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Basic principles of mathematical modelling

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BASIC PRINCIPLES OF MATHEMATICAL MODELING

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Glossary

Deterministic: One says that the evolution of a system is deterministic when once the initial state is well known, the future evolution is known.

Differential system: a mathematical representation, by the mean of differentials, partial derivatives, integrals of the evolution of systems.

Euclidian spaces: Mathematical generalization to arbitrary dimension of the concepts of two and three dimensional paces of elementary geometry.

Feed-back: The way by which some output of a system can act as an input.

Infinite dimension spaces: Linear spaces which are used very often to represent spaces of functions.

Input: All the quantities which can act and modify a system.

Linear vector spaces: The central concept of linear algebra. In a linear vector spaces one can add vectors and multiply tem by scalars.

Linear operator : It is a mapping between two euclidian spaces or more general linear spaces which respect the linear structure.

Manifold : Mathematical generalization of the notion of surface.

Matrix: It is a rectangular table which entries are real or complex numbers. Matrices are the best way to represent linear operators in euclidian spaces.

Output: Quantities which are produced by a system.

Random variable: It is a variable which is not well determined and which value is the result of some experiment. (Hed, Tail) is the random issue of coin tossing. As a mathematical objec a random variable is a function defined oa probability space.

Serie expansion: It is a way of representing functions as an infinite sum of successive powers of the variable.

State : For a system which evolves during time, the state is the set of quantitative parameters (e.g. position and velocity for a mass point) which are necessary to write down the equations of the motion.

Stochastic: In some circumstances it is not possible to predict the future evolution of a system, for instance the issue of a coin tossing. Such a phenomenon is called stochastic.

Summary
In this article try to introduce the reader to the main features of Mathematical System Theory. Mathematical system theory was developed, mostly during the twentieth century for the purposes of understanding the dynamics of complex man made devices. It was a theory for engineers and was mainly developed in electrical engineering departments. It is a mathematically sophisticated science. By the middle of the twentieth century two major events took place: The development of computer facilities and the interest for the dynamics of life systems. For these reasons mathematical system theory became more and more used for the modeling of natural systems as opposed to artificial ones. The first ingredient of mathematical system theory is the classical mathematical concept of dynamical system, either continuous or discrete with respect to time, finite or infinite dimensional with respect to the state space, deterministic or stochastic. The second ingredient are the concepts of input and output which are absolutely necessary to define what a feed back is.

We will see how these concepts are useful for the modeling of few natural systems. After this introduction to the major concepts of system theory we shall give a very brief account on controllability, observability and stabilizability. We shall explain the major results for the case of linear systems and give few indications on the possible generalization to nonlinear systems. In the conclusion we come back to the philosophy of modeling and propose three different possible acceptation of the word model.

1. Introduction

1.1 A fashionable word

The use of the word "modeling" in sciences is relatively recent. Scarcely used during the last century it is now a fashionable word, but the "mathematical modeling" is not a new activity, even if did not appeared under this name before. As we know, the classical theories of physics express their law through a mathematical apparatus, the "equations" as we used to say formerly, that could be called a "mathematical model" in our present language. But we are not mainly concerned with this kind of model in the present article, despite the fact we often shall refer to them for comparison purposes.

We shall consider the word "model" in the engineer tradition of the last century, who, before the computer achievement, used to built "reduced models". Nowadays, the models used by engineers are symbolic representations, expressed in a language which is possibly recognized by a computer, of some reality complex and changing. The computer simulations of the dynamic generated by the model are used to solve a great variety of practical questions. In its classical mathematical text-book, "Ordinary Differential Equations", the famous mathematician Pontryagin, relates an important practical problem which was solved thanks to a mathematical model: The question of the stability of Watt's governor for steam engines. The Watt governor was invented by the end of the eighteenth century and was perfectly suitable for its purposes for some time. It turned out that, by the middle of the nineteenth century, the functioning was worse and worse. A mathematical model of the motion of the steam engine with its governor was elaborated, and a mathematical theory of the stability of motion was simultaneously elaborated by the Russian Wischnegradsky and the famous physicist J. C. Maxwell around 1870. In that case the mathematical model was a set of three differential equations established on the basis of Newton's laws of mechanics for an artificial apparatus designed by man. Since that time, "Automatic Control Theory" is the science which aim is to provide to engineers the tools for achieving the regulation of systems more and more complex like modern aircraft, industrial processes, electrical networks .....During a little bit more than one century, automatic control theory developed for its own use a theory of mathematical modeling with efficient concepts and highly mathematically sophisticated developments. This theory was well developed when the computer appeared at the middle of the twentieth century and was able to incorporate this revolution harmoniously. We call this theory *Mathematical System Theory.*

More recently, say for half a century, there is a need for a more quantitative theory of the dynamics of complex natural systems. (The existence of the EOLSS is a good example of this new important challenge facing humanity and we do not insist on this point here). The representation of ecological systems dynamics through mathematical equations was present long time ago, at the beginning of the century, but the nonlinear equations where too complex for a mathematical treatment. Thanks to the fantastic computing power of modern computers one can simulate more and more complex systems of equations. Thanks to the low cost of modern personal computers and to the facilities of new computer languages, more and more people do simulations for various purposes. What is the scientific value of such simulations? This is a big issue and mathematical modeling is a way, if not the way, to address it.

In this introduction we shall describe what mathematical modeling is in the spirit of the traditional training of automatic control engineers and try to see to what extend this methodology is suitable for other area of research than industrial production.

1.2 Modeling : A complex activity

A picture like the one in Fig. 1 is by itself all a philosophical program ! We begin by some comments about it. On this picture we observe a first ensemble called "piece of reality" which feeds (arrow 1) another ensemble called "discourse about reality". This possibility of a crude separation between an objective "reality" which exists, independently and previously to any discourse, and the "discourse about this reality" is quite questionable. The question of the existence of an objective world, and the possibility of a non subjective analysis of it, is a formidable philosophical question that we shall not consider here. We shall adopt the following pragmatic point of view. We accept that there are circumstances where the existence of an object to be studied, the piece of reality, is quite clear. For instance if the problem is to realize a flight simulator for an aircraft, say an Airbus A 320, the piece of reality is well defined. Small ambiguities, like what kind of engine or instrumentation are used, will be easily clarified. An other example: Make a model of the growth of the temperate climate oak. In this case also, even if it seems a little bit more difficult to agree on what an oak is and what means its growth, clearly a general consensus is possible.

Fig.1: The model and the real world

But, "do a model of the functioning of the tropical forest" is clearly a different question. If the rain forest is a relatively well understood object, what does mean "its functioning"? Are we speaking of productivity or are we interested by the biodiversity it can sustain? The question about the "good" use of such a tropical forest might be different if you are a poor peasant of Brazil or a rich citizen of an industrialized country. Different points of view, even contradictory points of view are to be expected. Even more difficult is the problem of the separation between a subject and an object when considerations about human psychology are concerned. Is it possible to make a model of the behavior the Stock Exchange, since people who knows the model will change their behavior? Mathematical modeling assumes that there is an "object" and a "subject" making a discourse on this objet. We know that it is not always the case but we do as if it was the case.

On fig. 1 we have drawn some rectangles to indicate "models". There are several models, n°1, n°2... recalling the fact that, for the same object, there are many different viewpoints and by the way many different models. By this we mean that, in mechanics for instance, we do not use the full equations of relativity to represent the motion of an air plane, but Newton equations which is a simplified model. But we do not mean only this. We mean that some object may have several models which are both necessary to describe it and not reducible one to each other. The best instance for this fact is probably quantum mechanics where wave and particle models are both necessary to describe the same reality.

The arrows 2.1, 2.2, ... define a connection between the model and the discourse about the reality. This is the work of interpretation. By the interpretation we mean the following point. Suppose that we are interested in two interacting populations, say a prey and a predator. The number of prey is of the order of magnitude of 10^9 and the predators of 10^6 (this is plausible in the case of a relation between fishes and small plankton) and choose the 10^9 as unity for the prey and 10^6 for the predators. Denote by $x(t)$ and $y(t)$ the quantity of prey and predators in these units. Because of this choice $x(t)$ and $y(t)$ appears to be real numbers, (i.e. numbers with decimals). Assume that we built a model in term of differential equations and that the model has the property that, when t tends to infinity, $x(t)$ tends to zero. The "first degree" interpretation of this affirmation of the model is that the prey population extinguishs. But one must be careful since the justification for working with real numbers was the fact that the numbers of preys was large, of the order of 10^9 which will no longer be the case if $x(t)$ tends to 0. So, when $x(t)$ decreases, it appends that at some moment the model is no longer appropriate to represent reality and we must think to it. Thus $x(t)$ tends to 0 in this model means that the population of prey decreases under some threshold, no more. In some sense, what we call interpretation is the art of modeling itself, but we prefer to keep the word modeling for the more complete process that we are presently describing than just for the activity of interpretation of mathematical results. An other reason why we do not want to call modeling the activity of giving sense to mathematical facts because with this respect astrology could also be called modeling since it is an activity in which one gives interpretations of fact about celestial objects that can be predicted mathematically.

In the scientific modeling activity the predictions of the model are to be compared to reality by means of empirical data. This is shown through arrows 4.1, 4.2... which concern empirical data which are produced in laboratories where the "piece of reality" is extrapolated, or data produced directly by the "piece of reality itself" (arrows 5.1, 5.2...). We make a distinction between data which are produced directly by reality and those which are produced in a laboratory. The second are much secure while the first are subjected to errors of various origin. Most of "life support systems" are complex systems in the sense that we cannot reduce them to the laboratory and by the way they often lake of good data. This is a reason why we try to understand large complex systems but we must remember that no model can replace good data.

1.3 The need for mathematics

By definition a *mathematical model* must have a strong connection with mathematics! This is represented by the arrow 3. A mathematical model is by itself an object, which is inserted in a theory more or less developed. It is this mathematical theory which gives to the model its utility. Let us consider an example. In a lake (which water is supposed to be at rest), one represents by $U(t,x,y)$ the concentration of some pollutant at the point of coordinates (x,y) at time t. The pollutant diffuses in the water, and the evolution of this concentration during time can be represented by the system of equations :

$$\frac{\partial U(t,x,y)}{\partial t} = k^2 \{ \frac{\partial^2 U(t,x,y)}{\partial x^2} + \frac{\partial^2 U(t,x,y)}{\partial y^2} \} + \phi(t) \delta_{x_0, y_0}$$

$$(x,y) \in \partial\Omega \Rightarrow \quad \frac{\partial U}{\partial \vec{n}} = 0$$

which express that the flux at the boundary of the lake is null and that the pollutant source is located at the point (x_0, y_0). From the knowledge of the function $\phi(t)$, one can compute (analytically in some cases, with computer simulations in other cases) the value of U at any point at time t. From the knowledge of the position of the emission of pollutant one can deduce the future pollution at each point. This is not much surprising, we are accustomed from longstanding successes of celestial mechanics, that mathematical models predict the future motion. May be, more surprising is the following: Assume that we know the intensity of the pollution at one, or some points, in the lake; is it possible to recover, from this information, the position and the intensity of the emission of pollutant? The answer is yes. This kind of problems are known as inverse problem in mathematics, have a long history, and have been solved recently for wide classes of useful equations. It must be noticed that if the question of prediction of future pollution from the knowledge of the pollutant source requires relatively little mathematical technicalities and can be implemented on a computer quite easily, the solution of the inverse problem requires high mathematical sophistication.

There is an other reason why a mathematical theory is necessary when we develop models. As we said we need to make simulations and need to interpret them. Since the simulation is done on a computer which is not able to compute with ideal real numbers there are numerous artifacts related to the computation by itself, which must not be interpreted as properties of the model. The way to measure the distance between mathematical ideal solutions and the actual simulations consist in a wide body of knowledge which was developed for the use of computers and is called *numerical analysis*.

More and more frequent are models of simulation, *computer models*, which are built directly from the discourse about reality, using high level languages which fit perfectly with the purpose of modeling. The temptation, for non mathematicians, is important to try to avoid mathematics. Our position here is that a model, by itself, needs a theory. The physical theory which explain how the electronic microscope works has nothing to do with the biology of cells that are observed but it is necessary to understand the image that one sees. Computer simulation is to the knowledge of the dynamics of complex systems what the microscope is to the vision of the infinitely small: It is a tool which also needs its own theory. The theory of the model, which must not be confused with the theory of the piece of reality for which the model contributes, must be done in the language of mathematics.

1.4 Orientation of the article

We first (section I) give an account of the concept of mathematical dynamical systems, including infinite dimensional systems and stochastic processes. The importance that we give to each aspect of dynamical systems does not reflect the importance they occupy in the mathematical world because our objective is not to give a fair description of the subject but to introduce to the mathematical tools that are useful for modeling. We tried to explain concepts through examples that we choose more in the life sciences than in physics since we think that people trained to exact sciences have already a general understanding of the subject.

In the second section we give a large account of mathematical system theory for the reasons we explained earlier. The basic concept developed in this section is the concept of input-output system.

In the third and last section we ask the question: A model for what purposes? We suggest the following classification.

- Models for understanding, where the model has no pretension of being in accordance with empirical dada but just pretends to give light on the discourse.
- Models for description and prediction which have qualitative and quantitative connections with the real world
- Models for purposes of control.

2. The mathematical concept of dynamical system

Every scientist has its own idea of what an actual *dynamical system* is. For a physicist it could be a set of punctual masses interconnected by strings, occupying various places along time. For an oceanograph it could be the displacement of masses of oceanic water. For the agricultural engineer, which fight against insects it could be the evolution during the season of their number in correlation to its action. For an economist it could be the evolution of prices of goods related to their production. It is not useful to multiply these kinds of examples. But now consider the following sentence : "The spirit of this man has changed : he was careless and casual but now we can be confident in him". This sentence says that something changed during the time : the spirit of a man. But, opposed to the preceding examples, where there was something to quantify, the position, the production, e.t.c... there is hardly something comparable in the case of the spirit. This is the reason why mathematical concept of dynamical system is not very useful in such questions. In the game of love mathematicians have no more nor less success than anyone!

The mathematical concept of dynamical system was certainly elaborated in view of the development of classical mechanics since the discovery of its laws by Newton. We shall not attempt in this article to recall the history of this mathematical concept, which we assume to be more or less familiar to the reader. We just want to recall what are the various type of dynamical systems and fix some notations. A most important concept in the theory of dynamical system is the concept of state and state space. We try to make this concept clear in this section.

2.1 Deterministic Systems

A *deterministic system* is a set of mathematical rules, one could say an algorithm, which, once the state of a certain system is known at some time t_0, the future sates of the system are defined in a unique way.

2.1.1 Dynamical system on a finite set

The simplest mathematical system that one can consider is constituted by a finite set of points (or numbers, or letters). For each point (number, letter) a rule indicates a new point (number, letter). The rule can be given by a set of arrows, like in Fig. 2. below.

Fig.2 : A dynamical system on a finite set

A set of points connected with the arrows constitute what is called a graph and can be also represented by the table below :

a	b	c	d	e	f	g	h	i	j	k	l	m
d	a	f	f	g	g	h	f	g	m	j	k	k

This can be also summarized in a table which at the first glance look more complicated but reveals to be very efficient (See Fig 3)

Fig.3 : The matrix associated to a dynamical system on a finite set

Starting from the point e we see that the corresponding point is g, the successor of g is then h and son on. We have the succession e, g, h, f, g, h, ...This succession is called the *trajectory* from *the initial condition* e. For all the possible initial conditions the trajectories are given below :

a, d, f, g, h, f, g, h, f, g, h, f, g, h, , f, g, h, ...
b, a, d, f, g, h, f, g, h, f, g, h, f, g, h, f, g, ...
c, f, g, h, , f, g, h, f, g, h, f, g, h, f, g, h, , f, g, ...
d, f, g, h, , f, g, h, f, g, h, f, g, h, f, g, h, , f, g, ...
e, g, h, , f, g, h, f, g, h, f, g, h, f, g, h, f, g, ...
f, g, h, f, g, h, f, g, h, f, g, h, f, g, h, f, g, ...
g, h, f, g, h, f, g, h, f, g, h, f, g, h, , f, g, ...
h, f, g, h, f, g, h, f, g, h, f, g, h, f, g, h, f, g, ...
i, g, h, f, g, h, f, g, h, f, g, h, f, g, h, f, g, ...
j, m, k, j, m, k, j, m, k, j, m, k, j, m, k, j, m,...
k, j, m, k, j, m, k, j, m, k, j, m, k, j, m, k, ...
l, k, j, m, k, j, m, k, j, m, k, j, m, k, j, m, k....
m, k, j, m, k, j, m, k, j, m, k, j, m, k, j, m, k, j...

Denote by F the mapping from the set {a, b, ..., l, m} into itself which is defined by . The successions above can be represented by the algorithm :

$$x(0) = x_0 \qquad x_0 \in [a,\ b,\ ....,\ m]$$

$$x(n+1, x_0) = F(x(n, x_0))$$

which is a most efficient way to represent the infinite succession defined by F. If one recalls the way one must do the product of matrices, row by column, the mapping F can be represented by the matrix product below :

$$\begin{bmatrix} d \\ a \\ f \\ f \\ g \\ g \\ h \\ f \\ g \\ m \\ i \\ k \\ k \end{bmatrix} = \begin{bmatrix} 0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \\ k \\ l \\ m \end{bmatrix}$$

If we call A the matrix and denote by

$$X(n) = \begin{bmatrix} x(n,a) \\ x(n,b) \\ \cdots \\ \cdots \\ x(n,m) \end{bmatrix}$$

the column consisting of the trajectories issued from a, b, ..., m one sees that we have:

$$X(n+1) = AX(n)$$

One sees that every trajectory ends with a periodic succession. On Fig. 2, one can see why that there are in our example exactly two such periodic succession, namely f, g, h and j, m, k. This is a general fact on a finite set. Every trajectory must be ultimately periodic because, since the set is finite, after some time one must go through a point which has been already explored.

In principle, every algorithm on a computer is a dynamical system on a finite set since the complete collection of numbers (or any object) that can be represented on a computer is finite. But this "finite" is so large that the possible dynamical systems that can be implemented on a computer cannot be explored systematically. We shall come back to this point in section 2.3

2.1.2 Discrete dynamical systems on R

A slightly more rich dynamical system is given by the consideration of the set of real numbers R and a mapping F from R into itself and, as previously, a trajectory starting from x_0 will be defined by : $x(n+1,x_0) = F(x(n,x_0))$. Since the set of real numbers is not a finite set there is no reason, like in the previous section, for a trajectory, to be ultimately periodic. On Fig. 4.a one sees how one can use the graph of the function F in order to get graphical construction of the trajectory from a point. One sees on the example shown that it seems (and actually it is the case) that every trajectory converges to a fixed point. This point is called a *stable equilibrium*, On Fig. 4.b one sees that there is an equilibrium which is unstable, and on Fig. 4.c one sees that the trajectory might be complicated.

Fig.4 : Construction of trajectories for a dynamical system on R

Actually, thank to the first computers, it was discovered by numerical simulations that the behavior of such systems might be very rich, and they started to be studied very extensively in the seventies. The mathematical theory of such mathematical systems is still in progress. It was proved that trajectories may have a behavior which is very complicated and unpredictable. This was called *deterministic chaos* and turned out to become very popular in many sciences (*See Complexity, Pattern recognition and Neural network*).

When the state is a vector with n components and the mapping F is a linear application defined by a matrix A one have a *discrete linear system*. These systems are very useful in many applications, from ecology to economics, especially in population dynamics. We exemplify in details an application to population dynamics.

A population is structured in n stages and the "life cycle" of the organism is represented in Fig.5 :

Fig.5 : Life cycle

The weights p_i supported by the arrows mean that at each time step, a proportion p_i of the population which is in stage i goes to stage i+1 and the weights μ_i are birth rate coefficients. We represent by the vector :

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \cdot \\ x_n(t) \end{bmatrix}$$

of all the quantities of the individuals in each stage and by A the matrix :

$$A = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \cdot & \mu_n \\ p_1 & 0 & 0 & \cdot & 0 \\ 0 & p_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & p_{n-1} & 0 \end{bmatrix}$$

one has clearly :

$X(t+\Delta t) = AX(t)$

In this example we decided that the time step is not 1 but Δt. From the mathematical point of view this does not make a difference. The matrices with the specific structure of the matrix A are called Leslie matrices. Since all the entries of the matrix are nonnegative these matrices have interesting specific properties (See *Basic Methods of the Development and Analysis of Mathematical Models*)

2.1.3 Systems of differential equations

A particular case of discrete dynamical system is the one where the function F is of the form below:

$$F(x) = x + f(x)dt$$

and thus the trajectory from an initial condition is given by the following equation :

$$x(t+dt) = x(t) + f(x(t))dt$$

to which we give the following interpretation : During a very short interval of time, which is denoted by dt, some quantity which is represented by $x(t)$ varies of a quantity denoted by $f(x(t))dt$. One sees that the variation of $x(t)$ is small, depends on $x(t)$ and is proportional to dt. The relation below which defines the trajectory from an initial condition :

$$x(t+dt) = x(t) + f(x(t))dt$$
$$x(0) = x_0$$

can be rewritten as follow :

$$\frac{x(t+dt) - x(t)}{dt} = f(x(t))$$
$$x(0) = x_0$$

In the left hand side of the above equality one remarks the classical formula defining the derivative. Thus if we go to the limit when dt goes to 0 we can write :

$$\frac{dx(t)}{dt} = f(x(t))$$
$$x(0) = x_0$$

The above set of two equations is called the *Cauchy problem* for the *differential equation* with right hand side f and *initial condition* x_0. This means that we are looking for a function $x(t)$ which is continuous and has a derivative at each point and satisfy identically the above equality. Under technical, but widely satisfied conditions, it is proved that the Cauchy problem admits a unique solution defined on an open interval containing 0. This solution is denoted by $: t \to x(t, x_0)$ and it is called the trajectory through x_0.

In the above notation the symbol x represents a real number and we speak of differential equation. We can generalize this to the case where the mapping f is a mapping from R^n to R^n and x represents a vector. In this case we say that we have a system of differential equations or a *differential system.*

A system of differential equations can be used to describe the evolution during time of a set of n distinct quantities like, for instance, the position of punctual masses in the space (mechanics), or the concentrations of various chemical species interacting in a well stirred tank (chemical kinetics), or the biomass of various specie interacting in an ecosystem.

The study of systems of systems ordinary differential equations have been, and is still, the object of a considerable effort of mathematicians. Except for the case of linear systems :

$x'(t) = Ax(t)$

which is well understood most of the theoretical results concerns the case of comparatively small dimensions. Two dimension nonlinear systems benefit of a fairly good understanding, thanks to a geometrical method called phase plane analysis, three, and more, dimensional systems may exhibits complex behaviors like strange attractors. Motivated by application in life sciences, a lot of work on differential equations of different type from those of classical mechanics, was done since 1970. So called *positive systems* are of particular importance in the study of ecological systems. (See *Basic Methods of the Development and analysis of Mathematical Models*).

2.1.4 Dynamical systems on manifolds

It is not always suitable to take a vector space as the state space for a differential system. For instance if one considers a system representing the evolution of some populations the quantities of individuals are intrinsically positive variables and one must restrict to the positive hortant of R^n. An other example is when a state variable is an angle. Two angles are identical when they differ of 2π and the set of angles is commonly represented as a circle of radius one. These two examples show the need for something different from an euclidian space. The right object is a manifold.

A simple example of manifold is the sphere $S^2$of radius one in the three dimensional space. At each point m a sphere possesses a tangent plane denoted by $T_m S^2$. If one considers a smooth curve $t \to \gamma(t)$ at the surface of the sphere, the derivative of this map with respect to t, the tangent to the curve, belongs to the tangent plane :

$$\frac{d\gamma(t)}{dt} \in T_{\gamma(t)} S^2$$

This creates some complications in order to define what is a differential system. One must define at each point of the sphere, a vector in the tangent plane to the sphere at this point. This is called a vector field and is usually denoted by some capital letter like X. By a solution of the differential equation (on the manifold) it is meant any curve $t \to \gamma(t)$ such that:

$$\frac{d\gamma(t)}{dt} = X(x) \in T_{\gamma(t)} S^2$$

It can be proved that for every initial condition x, and sufficiently small duration, it exists a unique solution, which is at the initial condition for t equal to zero. The value at time t of this solution is denoted by: $X_t(x)$. It is a useful notation, which contains all the necessary information: The name of the vector field, which defines the differential system, the name of the initial condition and the duration. The mapping:

$$(x,t) \to X_t(x)$$

A general manifold is something like a sphere, or a surface in R^n, but of arbitrary dimension and by the way is difficult to visualize since dimension three. Manifolds are important for applications because they have topological properties which are very different from that of the euclidian space. For instance the sphere is compact. More important are topological obstructions that a manifold may induce. For instance imagine that for some reason we want to model a system which is never at rest on the sphere of dimension two. For this purpose we want to take a vector field which is never equal to zero. It turns out that this is impossible. A continuous vector field on the sphere of dimension two must be equal to zero at least in one point. This is a result of algebraic topology.

2.1.5 Infinite dimensional systems

All the systems we considered up to now, discrete systems, differential equations, systems of differential equations have in common the essential fact that, once an initial condition is fixed, then, all the future behavior of the system is fixed in a unique way. This is determinism ! The quantity, which we noted x, is the *state* variable. It was successively a point in a finite set, a real number, a vector and a point in a manifolfd. This is not sufficient to cover all interesting applications. Let us come back to the question of the diffusion of a pollutant that we evoked earlier. Speaking of diffusion supposes that the concentration of the pollutant is different at each point of the physical space which is a three dimensional space or a two dimensional one, if we assume that the thickness is negligible. As opposed to the previous cases, where the knowledge of a finite number of values is enough to define the future behavior, in this case one must know the values of all the concentrations at every point of the medium, that is to say the knowledge of some function :

$$(x,y) \to U(t,x,y)$$

which determines the future. In this case the state variable belongs to an infinite dimensional vector space, for instance the space of two time continuously differentiable functions. In this context the concept of Hilbert space is of a tremendous importance since it is both a natural and intuitive extension of the concept of finite dimensional vector space and has numerous applications in physics and engineering techniques. A good example of Hilbert space is the space of trigonometric series, that is to say the set of functions of the form :

$$u(x) = a_0 + \sum_{1}^{+\infty} a_n \cos(nx) + b_n \sin(nx) \quad \text{with} \qquad \sum_{0}^{+\infty} a_n^2 + b_n^2 < \infty$$

A scalar product is defined as :

$$< v,v >= \int_0^{2\pi} u(x)v(x)dx$$

and the functions $x \to \cos(nx)$, $\quad x \to \sin(nx)$ define an "orthogonal basis" of the space in the sense that every function which is square integrable has a unique expansion as a trigonometric series. Let us come back to our *diffusion equation* in one dimension without a source term :

$$\frac{\partial U(t,x)}{\partial t} = k^2 \frac{\partial^2 U(t,x)}{\partial x^2}$$
$$U(t,0) = U(t,\pi) = 0$$
$$U(0,x) = \varphi(x)$$

If one uses the Hilbert space representation, that is to say if we write the trigonometric expansion of $U(t,x)$ with respect to x :

$$U(t,x) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(nx)$$

(the sine terms do not appear because of the limit conditions) the partial differential equation reduces to:

$$a'_n(t) = -k^2 n^2 a_n(t) \quad ; \quad n = 0,1,...\infty$$

which looks very much like a system (infinite) of ordinary differential equations. One defines abstract *evolution equations*, in infinite dimensional spaces like Hilbert spaces or more abstract spaces and the same type of notation than in the finite dimensional case is used to denote them :

$$\frac{dx(t)}{dt} = f(x(t)) \qquad x \in E$$

but one must have in mind that in this case, despite of simple notations, one manipulates highly abstract objects.

2.1.6 Miscellanies

All the dynamical systems that we have considered up to now are *autonomous*, which means that the function *f* which describes the dynamics depends only on x and no other variable. This means that the laws which govern the motion of x are everlasting, do not change with time. It is easy to imagine situations where these laws depend on time. For instance the growth of a photosynthetic organism will depend of the quantity of light it receives, quantity which varies during the day. Essentially, everything which can be said on dynamical systems can be said for non autonomous dynamical systems and the notations used is the following :

$$\frac{dx(t)}{dt} = f(x(t),t) \qquad x \in E$$

There are also classes of dynamical systems which are dot differential systems, nor partial differential equations but can be considered as evolution equations on an infinite dimensional space. Namely *delay equations* and *integro - differential equations.*

A delay equation is an equation of the form :

$$\frac{dx(t)}{dt} = f(x(t), x(t-h))$$

where the variation of x at time t depends on the value of x at time t but also at a previous time t-h. The number h is the delay. This kind of models is important in demography or transmission of diseases where an event - birth, illness - depends of events that occurred some time ago - female-male meeting, infected-susceptible meeting. The harmless aspect of the delay equation hide the fact that it is actually an evolution equation in an infinite dimensional space, since the initial condition to be specified must be a whole function defined all over the interval [-h, 0].

In an integro - differential equation the variation might depend on the whole past of the object :

$$\frac{dx(t)}{dt} = \int_{-\infty}^{t} f(x(s))dt$$

or depend, in an integral way, of its spatial extension.

$$\frac{dU(t,x)}{dt} = \int_{\Omega} f(U(t,x)dx$$

(See *Classification of models*)

So far we have considered differential equations where the derivative of the state, say :

$$\frac{dx}{dt}$$

is equal to some value computable from the state. There are many circumstances where the exact value of the rate of change of the phenomenon is not known with precision. We just know that the derivative belongs to some set. We write this :

$$\frac{dx}{dt} \in \Gamma(x)$$

where $\Gamma(x)$ is some set of admissible velocities. Such a system is called *a differential inclusion.* From a given initial conditions a differential inclusion has many solutions. Suppose moreover that we are interested only by solutions which remain in a certain set; the study of this kind of dynamical system is called *viability theory* and was recently the object of many theoretical investigations. A differential inclusion does not define a deterministic dynamical system since the future is not unique, but is not a stochastic system since no probabilistic assumptions are made.

2.2 Stochastic dynamical systems

In the nature there are many situations where the future is not predictable from the initial conditions. It might depend on our imperfect knowledge of the laws or of the initial conditions, or to more intrinsic reasons. The most common instance of this situation is coin tossing. When one throw a coin, nobody knows whether it will turn head or tail. One may argue that this is due to the imprecision of our exact knowledge of the initial condition, the exact force exerted by the thumb of the player, the state of atmosphere etc... But this is not of great help since there is no way to have this knowledge. Thus we must accept that, in some circumstances, we cannot predict the future. But an interesting point is that if we cannot predict the future for one toss, we can for a great number, say 10000. In this case we are "almost sure" that the number of tails will be between 4850 and 5150. A mathematical theory gives a precise meaning to what is pure random and tells us that, if our coin is fair, i.e. obey to the rules of pure random, then there is one chance over one thousand to observe a deviation greater than 150 from 5000. Probability theory, which gives precise definition to "perfect randomness" has a long history. So there is a mathematical model of "pure randomness" and this model proves to be useful since, at least in physics, it gives unified explanations of such phenomena like diffusion processes, laws of thermodynamics, percolation etc.... This model gives also the foundations of the widely spread method of statistical inference.

A *stochastic process* is a system where the future of one individual is not predictable, but the average behavior of a great number of individuals is predictable to some extent. From this viewpoint a stochastic situation can reduce to a deterministic one. It is the case, for instance, when we are interested in the growth of some cell population. We know that at some time some cell will duplicate, but we do not know exactly when. But at the population level the random variations are not seen, at least if the population is large enough, the growth rate is a well defined number and the population dynamic is perfectly described by some differential equation of logistic type. But not all stochastic situation are reducible to deterministic ones. For instance it is the case when we are interested in the survival of a population of a few number of individuals like whales in the ocean.

There is another extremely important reason to be strongly interested by stochastic models. Most of actual dynamical systems, even if they are deterministic, are corrupted by small irregularities that we do not want to include in the model, but that we cannot completely neglect. These small irregularities are considered as if they where produced by a process of pure randomness called noise. The irregularities can affect also the precision of some measurement process that are included in the model. Under the name of *filtering* engineers from automatic control and data processing have developed highly sophisticated mathematical devices implemented on high speed computer which improve considerably the performances of any sensor.(See *Classification of models*)

In this section we follow the same lines than for the previous one, starting from the simplest possibility, stochastic system on a finite set, looking to the random walk on an infinite lattice and then introducing diffusion processes and stochastic differential equations.

2.2.1 Markov Chain

We consider a particle π which moves on a finite set denoted by $\{1, 2, \ldots, n\}$ and jumps from the point i to any other point j including i itself. We assume that if the particle is at the point i at time t it will jump to the point j with the probability $p_{i,j}$ which is independent of the past of the particle. This is a *finite Markov chain.* If we introduce the notation $x_i(t)$ for the probability for π of being at the point i :

$$x_i(t) = P(\text{"}\pi \in i \text{ at time } t\text{"})$$

we have clearly the relation :

$$x_j(t+1) = \sum_{i=1}^{n} x_j(t) p_{ij}$$

If we denote by $X(t)$ the row :

$$X(t) = [x_1(t), \quad x_2(t), \quad . \quad . \quad x_n(t)]$$

and by A the matrix :

$$A = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdot & p_{1,j} & \cdot & p_{1,n-1} & p_{1,n} \\ p_{2,1} & p_{2,1} & \cdot & p_{2,j} & \cdot & p_{2,n-1} & p_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{i,1} & p_{i,2} & \cdot & p_{i,j} & \cdot & p_{i,n-1} & p_{i,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{n-1,1} & p_{n-1,2} & \cdot & p_{n-1,j} & \cdot & p_{n-1,n-1} & p_{n-1,n} \\ p_{n,1} & p_{n,2} & \cdot & p_{n,j} & \cdot & p_{n,n-1} & p_{n,n} \end{bmatrix}$$

we have the relation : $X(t+1) = X(t)A$. The matrix A has the property that all its entries are nonnegative and the sum of each row is equal to 1. It is called the *transition matrix*. Formally the study of the probability vector $X(t)$ of a finite Markof chain is equivalent to the study a linear discrete deterministic system on R^n, (except the fact that the tradition in probability is to write the vector of probability as a row vector). Thus the evolution of the jumping point is not deterministic but the evolution of the probability of its presence is. The state space for the vector of probability is R^n while the state space for the particle is a finite set of n points. If we imagine that we have a great number of particles jumping with the same rules, the number of particles at each point will be proportional to the probability of presence of one single particle. If now we do not think in term of particle jumping from one place to another but in term of population leaving one stage for an other we have an analogous model to the one that we already discussed in the deterministic section, which is not surprising.

On fig. 4 below we have represented a Markov chain with four states by four points with arrows and the corresponding transition matrix. As a rule we do not represent the arrows with transition probability 0.

Fig.6 : A Markof chain and its transition matrix

2.2.2 Random walk and Wiener process

We suppose now that we have a stochastic process with all the features of finite Markov chain, except that the number of possible states is not finite. The simplest case of this situation is the random walk where the states are the relative integers ($\pm 1, \pm 2, \pm 3, \ldots$) and the probability of a jump from i to $i+1$ is equal to the probability of a jump from i to $i-1$ and equal to $\frac{1}{2}$. Assume that the particle is at point 0 at time 0. Denote by $X_k \quad k = 1,2,\ldots,n$ a sequence of independent random variable taking values ± 1 with probability $\frac{1}{2}$, then the random variable describing the position of the particle at time n is :

$$Y_n = \sum_{k=1}^{n} X_k$$

The law of Y_n is well known since Newton and Pascal and is given by :

$$P(Y_n = k) = \frac{1}{2^n} C_n^k = \frac{1}{2^n} \frac{n.(n-1).(n-2).....(n-k+1)}{k!}$$

Imagine now a process where the random do not operate each second but each very short time *dt* and that during this short time the particles jump on right or left, with probability $\frac{1}{2}$, for a jump of length *l*. If we denote by Y_t the probability of presence at time t one has :

$$Y_{t+dt} = Y_t + l\, X_t$$
$$Y_{ndt} = \sum_{k=1}^{n} l X_t$$

It is very easy to compute the mean (it is 0 !) and the standard deviation of Y_t which is

$$\sigma(Y_{ndt}) = l\sqrt{n}$$

The standard deviation at the time $ndt = 1$ is by the way equal to $\frac{l}{\sqrt{dt}}$. If *dt* is a very small quantity (or if we intend to consider the limit case where *dt* tends to 0) and if we want to have a non trivial process, that is to say with a standard deviation which is neither infinitesimal nor infinitely large the length *l* must be of the order of magnitude of $\sqrt{dt}$ which we express by rewriting the process :

$$Y_{t+dt} = Y_t + \sigma\sqrt{dt}\, X_t$$
$$Y_0 = \delta_0$$

We are in a very similar situation than that of the discrete deterministic system when *dt* is small :

$$x(t+dt) = x(t) + f(x(t))dt$$
$$x(0) = x_0$$

where we made the derivative appear like below:

$$\frac{x(t+dt) - x(t)}{dt} = f(x(t))$$
$$x(0) = x_0$$

Unfortunately, if we do this manipulation here we get a division by $\sqrt{dt}$ which makes the right hand side going to infinity when *dt* goes to zero:

$$\frac{Y_{t+dt} - Y_t}{\sqrt{dt}} = \sigma\, X_t$$
$$Y_0 = \delta_0$$

This tells us that a pure random process indexed by a real number has no derivative and by the way the theory of ordinary differential equations does not apply.

It is not very easy to go to the continuous time process, like in the deterministic case. In some sense this is easy to understand. We have the concept of pure random walk in which we run strait until the **next step** time where we choose at random the direction of our velocity. If time is indexed by a real number *t*, there is no "next step" and the idea of choosing at random in a continuous way is necessary rather abstract.

This problem is solved by the so called Wiener process (introduced by Bachelier in 1900 and studied by Wiener in 1922 and Levy 1937). The Wiener process is a family of random variables W_t indexed by the real number *t* which satisfies the following conditions :

- For every pair $t > s$ the law of the variable $W_t - W_s$ is defined by the gaussian density:

$$N(0,\sigma\sqrt{(t-s)}) = \frac{1}{\sigma\sqrt{t-s}\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2(t-s)}}$$

- For every two pairs $t > s$ and $t' > s'$ the random variables $W_t - W_s$ and $W_{t'} - W_{s'}$ are independent.

One recognize in the above formula the fundamental solution for the heat equation :

$$\frac{\partial}{\partial t}u(x,t) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}u(x,t)$$

Indeed this is not surprising. Consider our previous random walk with infinitesimal step:

$$Y_{t+dt} = Y_t + \sigma\sqrt{dt}\ X_t$$
$$Y_0 = \delta_0$$

and denote by $P(x,t)$ the probability of being in the position x at time t. We have the following formula :

$$P(x,t+dt) = \frac{1}{2}P(x-\sigma\sqrt{dt}) + \frac{1}{2}P(x-\sigma\sqrt{dt})$$

since if we are at x at time $t+dt$ we must be at $x \pm \sigma\sqrt{dt}$ at time t. From this formula we get by adding $P(x,t)$ to each side :

$$P(x,t+dt) - P(x,t) = \frac{1}{2}(P(x-\sigma\sqrt{dt}) - P(x,t)) + \frac{1}{2}(P(x-\sigma\sqrt{dt}) - P(x,t))$$

and taking Taylor expansion to the order one with respect to t in the left had side, and to the order two with respect to x in the right hand side we get the heat equation :

$$\frac{\partial}{\partial t}P(x,t) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}P(x,t)$$

2.3 Discrete versus continuous models

In sections 2.2.1 and 2.2.2, we started from the simplest dynamical system, a discrete deterministic system on a finite set or a Markof and reviewed various more and more abstract systems and concluded by systems where time is a continuous variable and state is an infinite dimensional vector space of functions. On the other hand we do know that, most of the time, these dynamical systems will be simulated on a computer through an appropriate program of discretization. Since real numbers are actually represented by a finite set of numbers, it turns out that all our simulations will be simulations of some discrete finite dynamical system.

Consider now the question of the fitness of our models with reality. For instance we have considered the question of the diffusion of a pollutant in some medium and decided that the concentration at each point was a continuous variable? Why do we not consider the number of molecules which are in some volume? This would appear more realistic. Moreover, since reality is observable only through a finite number of sensors, would it not be more simple and adequate to simulate directly reality by discrete systems on finite sets. We end this list of questions by asking: mathematics where useful before the high speed computers, but now? What proves that mathematicians are not defending the old privilege of being essential in the process of describing nature? This is not the case but just the opposite. The development of high speed computing creates a big need of both traditional and new mathematics. Let us explain why.

2.3.1 Discrete versus continuous time models

Thanks to mathematics we can replace simple, but lengthy, computations by shot ones. Let us try to explain this point on a very simple example. Imagine that we are in old times, before the achievement of computer, when only paper and pencil where available.

A man put money on a bank account which is remunerated every two weeks at the rate of 0,13%. This man put a capital of C and wants to know how much money he will own 30 years later, which means 720 steps of time later. This defines a perfect discrete dynamical system on R. Namely we make the following model.

One denote by S(n) the amount of money owned at step n. From the definition of the process adding 0,13% of interest one has :

S(n+1) = S(n) + 0,0013 S(n)
S(0) = C

or in other words :

S(n+1) = (1,0013) S(n)
S(0) = C

If we decide to make the computation with two digit accuracy at each step, it takes about one minute to do this multiplication by hand, and about 12 hours to do the complete computation. If we intend to do it safely it is reasonable not to go further than 4 hours of computations each day and thus it takes about three days to get the result.

From the mathematical theory of ordinary differential equations we know that the numbers defined by the 720 multiplication above and the value a time t = 1 of the Cauchy problem:

$$\frac{dy(t)}{dt} = 0{,}936y(t)$$
$$y(0) = C$$

are very close together. Indeed we have:

$$S(n+1) = S(n) + 0{,}0013S(n)$$

which we can rewrite:

$$S(n+1) = S(n) + kS(n)dt$$

with:

$$dt = \frac{1}{720}$$
$$k = 0.0013 \times 720 = 0{,}936$$

and since dt can be, in that case, considered as small, we can replace the difference equation by the differential equation.

Every body knows that $t \to e^{0{,}936t}$ is the solution of this equation and one have just to compute the number exp(0;936). One also knows that the function exp(z) has the series expansion :

$$1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^n}{n!} + ...$$

from which the sum of the ten first terms is a very accurate approximation. The computation of sum :

$$1+\frac{0,936}{1!}+\frac{0,936^2}{2!}+\frac{0,936^3}{3!}+\frac{0,936^4}{4!}+\frac{0,936^5}{5!}+\frac{0,936^6}{6!}+\frac{0,936^7}{7!}+\frac{0,936^8}{8!}+\frac{0,936^9}{9!}$$

will takes at most one hour.

Actually at the calculus with logarithms tables is much more efficient than our present procedure through a Cauchy problem. We choose this illustration because our topic is differential equations and the important point in the above example is that, via the approximation of our (exact) discrete model by a continuous differential equation we found a much more efficient algorithm to compute the desired solution.

Let us come back to more realistic situations. We know that in one liter of some gas there are about 10^{22} molecules for which we have a good model in the consideration of 10^{22} point masses reacting under the Newton's laws. The simulation of a set of 10^{22} differential equations is definitely impossible since, if we imagine that each human has a personal computer with each ten Giga octets of memory we get less than 10^{22} kilo octet of memory which is still not enough to perform such a simulation. Thus good models for gas are macroscopic ones with continuous variables.

The art of dealing with order of magnitudes, change of scale and invent good macroscopic variables is a difficult one. In particular the modeling in fluid mechanics has developed this art to a degree of high sophistication, but in many domains of modeling pertinent for Life Support Systems, it has to be improved and is a matter of current research.

2.3.2 Cellular automata and individual oriented models

A cellular automaton is a popular discrete time model used to represent spatio-temporal interactions. Let us explain it on the example of so called excitable media. We consider a grid of squares in the plane, denoted by Cij. The state of each cell is an integer denoted Cij(t) which belongs to the set : $\{0,1,2,...r-1,r,r+1,.....n-1\}$ where 0 is called the *rest* state $\{1,2,...r-1\}$ the *excited* states and $\{r,r+1,...n-1\}$ the *refractory* states.

The dynamic is defined by the following rules :

If $C_{i,j}(t)\neq 0$ then :

$$C_{i,j}(t+1)=C_{i,j}(t)+1 \qquad \mathrm{mod}(n)$$

else if $C_{i,j}(t)=0$ and if at least one neighbor of $C_{i,j}$ is in an excited state then :

$$C_{i,j}(t+1)=1$$

else

$$C_{i,j}(t+1)=0$$

In other words, when a cell is in an excited state, it communicates its excitation to a neighboring cell which is no longer in a refractory state.

Such a model is a very crude approximation of what appends in a chemical reaction in a reactor which is not well stirred and thus supports diffusion phenomena. A more realistic model for such a situation is given by so called "reaction-diffusion" equations which turns out to be systems of partial differential equations of parabolic type. Unfortunately mathematical theory of such models is very difficult and is still in progress. Opposed to these realistic models, cellular automata models, which retain only the essential features of the phenomenon, where easy to observe through computer simulation and we got much insight from this experimental work.

In a cellular automaton one can consider each cell as an "individual" which interacts with other individuals, its four neighbors in this case. An individual oriented model is a generalization of this situation. One considers a set of individuals indexed by the integer i. Each individual has a state which can be anything one wants, for instance position, age, some chemical composition, etc, each individual changes its state at each step of time, according to the state of the others individuals.

Thanks to modern computer languages facilities the kind of models are very popular because one can simulate directly any discourse about reality, avoiding mathematical formulations. A draw back of these models is that they can produce phenomena which are just computer artifacts which where not intended to be in the model.

We see that there is a strong connection between stochastic processes (microscopic viewpoint) and continuous partial differential equations (macroscopic view point). In our example the laws of random do not depend on position and time. When it is not the case the connection between the microscopic and the macroscopic viewpoints deserves very difficult problems of modeling. We shall give an idea of the origin of such problems in the next paragraph but before we come back to the problem of definition of the pure random walk.

There is an alternative way to the definition of the Wiener process when one wants to define an idealized pure random walk of infinitesimal step. It is to use the so called "Non Standard Analysis". Non Standard Analysis is a perfectly rigorous mathematical language in which the notion of fixed infinitesimal dt (as opposed to dt tending to 0) is perfectly defined. Historically the use of fixed infinitesimals was considered as correct up to the end of the 18th century when some unacceptable paradoxes where discovered. This is the reason why the ε, δ method was developed by Weirstrass and Cauchy during the 19 th century. It is only during the second half of the 20th century that, thanks to the results of mathematical logic, the possibility of dealing with fixed infinitesimals, with complete mathematical rigor was invented by Robinson. In this framework, the equivalent of the Wiener process is just the random walk with dt infinitesimal, infinitesimal being taken in the technical sense of NSA. This way of dealing with the matter was not considered by many mathematicians who where reluctant to learn the basis of NSA until recently. The mathematician and physicist E. Nelson wrote a pioneering paper in NSA small book (less than 100 pages) "Radically elementary probability theory" in which one find explained a soft version of NSA which is sufficient for doing probabilities.

2.2.3 Stochastic differential equations

We first consider a random walk with infinitesimal steps of a more general type than the one considered in the previous section. Namely :

$$Y_{t+dt} = Y_t + f^0(Y_t)dt + \sum_{i=1}^{p} f^i(Y_t)Z_t^i \sqrt{dt}\ X_t$$

$$Y_0 = \delta_{y_0}$$

where Y_t is a vector of R^n, f^0 and f^n are function from R^n in itself and Z_t^i are independent random variables taking values ± 1 with probability $\frac{1}{2}$. We want to go to the limit for dt going to 0 like we did with the pure random walk. First assume that f^i is null for i positive. Then there is no random term and we recognize the discrete deterministic walk that led us to the concept of differential equation :

$$\frac{dY(t)}{dt} = f^0(Y(t))$$

Conversely, if $f^0 = 0$, $p = 1$ and $f^1 = 1$ we recognize the pure random walk. Thus we try to do something similar to what we did for the random walk but there is a problem due to the fact that f^1 is not constant. Let us explain it on an example. Consider the system of differential equations :

$$\frac{dx_1(t)}{dt} = x_2(t)u(t)$$

$$\frac{dx_2(t)}{dt} = -x_1(t)u(t)$$

where $u(t)$ is some arbitrary function. Consider the quantity $E(t) = x_1^2(t) + x_2^2$. It is easily checked that its derivative is 0 which proves that the trajectory issued from a point is contained in the circle centered at 0 passing through this point, since the distance to the origin is constant. The deterministic discrete process which converges to this differential system is:

$$x_1(t+dt) = x_1(t) + x_2(t)u(t)dt$$
$$x_2(t+dt) = x_2(t) - x_1(t)u(t)dt$$

Compute $E(t+dt)$ for this discrete process. One obtains:

$$E(t+dt) = E(t) + E(t)u^2(t)dt^2$$

and if we neglect the terms of the order of dt^2 as it is the rule we see that $E(t)$ is constant. Now replace the deterministic $u(t)dt$ by our usual stochastic term $Z_t\sqrt{dt}$:

$$x_{1t+dt} = x_{1t} + x_{2t}Z_t\sqrt{dt}$$
$$x_{2t+dt} = x_{2t} - x_{1t}Z_t\sqrt{dt}$$

and compute $E(t+dt)$:

$$E_{t+dt} = x_{t+dt}^2 + x_{t+dt}^2 = (x_{1t} + x_{2t}Z_t\sqrt{dt})^2 + (x_{2t} - x_{1t}Z_t\sqrt{dt})^2 = E(t) + E(t)dt$$

The distance $E(t)$ is no longer constant and when we go to the limit we have $E\,(t) = E(t)$. In general, if we go to the limit like we did for the pure random walk and we consider our general random walk in R, for any differentiable function $x \rightarrow \varphi(x)$ we obtain, if we neglect the terms which are small with respect to dt:

$$\varphi(Y_{t+dt}) = \varphi(Y_t) + \varphi'(Y_t)f^0(Y_t)dt + \frac{1}{2}[\sum_{i=1}^{p}\varphi''(Y_t)f^{i2}(Y_t)]dt + \sum_{i=1}^{p}\varphi(Y_t)f^i(Y_t)Z_t^i\sqrt{dt}$$

The continuous version of this formula is called Ito formula. It tells us that the usual rules of differentiation do not work for stochastic differential calculus.

It is possible to avoid this spurious deterministic term by using a differential scheme for the random walk which is the following:

$$Y_{t+} = Y_t + f^0(Y_t)dt + \sum_{i=1}^{p} f^i(Y_t)Z_t^i\sqrt{dt}\ X_t$$

$$Y_{t+dt} = Y_t + f^0(Y_t)dt + \sum_{i=1}^{p} f^i(Y_t + \frac{1}{2}Y_{t+})Z_t^i\sqrt{dt}\ X_t$$

$$Y_0 = \delta_{y_0}$$

If one adopt the first scheme and go to the limit one obtains stochastic differential equations in the sense of Ito while if one uses the second one obtains stochastic differential equations in the sense of Stratonowitch. What is the "good model" depends on what kind of real object we are looking for and is never a trivial matter (See *Classification of Models*).

3 Modeling in automatic control (Mathematical system theory)

As we mentioned in the introduction of this article we want to develop the concepts of mathematical modeling in the spirit of engineers of automatic control. We did it when we tried to present what are dynamical systems in mathematics and their use for modeling. Here we go a little bit further in presenting the mathematical framework which is used by people in automatic control. As we said before, this formalism was first developed for the purpose of the control of mechanical systems and as time elapsed, was used for wider purposes. We do believe that these concepts will be very useful for modeling of Life Support Systems.

Roughly speaking an *input-output system* is a mathematical system with some additional features. So there are discrete, continuous, infinite dimensional, deterministic and stochastic input-output systems. We decided to put the emphasis on continuous deterministic ones because they have the largest number of potential application and are not too mathematically sophisticated. Nevertheless we shall pay some attention to stochastic systems related to the question of filtering and identification since these questions are of tremendous interest in applications.

3.1 The deterministic input - output system.

When one considers natural life support system one is often tempted to use the word retroaction. It is very popular. For instance in the case of greenhouse effect when one says that the heat increases the production of clouds which have a *feed back* effect in stopping light from the sun to earth . More precisely in the case of atmospheric convection we have :

- The sun warms the soil
- The hot soil creates a convection movement in the atmosphere
- Hot air which arrives at a certain altitude (where there is an inversion in the evolution of temperature) where it condenses into a cloud (a cumulus)
- The cumulus hides the sun, the sun does no longer warms the soil, convection is stopped, and we say that there is a feed back.

The Watt governor that we mentioned earlier creates also a feedback.

It turns out that this word cannot be used safely if one does not define where is the" beginning" and the "end" of the system in order to "feed it back". The input - output concept solves the problem.

3.1.1 Input - output systems

An *input - output system* (IO system) is a dynamical system :

$$\frac{dx(t)}{dt} = f(x(t),u)$$

where the right hand side depends on a parameter u (which is a vector) which varies during time and is called the *input*. To these data is added a function, called observation function, which associate to every state x a new vector y called the *output*. The complete system is written :

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t),u) & x \in R^n; u \in R^p \\ y(t) = \varphi(x(t)) & y \in R^q \end{cases} \qquad (\Sigma)$$

and is also symbolized by the well known diagram below :

The vocabulary and the most common notations are the following :

- The variable x is the *state* of the system. The state must be understood in the sense it had in the section on dynamical systems. Once the state is completely known at time t°, the future of *x(t)* is completely determined. The variable x is, most of the time, taken in a vector space of dimension n. In various questions it is more suitable to take it on in a manifold.
- The variable u represents the *input* of the system. Inputs can be, known or unknown, controlled or uncontrolled, deterministic or random. The case of random inputs is considered later. When the input is a parameter which is completely controlled by an operator, one usually says a *control*. On the opposite, when the input is unknown and random it is called a *noise*.
- The variable y is the *output*, or the *observation* of the system. The idea comes from the fact that, as a rule, we have only access to some components of the state, or to some combinations of them. In mechanical systems we may have sensors for acceleration but not for velocity and position or conversely we have sensors for position but not for velocity. In life support systems like fisheries, if we model some population of fishes by a stage structured population it may append that we have only access to an estimation of the whole population, but not to its repartition in the different classes.

A mapping $t \longrightarrow u(t)$ is called an open loop control and is opposed to closed loop controls or feed - back.

3.1.2 Feed - Back

We give, in this section, some definitions for a feed back. As we shall see there are different types of feed back and this diversity is not a consequence of the taste of mathematicians for completeness but corresponds to modeling needs.

- Static state feed back: A mapping $x \mapsto U(x)$ is called a static state feed back for the system Σ. The associated closed loop system is :

  $$\frac{dx(t)}{dt} = f(x(t), U(x))$$

- Static output fee back : A mapping $y \longrightarrow U(y)$ is called a static output feed back for the system S. The associated closed loop system is :

  $$\frac{dx(t)}{dt} = f(x(t), U(y)) = f(x(t), U(\varphi(x)))$$

- Dynamic state feed back : Consider the input out put system :

  $$(\Sigma) \qquad \begin{cases} \dfrac{dx(t)}{dt} = f(x(t), u) & x \in R^n; u \in R^p \\ y(t) = \varphi(x(t)) & y \in R^q \end{cases}$$

  and associate to it a new system of the same type :

  $$(\Psi) \qquad \begin{cases} \dfrac{dz(t)}{dt} = g(z(t), v) & z \in R^m; v \in R^n \\ o(t) = \psi(z(t)) & o \in R^p \end{cases}$$

  where the dimensions of the inputs and outputs of the second system are chosen in accordance to those of the first one. One says that Ψ is a dynamic state feed back for Σ and the closed loop system associated is the following:

  $$(\Sigma\Psi) \qquad \begin{cases} \dfrac{dx(t)}{dt} = f(x(t), \psi(z(t))) \\ \dfrac{dz(t)}{dt} = g(z(t), \varphi(x(t))) \end{cases}$$

- <u>Dynamic output feed back</u> : Consider the input output system :

$$(\Sigma) \qquad \begin{cases} \dfrac{dx(t)}{dt} = f(x(t),u) & x \in R^n; u \in R^p \\ y(t) = \varphi(x(t)) & y \in R^q \end{cases}$$

and associate to it the system:

$$(\Psi) \qquad \begin{cases} \dfrac{dz(t)}{dt} = g(z(t),v) & z \in R^m; v \in R^q \\ o(t) = \psi(z(t)) & o \in R^p \end{cases}$$

where the dimensions of the input and output variables are chosen in accordance with S. This is a dynamic output feed back and the associated closed loop system is :

$$(\Sigma\Psi) \qquad \begin{cases} \dfrac{dx(t)}{dt} = f(x(t),\psi(z(t))) \\ \dfrac{dz(t)}{dt} = g(z(t),\varphi(x(t))) \end{cases}$$

A feed back is a "policy" which is based on the" observation" of the state (if the whole state is observed we forget the word observation) which is designed to achieve certain goals. This vocabulary comes from engineering purposes where the input is supposed to be some control parameter of the system. But in the perspective of the EOLSS one must have a broader view. Consider, for instance, a population of flowers growing on an island. The rate of growth is some function of various parameters that we suppose to be constant, and depends on the room with is available. If we call x the size of the population and u the available room, we can write:

$$\frac{dx(t)}{dt} = k\, u.x(t)$$

Since we are on an island, the remaining room is just a function of the population (remaining room is just the total space minus the room occupied by the population) and thus we have a static feed back and the model looks like:

$$\frac{dx(t)}{dt} = k(S - x(t))x(t)$$

where S denotes the total area of the island. Thus the classical logistic model can be interpreted as a kind of input output system with a static fee back.

But if in place of room we consider an other limiting resource for the population growth like the presence of some nutrient in the soil. The total quantity of nutrient is not a direct function of the population but is consumed by the population and by the way its rate of disappearance is proportional to the population. This is a dynamic feed back.

A closed loop system is no longer an input output system but just a system of differential equations. Most of the time the feed back is designed in such a way that the closed loop system be able to achieve some goal. The most often desired property is the stability around some specific point.

All the definitions we have given here are definitions for stationary feed back in the sense that the mappings considered here are independent of time. A non *stationary static state feed back* is a mapping $(x,t) \to u(x,t)$. It is important to say here that stabilization objectives which are not achievable with stationary feed back are with non stationary ones (see Controllability, Observability, Sensitivity and Stability of Mathematical Models).

Most of the theory for input output systems is known for systems of the above form but one must say that, in some cases, it is necessary to include the input in the expression of the observation. We have the more general form :

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t),u) & x \in R^n; u \in R^p \\ y(t) = \varphi(u,x(t)) & y \in R^q \end{cases}$$

3.2 Examples of deterministic input output systems

In this section we show some examples of input output models.

3.2.1 An artificial system : The inverted pendulum

The inverted pendulum is an example of mechanical system where one likes to explain concepts in automatic control. The device is the following. A moving carriage of mass M can run on rails. In the middle of the vehicles there is a rigid bar articulated of total mass m supposed to be small with respect to M. A force u is applied on the vehicle.

Fig.7 : The inverted pendulum

We make the hypothesis that the angle of the bar with the vertical remain small and thus, applying laws of mechanics one obtains:

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{1}{M}u$$
$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\omega}{dt} = \frac{g}{L}\theta - \frac{1}{ML}u$$

where g is the gravitational constant and $L = \dfrac{J + ml^2}{ml}$, where l is the distance of the center of mass of the bar to the pivot. Since the system is linear it can be written:

$$X' = AX + bu$$

with:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{L} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{ML} \end{bmatrix}$$

The input is reduced to the force u, the state variables are: $[x_1, x_2, \theta, \omega]$. As output we can imagine the position of the carriage, the angle of the bar, or both the position and the angle. We shall write:

$$X' = AX + bu$$
$$y = cX$$

with :

$c = [1,0,0,0]$ ou $[0,0,1,0]$ *ou* $[1,0,1,0]$

In this mechanical system the equilibrium $x = 0$ and $\theta = 0$ is unstable. If the bar is slightly moved of its vertical position it falls but we see that it is likely to keep it vertical in moving in the right direction the carriage. This is the game that children play with a stick on the end of their finger. In order to keep the stick vertical one has to move accordingly to observation of the stick, which means to design a feed back. We shall use this example later in order to illustrate some results.

3.2.2 A natural system: Bacterial growth

It is not surprising that the formalism of automatic control was adapted to the modeling of mechanical since it was developed for this purpose! In the following example we show that it is also adapted to very different situations.

We consider, in some aquatic medium, a quantity of bacteria large enough to be represented adequately by a real number x(t). The bacteria absorb some substrate, say glucose, which quantity in the medium is denoted by u, and we assume that all the glucose is transformed into bacteria during the biological process. During a short time interval one has:

$$x(t+dt) = x(t) + rx(t)udt$$

an equation which expresses that the quantity of new bacteria produced during the interval of time dt is proportional to the product of the quantity in substrate by the quantity of bacteria. The hypothesis of perfect mixing is underlying that proportionality. If we go to the limit we obtain an input output system:

$$\frac{dx(t)}{dt} = rux(t)$$

where u, the substrate concentration is supposed to be the input. This is one the simplest input output system one can imagine. If we assume that there is no source of substrate the quantity of substrate diminish according to:

$$\frac{du(t)}{dt} = -ru(t)x(t)$$

This last equation can be interpreted as a dynamic state feed back. The closed loop system is:

$$\frac{dx(t)}{dt} = u(t)x(t)$$
$$\frac{du(t)}{dt} = -u(t)x(t)$$

Denote now by $u(0)$ the quantity of substrate at time 0 and by $x(0)$ the initial quantity of bacteria. From the equations of the closed loop system we get that $x(t) + u(t)$ is constant. So we have:

$$u(t) = x(0) + u(0) - x(t) = K - x(t)$$

hence

$$\frac{dx(t)}{dt} = r(K - x(t))x(t)$$

and we get the the logistic model.

3.2.3 A natural system: A structured population

The following is a generalization to more complex populations of the considerations introduced above for the growth of bacteria. We consider a population which is structured in two stages, say larvae and adults. We assume that in each stage all individuals are the same and submitted to exactly the same environmental conditions and are very numerous in order that the approximation by real numbers is valid. Thus the vector:

$$x(t) = (x_1(t), x_2(t))$$

represents the number of individuals in each stage at time t. During a very short time one has:

$$x_1(t+dt) = x_1(t) - \alpha_1 x_1(t)dt + \mu_2 x_2(t)dt$$
$$x_2(t+dt) = x_2(t) + \beta_1 x_1(t)dt - \alpha_2 x_2(t)dt$$

which we interpret in the following way. When *dt* is very small, during the time *t* and the time *t+dt* the population remains almost constant and also the environmental conditions. The term $-\alpha_1 x_1(t)dt$ expresses that the number of individuals that leave stage 1 for stage 2 or death is proportional to the size of the population and the duration. The term $-\alpha_2 x_2(t)dt$ expresses the death for population in stage 2. Since the term $+\beta_1 x_1(t)dt$ expresses the number of larvae that became adults during the elapsed time we must have $\beta_1 \leq \alpha_1$. The term $+\mu_2 x_2(t)dt$ expresses the birth process.

All the coefficients depend of some environmental parameters that we assume to be able to quantify through some vector $u = (u_1, u_2, ..., u_p)$. If we go to the limit when *dt* goes to 0 we get the system:

$$\frac{dx_1(t)}{dt} = -\alpha_1(u)x_1(t) + \mu_2(u)x_2(t)$$
$$\frac{dx_2(t)}{dt} = +\beta_1(u)x_1(t) - \alpha_2(u)x_2(t)$$

For more general situation with more than two stages we would write, with evident notations:

$$\begin{bmatrix} x'_1(t) \\ x'_2(t) \\ \\ x'_i(t) \\ \\ x'_{n-1}(t) \\ x'_n(t) \end{bmatrix} = \begin{bmatrix} -\alpha_1(u) & \mu_2(u) & . & \mu_{i-1}(u) & \mu_i(u) & . & \mu_n(u) \\ +\beta_1(u) & -\alpha_2(u) & . & 0 & 0 & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & +\beta_{i-1}(u) & -\alpha_i(u) & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & 0 & +\beta_{n-2}(u) & -\alpha_{n-1}(u) & 0 \\ 0 & 0 & . & . & 0 & +\beta_{n-1}(u) & -\alpha_n(u) \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \\ \\ x_i(t) \\ \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}$$

which can be rewritten in a condensed form:

$$\frac{dx(t)}{dt} = A(u)x(t)$$

In this model we can say that the particular structure of the matrix is very secure since it expresses merely mass conservation. At the opposite, in such biological models, the form of the dependence of the parameters with respect to the input *u* is very badly known if not known at all.

A possible output for this population might be $[x_p, x_{p+1}, ..., x_n]$ for some *p* strictly greater than if we imagine that we are only able to observe some old individuals, but not too small ones. An interesting example is the case of fisheries. Suppose that we have some mono-species fishery and that the above model of multi stage population is valid. Denote by *E* the fishing effort (roughly speaking the number of boats if they identical) and consider it as an input. The output is the result of the fishing which can be modeled by:

$$y = E\sum_{p=1}^{n} c_i x_i$$

where the coefficients express the more or less ability of the fishermen to capture the stage. For instance it can be 0 for small fishes which pass through the net. This is an example where the output depends directly on the input.

3.2.4 Continuous culture of micro-organisms

We have presented an artificial system (a mechanical system), natural systems (populations), we want to present now some intermediary system. The chemostat of Monod and Szilard. The chemostat is a famous laboratory device (the publication of Monod describing the apparatus is one of the most quoted in the scientific literature). The experimental device is described on the picture below:

Fig.8 : The chemostat

A liquid, containing a substrate is introduced in a tank where it is stirred with certain micro-organism like bacteria or plankton. The inflow is equal to the outflow. The inflow contains substrate, the outflow contains substrate and microorganisms. A classical model for this device is the following:

$$\frac{dS(t)}{dt} = d(S_i - S(t)) - V(S(t))x(t)$$
$$\frac{dx(t)}{dt} = cV(S(t))x(t) - dx(t)$$

in which the variables x(t) and S(t) represent respectively the concentration in micro-organisms and substrate in the tank. The constant S_i represents the concentration of substrate in the inflow and d the flux of the flow crossing the tank. The function $V(S)$ is the absorbing rate of the substrate by the micro-organism and c is a constant which expresses that a part of the substrate is used to insure the metabolism of the organism the other part being used for the growth.

In this model two inputs are possible : d and S_i. A usual output is $x(t)$.

3.2.5 Examples of Infinite dimensional systems

So far we have considered finite dimensional systems :

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t),u) & x \in R^n \quad , \quad u \in R^p \\ y(t) = \varphi(x(t) & \qquad\qquad y \in R^q \end{cases}$$

but there is no objection to consider infinite dimensional ones. At a formal level all the concepts are exactly the same, one has just to consider that the space belongs to some abstract infinite dimensional space. Thus one write :

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t),u) & x \in E \quad , \quad u \in F \\ y(t) = \varphi(x(t) & \qquad\qquad y \in G \end{cases}$$

where E, F and G are suitable Banach or Hilbert spaces. But this formal setting does not help much to understand what are the particular features of the infinite dimension systems. One point is of importance : The way the controls acts on the system. Let us consider few examples.

a) **Parabolic type input output systems**

Consider a metallic bar of length l and consider a general heat equation like the following :

$$\frac{\partial T(t,x)}{\partial t} = k^2 \frac{\partial^2 T(t,x)}{\partial x^2} + f(t,x) \qquad 0 \le x \le l$$
$$T(t,0) = \alpha(t)$$
$$T(t,l) = \beta(t)$$

with Dirichlet conditions at the boundary of the domain. The function f expresses an exchange of heat with the external world and the boundary conditions the fact that the two boundaries of the bar are maintained at some given temperature. We suppose that these exchange of heat with the external world and the extreme temperature of the bar are our inputs, and to emphasise this we write:

$$\frac{\partial T(t,x)}{\partial t} = k^2 \frac{\partial^2 T(t,x)}{\partial x^2} + u_1(t,x) \qquad 0 \le x \le l$$
$$T(t,0) = u_2(t)$$
$$T(t,l) = u_3(t)$$

It is quite clear that the input u_1 is of a rather different mathematical nature than the inputs u_2 and u_3. The first control is a function of x while the other ones are only scalars. The first control is called a *distributed* control, the other *controls* are boundary controls.

A more interesting equation, in view of environmental problems is the equations of diffusion of a pollutant in a lake that we already considered:

$$\frac{\partial U(t,x,y)}{\partial t} = k^2 \Delta_{x,y} U(t,x,y) + u(t)\delta_{x_0,y_0} \qquad (x,y) \in \Omega$$
$$(x,y) \in \partial\Omega \Rightarrow \frac{\partial U(t,x,y)}{\partial \vec{\eta}} = 0$$

and add to these equations the output:

$$y_{x_1,y_1}(t) = U(t,x_1,y_1)$$

which expresses that the pollution is observed just at some point in the lake. This kind of output is very different from the knowledge of the whole state, which would mean the complete knowledge of the concentration of the pollutant at each point in the lake.

b) **An hyperbolic type input output system**

A typical hyperbolic system is the one where we try to control the vibrations of a rope, or any vibrating medium with the help of some boundary control. The one dimensional problem is given by the equations :

$$\frac{\partial^2 W(t,x)}{\partial t^2} = k^2 \frac{\partial^2 T(t,x)}{\partial x^2} \qquad 0 \le x \le l$$
$$W(t,0) = u_1(t)$$
$$W(t,l) = u_2(t)$$

where the input u is the position of the rope at the origin of the rope. For a two dimension medium the same type of boundary control model becomes:

$$\frac{\partial^2 W(t,x,y)}{\partial t^2} = k^2 \Delta_{x,y} U(t,x,y) \qquad (x,y) \in \Omega$$

$$(x,y) \in \partial\Omega_1 \Rightarrow \frac{\partial W(t,x,y)}{\partial \vec{\eta}} = 0$$

$$(x,y) \in \partial\Omega_2 \Rightarrow W(t,x,y) = u(t,x,y)$$

where the boundary control u is exerted only on some part of the boundary, the other boundary condition being fixed.

c) **A structured population model**
We consider the chemostat model already presented in 3.2.4 but now we decide to take care in the model of the size of the micro-organism. Thus we introduce the state variable $W(t,a)$ which is the density at time t of the individuals which age is a. This density is governed by the equation of von Foester Mac Kendrick :

$$\frac{\partial W(t,a)}{\partial t} + \frac{\partial W(t,a)}{\partial a} = -um(a)W(t,a)$$

$$W(t,0) = \int_0^\infty n(a)W(t,a)da$$

where $m(a)$ is the mortality rate for individuals of age a, $n(a)$ the birth rate at age a and u is an input which expresses the intensity of the mortality rate. It is reasonable to imagine a model where u is an increasing function Φ of the total quantity of individuals leading to the closed loop model :

$$\frac{\partial W(t,a)}{\partial t} + \frac{\partial W(t,a)}{\partial a} = -um(a)W(t,a)$$

$$W(t,0) = \int_0^\infty n(a)W(t,a)da$$

$$u = \Phi\left(\int_0^\infty W(t,a)da\right)$$

Such models are called "density dependent" in the literature devoted to populations dynamics.

3.3 Mathematical theory of input output systems

There are many text books on "Deterministic linear systems", "stochastic systems", "deterministic non linear systems" etc... It is not in the scope of the EOLSS to provide new text book on the subject but rather to explain to the reader, who is not supposed to become a specialist of input- output systems, why we think that this theory is relevant for him. For this reason this section does not reflect with fidelity the present development of the theory but its most appealing aspects for a wide audience.

We decided to put the light on two structural properties of finite dimensional linear systems: *controllability* and *observability*, because they are understandable only with an elementary training in linear algebra and to look for the generalization of these properties to no linear finite dimensional systems since most relevant problems from LSS are non linear.

3.3.1 Controllability
Given the general IO system

$$\frac{dx(t)}{dt} = f(x(t),U)$$

$$y(t) = \varphi(x(t))$$

we consider an *open loop* control $t \rightarrow u(t)$ and an initial condition x_0 at time 0. These data defines the Cauchy problem:

$$\frac{dx(t)}{dt} = f(x(t), u(t))$$
$$x(0) = x_0$$

which has a unique solution. This solution is called the *response* to the open loop control u and is denoted by :

$$t \rightarrow x(t, x_0, u(.))$$

and the corresponding output is denoted by:

$$t \rightarrow y(t, x_0, u(.)) = \varphi(x(t, x_0, u(.)))$$

The notation *u(.)* is used to recall that the state at time t does not depend on the value at time t of *u(t)* but on the whole function from 0 to t.

The class of allowed controls is generally specified by the problem. For instance in mechanical systems the inputs are forces and are bounded, in fisheries the fishing effort is positive (one can subtract fishes, not add fishes). Moreover the class of regularity of the function is specified $t \rightarrow u(t)$ (continuous, integrable etc...). This class of controls is called the class of admissible controls and is denoted by U_{ad}. Now we specify our control problem by :

$$\frac{dx}{dt} = f(x, u)$$
$$u(.) \in U_{ad}$$

and we assume, without other mention, that any open loop control considered is an admissible control.

We say that a point x_1 is accessible from x_0 if there exist an admissible control such that the corresponding response is x_1 at some time t. The set of accessible points from x_0 is denoted $A(x_0, U_{ad})$ or simply $A(x_0)$ if no confusion is possible and is called the *reachable* set. When the set $A(x_0)$ is equal to the whole state space one says that the system is *controllable* from x_0 and when this property is true for every initial point x_0 one say that the system is *completely controllable.*

The question of the description of the reachable set of some system is clearly of great importance for applications because it tells us what can be done by the system. In the case of an artificial control system this description specifies the performances of the system, in the case of a natural system it tells us what are the possible outcomes of the system.

At this point we can understand the great difference in the philosophy of mathematical modeling when it is considered more with the view point of control system theory than the point of view of dynamical system. In the philosophy of dynamical system the problem is to describe all the individual trajectories of a given system, in system theory the goal is more to describe the accessible sets.

We consider now an example, which gives a flavor of the subject. Consider a swimmer in a river.

Fig.9 : Two islands in the river

The swimmer is able to swim in every direction with a maximal speed V. Two islands I_1 and I_2 are located in the middle of the river and I_1 is supposed to be down stream of I_2. The profile of the strength of the stream is some symmetric function with the maximum speed W in the center, and 0 on the boundary. The question is the following: is it possible for the swimmer to go from island I_1 to I_2. The answer is yes but the trajectory will be very different if V is greater than W or not. If V is greater than W there is a very simple strategy which consists in swimming against the stream along a strait line from I_1 to I_2. If W is greater than V this short sighted strategy (one just swims in the direction of the objective) is no longer possible. One has to swim, not against the stream, but across the river to go near the boundary where the stream is small enough in order that you can swim against it. This example tells us that questions of controllability are not of local nature and, for this reason, are very often difficult to solve.

The first important result about controllability is due to Kalman . Consider the linear system:

$$\frac{dx}{dt} = Ax(t) + Bu \qquad\qquad x \in R^n; \quad u \in R^p$$

where the set of admissible controls is the set of integrable functions with values in the whole space R^p and built the matrix:

$$M = [B, AB, A^2B, ..., A^pB, ..., A^{n-1}B]$$

then the system is completely controllable if and only if the rank of the matrix M is equal to the dimension n of the state space. In this result it is important to notice that the dimension of the input space can be strictly smaller than the dimension of the state space. For instance, consider the inverted pendulum example described by:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ \theta' \\ \omega' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{L} & 0 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{ML} \end{bmatrix} u$$

This system is completely controllable since the matrix:

$$\left[B, AB, A^2B, A^3B\right] = \begin{bmatrix} 0 & \frac{1}{M} & 0 & 0 \\ \frac{1}{M} & 0 & 0 & 0 \\ 0 & -\frac{1}{ML} & 0 & -\frac{g}{ML^2} \\ -\frac{1}{ML} & 0 & -\frac{g}{ML^2} & 0 \end{bmatrix}$$

is clearly of rank four.

Since the property of complete controllability is often impossible to establish one is satisfied with the property of local controllability for which we ask only that the reachable set $A(x_0)$ be a neighborhood of x_0.

If we turn now to nonlinear systems like

$$\frac{dx}{dt} = f(x,u)$$
$$u \in U \subset R$$

the problem of controllability is addressed in the following way. One first remark that there is not a great lost in generality in considering only piecewise constant controls in place of general measurables ones since every point you can reach using general controls can be reached as close as you want using piecewise constant ones. This is a direct consequence of the fact that piecewise constant functions are dense in the set of measurable ones. Now, for each constant control *u* in *U*, consider the vector field :

$$x \to f(x,u) = f^u(x)$$

and its associated one parameter group $(x,t) \to f_t^u(x)$. Consider now the formula:

$$(t_1, t_2, ..., t_i, ...t_{n-1}, t_n) \to f_{t_n}^{u_n} \circ f_{t_{n-1}}^{u_{n-1}} \circ .... \circ f_{t_i}^{u_i} \circ .... \circ f_{t_1}^{u_1}(x)$$

if the sequence or real numbers t_i is composed uniquely of strictly positive numbers it is easy to see that this defines the response to the control defined by the graph below (one sees that it is essential that thet_i's are positive)

Fig. 10 : Piecewise constant control

This means that the question of controllability is equivalent to the question of the transitivity of the semi group of diffeomorphisms generated by the one parameter semi groups associated to the family of vector fields $f^u(x)$; $u \in U$. It turns out that this way of looking to controllability is useful and one can see that the Lie algebra generated by the family of vector fields is of crucial importance (see *Controllability, Observability, Sensitivity and Stability*)

If controllability is fairly well understood for finite dimensional systems it is still a subject of active research for infinite dimensional systems. We briefly mention here that controllability is a too strong requirement for parabolic type systems, or systems governed by delay equations, because these systems have the property of regularize the solutions (for instance, for the heat equation, as soon as time elapsed is strictly positive the solution is smooth) and thus controllability to a non smooth solution is not possible. This is the reason why in infinite dimensional systems one have to make the distinction between *exact controllability*, which is actually the extension of the concept of controllability, and *approximate controllability* where one is satisfied with the fact that only any small neighborhood of the desired state is attainable.

3.3.2 Observability

Let us come back to the standard IO. system:

$$\frac{dx(t)}{dt} = f(x(t),U)$$
$$y(t) = \varphi(x(t))$$

and consider the following question: Is it possible to recover the state of the system just from the knowledge of the output ? As on can see this question is of very great importance since, if the answer is positive, one will be able to replace physical sensors by mathematical (computerized) calculations. The answer is essentially yes. To see this let us consider just a linear system with output but no input.

$$\frac{dX(t)}{dt} = AX(t) \qquad X \in R^n$$
$$y(t) = [1,0,0,......,0]X(t) = CX(t)$$

So, we assume that the function $t \to y(t)$ is known. If it is known we can compute its derivatives of all orders. Let us write them down:

$$y(t) = CX(t)$$
$$y'(t) = CX' AX'(t) = CA$$
$$y''(t) = CAX'(t) = CA^2X(t)$$
$$...............$$
$$y^{(n-1)}(t) = CA^{n-1}(t)$$

If we write all the row vectors *C, CA, ...* in a matrix form we get:

$$M = \begin{bmatrix} C \\ CA \\ CA^2 \\ ..... \\ CA^{n-1} \end{bmatrix}$$

and we have the relation:

$$MX(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ ...... \\ y^{(n-1)}(t) \end{bmatrix}$$

From this last relation we see that, if the matrix M is invertible we can compute the state vector from the known data.

From this example we understand why the information about the state is actually contained in the output, provided certain conditions are realized. This possibility of recovering the state from the output is defined to be *observability* , and, in the case of linear systems, we have a criterion very similar to the one we got for controllability. Namely, the system:

$$\frac{dx}{dt} = Ax + Bu \qquad x \in R^n; \quad u \in R^p$$

$$y = Cx \qquad y \in R^q$$

is observable if and only if the matrix:

$$O = \begin{bmatrix} C \\ CA^2 \\ \dots \\ CA^{n-1} \end{bmatrix}$$

is of rank n.

Consider this criterion on the example of the inverted pendulum:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ \theta' \\ \omega' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{L} & 0 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{ML} \end{bmatrix} u$$

for which we assume that the output is the position of the carriage. In this case the output is defined by: $y_1 = [1,0,0,0]X$ and the corresponding matrix O is:

$$O_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is not of full rank 4. If we observe just the angle of the bar with the vertical, the output is $y_2 = [0,0,1,0]$ and the corresponding matrix O is:

$$O_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{L} & 0 \\ 0 & 0 & 0 & \frac{g}{L} \end{bmatrix}$$

which is not of rank four. But if we consider the two outputs $[y_1, y_2]$ we get the matrix:

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{g}{L} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g}{L} \end{bmatrix}$$

which is of rank four.

We can also consider an example from natural sciences in considering the Leslie type equations for the dynamics of a structured population, where the entries of the matrix are supposed to be independent of the environment. Let it be:

$$
\begin{bmatrix} x'_1(t) \\ x'_2(t) \\ \\ x'_i(t) \\ \\ x'_{n-1}(t) \\ x'_n(t) \end{bmatrix} = \begin{bmatrix} -\alpha_1 & \mu_2 & . & \mu_{i-1} & \mu_i & . & \mu_n \\ +\beta_1 & -\alpha_2 & . & 0 & 0 & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & +\beta_{i-1} & -\alpha_i & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & 0 & +\beta_{n-2} & -\alpha_{n-1} & 0 \\ 0 & 0 & . & . & 0 & +\beta_{n-1} & -\alpha_n \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \\ \\ x_i(t) \\ \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}
$$

and assume that we observe only the first or the last stage, which means that:

$y_1 = [1,0,0,...,0]X$ or $y_2 = [0,0,..,0,1]X$

Let us compute the matrix O in these two cases for the particular system:

$$
\begin{bmatrix} -\alpha_1 & 0 & \mu_3 \\ \beta_1 & -\alpha_2 & 0 \\ 0 & \beta_2 & -\alpha_3 \end{bmatrix}
$$

one gets:

$$
O_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \beta_2 & -\alpha_3 \\ \beta_1\beta_2 & -\beta_2(\alpha_1+\alpha_2) & \alpha_3^2 \end{bmatrix} \qquad O_2 = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha_1 & 0 & \mu_3 \\ \alpha_1^2 & \mu_3\beta_2 & -\mu_3(\alpha_1+\alpha_3) \end{bmatrix}
$$

The conditions for the matrices being of rank 3 are not the same. For the first one the conditions are that neither β_1 nor β_2 be null and for the second the conditions are that neither μ_3 nor β_2 be null. We leave to the reader the task interpreting these conditions.

Like in the case of controllability, observability of nonlinear systems have been extensively studied and is still an object of current research. The case of cyclic monotone positive systems is of interest in population dynamics and is detailed (*See Controllability, Observability, Sensitivity and Stability of Mathematical Models*)

3.3.3 Realization theory

We consider now a question which is a big issue for modeling. Many concrete systems can be considered as a system which produces an input-output relation. Is it possible, from the input-output relation, to recover the system in a unique way. Let us be more specific in the case of linear system.

Given a triple A, B, C of matrices with compatible dimensions we can define the linear system:

$$
\Sigma_{(A,B,C)} \qquad \begin{cases} \dfrac{dX}{dt} = AX + BU \\ Y = CX \end{cases}
$$

and consider its response to any input from the initial condition 0. (This is not a restriction if we assume that the matrix A is stable and that before applying any input to the natural system we wait for some time, in order that the system is at rest). From the elementary theory of linear differential equations we have:

$$y(t,0,u(.)) = C\int_0^t e^{(t-s)A} Bu(s)ds$$

and we define the input-output map $T_{(A,B,C)}$ as the map which associate to each input the corresponding output. Thus we have:

$$T_{(A,B,C)}u(.) = y(t,0,u(.)) = C\int_0^t e^{(t-s)A} Bu(s)ds$$

The input-output map is a linear map from the space of input functions to the space of output functions which satisfy the condition of *non anticipativity*, which means that if two inputs are equal up to time t then the corresponding outputs will be equal up to time t.

Conversely, given a non anticipative input-output map T, is there a (unique ?) triple (A,B,C) such that:

$$T = T_{(A,B,C)}$$

Such a triple (A,BC) is called a realization of T.

The question of uniqueness is easily answered. Consider an n x n invertible matrix P. Consider:

$$A' = P^{-1}AP$$

$$B' = P^{-1}B$$

$$C' = CP$$

it is easily checked that $T_{(A,B,C)} = T_{(A',B',C')}$. We say that the two systems $\sum_{(A,B,C)}$ and $\sum_{(A',B',C')}$ are isomorphic. Thus the question is not to ask for uniqueness, but to ask if two realizations of the same input-output map T are isomorphic. The answer is no for the following reason. Consider the two systems:

$$\Sigma_1 \quad \begin{cases} \dfrac{dx_1}{dt} = -x_1 + u \\ y = x_1 \end{cases} \qquad\qquad \Sigma_2 \quad \begin{cases} \dfrac{dx_1}{dt} = -x_1 + u \\ \dfrac{dx_2}{dt} = 3x_2 + 2u \\ y = x_1 \end{cases}$$

they define clearly the same input-output map since the second equation in the second system is definitely disconnected from the other state variable. In some sense the dimension of the state space of the second system is too large. This is the reason why we define a minimal realization of an input-output map as a realization for which the dimension of the state space is minimal. Now we can state the result of the theory: A minimal realization is both controllable and observable, two minimal realizations of an input-output map are isomorphic.

Since all the matrices :

$$P^{-1}AP$$

$$P^{-1}B$$

$$CP$$

are equivalent from the input-output behavior it is important to find those for which the matrices have the most appealing form. This is the problem of canonical forms (*See Identification, Estimation and Resolution of Mathematical Models*)

Realization theory was also considered for nonlinear systems and the theory is similar but the technicalities involved are beyond the scope of the EOLSS.

3.3.4 Stabilization and observers

Let us recall that a dynamical system (or a vector field) :

$$\frac{dx}{dt} = f(x)$$

is *(Globally) Asymptotically Stable* (GAS) at the point x_0 the following properties hold :

- For every neighborhood U of x_0 it exists a neighborhood V of x_0 such that for every initial condition x in V the whole positive trajectory issued from x remains in U
- For every initial condition the limit when t tends to $+\infty$ of the corresponding trajectory is x_0

One says that the system is locally asymptotically stable at x_0 when the system restricted to some neighborhood U of x_0 is GAS considered as a dynamical system on U. A necessary condition for x_0 to be an asymptotically stable equilibrium is that $f(x_0)$ be equal to zero. For a linear system GAS is equivalent to the fact that every eigenvalue must have a strictly negative real part.

When one considers an artificial input system, where the inputs are controls, a very important question is the question of stabilizability, which is the following. Assume for simplicity that the system is defined on R^n by the equations:

$$\frac{dx}{dt} = f(x,u) \qquad u \in U$$

such that $f(0,0) = 0$. The question is to design a feed back $x \to V(x)$ such that the closed loop system :

$$\frac{dx}{dt} = f(x, V(x))$$

is GAS. This question is fairly well understood for the case of linear systems and is still an active research area for the case of nonlinear systems. When the system is linear :

$$\frac{dx}{dt} = Ax + Bu \qquad x \in R^n, \quad u \in R^p$$

the system is stabilizable at 0 if and only if it is completely controllable. One can actually prove that for a completely controllable linear system, for every previously given spectrum (and particularly for a spectrum σ which is entirely contained in the half plane of complex numbers with strictly negative real parts), one can find a matrix $K(\sigma)$ such that the spectrum of the matrix $(A+BK(\sigma))$ is precisely σ. This means that if one considers the linear feed back : $x \to x(x) = Kx$ then the closed loop system :

$$\frac{dx}{dt} = Ax + BKx = (A + BK)x$$

is GAS.

Suppose now that our system is a real system, that we have to compute concretely our control. For this purpose we need the value of the actual state x. But suppose that we are in a situation where the state is not available from sensors, but only an observation is given, namely :

$$y = Cx \qquad y \in R^p$$

We assume that our system is observable. We know that in this case the state is completely determined by the knowledge of the output signal $t \to y(t) = Cx(t)$. The question is to have a concrete way of determining x(t) from the output in order to use it as a value of the state for the computation of the feed back. This is done by a device called the *Luenberger observer*.

A Luenberger observer is a copy of our system, running on a computer, which we denote by :

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu$$
$$\hat{y} = C\hat{x}$$

since this system is running on a computer everything is known and particularly, at any time, $\hat{x}(t)$ is known. The idea is now to compare the output of the observer with the output of the actual system and to make a correction base on the observed difference. Everything being linear the correction is and the composite system is given by :

$$\frac{dx}{dt} = Ax + Bu$$
$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + \theta(Cx - C\hat{x})$$

One checks easily that the difference between the actual state and the state of the observer satisfies the following differential system :

$$e = x - \hat{x}$$
$$\frac{de}{dt} = Ae + \theta Ce$$

and since the system is observable, the matrix θ can be computed in order that the system for the error *e* be GAS, which means that the error tends to zero. Thus the Luenberger observer is a device which is able to give asymptotically the value of the state of the system. In other the value $\hat{x}$ is a more and more accurate estimation of the state.

Consider now the system

$$\frac{dx}{dt} = Ax + Bu \qquad x \in R^n, \quad u \in R^p$$

and assume that the matrix K is chosen such that the closed loop system :

$$\frac{dx}{dt} = Ax + BKx$$

is GAS, and consider that in place of x for the determination of the feed back we put $\hat{x}$. The composite system is defined by :

$$\frac{dx}{dt} = Ax + BK\hat{x}$$
$$y = Cx$$
$$\frac{d\hat{x}}{dt} = A\hat{x} + \theta(y - C\hat{x})$$

The second equation in this system can be interpreted as a dynamic feed back for the first one. It can be proved that the system is GAS under the action of this feed back. The two matrices K and θ are called gain matrices.

There is a great choice in the determination of K and θ but there are ways to precise things. First assume that we have to pay for our control according to :

$$J(u(.))\int_0^t {}^t x(s)Px(s) + {}^t u(s)Qu(s)ds$$

We can find a matrix K which makes the system GAS and minimizes J. For the choice of θ we assume that the output y is subjected to some noise with known statistics and try to obtain the best possible estimation (with no bias, and minimal standard deviation). This is possible and the specific gain that one obtains for the Luenberger observer specifies what is called the *Kalman filter*. In both cases the problem turns out in the resolution of matrix Ricati equations.

The choice of the gain matrices optimizing both the cost and the estimation error is known as the linear quadratic regulator problem. Because of its tremendous importance it deserved much attention during last half century . We have now efficient methods for the computation of the gain matrices in the case of high dimension state spaces. Much was done during last twenty years to extend these results to the nonlinear case (*See Controllability, Observability, Sensitivity and Stability of Mathematical Models*)

4. Conclusion. Mathematical models for what purposes

In the preceding sections we described what model is from the view point of the mathematician for whom a model is a dynamical system and for the system theorist for whom a model is an input-output system. We conclude now with a section which address the question : For what purposes do we built mathematical model ? We propose three kinds of models.

- Models for understanding, which are models which have not necessarily a good fit with reality but help to understand it.
- Models to describe and predict which are models with strong connections with reality.
- Models that we can use to modify, control, the future process of the reality.

The last class is certainly the most appealing in view of the problem of having a sustainable development of life support systems, but, as we shall see they are not easily available.

4.1 Models for understanding

The best way to explain what we have in mind is to detail an example. We consider a population divided in two stages modeled by the differential system:

$$\frac{dx}{dt} = -x + \mu(u)y$$
$$\frac{dy}{dt} = p(\mu)x - my$$

where the growth rate and the rate of transfer from the first stage to the second one are dependent of the environmental parameter u. We assume that the mortality rate m is constant and also the rate of departure from the first stage, which is chosen equal to -1 by a suitable choice of units. In a constant environment this is a linear model. In a constant environment the growth is given by the consideration of the sign of the biggest eigenvalue and it turns out that the total population tends to 0 if $p(u)\mu(u) < m$ and tends to infinity otherwise, a condition which has an easy interpretation. Let us say that the environment is <u>favorable</u> when the population tends to infinity, <u>unfavorable</u> in the other case. We assume now that m = 1 and that for a first environmental condition the parameters are: $p(u_1) = 0{,}1 \quad \mu(u_1) = 9$ and that for a second one they are: $p(u_2) = 9 \quad \mu(u_2) = 0{,}1$. These two environments are unfavorable since $p(u)\mu(u) < m$ for both values of u. Assume now that for some reason the environment switches from one value to the other (the reason could be an external forcing like night and day). Since both environments are unfavorable, apparently the succession will be unfavorable. This is not necessarily the case; if the succession is fast enough it can be proved that the system is well approximated by the constant system with parameters:

$$p = \frac{p(u_1) + p(u_2)}{2} = 4{,}55 \quad \mu = \frac{\mu(u_1) = \mu(u_2)}{2} = 4{,}55$$

For this system the condition $4{,}55 \times 4{,}55 > m$ is realized and thus the total population tends to infinity! This is a rather counter intuitive result.

What does it proves? Certainly nothing about any actual natural system since our population is purely speculative and no attempt is done to adjust parameters to real data. Does that mean that this result is useless? Certainly no !In the introduction of the article we said that the first step of modeling is to have a discourse in the natural language about a piece of the real world.

Suppose that we are considering some real population which posses a larval state and an adult state. Consider the sentence:

> "If the total population increases, the environment is said <u>*favorable*</u>, if the total population decreases the environment is said <u>*unfavorable*</u> , then in a variable environment, if the environment is always <u>unfavorable</u> the population will decrease"

This is a discourse in the natural language which sounds true. Our mathematical model is a model of this discourse, it has the same postulates and the conclusion is in contradiction with that of our discourse. The mathematical model produces a falsification of the discourse and by the way tells us that there is something wrong in the discourse. If we come back to it we will see that the point is in "*the total population increases*". Since the population is composed simultaneously of larvae and adults it may happen that one is increasing and the other decreasing; in this case it is not clear whether the environment is favorable or not and further investigations must be done to precise what we mean by "*favorable environment*".

This is what we call a model for understanding:

- First step: we have a discourse in the natural language about some reality
- Second step: we built a mathematical, model which can be considered as a realization of our discourse, a *model of the discourse*.
- Third step: we explore the mathematical implications of the model
- Forth step: we come back to the discourse and improve it according to the light given by the model

These model are very useful because they help to give internal coherence to discourse in the natural language but it is important to remind that, unless their predictions are compared to actual data, we have no reason to believe that they tell something about the real world. Internal coherence is just a necessary condition for a good theory of the real world but by no means sufficient. After all astrology is a coherent discourse! There are many circumstances where data are very scarce and experiments are difficult to implement. This is the case for very large natural systems including human activity. In that cases models for understanding are useful but they certainly prove nothing as long as they are not supported by data.

4.2 Models for description and prediction

Fortunately there are also circumstances where data are available. Consider for instance the basin of some river, which has the tendency to overflow. The problem is to predict overflows few hours before they occur. For that purpose we have access real time data concerning rainfall in few places of the basin and we possess very long (say one century) time series of the rainfall and the eight of the water in the river. This is a typical input-output situation.

Fig. 11 : Example of input-output empirical data

Two possible attitudes are possible. The first one is to model in very tiny details the landscape of the basin, consider the different aptitudes of the soil to retain water and so on. By this way we get a big model with many parameters with physical significance that we shall try to adjust to real data. The second attitude is quite different. It consists in forgetting everything about the physical process and to concentrate just on the empirical data of the input (rainfall) output (flow of the river) system and to adjust to it some model of a predefined class. From realization theory we know that, if data are rich enough we have some chance to find a unique model - up to an isomorphism - which fits the data. Most popular models for this type of modeling are the so called ARMA (Auto Regressive Moving Average) models and their avatars. Very often, in such situation the data are corrupted by noise and before the search for a model there is a phase of filtering where one try to separate the signal from the noise. These procedures of *data processing* where very much developed, mainly in electrical engineering and remote sensing, and they are now adapted to various class of situation like ecological processes and finance (*See Measurement in Data Processing*).

To what extend this model can predict future overflows relies on an important hypothesis that we have not yet mention: *Stationarity*. We must assume that the physical process which determines the flow in the river, did not changed during the century of recorded data and will not change in the future where we want to predict. This assumption is very often quite questionable. It is the case in our example. It can append that during the last decades human populations have built roads, houses, big parking places for super markets have changed the physical process. As a mater of fact, in the sixties "experts" where asked to predict the future total catch of fishes an other food from the sea. From the data and assuming that the process is stationary they predicted that the catch in 2000 will be two hundred millions tons a year. It is actually less than one hundred million!

If we suspect that the data we observe are not produced by a stationary process we must turn to more physical models. When we try to built a physical model some pieces of the model comes from well known theories with very accurate equations and well known parameters. Let us be a little bit more specific by the consideration of an ecosystem like a bay in the ocean. We probably will model the physical processes concerning the water : heat, salinity, streams... All these variables are constrained by well known physical equations of hydrodynamics and, provided that we have a good knowledge of the shape of the bay it is possible to produce a good numerical model of these variables (*See Mathematical Models in Water Sciences*). After that we will try to add to the model some biological state variable, let us say, for instance, plankton. Here the situation is quite different. There is no "equations" of plankton growth in the same sense than equations of hydrodynamics but just models that where developed in laboratory conditions, like in the chemostat. It is well known that what is true in vitro is not necessarily true in vivo. Imagine that from our chemostat experiments we found that the growth rate with respect to the concentration of nutrient S of some kind of plankton is given by the formula:

$$V(S) = \frac{V_m S}{e + S}$$

where the constants V_m and e where determined experimentally. We can expect that general form our model is correct for the natural conditions but we suspect that the constant are different. For this reason we introduce the constant as parameters in the model and we give the following more general form to our input output model:

$$(\Sigma_p) \qquad \begin{cases} \dfrac{dx(t)}{dt} = f(x(t), u, p) & x \in R^n; u \in R^p \\ y(t) = \varphi(x(t), p) & p \in R^m; y \in R^q \end{cases}$$

Here the parameter p is something very different from an input. It does not model actions on the system, but to some extent, our ignorance or the "good" model. So we are not dealing with one model of the piece of real world but with a family of models and we want to adjust the best one to available data.

- The first point to check is *structural stability*. Structural stability at value p_0 of the parameter means that small changes in the parameter will not change drastically the qualitative behavior of the system. What structural stability is can be given precise mathematical *meaning (See Classification of Models*) and was popularized by R. Thom under the name of Catastrophe Theory.
- The second point is to look for *identifiability*. The model is identifiable when two different values of the parameter p_0 an p_1 give rise to systems Σ_{p_0} and Σ_{p_1} with different input output behavior. If it is not the case this means that we will not be able to identify parameters and that the way our model is parametrized is perhaps not the best one.
- The *sensibility analysis* measures the rate of dependence of the model to parameters.

Identification is a way to determine the best set of parameters with respect to some criterion. The most well known identification procedure is certainly linear regression. A line $y = at + b$ can be considered to be the output of the trivial differential model:

$$\frac{dx}{dt} = a$$
$$x(0) = b$$

and the goal is to find a and b in such a way that the error between the predictions of the model is minimum in the sense of least square error. This very basic procedure is now generalized to many kinds of linear and nonlinear systems, least square and other estimators. Thanks to high speed computer and efficient algorithms one can estimate the parameters of very general models(*See Identification, Estimation and Resolution of Mathematical Models*)

Before closing this section let us emphasize once more that, if high speed computers and ready to use software are a formidable tool, they also might be very dangerous. Consider the empirical of Fig. 12 and the predictions of two models A and B.

Fig. 12 : Two models fitting the same empirical data

According to least square error estimation the first model is better, but according to qualitative behavior, a feature which is not included in ready to use software, the second one is better, and common sense indicates that we have good reasons to come back to the model B and try to improve it.

4.3 Models for control

There is an expression which is popular because it is appealing : *Optimal control*. This is the dream of everybody, to control the process in an optimal way! Technically speaking, optimal control is a mathematical chapter of a very old discipline: The calculus of variation. An optimal control problem is defined when one considers an input (or a controlled) system in its standard form:

$$\frac{dx}{dt} = f(x,u) \qquad x \in R^n; \quad u \in U \subset R^p$$

and ask for a control $u(.)$ which transfers some initial point x_0 to a final point x_1 in such a way that the criterion:

$$J(u(.)) = \int_{t_0}^{t_1} g(x(t,x_0,u(.)),u(t))dt$$

is minimized. The theory of optimal control had a great achievement in the sixties when the *maximum principle*, which is a necessary condition for optimality was proved. The maximum principle is analytically tractable when the state space is of low dimension or the problem is linear. An other way to attack the optimal control problem is to solve the Bellman-Hamilton-Jacobi equation associated to the problem. This equation is a partial differential equation that generalizes the classical Hamilton-Jacobi equation of optics and is very difficult to solve. Much progress in this domain was done during the last years, in connection with mathematical physics. Of course these techniques are available for LSS but one must remind that they are useful only when we have an accurate model. When the model is a crude approximation sub optimal policies, that are more easily computed are of better interest.

Actually when the model is *uncertain*, which roughly speaking means that instead of knowing the parameters we just know an interval where we are sure that the parameters belong. In this case what we try to do is *robust control*. A robust control is a strategy developed to achieve a certain goal which is such that it is still acceptable in the worse case. The philosophy of robust control is close to that of *viability theory*. In viability theory our uncertainty about the reality is modeled by a *differential inclusion* . These topics where introduced somewhat recently in control theory, mostly motivated by numerous questions related to LSS. Since the subject is young there are hopes for useful developments.

A more modest objective than optimal control is *stabilization* and *regulation*. In this case we still have an input-output system and look for a feed back, like it was explained in the section 3.3.4, that stabilizes the system at some point, fix or varying in time. Regulation begins when one wants to do it at the lowest possible cost.

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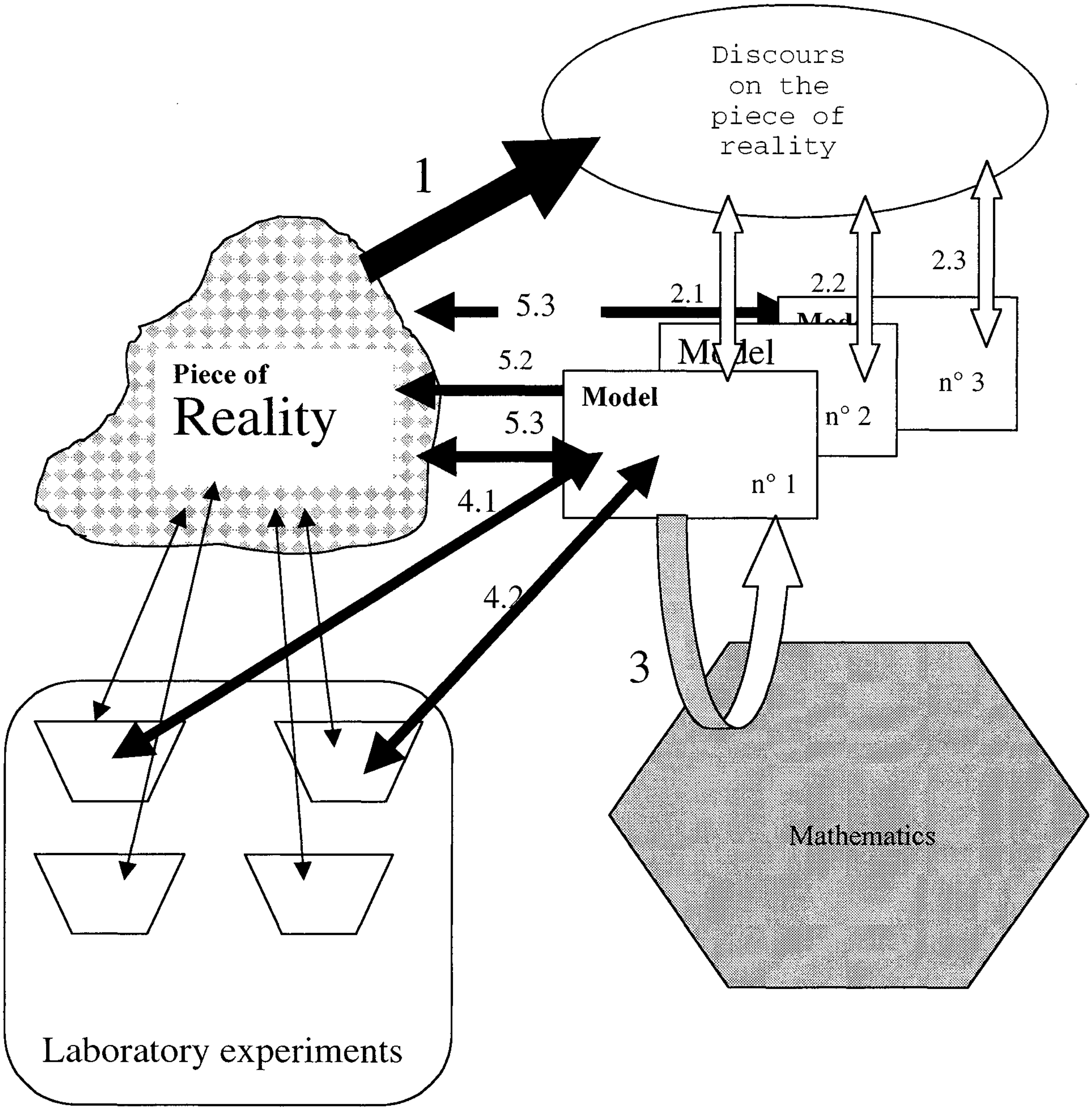


Fig.1: The model and the real world

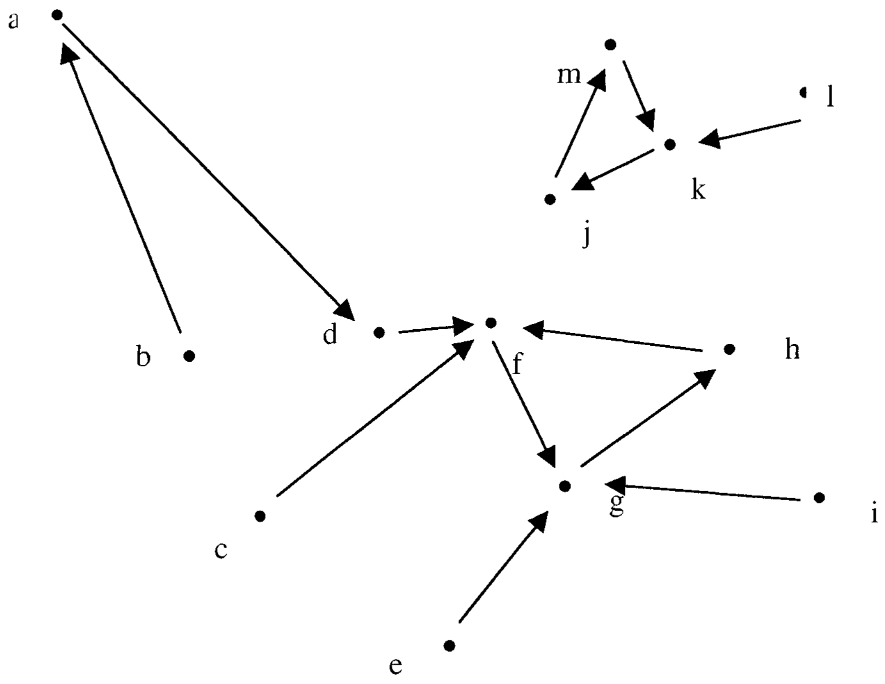
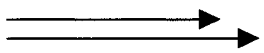


Fig.2 : A dynamical system on a finite set



We consider a 13 x 13 table where each line corresponds to a, b, … m, and each column also to a, b, …m. In the *a* line we put a 1 in the column corresponding to *d* and 0 elsewhere, in the line corresponding to *b* we put 1 in the column corresponding to *a* and 0 elsewhere, etc…Such a table is called a matrix. Taking an initial condition, say e for instance, we look for the successors of e when we apply the rules defined by the table.

$$\begin{bmatrix}
0&0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0
\end{bmatrix}$$

Fig. 3 : The matrix associated to a dynamical system on a finite set

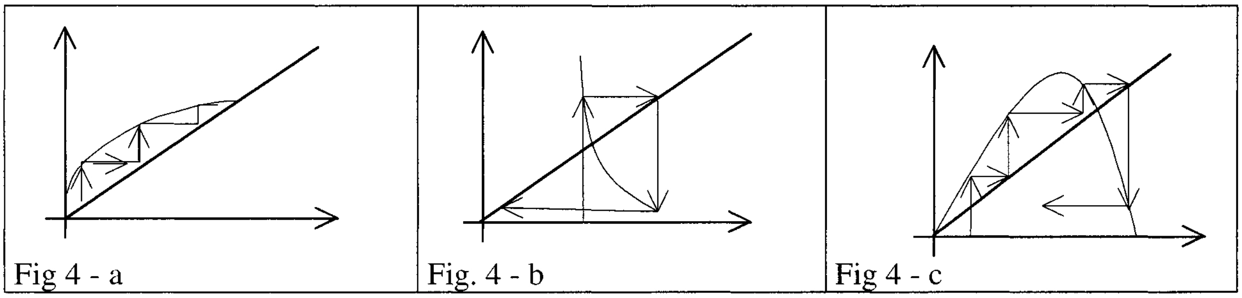


Fig.4 : Construction of trajectories for a dynamical system on *R*

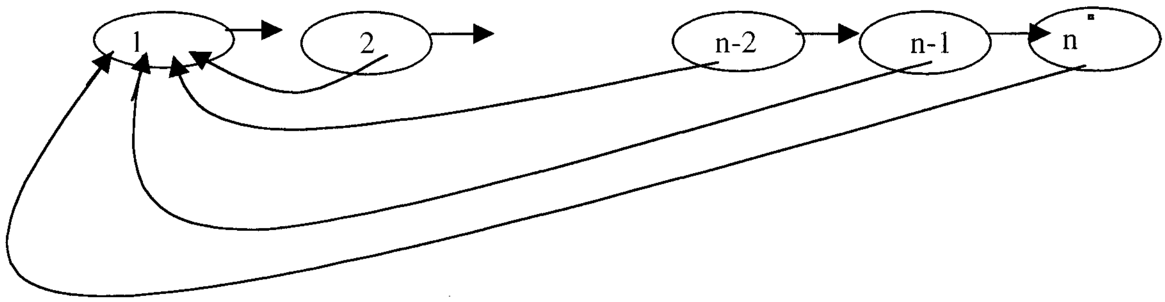
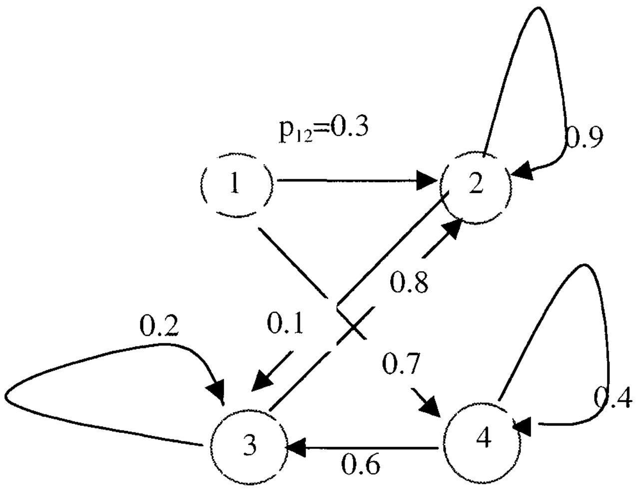


Fig.5 : Life cycle



$$\begin{bmatrix} 0 & 0.3 & 0 & 0.7 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0.6 & 0.4 \end{bmatrix}$$

Fig.6 : A Markof chain and its transition matrix

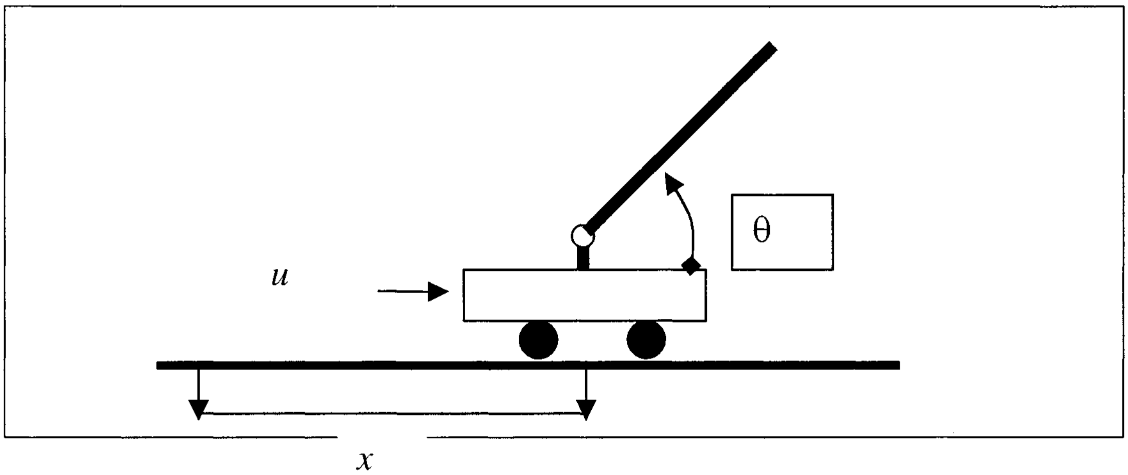


Fig.7 : The inverted pendulum

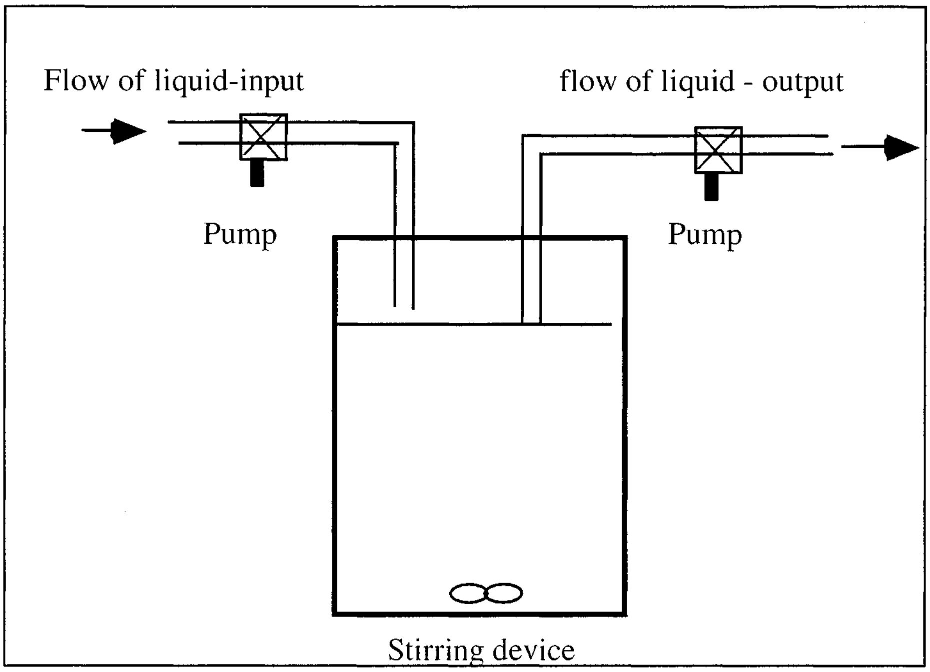


Fig.8 : The chemostat

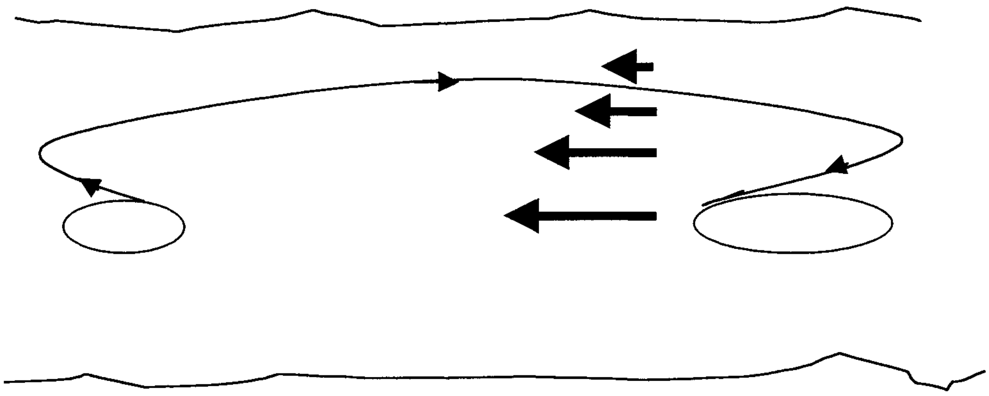


Fig.9 : Two islands in the river

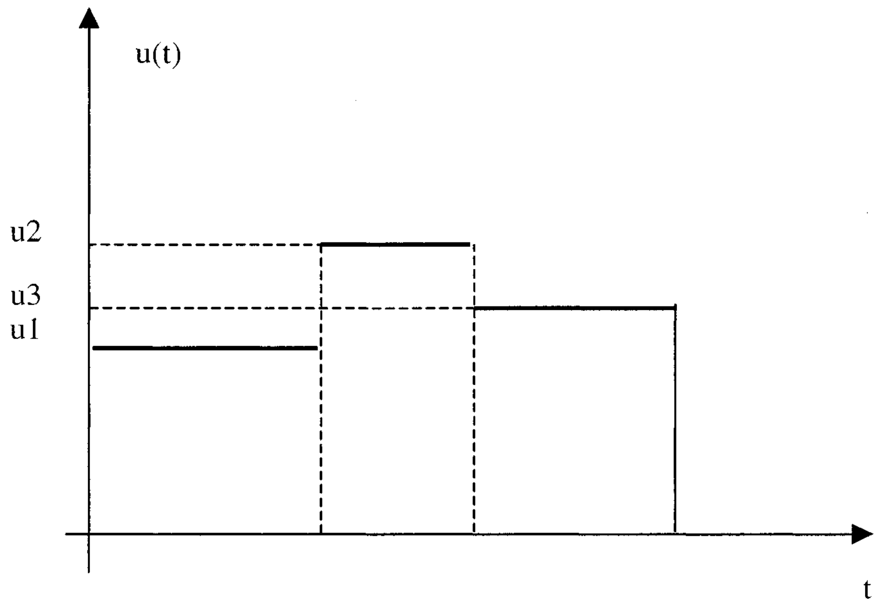


Fig.10 : A piecewise constant control