Annex to SMR1327/17

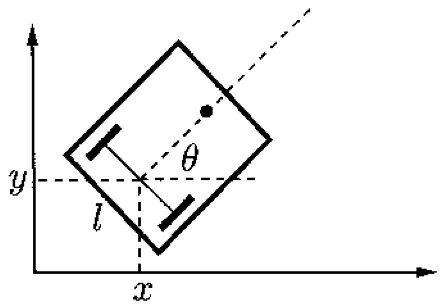
by P. Rouchon

Summer School on Mathematical Control Theory, September 2001, Abdus Salam ICTP

Flatness based design

Pierre Rouchon (*rouchon@cas.ensmp.fr*)

Problem no 1: the mobile robot



Many mobile robots admit on the same axis two wheels independently actuated via two electrical drives. Denote by $(x, y) \in \mathbb{R}^2$ the Cartesian coordinates of the middle of this axis, $\theta \in [0, 2\pi[$ the orientation of the robot. The rolling without slipping conditions yield the following dynamics:

$$\begin{cases} \frac{dx}{dt} = v\cos\theta \\ \frac{dy}{dt} = v\sin\theta \\ \frac{d\theta}{dt} = \omega \end{cases} \tag{1}$$

where (x, y, θ) is the state and $u = (v, \omega) \in \mathbb{R}^2$ the control (v corresponds to the average wheel velocities and ω to their difference).

1. What are the equilibrium points of the system ? Write down the tangent linear system around any equilibrium and study its controllability.

2. In this question, the goal is to follow the x-axis with a constant velocity $a > 0$. We have thus the following reference trajectory:

$$x_r(t) = at, \quad y_r(t) = 0, \quad \theta_r(t) = 0, \quad v_r(t) = a, \quad \omega_r(t) = 0.$$

   We set $x = x_r + \Delta_x$, $y = y_r + \Delta_r$, $\theta = \theta_r + \Delta_\theta$, $v = v_r + \Delta_v$ and $\omega = \omega_r + \Delta_\omega$ where the errors Δ_σ, $\sigma = x, y, \theta, v, \omega$, are assumed to be small.

(a) Show that, up to second order terms, the linear equations satisfied by the Δ_σ's are

$$\begin{cases} \dfrac{d\Delta_x}{dt} = \Delta_v \\ \dfrac{d\Delta_y}{dt} = a\Delta_\theta \\ \dfrac{d\Delta_\theta}{dt} = \Delta_\omega \end{cases} \tag{2}$$

with $(\Delta_v, \Delta_\omega)$ as control.

(b) Prove that (2) is controllable. Compute its Brunovsky output (flat-output) and give the static feedback that stabilizes the errors dynamics. We will denote by (p_1, p_2, p_3) the poles of the closed-loop system. Discuss their choices with respect to characteristic quantities such as a and the distance $l > 0$ between the wheels.

3. In this question the goal is to follow a smooth curve defined by its arc length parameterization $s \mapsto (x_r(s), y_r(s))$. Denote by $\theta_r(s)$ its tangent angle and $\kappa_r(s)$ its curvature. We recall the Frénet formulae

$$\frac{dx_r}{ds} = \cos\theta_r, \quad \frac{dy_r}{ds} = \sin\theta_r, \quad \frac{d\theta_r}{ds} = \kappa_r.$$

The tracking velocity $a > 0$ is still constant. Instead of the cartesian errors (Δ_x, Δ_y) used previously, we introduce the tangent $\Delta_\parallel$ and normal $\Delta_\perp$ errors defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_r \\ y_r \end{pmatrix} + \Delta_\parallel \begin{pmatrix} \cos\theta_r \\ \sin\theta_r \end{pmatrix} + \Delta_\perp \begin{pmatrix} -\sin\theta_r \\ \cos\theta_r \end{pmatrix}.$$

(a) Prove that the reference control is

$$v_r(t) = a, \quad \omega_r(t) = a\kappa_r(at).$$

Prove that, up to second order terms, the tracking errors Δ_σ $(\sigma = \parallel, \perp, \theta, v, \omega)$ satisfy

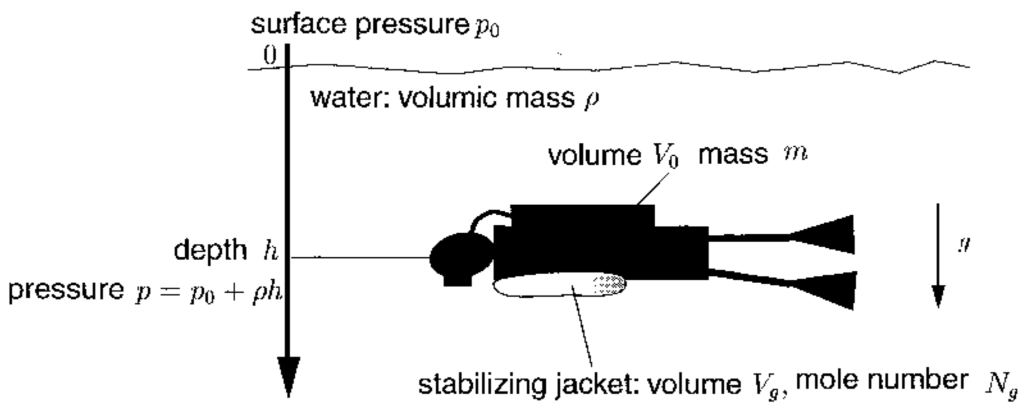
$$\begin{cases} \dfrac{d\Delta_\parallel}{dt} = a\kappa_r(at)\Delta_\perp + \Delta_v \\ \dfrac{d\Delta_\perp}{dt} = -a\kappa_r(at)\Delta_\parallel + a\Delta_\theta \\ \dfrac{d\Delta_\theta}{dt} = \Delta_\omega \end{cases} \tag{3}$$

with $(\Delta_v, \Delta_\omega)$ as control .

(b) We will assume here that, in (3), the curvature $\kappa_r(s)$ varies slowly: $\kappa_r(at) \approx \bar{\kappa}_r$ is assumed to be independent of t (in a first approximation). Show that (3) is controllable. Give its Brunovsky output. Design the static feedback that stabilizes the tracking errors (we still denote by (p_1, p_2, p_3) the closed-loop poles).

(c) How to exploit the previous tracking controller if the goal is still to follow the same curve $s \mapsto (x_r(s), y_r(s))$ but with a time varying reference velocity $a(t) = \frac{ds_r}{dt}$ corresponding to a prescribed time parameterization $t \mapsto s_r(t)$?

Problem no 2: diving with a stabilizing jacket



We study here the vertical dynamics of a person diving with a stabilization jacket admitting a varying air quantity N_g (flush valve for $\dot{N}_g = u < 0$, and air bottle for $\dot{N}_g = u > 0$). With the figure notation, the depth h dynamics is given by the Newton equation along the vertical. It involves the Archimedean force $\rho g(V_0 + V_g)$ where V_g is obtained as a function of the pressure $p = p_0 + \rho h$ at h via $pV_g = N_g R\theta$, the perfect gas law (R and θ are constants). We have thus

$$\begin{cases} m\ddot{h} = \left(m - \rho\left(V_0 + \dfrac{R\theta N_g}{p_0 + \rho h}\right)\right) g \\ \dot{N}_g = u. \end{cases} \qquad (4)$$

In the sequel, $h > 0$, $N_g \geq 0$ and $m > \rho V_0$. We denote by $x = (h, dh/dt, N_g)$ the state.

Constant control

1. We assume here $u = 0$.

   (a) Compute the equilibrium state $\bar{x}$ as a function of the depth $h = \bar{h} > 0$.

(b) Show that

$$W(x) = \frac{m}{2}\dot{h}^2 + (\rho V_0 - m)gh + gR\theta N_g \log\left(1 + \frac{\rho h}{p_0}\right)$$

is a first integral .

(c) Draw in the phase space $(h, \dot{h})$ the behavior of the trajectories around $(\bar{h}, 0)$ (phase portrait). The equilibrium state $\bar{x}$ is it stable or unstable ?

2. Show that the matrices A and B of the tangent system around $x \approx \bar{x}$ and $u \approx 0$ admit the following structure

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \alpha & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Compute α and β with respect to $\bar{h}$. Give the eigenvalues of A. Recover that the open-loop system is unstable.

3. Shows that around $\bar{x}$ the system cannot be stabilized via a simple output feedback on h, i.e., for all $k \in \mathbb{R}$ the closed-loop system with $u = k(h - \bar{h})$ is not asymptotically stable (This explains why the control is not so easy in practice....)

Motion planing and tracking

The goal is to start at the equilibrium at depth $\bar{h}$ at $t = 0$ and to arrive at time $t = T > 0$ at the equilibrium with depth $\tilde{h} < \bar{h}$.

1. We suppose that $\tilde{h}$ and $\bar{h}$ are close enough such that the tangent model around $\bar{h}$ remains valid for h between $\tilde{h}$ and $\bar{h}$.

   (a) Show that the linear tangent dynamics around $\bar{x}$ is controllable and give its Brunovsky output (flat output).

   (b) Compute an open-loop control $[0, T] \ni t \mapsto u_r(t)$ and a reference trajectory $[0, T] \ni t \mapsto x_r(t)$ (we still denote by x the state of the linear tangent model) that provides the transition from $\bar{h}$ to $\tilde{h}$

   (c) Construct the tracking feedback.

2. We do not suppose that $\tilde{h}$ and $\bar{h}$ are close. Solve the question 1b with the nonlinear system instead of the linear one. How to chose T in order to respect the physical constraint $V_g > 0$?