{\mathcal{C}(t)} y(\cdot,t) \cdot ds = \int_{\mathcal{C}_0} y_0 . ds, \tag{4.48}$$

where $\mathcal{C}(t)$ is the Jordan curve obtained, at time t, from the points of the fluids which at time 0 where on $\mathcal{C}_0$; in other words $\mathcal{C}(t)$ is the image of $\mathcal{C}_0$ by the flow map associated with the time-varying vector field y and the time interval $[0,T]$. But (4.44), (4.46), (4.47) and (4.48) are in contradiction.

Conversely, if (4.45) holds, then the Euler control system is controllable:

Theorem 4.3 *Assume that $\Gamma^{\#}$ intersects any connected component of $\partial\Omega$. Then the Euler control system is exactly controllable.*

Theorem 4.3 has been proved in

- [27] when Ω is simply connected and $l = 2$,
- [28] when Ω is multi-connected and $l = 2$,
- [55] when Ω is contractible and $l = 3$,
- [56] when Ω is not contractible and $l = 3$.

The strategy of the proof of Theorem 4.3 relies on the "return method". For the case of the Euler control system, it consists in looking for $(\bar{y},\bar{p})$ such that (4.37), (4.36), (4.43), (4.44) hold, with $y = \bar{y}, p = \bar{p}$, $y_0 = y_1 = 0$, and such that the linearized control system around the trajectory $\bar{y}$ is controllable under the assumptions of Theorem 4.3. With such a $(\bar{y},\bar{p})$ one may hope that there exists (y,p) close to $(\bar{y},\bar{p})$ satisfying the required conditions, at least if y_0 and y_1 are "small". Finally, by using some scaling argument, one can deduce from the existence of (y,p) when y_0 and y_1 are "small" the existence of (y,p) even if y_0 and y_1 are not "small".

Let us emphasize that one cannot take $(\bar{y},\bar{p}) = (0,0)$. Indeed, with such a choice of $(\bar{y},\bar{p})$, (4.37), (4.36), (4.43), (4.44) hold, with $y = \bar{y}$, $p = \bar{p}$, $y_0 = y_1 = 0$, but the linearized control system around $\bar{y} = 0$ is not at all controllable. Indeed the linearized control system around $y = 0$ is

$$\operatorname{div} z = 0 \text{ in } \overline{\Omega} \times [0,T], \tag{4.49}$$

$$\frac{\partial z}{\partial t} + \nabla\pi = 0 \text{ in } \overline{\Omega} \times [0,T], \tag{4.50}$$

$$z(x,t) \cdot n(x) = 0 \, \forall (x,t) \in (\Gamma \setminus \Gamma^{\#}) \times [0,T].$$

Taking the curl of (4.50), one gets

$$\frac{\partial \text{curl } z}{\partial t} = 0,$$

which clearly shows that the linearized control system is not controllable. So one needs to consider other $(\bar{y},\bar{p})$. Let us briefly explain how one constructs a "good" $(\bar{y},\bar{p})$ when $l = 2$

and Ω is simply connected. In such a case one easily checks the existence of a harmonic function θ in $C^\infty(\bar{\Omega})$ such that

$$\nabla\theta(x) \neq 0 \ \forall x \in \overline{\Omega},$$
$$\frac{\partial\theta}{\partial n} = 0 \text{ on } \Gamma \setminus \Gamma^{\#}.$$

Let $\alpha \in C^\infty(0, T)$ be vanishing 0 and T. Let

$$(\bar{y}, \bar{p})(x,t) = (\alpha(t)\nabla\theta(x), -\alpha'(t)\theta(x) - \frac{1}{2}\alpha^2(t) \mid \nabla\theta(x) \mid^2).$$

Then (4.37), (4.36), (4.43), (4.44) hold, with $y = \bar{y}$, $p = \bar{p}$, $y_0 = y_1 = 0$. Moreover using arguments relying on an extension method analogous to the one introduced by Russell in [110] one can see that the linearized control system around y is controllable.

When $\Gamma^{\#}$ does not intersect all the connected components of $\Gamma^{\#}$ one can get, if $l = 2$, approximate controllability and even exact controllability outside any arbitrarily small neighborhood of the union Γ^* of the connected components of Γ which do not intersect $\Gamma^{\#}$. More precisely, one has

Theorem 4.4 ([28]) *Assume that $l = 2$. There exists a constant c_0 depending only on Ω such that, for any $\Gamma^{\#}$ as above, any $T > 0$, any $\varepsilon > 0$, and any y_0, y_1 in $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying* (4.39), (4.40), (4.41) *and* (4.42), *there exists a trajectory y of the Euler control system on* $[0, T]$ *satisfying* (4.43) *such that*

$$y(x,T) = y_1(x), \ \forall x \in \overline{\Omega} \text{ with } \operatorname{dist}(x, \Gamma^*) \geqslant \varepsilon, \tag{4.51}$$

and

$$|y(.,T)|_{L^\infty} \leqslant c_0(|y_0|_{L^2} + |y_1|_{L^2} + |\operatorname{curl} y_0|_{L^\infty} + |\operatorname{curl} y_1|_{L^\infty}). \tag{4.52}$$

In (4.51), $\operatorname{dist}(x, \Gamma^*)$ denotes the distance of x to Γ^*, i.e.

$$\operatorname{dist}(x, \Gamma^*) = \operatorname{Min}\{|x - x^*|; x^* \in \Gamma^*\}. \tag{4.53}$$

We use the convention $\operatorname{dist}(x, \emptyset) = +\infty$ and so Theorem 4.4 implies Theorem 4.3. In (4.52) $|\ |_{L^r}$ denotes the L^r-norm on Ω for $r \in [1, +\infty]$. Let us point out that, y_0, y_1, and T as in Theorem 4.4 being given, it follows from (4.51) and (4.52) that for any r in $[1, +\infty)$,

$$\lim_{\varepsilon\to 0^+} |y(0,T) - y_1|_{L^r} = 0; \tag{4.54}$$

that is, Theorem 4.4 implies approximate controllability in the L^r-space for any r in $[1, +\infty)$. Let us notice that, if $\Gamma^* \neq \emptyset$, then, again by Kelvin's law, approximate controllability for the L^∞-norm does not hold.

Remark 4.5 One can find recent results on the controllability of the Navier-Stokes equations of incompressible fluids (i.e. with $-\nu\Delta$ added on the left hand side of (4.36)) in [29, 42, 43, 44, 49, 50, 51, 52, 53, 70].

Remark 4.6 The return method has also been used by Horsin to study the controllability of the Burger equation in [69]. He has also introduced a new tool, namely "variations of domain" to the study the controllability around an analogous "$\bar{y}$" for the Burger equation.

4.2.2 Stabilization of incompressible inviscid fluids

Let us first notice that, as in the counterexample (1.1) of [9] to the stabilization by means of stationary feedback laws, the linearized control system of the Euler equation around the origin is not controllable.

Therefore it is natural to ask what is the situation for the asymptotic stabilizability of the origin for the Euler equation of an incompressible inviscid fluid in a bounded domain Ω when the controls act on an arbitrary small open subset $\Gamma^{\#}$ of the boundary which meets any connected component of this boundary. In this section we give explicit feedback laws which globally asymptotically stabilize the origin in dimension 2 ($l = 2$) when the domain Ω is simply connected. Since Ω is assumed to be simply connected, y is completely characterized by $\omega := \text{curl } y$ and $y \cdot n$ on Γ, where n denotes the unit outward normal to Γ. For the problem of controllability, one does not really need to specify the control and the state: one considers the "Euler control system" as an under-determined system by requiring $y \cdot n = 0$ on $\Gamma \setminus \Gamma^{\#}$ instead of $y \cdot n = 0$ on Γ as for the uncontrolled usual Euler equation. For the stabilization problem, one needs to specify more precisely the control and the state. In this paper the state is ω. For the control there are at least two natural possibilities:

(a) The control is $y \cdot n$ on $\Gamma^{\#}$ and the time derivative $\partial\omega/\partial t$ of the vorticity at the points where $y \cdot n < 0$, i.e. at the points where the fluid enters into the domain Ω;

(b) The control is $y \cdot n$ on $\Gamma^{\#}$ and the vorticity ω at the points where $y \cdot n < 0$.

Let us point out that, by (4.34), $y \cdot n$ has to satisfy $\int_\Gamma y \cdot n = 0$ in both cases. In this paper we give stabilizing feedback laws for case (a); for case (b), see [30]. Let $g \in C^\infty(\Gamma)$ be such that

$$\text{Support } g \subset \Gamma^{\#}, \tag{4.55}$$

$$\Gamma^{\#}_{+} := \{g > 0\} \text{ and } \Gamma^{\#}_{-} := \{g < 0\} \text{ are connected}, \tag{4.56}$$

$$g \neq 0, \tag{4.57}$$

$$\overline{\Gamma^{\#}_{+}} \cap \overline{\Gamma^{\#}_{-}} = \emptyset, \tag{4.58}$$

$$\int_\Gamma g = 0. \tag{4.59}$$

For any compact set K of $\mathbb{R}^q$ and any $f \in C^0(K; \mathbb{R}^m)$, we denote

$$|f|_{0,K} = \text{Max } \{|f(x)|\,; x \in K\}.$$

For simplicity, we write $|f|_0$ instead of $|f|_{0,\overline{\Omega}}$. Our stabilizing feedback laws are

$$y \cdot n = M\, |\omega|_0\, g \quad \text{on } \Gamma^{\#},$$
$$\tfrac{\partial\omega}{\partial t} = -M\, |\omega|_0\, \omega \quad \text{on } \Gamma^{\#}_{-} \text{ if } |\omega|_0 \neq 0,$$

where $M > 0$ is large enough. With these feedback laws, a function $\omega : I \times \overline{\Omega} \to \mathbb{R}$, where I is an interval, is a solution of the closed loop system Σ if

$$\frac{\partial \omega}{\partial t} + \text{div } (\omega y) = 0 \quad \text{in } \overset{\circ}{I} \times \Omega, \tag{4.60}$$

$$\text{div } y = 0 \quad \text{in } \overset{\circ}{I} \times \Omega, \tag{4.61}$$

$$\text{curl } y = \omega \quad \text{in } \overset{\circ}{I} \times \Omega, \tag{4.62}$$

$$\text{curl } y = \omega \quad \text{in } \overset{\circ}{I} \times \Omega, \tag{4.63}$$

$$y(t) \cdot n = M \left|\omega(t)\right|_0 g \quad \text{on } \Gamma, \forall t \in I, \tag{4.64}$$

$$\frac{\partial \omega}{\partial t} = -M \left|\omega(t)\right|_0 \omega \quad \text{on } \{t; \omega(t) \neq 0\} \times \Gamma^{\#}_{-}, \tag{4.65}$$

where, for $t \in \Omega$, $\omega(t) : \overline{\Omega} \to \mathbb{R}$ is defined by requiring $\omega(t)(x) = \omega(t,x), \forall x \in \overline{\Omega}$. More precisely, the definition of a solution of system Σ is

Definition 4.7 Let I be an interval. A function $\omega : I \to C^0(\overline{\Omega})$ is a *solution of system* Σ if

(i) $\omega \in C^0(I; C^0(\overline{\Omega}))(\cong C^0(I \times \overline{\Omega}))$;

(ii) for $y \in C^0(I \times \overline{\Omega}; \mathbb{R}^2)$ defined by requiring (4.61) and (4.63) in the sense of distributions and (4.64), one has (4.60) in the sense of distributions;

(iii) in the sense of distributions on the open manifold $\{t; \omega(t) \neq 0\} \times \Gamma^{\#}_{-}$ one has

$$\partial \omega / \partial t = -M \left|\omega(t)\right|_0 \omega.$$

Our first theorem, which is proved in [30] says that, for M large enough, the Cauchy problem for system Σ has at least one solution defined on $[0, +\infty)$ for any initial data in $C^0(\overline{\Omega})$. More precisely one has:

Theorem 4.8 *There exists $M_0 > 0$ such that, for any $M \geqslant M_0$, the following two properties hold*

(i) *for any $\omega_0 \in C^0(\overline{\Omega})$, there exists a solution of system Σ defined on $[0, +\infty)$ such that $\omega(0) = \omega_0$;*

(ii) *any maximal solution of system Σ defined at time 0 is defined on $[0, +\infty)$ (at least).*

Remark 4.9 (a) In this theorem, property (i) is in fact implied by property (ii) and Zorn's lemma. We state (i) in order to emphasize the *existence* of a solution to the Cauchy problem for system Σ.

(b) One does not know if the solution to the Cauchy problem is unique for positive time. (For negative time, one does not have uniqueness since there are solutions ω of system Σ defined on $[0, +\infty)$ such that $\omega(0) \neq 0$ and $\omega(T) = 0$ for $T \in [0, +\infty)$ large enough.) But let us emphasize that already for control systems in finite dimension, one considers feedback laws which are merely continuous; with these feedback laws, the Cauchy problem for the closed loop system may have many solutions. It turns out that this lack of uniqueness is not a real problem. Indeed, let us recall that by Kurzweil's theorem [87], in finite dimension at least, if a point is asymptotically stable for a continuous vector field, then as in the case of regular vector fields, there exists a smooth Lyapunov function. It is tempting to conjecture that a similar result holds in infinite dimension under reasonable assumptions. The existence of this Lyapunov function ensures some robustness to perturbations. It is precisely this robustness which makes the interest of feedback laws compared to open loop controls. One can prove (see [30]) that for the above feedback laws there exists also a Lyapunov function. Therefore the above feedback laws provide some kind of robustness.

The next theorem, proved in [30] shows that, at least for M large enough, our feedback laws globally and strongly asymptotically stabilize the origin in $C^0(\overline{\Omega})$ for system Σ.

Theorem 4.10 *There exists a positive constant $M_1 \geqslant M_0$ such that, for any $\varepsilon \in (0,1]$, any $M \geqslant M_1/\varepsilon$ and any maximal solution ω of system Σ defined at time 0,*

$$|\omega(t)|_0 \leqslant \ Min \ \{|\omega(0)|_0, \frac{\varepsilon}{t}\}, \ \forall t > 0. \tag{4.66}$$

Remark 4.11 Due to the term $|\omega(t)|_0$ appearing in (4.64) and in (4.65), our feedback laws do not depend only on the value of ω on $\Gamma^{\#}$. Let us point out that there is no asymptotically stabilizing feedback law depending only on the value of ω on $\Gamma^{\#}$ such that the origin is asymptotically stable for the closed loop system. In fact, given a nonempty open subset Ω_0 of Ω, there is no feedback law which does not depend on the values of ω on Ω_0. This phenomenon is due to the existence of "phantom vortices": there are smooth stationary solutions $\bar{y} : \overline{\Omega} \to \mathbb{R}^2$ of the 2-D Euler equations such that Support $\bar{y} \subset \Omega_0$ and $\bar{\omega} :=$ curl $\bar{y} \neq 0$; see, e.g., [96]. Then $\omega(t) = \bar{\omega}$ is a solution of the closed loop system if the feedback law does not depend on the values of ω on Ω_0 and vanishes for $\omega = 0$.

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