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SMR1327/2

Summer School on Mathematical Control Theory

(3 - 28 September 2001)

Stability Analysis based on direct Liapunov Method

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These are preliminary lecture notes, intended only for distribution to participants

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STABILITY ANALYSIS BASED ON DIRECT LIAPUNOV METHOD

Acknowledgements and apology. These notes have been prepared for a short summer course at SISSA (Trieste, Italy), September 2001. I wish to thank Professor A. Agrachev and Professor B. Jakubczyk for the invitation. The text is the result of a computer aided patchwork of previous papers, reports and a monograph. For this reason, I am indebted with some co-authors, and in particular with Lionel Rosier and Francesca Ceragioli. For the same reason, inconsistencies in the notation and repetitions are possible (in fact, very probable). I apologize with the readers.

Apart from a few of additions, the list of references in taken from [17] and it is therefore overstated for the purposes of these notes. Anyway, it can be useful for readers interested in some developments.

Contents

Introduction

We are interested in time-invariant control systems of the form

$$\dot{x} = f(x, u) \tag{0.1}$$

where $x \in \mathbb{R}^n$ represents the physical state of the system, and $u \in \mathbb{R}^m$ represents the input from the exterior world. In general, the input is decomposed as a sum $u = u_c + u_r + u_d + \dots$ (u_c =control, u_r =reference signal, u_d =disturbance, ...). The action of the control consists of finding u_c in such a way that the system evolves according to some prescribed goals. Usually two typical control actions can be performed

- *open loop control:* $u_c = c(t)$ (it may also depend on the initial state)
- *closed loop (automatic, feedback) control:* $u_c = k(x)$.

Just in order to fix the notation, assume that a notion of solution has been specified. Then, we denote by $\mathcal{S}_{x_0,u(\cdot)}$ the set of all solutions of (0.1) corresponding to a given initial state x_0 and a given input $u = u(t)$. When we want to emphasize the dependence of a particular solution $\varphi(t) \in \mathcal{S}_{x_0,u(\cdot)}$ on the initial state and the input, we may also write $x = \varphi(t; x_0, u(\cdot))$. When the only input is provided by a feedback $u = k(x)$, solutions of (0.1) are denoted by $x = \varphi_{k(\cdot)}(t; x_0)$.

Clearly, to every feedback $u = k(x)$ and every initial state there corresponds an open loop control $u = k(\varphi_{k(\cdot)}(t; x_0))$, but not vice versa.

Preliminary to control synthesis is system analysis; that is, the analysis of the way the solutions $x = \varphi(t; x_0, u(\cdot))$ are affected by the choice of the input $u = u(t)$. A first step in this direction is the investigation of the so-called *unforced system*

$$\dot{x} = f(x, 0)\ . \tag{0.2}$$

Since there is no energy supply, we expect that the initial energy is dissipated during the evolution, so that any solution converges to some equilibrium position. However, this is not necessarily the case because of possible unmodeled effects. The behavior could be also affected by undesired phenomena (resonance, multiple equilibrium positions, limit cycles, bifurcations, etc.). The stabilizability problem consists of finding a feedback $u = k(x)$ such that the closed loop system

$$\dot{x} = f(x, k(x)) \tag{0.3}$$

exhibits improved stability performances. As we shall see, stability of the unforced system is related to a better behavior of (0.1) with respect to external unpredictable inputs.

Prerequisites. We assume that the reader is familiar with the theory of linear systems of ordinary differential equations, and with basic facts about existence, uniqueness and continuous dependence of (classical) solutions of nonlinear ordinary differential equations.

Chapter 1

Unforced systems

1.1 Basic stability notions

The mathematical formalization of stability concepts is due to A.M. Liapunov (1892). For convenience, we refer to a system of ordinary differential equation

$$\dot{x} = f(x) \qquad (x \in \mathbb{R}^n) . \tag{1.1}$$

For the moment, we assume that f is continuous on the whole of $\mathbb{R}^n$, so that for each measurable, locally bounded input and each initial condition a (classical) solution exists, but it is not necessarily unique. Solutions of (1.1) will be denoted by $x = \varphi(t; x_0)$; we shall also write $\mathcal{S}_{x_0}$ instead of $\mathcal{S}_{x_0,0}$.

Definition 1 *We say that (1.1) is* (Liapunov) stable *at the origin (or that the origin is stable for (1.1)) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each x_0 with $||x_0|| < \delta$ and all the solutions $\varphi(\cdot) \in \mathcal{S}_{x_0}$ the following holds: $\varphi(\cdot)$ is right continuable for $t \in [0, +\infty)$ and*

$$||\varphi(t)|| < \varepsilon \quad \forall t \geq 0 .$$

Problem 1 *Prove that if the origin is stable, then it is an equilibrium position for (1.1) i.e., $f(0) = 0$.*

Definition 2 *We say that (1.1) is* Lagrange stable *(or that it has the property of uniform boundedness of solutions) if for each $R > 0$ there exists $S > 0$ such that for $||x_0|| < R$ and all the solutions $\varphi(\cdot) \in \mathcal{S}_{x_0}$ one has that $\varphi(\cdot)$ is right continuable for $t \in [0, +\infty)$ and*

$$||\varphi(t)|| < S \ , \quad \forall t \geq 0 .$$

A very special (but very important for engineering applications) case arises when the system is linear i.e.,

$$\dot{x} = Ax \tag{1.2}$$

where A is a square matrix with constant entries.

Problem 2 *Prove that in the linear case Liapunov stability and Lagrange stability imply each other; give an example to prove that in general Liapunov stability and Lagrange stability are distinct properties.*

Definition 3 *We say that system (1.1) is* locally asymptotically stable *at the origin (or that the origin is locally asymptotically stable for (1.1)) if it is stable at the origin and, in addition, the following condition holds: there exists $\delta_0 > 0$ such that*

$$\lim_{t\to+\infty} ||\varphi(t)|| = 0$$

for each x_0 *such that* $||x_0|| < \delta_0$*, and all the solutions* $\varphi(\cdot) \in \mathcal{S}_{x_0}$.

The origin is said to be globally asymptotically stable *if* δ_0 *can be taken as large as desired.*

Problem 3 *Prove that for linear systems, the Liapunov stability requirement can be dropped in the previous definition (in the sense that it is implied by the remaining conditions).*

Problem 4 *Find an example which shows that in general, the Liapunov stability requirement cannot be dropped in the previous definition (difficult: see [21], [63]).*

Problem 5 *Find an example of a system which is Liapunov stable but not asymptotically stable (easy: there are linear examples).*

Problem 6 *Prove that every linear system which is locally asymptotically stable is actually globally asymptotically stable.*

Remark 1 When dealing with systems without uniqueness, one should distinguish between weak and strong notions. The previous definitions are *strong* notions in the sense that the properties are required to hold for all the solutions, and not only for some of them (see also Remark 5, next chapter).

Remark 2 Definitions 1 and 3 can be referred to any equilibrium position, that is any point x_0 such that $f(x_0) = 0$. The choice $x_0 = 0$ implies no loss of generality. ■

1.2 Liapunov functions

Liapunov functions are energy-like functions which can be used to test stability. Actually, for each concept of stability there is a corresponding concept of Liapunov function.

Notation: $B_r = \{x \in \mathbb{R}^n : ||x|| < r\}$ and $B^r = \{x \in \mathbb{R}^n : ||x|| > r\}$.

Definition 4 *A* smooth weak Liapunov function in the small *is a real map* $V(x)$ *which is defined on* B_r *for some* $r > 0$*, and fulfills the following properties:*

(i) $V(0) = 0$

(ii) $V(x) > 0$ *for* $x \neq 0$

(iii) $V(x)$ *is of class* C^1 *on* B_r

(iv) $\nabla V(x) \cdot f(x) \leq 0$ for each $x \in B_r$.

When a real function $V(x)$ satisfies (ii), it is usual to say that it is *positive definite.* The function

$$\dot{V}(x) \underset{\text{def}}{=} \nabla V(x) \cdot f(x)$$

is called the *derivative of* V *with respect to (1.1).* Condition (iv) means that $\dot{V}$ is *semi-definite negative.*

A real function $V(x)$ is said to be *radially unbounded* if it is defined on B^r for some $r > 0$, and

$$\lim_{||x||\to+\infty} V(x) = +\infty \ .$$

Problem 7 *Radial unboundedness is equivalent to say that the level sets* $\{x \in \mathbb{R}^n : V(x) \leq a\}$ *are bounded for each* $a \in \mathbb{R}$.

Definition 5 *A function* $V(x)$ *defined on* B^r *for some* $r > 0$, *which is radially unbounded and fulfills (iii) and (iv) of Definition 4 (with* B_r *replaced by* B^r*), will be called a* smooth weak Liapunov function in the large.

Definition 6 *A* smooth strict Liapunov function in the small *is a weak Liapunov function such that* $\dot{V}(x)$ *is* negative definite; *in other words, it satisfies, instead of (iv),*

(v) $\nabla V(x) \cdot f(x) < 0$ *for each* $x \in B_r$ $(x \neq 0)$.

A function $V(x)$ *defined for all* $x \in \mathbb{R}^n$, *which is radially unbounded and fulfills the properties (i), (ii), (iii), (v) with* B_r *replaced by* $\mathbb{R}^n$, *will be called a* smooth global strict Liapunov function.

Remark 3 As far as Liapunov functions are assumed to be of class (at least) C^1, condition (iv) is clearly equivalent to the following one:

(iv') for each solution $\varphi(\cdot)$ *of (1.1) defined on some interval* I *and lying in* B_r, *the composite map* $t \mapsto V(\varphi(t))$ *is non-increasing on* I.

Such a monotonicity condition can be considered as a "nonsmooth analogous" of properties (iii), (iv). Indeed, it can be stated without need of any differentiability (or even continuity) assumption about V.

■

Definition 7 *Let* $r > 0$. *A function* $V : B_r \to \mathbb{R}$ *is called a* generalized weak Liapunov function in the small *if it satisfies (i), (iv') and, in addition, the following two properties:*

(ii') for some $\eta < r$ *and for each* $\sigma \in (0, \eta)$ *there exists* $\lambda > 0$ *such that* $V(x) > \lambda$ *when* $\sigma \leq ||x|| \leq \eta$

(iii') $V(x)$ *is continuous at* $x = 0$.

The existence of a generalized Liapunov function is sufficient in order to achieve Liapunov stability (analogous generalization about Lagrange stability).

1.3 Sufficient conditions

Theorem 1 *If there exists a smooth weak Liapunov function in the small, then (1.1) is stable at the origin.*

Theorem 2 *If there exists a smooth strict Liapunov function in the small, then (1.1) is locally asymptotically stable at the origin.*

If there exists a smooth global strict Liapunov function, then (1.1) is globally asymptotically stable at the origin.

These theorems are respectively called First and Second Liapunov Theorem. Next theorem is due to Yoshizawa.

Theorem 3 *If there exists a smooth weak Liapunov function in the large, then (1.1) is Lagrange stable.*

Problem 8 *Prove that if there exists a symmetric, positive definite real matrix* P *such that*

$$A^{\mathbf{t}}P + PA \leq 0$$

then $V(x) = x^{\mathbf{t}}Px$ *is a weak Liapunov function for the linear system (1.2), so that the system is stable.*

Problem 9 *Prove that if* P *and* Q *are symmetric, positive definite real matrices such that*

$$A^{\mathbf{t}}P + PA = -Q \tag{1.3}$$

then $V(x) = x^{\mathbf{t}}Px$ *is a strict Liapunov function in the large for (1.2).*

1.4 Converse theorems

From a mathematical point of view, the question whether Theorems 1, 2 and 3 are invertible is quite natural. Recently, it has been recognized to be an important question also for applications to control theory.

1.4.1 Asymptotic stability

Great contributions to studies about the invertibility of second Liapunov Theorem were due to Malkin, Barbashin and Massera, around 1950. In particular, in [95] Massera proved the converse under the assumption that the vector field f is locally Lipschitz. For such vector fields, he proved that asymptotic stability actually implies the existence of a Liapunov function of class C^∞. In 1956, Kurzweil ([89]) proved that the regularity assumption about f can be relaxed.

Theorem 4 *Let f be continuous. If (1.1) is locally asymptotically stable at the origin then there exists a C^∞ strict Liapunov function in the small.*

If the system is globally asymptotically stable at the origin, then there exists a C^∞ global strict Liapunov function.

It is worth noticing that Kurzweil's Theorem provides a Liapunov function of class C^∞ in spite of f being only continuous.

1.4.2 Stability

The invertibility of first Liapunov theorem is a more subtle question.

Problem 10 *Find an example in order to prove that a system with a stable equilibrium position may admit no continuous Liapunov functions (difficult: see [8], [85]).*

For one-dimensional systems with a stable equilibrium position it is proven in [16] there may be a variety of situations.

- continuous but not locally Lipschitz Liapunov functions
- locally Lipschitz but not C^1 Liapunov functions.

However, if there exists a C^1 Liapunov function then there are also C^∞ Liapunov functions. For two-dimensional systems the situation is still worse. We may have Liapunov functions of class C^r but not of class C^{r+1} $(0 \leq r \leq \omega)$. All this can be done with $f \in C^\infty$.

The following result concerns generalized Liapunov functions.

Theorem 5 *System (1.1) is Liapunov stable at the origin if and only if there exists a generalized weak Liapunov function in the small.*

Theorem 6 *Assume that the right hand side of (1.1) is locally Lipschitz continuous. Then, if (1.1) is Liapunov stable at the origin there exists a lower semi-continuous generalized weak Liapunov function in the small.*

1.5 Time-dependent Liapunov functions

Another possible approach to the invertibility of first Liapunov theorem is to seek time-dependent Liapunov functions. Recall that $a \in \mathcal{K}_0$ means that $a : [0, +\infty) \to \mathbb{R}$ is a continuous, strictly increasing function such that $a(0) = 0$. If in addition $\lim_{r\to+\infty} a(r) = +\infty$, then we write $a \in \mathcal{K}_0^\infty$.

Definition 8 *A* time-dependent weak Liapunov function in the small *for (1.1) is a real map* $V(t, x)$ *which is defined on* $[0, +\infty) \times B_r$ *for some* $r > 0$*, and fulfills the following properties:*
(i) there exist $a, b \in \mathcal{K}_0$ *such that*

$$a(||x||) \leq V(t, x) \leq b(||x||) \qquad \text{for } t \in [0, +\infty), \ x \in B_r$$

(ii) for each solution $\varphi(\cdot)$ *of (1.1) and each interval* $I \subseteq [0, +\infty)$ *one has*

$$t_1, t_2 \in I, \ t_1 < t_2 \Longrightarrow V(t_1, \varphi(t_1)) \geq V(t_2, \varphi(t_2))$$

provided that $\varphi(\cdot)$ *is defined on* I *and* $\varphi(t) \in B_r$ *for* $t \in I$.

From (i) it follows $V(t, 0) = 0$. The existence of a time-dependent weak Liapunov function is sufficient to prove stability of the origin for (1.1). The following statement is a particular case of a theorem independently proved by Krasovski, Kurzweil and Yoshizawa around 1955.

Theorem 7 *Consider the system (1.1), and assume that* $f(x)$ *is locally Lipschitz continuous. If the origin is stable, then, there exists a weak Liapunov function in the small of class* C^∞.

Unfortunately, the conclusion fails if f is only continuous.

Chapter 2

Stability and nonsmooth analysis

In control theory, one often needs to resort to discontinuous feedback. For this reason, we are interested in the extension of stability theory to systems

$$\dot{x} = f(x) \tag{2.1}$$

with discontinuous right-hand-side. More precisely, we assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded and (Lebesgue) measurable. Under these assumptions, the existence of *classical* (i.e., differentiable everywhere and satisfying (2.1) everywhere) is not guaranteed.

We say that $\varphi(t)$ is a *Carathéodory* solution if $\varphi \in AC$ and it satisfies (2.1) a.e..

We say that $\varphi(t)$ is a *Filippov* solution if $\varphi \in AC$ and it satisfies a.e. the differential inclusion

$$\dot{x} \in F(x)$$

where

$$F(x) = \mathbf{F}f(x) \underset{\text{def}}{=} \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\text{co}}\,\{f(B_\delta(x)\backslash N)\} \tag{2.2}$$

where $\overline{\text{co}}$ denotes the convex closure of a set and μ is the usual Lebesgue measure of $\mathbb{R}^n$.

We say that $\varphi(t)$ is a *Krasowski* solution if $\varphi \in AC$ and it satisfies a.e. the differential inclusion

$$\dot{x} \in K(x)$$

where

$$K(x) = \mathbf{K}f(x) \underset{\text{def}}{=} \bigcap_{\delta>0} \overline{\text{co}}\,\{f(B_\delta(x))\}\ . \tag{2.3}$$

Problem 11 *Compute* $\mathbf{F}f$ *and* $\mathbf{K}f$ *in the following cases:*

$$f(x) = \operatorname{sgn} x\ , \quad f(x) = |\operatorname{sgn} x|\ , \quad f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} .$$

Every Filippov solution is a Krasowski solution but there may be Carathéodory solutions which are not Filippov solution (find an example).

It is not yet clear what type of solution is the best for control theory applications. Here, we focus on Filippov solutions. In particular, we want to give criteria for stability which apply to discontinuous systems and involves nonsmooth (say, locally Lipschitz continuous) Liapunov functions.

We recall that if $f(x)$ is measurable and locally bounded, then the multivalued map $F(x) = \mathbf{K}_x f(x)$ enjoys the following properties

$\mathbf{H}_1$) $F(x)$ is a nonempty, compact, convex subset of $\mathbb{R}^n$, for each $x \in \mathbb{R}^n$

$\mathbf{H}_2$) $F(x)$, as a multivalued map of x, is upper semi-continuous i.e.,

$$\forall x\ \forall \varepsilon\ \exists \delta\ :\ ||\xi - x|| < \delta \Longrightarrow F(\xi) \subseteq F(x) + B_\varepsilon$$

$\mathbf{H}_3$) for each $R > 0$ there exists $M > 0$ such that

$$F(x) \subset \{v:\ ||v|| \leq M\}$$

for $0 \leq ||x|| \leq R$.

When $f(x)$ is locally bounded, there is also an equivalent (perhaps more intuitive) definition (see [100]). Indeed, it is possible to prove that there exists a set $N_0 \subset \mathbb{R}^n$ (depending on f) with $\mu(N_0) = 0$ such that, for each $N \subset \mathbb{R}^n$ with $\mu(N) = 0$, and for each $x \in \mathbb{R}^n$,

$$\mathbf{F}f(x) = \mathrm{co}\{v:\ \exists\{x_i\} \text{ with } x_i \to x \text{such that } x_i \notin N_0 \cup N \text{ and } v = \lim f(x_i)\}\ . \tag{2.4}$$

In [100], the reader will find also some useful rules of calculus for the "operator" $\mathbf{F}$.

Remark 4 A second, important reason to consider differential inclusions is given by the fact that a system with free inputs can be actually reviewed as a differential inclusion of a particular type.

Consider a system with a continuous right hand side $f(x,u)$. Let U be a given subset of $\mathbb{R}^m$, and assume that an input function $u(\cdot)$ is admissible only if it fulfills the constraint $u(t) \in U$ a.e. $t \geq 0$. Then, it is evident that every solution corresponding to an admissible input is a solution of a differential inclusion with right hand side defined by $f(x,U)$.

A celebrated theorem by Filippov states that also the converse is true, provided that $f(x,u)$ is continuous and U is a compact set. We recall that under the same assumptions on $f(x,u)$ and U, $f(x,U)$ turns out to be Hausdorff continuous[1]. On the other hand, if $f(x,u)$ is continuous and locally Lipschitz continuous with respect to x (uniformly with respect to u) then $f(x,U)$ is locally Lipschitz Hausdorff continuous with respect to x.

We can retain the following conclusion. From the point of view of control theory, it is interesting to consider differential inclusions

$$\dot{x} \in \mathcal{F}(x)$$

where either $\mathcal{F}$ satisfies assumptions $\mathbf{H}_1$, ..., $\mathbf{H}_3$ or $\mathcal{F}$ is locally Lipschitz Hausdorff continuous.

Remark 5 Let us recall that in the literature about differential inclusions, there are two possible way to interpret the classical notions of stability. The notions labelled "weak" (local, asymptotic, Lagrange stability) are deduced by asking that the respective conditions are satisfied for <u>at least</u> one solution corresponding to prescribed initial data. These notions are not irrelevant from a control theory point of view: indeed, they are related to controllability problems, feedback stabilization, viability theory and so on.

[1] Hausdorff continuity is continuity of set valued maps with respect to Hausdorff distance; the Hausdorff distance between nonempty, compact subsets of $\mathbb{R}^n$, usually denoted by h, is given by

$$h(A,B) = \max\{\sup_{a\in A} \mathrm{dist}\,(a,B), \sup_{b\in B} \mathrm{dist}\,(b,A)\}$$

where $\mathrm{dist}\,(a,B) = \inf_{b\in B} ||a-b||$.

On the contrary, the notions labelled "strong" (local, asymptotic, Lagrange stability) imply that all the solutions corresponding to the prescribed initial data satisfy the respective conditions. From our point of view, this type of stability is the ideal one we can look for, when the inputs are interpreted as disturbances. Indeed, it is obviously desirable that the effect of a disturbance is quickly absorbed and that it does not affect too much the evolution of the system. In the spirit of the present work, from now on we focus therefore on the strong notions, which can be reviewed as some forms of external stability.

2.1 Generalized derivatives

Let $N \geq 1$ be any integer number (in the sequel, we will focus in particular the case $N = n + 1$). Let $V(x) : \mathbb{R}^N \to \mathbb{R}$ be defined on an open subset Q of $\mathbb{R}^N$. For $x \in Q, v \in \mathbb{R}^N$ and $h \in \mathbb{R}$, we are interested in the difference quotient

$$\mathcal{R}(h, x, w) = \frac{V(x + hw) - V(x)}{h} .$$

Let finally $\bar{x} \in Q, \bar{w} \in \mathbb{R}^N$. The usual directional derivative at $\bar{x}$ with respect to $\bar{w}$ is defined as

$$DV(\bar{x}, \bar{w}) = \lim_{h \to 0} \mathcal{R}(h, \bar{x}, \bar{w})$$

provided that the limit exists and it is finite. When the existence of the limit is not guaranteed, certain notions of generalized derivatives may represent useful substitutes. The most classical type of generalized derivatives are *Dini derivatives*. The idea is as follows. To V, $\bar{x}$ and $\bar{w}$ we associate four numbers $\overline{D^+}V(\bar{x}, \bar{w})$, $\underline{D^+}V(\bar{x}, \bar{w})$, $\overline{D^-}V(\bar{x}, \bar{w})$, $\underline{D^-}V(\bar{x}, \bar{w})$. The former is defined as

$$\limsup_{h \to 0^+} \mathcal{R}(h, \bar{x}, \bar{w})$$

and the other are defined in similar way, taking the infimum instead of the supremum and the left limit instead of the right one, according to the notation. In this paper we shall make use of Dini derivatives, but we need also other types of generalized derivatives.

The *upper right contingent derivative* $\overline{D_K^+}(\bar{x}, \bar{w})$ is defined as

$$\limsup_{\substack{h \to 0^+ \\ w \to \bar{w}}} \mathcal{R}(h, \bar{x}, w) .$$

Analogously, one can define $\underline{D_K^+}V(\bar{x}, \bar{w})$, $\overline{D_K^-}V(\bar{x}, \bar{w})$, $\underline{D_K^-}V(\bar{x}, \bar{w})$.

Problem 12 *Show that the following relations hold:*

$$\underline{D_K^+}V(\bar{x}, \bar{w}) = \underline{D_K^-}(-V)(\bar{x}, -\bar{w}) = -\overline{D_K^-}V(\bar{x}, -\bar{w}) = -\overline{D_K^+}(-V)(\bar{x}, \bar{w}) .$$

Contingent derivatives are in some way related to the so-called contingent cone, introduced by Bouligand in 1930. Note that if V is locally Lipschitz continuous, then any contingent derivative coincides with the corresponding Dini derivative and the same is true if $N = 1$ and $\bar{w} \neq 0$.

More recently, *upper Clarke directional derivative* $\overline{D_C}V(\bar{x}, \bar{w})$ appears in the context of nonsmooth optimization theory ([33]). It is defined as

$$\limsup_{\substack{h \to 0 \\ x \to \bar{x}}} \mathcal{R}(h, x, \bar{w})$$

(in this case we do not distinguish between right and left limits, since they always coincide). Similarly, we can define $\underline{D_C}V(\bar{x},\bar{w})$. Note that $\underline{D_C}V(\bar{x},\bar{w}) = -\overline{D_C}V(\bar{x},-\bar{w})$.

It is not difficult to verify that the map

$$w \mapsto \overline{D^+}V(\bar{x},w)$$

from $\mathbb{R}^N$ to $\mathbb{R} \cup \{\pm\infty\}$ is positively homogeneous. The same is true for any other type of generalized (Dini, contingent or Clarke, upper or lower, left or right) derivative. In addition, $w \mapsto \overline{D_C}V(\bar{x},w)$ is subadditive (and hence a convex function).

In general, $\overline{D^+}V(\bar{x},\bar{w}) \leq \overline{D_C}V(\bar{x},\bar{w})$ and $\overline{D^+}V(\bar{x},\bar{w}) \leq \overline{D_K^+}V(\bar{x},\bar{w})$.

Problem 13 *It may happen that for some $\bar{x}$ and $\bar{w}$*

$$\begin{array}{ccc} \overline{D^+}V(\bar{x},\bar{w}) < \overline{D_C}V(\bar{x},\bar{w}) & , & \overline{D^+}V(\bar{x},\bar{w}) < \overline{D_K^+}V(\bar{x},\bar{w}) \ , \\ \overline{D_C}V(\bar{x},\bar{w}) > \overline{D_K^+}V(\bar{x},\bar{w}) & , & \overline{D_C}V(\bar{x},\bar{w}) < \overline{D_K^+}V(\bar{x},\bar{w}) \ . \end{array}$$

Give at least one example for each inequality.

Clarke gradient of V at x is given by

$$\partial_C V(x) = \{p \in \mathbb{R}^N : \ \forall w \in \mathbb{R}^N \text{ one has } \underline{D_C}V(x,w) \leq p \cdot w \leq \overline{D_C}V(x,w)\} \ .$$

The set $\partial_C V(x)$ is convex for each $x \in Q$. Moreover, if V is Lipschitz continuous, then $\partial_C V(x)$ turns out to be compact. The upper Clarke derivative can be recovered from Clarke gradient. Indeed,

$$\overline{D_C}V(x,w) = \sup_{p \in \partial_C V(x)} p \cdot w$$

(and, in a similar way, $\underline{D_C}V(x,w) = \inf_{p\in\partial_C V(x)} p \cdot w$).

If V is locally Lipschitz continuous, by Rademacher's Theorem its gradient $\nabla V(x)$ exists almost everywhere. Let S be the subset of $\mathbb{R}^N$ where the gradient does not exist. Then, it is possible to characterize Clarke generalized gradient as:

$$\partial_C V(x) = \text{co} \left\{ \lim_{i\to\infty} \nabla V(x_i), \ x_i \to x, \ x_i \notin S \cup S_1 \right\}$$

where S_1 is any subset of $\mathbb{R}^N$, with $\mu(S_1) = 0$. This suggests an analogy between Clarke gradient and Filippov's operator $\mathbf{F}$ (see [100]).

A map $V(x)$ is said to be *regular* if the usual one-side derivative

$$D^+V(\bar{x},\bar{w}) = \lim_{h\to 0^+} \mathcal{R}(h,\bar{x},\bar{w})$$

exists for each $\bar{x}$ and $\bar{w}$, and coincides with $\overline{D_C}V(\bar{x},\bar{w})$ (equivalently, $D^-V(\bar{x},\bar{w}) = \underline{D_C}V(\bar{x},\bar{w})$). Note that if V is regular,

$$\underline{D_C}V(\bar{x},\bar{w}) = -\overline{D_C}V(\bar{x},-\bar{w}) = -D^+V(\bar{x},-\bar{w}) = D^-V(\bar{x},\bar{w}) \ .$$

By analogy with Clarke's theory, we associate with the contingent derivatives the following two sets:

$$\underline{\partial}V(x) = \{p \in \mathbb{R}^n : \ \overline{D_K^-}V(x,w) \leq p \cdot w \leq \underline{D_K^+}V(x,w), \ \forall w \in \mathbb{R}^n\} \tag{2.5}$$

and

$$\overline{\partial}V(x) = \{p \in \mathbb{R}^n : \overline{D_K^+}V(x,w) \le p \cdot w \le \underline{D_K^-}V(x,w),\ \forall w \in \mathbb{R}^n\}\ .$$

These sets are both convex and closed and may be empty. In addition, they are bounded provided that the contingent derivatives take finite values for each direction. If one of them contains two distinct elements, the other is necessarily empty.

Note that since the contingent derivatives are not convex functions, it is not possible in general to recover their values for arbitrary directions from $\overline{\partial}V(x)$ and $\underline{\partial}V(x)$.

It turns out (see [59]) that $\overline{\partial}V(x)$ and $\underline{\partial}V(x)$ coincide respectively with the so-called *generalized super* and *sub-differentials*. They can be defined in an independent way, by means of a suitable extension of the classical definition of Fréchet differential. More precisely, one has

$$\overline{\partial}V(x) = \{p \in \mathbb{R}^n : \limsup_{h\to 0} \frac{V(x+h) - V(x) - p\cdot h}{|h|} \le 0\}$$

and

$$\underline{\partial}V(x) = \{p \in \mathbb{R}^n : \liminf_{h\to 0} \frac{V(x+h) - V(x) - p\cdot h}{|h|} \ge 0\}\ .$$

Using this representation, it is not difficult to see that if $\underline{\partial}V(x)$ and $\overline{\partial}V(x)$ are both nonempty, then they coincide with the singleton $\{\nabla V(x)\}$ and V is differentiable at x in classical sense.

Clarke gradient and generalized differentials are related by $\overline{\partial}V(x) \cup \underline{\partial}V(x) \subseteq \partial_C V(x)$.

In the class of locally Lipschitz functions, regularity can be characterized in terms of generalized differentials.

Proposition 1 *Let V be locally Lipschitz continuous. Then, V is regular if and only if $\partial_C V(x) = \underline{\partial}V(x)$ for all x.*

We finally recall the definition of the proximal gradient. In analytic terms, the *proximal subgradient* of V at x is the set of all vectors p which enjoy the following property. There exists $\sigma \ge 0$ and $\delta \ge 0$ such that for each z with $|z - x| < \delta$,

$$V(z) - V(x) \ge p \cdot (z - x) - \sigma|z - x|^2\ .$$

The proximal subgradient is denoted $\partial_P V(x)$. It is of course possible to define also the proximal supergradient $\partial^P V(x)$. For each x, $\partial_P V(x)$ is convex but not necessarily closed. In general, $\partial_P V(x) \subseteq \underline{\partial}V(x)$.

Relationship among these types of generalized derivatives, gradients and differentials, and comments on their possible geometric interpretation can be found in [36], [37].

2.2 Criteria for stability

The following result is well-known and easy to prove.

Theorem 8 *Let us consider system (2.1), with f measurable and locally bounded. Let $V(x)$ be positive definite and locally Lipschitz continuous. Assume that*

$$\underline{D^+}V(x,v) \le 0$$

for each $v \in F(x)$ and each $x \in \mathbb{R}^n$. Then, the origin is stable (with respect to Filippov solutions).

Since the upper Clarke's directional derivative majorizes the corresponding upper right Dini's derivative (and this in turn majorizes the lower one), it is clear that if

$$\overline{D_C}V(x,v) \leq 0$$

for each $x \in \mathbb{R}^n$ and $v \in F(x)$, then Theorem 8 applies. However, a criterion based on this inequality is too much conservative, since Clarke's gradient is a very large object and contains in general non-essential directions. On the other hand, Clarke's gradient possesses a rich amount of properties, so that its use could be advisable in view of certain applications. We obtain now a very sharp criterion which allows us to exploit the properties of Clarke's gradient: it avoids at the same time unnecessary verifications. The cost to be paid for this advantages is a new (mild) assumption on V.

Assume that V is a locally Lipschitz continuous and, in addition, a regular function. Let us define

$$\dot{\overline{V}}(x) = \{a \in \mathbb{R} : \ \exists v \in F(x) \text{ such that } \forall p \in \partial_C V(x) \text{ one has } v \cdot p = a\} \ .$$

It is easy to check that $\dot{\overline{V}}$ is closed, bounded and convex. Note that $\dot{\overline{V}}$ may be empty at some point.

Lemma 1 *Let V be locally Lipschitz continuous and regular, and let $\varphi : I \to \mathbb{R}^n$ be a Filippov solution. Let $N \subset I$ be the set of zero measure such that $N = N_0 \cup N_1 \cup N_2$ where:*

N_0 is the set where $\dot{\varphi}(t)$ does not exist

N_1 is the set where $\dot{\varphi}(t) \notin F(\varphi(t))$

N_2 is the set where $\frac{dV}{dt}(\varphi(t))$ does not exist.

Then, for $t \in I \setminus N$, we have $\frac{dV}{dt}(\varphi(t)) \in \dot{\overline{V}}(\varphi(t))$.

This lemma provides a chain rule for nonsmooth functions: it is essentially due to [123] (see also [11]). As an immediate consequence we obtain new stability criteria.

Theorem 9 *Assume that V is locally Lipschitz continuous and regular. Assume further that*

$$\dot{\overline{V}}(x) \subset (-\infty, 0]$$

for each x in some neighborhood of the origin of $\mathbb{R}^n$. Then, system (2.1) is stable at the origin, with respect to Filippov solutions.

Theorem 10 *Assume that V is radially unbounded, locally Lipschitz continuous and regular. Assume further that there exists a function $\omega \in \mathcal{K}_0$ such that*

$$\dot{\overline{V}}(x) \subset (-\infty, -\omega(||x||)]$$

for each $x \in \mathbb{R}^n$. Then, system (2.1) is globally asymptotically stable at the origin, with respect to Filippov solutions.

In fact, in the previous theorem it is sufficient to assume $\dot{\overline{V}}(x) \subset (-\infty, 0)$ for each $x \in \mathbb{R}^n$ (see [11]).

2.3 Converse theorems

For the purposes of this section, we find convenient to refer to any differential inclusion

$$\dot{x} \in \mathcal{F}(x) \ , \tag{2.6}$$

where $\mathcal{F}$ takes for each $x \in \mathbb{R}^n$ nonempty compact values. The first converse of second Liapunov theorem in this context has been given by Lin, Sontag and Wang in [92].

Theorem 11 *Assume that the origin is globally asymptotically stable for (2.6), where $\mathcal{F}$ is a locally Lipschitz continuous multivalued map, which takes nonempty compact values. Then there exists a C^∞ global strict Liapunov function V, which satisfies*

$$\langle \nabla V(x), v \rangle \leq -c(\|x\|) \qquad \forall x \in \mathbb{R}^n,\ \forall v \in \mathcal{F}(x),$$

for some function $c \in \mathcal{K}_0^\infty$.

Actually, Theorem 11 is stated in [92] in a somewhat different manner: (i) $\mathcal{F}$ takes in [92] the special form $\mathcal{F}(x) := \{f(x,d), d \in D\}$, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function and D is a compact set in $\mathbb{R}^m$; (ii) Theorem 2.9 in [92] deals with the asymptotic stability with respect to any *compact invariant set*, instead of the origin.

Another converse Liapunov theorem has been obtained a few years later by Clarke, Ledyaev and Stern in [35] for another class of multivalued maps.

Theorem 12 *Assume that the origin is globally asymptotically stable for (2.6), where $\mathcal{F}$ is an upper semi-continuous multivalued map, which takes nonempty compact convex values. Then there exists a C^∞ global strict Liapunov function V, which satisfies $\langle \nabla V(x), v \rangle \leq -W(x)$ for each $x \in \mathbb{R}^n$ and each $v \in \mathcal{F}(x)$, for some definite positive continuous function W.*

Chapter 3

Stabilization

3.1 Jurdjevic-Quinn method

One of the most popular approaches to the nonlinear stabilization problem (and probably the first that has been deeply studied from the mathematical viewpoint) is known as Jurdjevic-Quinn method in the western literature, and *speed gradient method* in the russian literature. In fact, it is not a general method for stabilization, but rather a method for improving stability performances. It can be described as follows. Let a nonlinear (affine) system be given. Assume that when the input is disconnected, the system has a stable (but not asymptotically stable) equilibrium position. If a (weak) Liapunov function $V(x)$ for the (unforced) system is known and some other technical assumptions are fulfilled, the system can be asymptotically stabilized at the equilibrium by a feedback law whose construction involves $\nabla V(x)$.

The idea can be reviewed as an extension of certain classical stabilization procedures of practical engineering. For instance, let us consider a mechanical system representing a nonlinear elastic force $\ddot{x} = -f(x) + u$ (with $f(x)x > 0$ for $x \neq 0$). In order to study its stability, it is natural to take $V(x, \dot{x}) = \frac{(\dot{x})^2}{2} + \int f(x)\,dx$ as a Liapunov function. Now, asymptotic stabilization can be achieved by "proportional derivative" control, which actually amounts to add friction to the system. It is not difficult to see that this is actually a particular case of feedback depending on the gradient of $V(x, \dot{x})$. For this reason, the method is sometimes also called *damping control.*

From now on, we restrict our attention to affine systems

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) = f(x) + G(x)u \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$. The vector fields $f, g_1, \ldots, g_m$ are required to be at least continuous, and $f(0) = 0$. Affine systems represent a natural generalization of the well-known linear systems

$$\dot{x} = Ax + Bu\ . \tag{3.2}$$

The basic assumption of the Jurdjevic-Quinn method is that the unforced system is stable at the origin, and that a smooth, weak Liapunov function $V(x)$ is known. Motivated by the previous discussion, we try the feedback

$$u = k(x) = -\frac{\gamma}{2}(\nabla V(x)G(x))^{\mathbf{t}} \tag{3.3}$$

where $\gamma > 0$ (the coefficient $1/2$ is due to technical reasons).

Problem 14 *Prove that the closed loop system is still Liapunov stable at the origin.*

The second typical assumption of the Jurdjevic-Quinn method is that the vector fields appearing in (4.5) are C^∞. Recall that the Lie bracket operator associates to an (ordered) pair f_0, f_1 of vector fields the vector field

$$[f_0, f_1] = Df_1 \cdot f_0 - Df_0 \cdot f_1$$

(here, Df_i denotes the jacobian matrix of f_i, $i = 0, 1$). The "*ad*" operator is iteratively defined by

$$ad^1_{f_0} f_1 = [f_0, f_1] \qquad ad^{k+1}_{f_0} f_1 = [f_0, ad^k_{f_0} f_1] \ .$$

Theorem 13 *(*JURDJEVIC-QUINN*) Assume that a weak Liapunov function of class C^∞ for the unforced system associated to (4.5) is known. Assume further that for $x \neq 0$ in a neighborhood of the origin*

$$\dim \operatorname{span} \{f(x), ad^k_f g_i(x), \ i = 1, \ldots m, \ k = 1, 2, \ldots \} = n \ .$$

Then, for any $\gamma > 0$, the system is stabilized by the feedback (3.3).

The proof of this theorem relies on LaSalle's invariance principle.

3.2 Optimality

We consider again affine systems, but now we assume that the vector fields $f, g_1, \ldots, g_m$ are locally Lipschitz continuous, so that uniqueness of solutions is guaranteed for any admissible input (but not under continuous feedback).

We need also to limit the class of admissible inputs. From now on, by an *admissible input* we mean any piecewise continuous, locally bounded function $u(t) : [0, +\infty) \to \mathbb{R}^m$. Without loss of generality, we always assume that any admissible input is right-continuous.

Assume that (3.1) can be asymptotically stabilized by a feedback law of the form (3.3). Then, an optimization problem can be associated to the stabilization problem. The solution of the optimization problem can be put in feedback form: it is exactly two times the feedback law (3.3). It follows some details.

3.2.1 The associated optimization problem

Let a continuous, positive definite and radially unbounded function $h(x)$ be given. We associate to (3.1) the following cost functional

$$J(x_0, u(\cdot)) = \frac{1}{2} \int_0^{+\infty} \left(h(\varphi(t)) + \frac{\|u(t)\|^2}{\gamma} \right) dt \tag{3.4}$$

where $\varphi(t) = \varphi(t; x_0, u(\cdot))$. For a given initial state x_0, we say that the minimization problem defined by (3.4) is solvable if there exists an admissible input, denoted by $u^*_{x_0}(t)$ such that

$$J(x_0, u^*_{x_0}(\cdot)) \leq J(x_0, u(\cdot))$$

for any other admissible input $u(t)$. The value function is defined by

$$V(x_0) = \inf_u J(x_0, u(\cdot)) \ .$$

$V(x_0)$ is actually a minimum if and only if the minimization problem is solvable for x_0.

3.2.2 From stabilization to optimality

Assume that there exist a radially unbounded, positive definite, C^1 function $V(x)$ and a positive number γ such that (3.1) is asymptotically stabilizable by means of the continuous feedback (3.3). Assume further that the closed-loop system admits $V(x)$ as a strict Liapunov function, with the additional requirement that the derivative of $V(x)$ with respect to the closed loop system is radially unbounded (this last assumption is not restrictive).

Set $h(x) = -2\nabla V(x)f(x) + \gamma\|\nabla V(x)G(x)\|^2$.

Then, the optimization problem has a solution for each x_0, the solution can be put in feedback form

$$u = k(x) = -\gamma(\nabla V(x)G(x))^{\mathbf{t}} \tag{3.5}$$

and the value function coincides with $V(x_0)$. Going from stabilization to optimality is called an inverse optimization problem in [122] (where the problem is treated with h positive semi-definite).

3.2.3 From optimality to stabilizability

Assume that there exist a continuous, positive definite, radially unbounded function $h(x)$ and a positive number γ such that the minimization problem (3.4) is solvable for each initial state x_0. Moreover, assume that the value function $V(x_0)$ is radially unbounded and of class C^1. Then, system (3.1) is asymptotically stabilizable by means of the continuous feedback

$$u = k(x) = -\alpha(\nabla V(x)G(x))^{\mathbf{t}} \tag{3.6}$$

for any $\alpha \geq \frac{\gamma}{2}$. Moreover, the value function V represents a strict Liapunov function for the closed loop system.

3.2.4 Hamilton-Jacobi equation

Solvability of the optimization problem (3.4) is equivalent to the following statement.

The first order partial differential equation (of the Hamilton-Jacobi type)

$$\nabla U(x)f(x) - \frac{\gamma}{2}\|\nabla U(x)G(x)\|^2 = -\frac{h(x)}{2} \tag{3.7}$$

has a solution $U(x)$ which is radially unbounded, positive definite and of class C^1.

Problem 15 *Prove that if the system is linear, $h(x) = 2\|x\|^2$ and $\gamma = 1/2$, then the Hamilton Jacobi equation reduces to the matrix equation (the so-called Algebraic Riccati equation)*

$$PA + A^{\mathbf{t}}P - PBB^{\mathbf{t}}P = -I \tag{3.8}$$

where I is the identity matrix of $\mathbb{R}^n$ and the unknown P is symmetric and positive definite.

3.3 Dissipation

So far we were mainly concerned with internal stability properties. However, there are also relevant notions of "stability" which relate the behavior of the output (or the state evolution) to the size of the external input. The most popular is probably the notion of ISS, due to E. Sontag. We report here the original definition (but many variants are known). For the sake of generality, we state the definition for the general system

$$\dot{x} = f(x, u) \tag{3.9}$$

although many applications are limited to the relevant case of affine systems. Recall that $\beta \in \mathcal{LK}$ means that $\beta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is decreasing to zero with respect to the first variable and of class $\mathcal{K}_0$ with respect to the second one.

Definition 9 *We say that (3.9) possesses the* input-to-state stability *(in short, ISS) property, or that it is an ISS system, if there exist maps $\beta \in \mathcal{LK}$, $\gamma \in \mathcal{K}_0$ such that, for each initial state x_0, each admissible input $u : [0, +\infty) \to \mathbb{R}^m$, each solution $\varphi(\cdot) \in \mathcal{S}_{x_0, u(\cdot)}$ and each $t \geq 0$*

$$||\varphi(t)|| \leq \beta(t, ||x_0||) + \gamma(||u||_\infty) \ .$$

Problem 16 *If the system is ISS and we set $u \equiv 0$, then we obtain a globally asymptotically stable system. Prove it.*

The following Liapunov-like characterization of the ISS property is very useful [137], [138]).

Theorem 14 *For the system (3.9) the following statements are equivalent:*

(i) the system possesses the ISS property

(ii) there exist a positive definite, radially unbounded C^∞ function $V : \mathbb{R}^n \to \mathbb{R}$ and a function $\rho \in \mathcal{K}_0$ such that

$$\nabla V(x) \cdot f(x, u) < 0$$

for all $x \in \mathbb{R}^n$ ($x \neq 0$) and $u \in \mathbb{R}^m$, provided that $|x| \geq \rho(|u|)$

(iii) there exist a positive definite, radially unbounded C^∞ function $V : \mathbb{R}^n \to \mathbb{R}$ and two functions $\omega, \alpha \in \mathcal{K}_0^\infty$ such that

$$\nabla V(x) \cdot f(x, u) \leq \omega(|u|) - \alpha(|x|)$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

A systems is said to be *IS-stabilizable* if the ISS property can be recovered by applying a suitable feedback law of the form $u = k(x) + \tilde{u}$. The following result concerns the affine system (3.1) ([126]).

Theorem 15 *Every globally asymptotically stable (or continuously globally asymptotically stabilizable) affine system of the form (4.5) is IS-stabilizable.*

In fact, ISS systems can be reviewed as special cases of dissipative systems. We proceed to introduce this notion. First of all, we complete the description of the system by associating with (3.9) an observation function $c(x) : \mathbb{R}^n \to \mathbb{R}^p$. In other words, we consider

$$\begin{cases} \dot{x} = f(x, u) \\ y = c(x) \ . \end{cases} \tag{3.10}$$

The variable y is called the *output*: it represents the available information about the evolution of the system. Let $w : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$ be a given function, which will be called the supply rate, and consider the following three dissipation inequalities.

(D1) (intrinsic version) For each admissible input $u(\cdot)$, each $\varphi \in \mathcal{S}_{0,u(\cdot)}$ and for each $t \geq 0$

$$\int_0^t w(c(\varphi(s)), u(s))\, ds \geq 0$$

(note the initialization at $x_0 = 0$).

(D2) (integral version) There exists a positive semidefinite function $S(x)$ (called a *storage function* such that for each admissible input $u(\cdot)$, each initial state x_0, each $\varphi \in \mathcal{S}_{x_0,u(\cdot)}$ and for each $t \geq 0$

$$S(\varphi(t)) \leq S(x_0) + \int_0^t w(c(\varphi(s)), u(s))\, ds\ .0$$

(D3) (differential version) There exists a positive semidefinite function $S(x) \in C^1$ such that for each $x \in \mathbb{R}^n, u \in \mathbb{R}^m$

$$\nabla S(x) f(x, u) \leq w(c(x), u)\ .$$

It is clear that **(D3)** $\Longrightarrow$ **(D2)** $\Longrightarrow$ **(D1)**. However, these conditions are not equivalent in general. The implication **(D1)** $\Longrightarrow$ **(D2)** requires a complete controllability assumption, while the implication **(D2)** $\Longrightarrow$ **(D3)** requires the existence of at least one storage function of class C^1.

In the literature, inequalities **(D1)**, **(D2)**, **(D3)** are alternatively used to define dissipative systems ([149], [72], [146], [130]). Moreover, several notions of "external" stability can be given by specializing the supply rate w. For instance we have

1) *passivity*, for $w = yu$

2) *finite L_2-gain*, for $w = k^2||u||^2 - ||y||^2$, where k is some real constant.

To explain the name given to the second property, observe that it implies the estimation

$$\int_0^t ||y(s)||^2\, ds \leq k^2 \int_0^t |u(s)||^2\, ds\ .$$

The ISS property can be interpreted as an extension of the finite L_2-gain property. Indeed, according to Theorem 14 (iii), ISS systems are dissipative in the sense of **(D3)**, with $c(x)$ =Identity and supply rate $w(x, u) = \omega(||u||) - \alpha(||x|))$. Hence, for zero initialization, the following estimation holds

$$\int_0^t \alpha(||\varphi(s)||)\, ds \leq k^2 \int_0^t \omega(|u(s)||)\, ds \tag{3.11}$$

(alternatively, we can set $c(x) = \sqrt{\alpha(||x||)}$, so that the integrand at the left hand side becomes $||y||^2$).

In the remaining part of this section we focus on the finite L_2-gain property, which has been deeply studied in [146]. Moreover, we limit to affine systems or, more precisely, systems of the form (3.10) where (3.9) is replaced by (3.1).

It is well known that if (3.10) possesses the finite L_2-gain property and a suitable observability condition is fulfilled, then the unforced part of the system is asymptotically stable at the origin. In particular, the required observability condition is automatically satisfied when $c(x)$ is positive definite. Vice-versa, assume that (3.1) is smoothly stabilizable. Then, by using a possibly different feedback the system can be rendered ISS (Theorem 15). As a consequence, we have an estimation of the form (3.11), but in general we cannot predict the nature of the functions ω and α. As an application of the theory developed in the previous sections, we now give a more precise result. For notational consistency, we put $k^2 = 1/(2\gamma)$. The starting point is the following important result ([72], [146])[1].

[1] The theorem is invertible under some restrictive assumptions, but here we need only the direct part

Theorem 16 *Assume that there exists a positive semidefinite function $\Phi(x) \in C^1$ which solves the equation (of the Hamilton-Jacobi type)*

$$\nabla\Phi(x)f(x) + \frac{\gamma}{2}||\nabla\Phi(x)G(x)||^2 = -||c(x)||^2 \tag{3.12}$$

for each $x \in \mathbb{R}^n$. Then, the affine system (3.10) has a finite L_2-gain.

Theorem 17 *Associated with the affine system (3.1) we consider the optimization problem (refJJ), where h is positive definite and continuous. Assume that the problem is solvable for each x_0, and that the value function $V(x)$ is C^1. Then, by applying the feedback*

$$u = k(x, \tilde{u}) = -\gamma(\nabla V(x)G(x))^{\mathbf{t}} + \tilde{u}$$

and choosing the observation function $c(x) = \sqrt{(h(x)/2)}$, the system (3.10) has a finite L_2-gain.

As a corollary, we see that if the affine system (3.1) is stabilizable by a damping feedback

$$u = k(x) = -\frac{\gamma}{2}(\nabla V(x)G(x))^{\mathbf{t}}$$

where $V(x)$ can be taken as a strict Liapunov function for the closed loop system, then the "doubled" feedback $u = 2k(x) + \tilde{u}$ gives rise to a system with finite L_2-gain.

3.4 The generality of damping control

It is well known that if a linear system is stabilizable by means of a continuous feedback, then it is also stabilizable by means of a linear feedback and in fact by a feedback in damping form ($u = -\alpha B^{\mathbf{t}}Px$, where $\alpha \geq 1/2$ and P is a solution of (3.8)). Surprisingly, this fact has an analogue for the nonlinear case.

The following result is basically due to [78].

Theorem 18 *Consider the affine system (3.1) and assume that*

$$|f(x)| \leq A|x|^2 + C \quad \text{and} \quad ||G(x)|| \leq D$$

for some positive constants A, C, D. Assume further that (3.1) admits a stabilizer $u = k(x)$ such that:

(i) $k(x)$ is of class C^1 and $k(0) = 0$,

(ii) $k(x)$ guarantees sufficiently fast decay: more precisely, we require that each solution of the closed loop system is square integrable i.e.,

$$\int_0^{+\infty} |\varphi_{k(\cdot)}(t; x_0)|^2 \, dt < +\infty \tag{3.13}$$

for each $x_0 \in \mathbb{R}^n$.

Then, there exists a map $V(x)$ such that the feedback law (3.3) is a global stabilizer for our systems. In other words, the system also admits a damping control.

Chapter 4

Control Liapunov functions

Consider for the moment a general system of the form

$$\dot{x} = f(x, u) \tag{4.1}$$

where f is continuous and $f(0,0) = 0$. The non-existence of continuous stabilizers for (4.1) is related to certain obstructions of topological nature. The most famous one is pointed out by the following result, usually referred to as Brockett's test (see [27], [118], [155]).

Theorem 19 *Consider the system (4.1) and assume that f is continuous and that $f(0,0) = 0$. A necessary condition for the existence of a continuous stabilizer $u = k(x)$ with $k(0) = 0$, is that for each $\varepsilon > 0$ there exist $\delta > 0$ such that*

$$\forall y \in B_\delta \ \exists x \in B_\varepsilon \,, \ \exists u \in B_\varepsilon \quad \text{such that} \quad y = f(x, u) \ .$$

In other words, f must map any neighborhood of the origin in $\mathbb{R}^{n+m}$ onto some neighborhood of the origin in $\mathbb{R}^n$ (note that in the linear case, the condition of this theorem reduces to rank $(A, B) = n$). There exist whole families of systems (typically, full rank nonholonomic systems with less inputs than states) which do not possess the property of Theorem 19. The most famous example of a system which does not satisfy Brockett's test is the so-called *nonholonomic integrator*

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_1 u_2 - x_2 u_1 \ . \end{cases} \tag{4.2}$$

The following interesting example is due to Z. Artstein. It passes Brockett's test. Nevertheless, it cannot be stabilized by a continuous feedback.

$$\begin{cases} \dot{x}_1 = u(x_1^2 - x_2^2) \\ \dot{x}_2 = 2u x_1 x_2 \end{cases} \tag{4.3}$$

(see [132] for a discussion).

4.1 Smooth control Liapunov functions

We need the following variant of the notion of Liapunov function (see [126], [124]).

Definition 10 *We say that (4.1) satisfies a* smooth global control Liapunov condition *(or that (4.1) has a smooth global control Liapunov function) if there exists a radially unbounded, positive definite, C^1*

function $V(x)$ vanishing at the origin and enjoying the following property: for each $x \in \mathbb{R}^n$ there exists $u \in \mathbb{R}^m$ such that

$$\nabla V(x) \cdot f(x,u) < 0 \ . \tag{4.4}$$

According to Kurzweil converse Theorem, it is clear that if there exists a continuous global stabilizer $u = k(x)$ for (4.1), then there exists also a smooth global control Liapunov function. The converse is false in general.

Problem 17 *Prove that the system*

$$\begin{cases} \dot{x}_1 = u_2 u_3 \\ \dot{x}_2 = u_1 u_3 \\ \dot{x}_3 = u_1 u_2 \end{cases}$$

possesses the control Liapunov function $V(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ but it does not pass Brockett's test.

However, it turns out to be true in the affine case

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{4.5}$$

where $f, g_1, \ldots, g_m$ are continuous vector fields of $\mathbb{R}^n$ (Z. Artstein [4]; but see also [127]). In order to state the theorem, we need to update the terminology. A feedback law $u = k(x)$ is said to be *almost continuous* if it is continuous at every $x \in \mathbb{R}^n \setminus \{0\}$. Moreover, we say that a control Liapunov function satisfies the *small control property*[1] if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in B_\delta$, (4.4) is fulfilled for some $u \in B_\varepsilon$.

Theorem 20 *If there exists a smooth global control Liapunov function for the affine system (4.5), then the system is globally stabilizable by an almost continuous feedback $u = k(x)$. If there exists a control Liapunov function which in addition satisfies the small control property, then it is possible to find a stabilizer $u = k(x)$ which is everywhere continuous.*

We do not report here the proof of this theorem, but some illustrative comments are appropriate. For sake of simplicity, we limit ourselves to the single input case ($m = 1$). If the vector fields f and g_1 are of class C^q ($0 \le q \le +\infty$) and a control Liapunov function of class C^r ($1 \le r \le +\infty$) is known, the stabilizing feedback whose existence is ensured by Theorem 20, can be explicitly constructed according to Sontag's "universal" formula

$$k(x) = \begin{cases} 0 & \text{if } b(x) = 0 \\ \frac{a(x) - \sqrt{a^2(x) + b^4(x)}}{b(x)} & \text{if } b(x) \neq 0 \end{cases} \tag{4.6}$$

where $a(x) = -\nabla V(x) \cdot f(x)$ and $b(x) = \nabla V(x) \cdot g_1(x)$ (see [127] for more details). We emphasize that such $k(x)$ is of class C^s (with $s = \min\{q, r-1\}$) on $\mathbb{R}^n \setminus \{0\}$. If the small control property is assumed, then the feedback law given by (4.6) turns out to be continuous also at the origin, but further regularity at the origin can be obtained only in very special situations.

It is worth noticing that the universal formula above has a powerful regularizing property. Indeed, if a continuous stabilizer for (4.5) is known, then Kurzweil's Converse Theorem applies. Hence, the existence of a C^∞ strict Liapunov function $V(x)$ for the closed loop system is guaranteed. It is not difficult to see that the same $V(x)$ is a control Liapunov function for (4.5). But then, the universal formula can be

[1] If the system admits a continuous stabilizer $u = k(x)$ such that $k(0) = 0$, then the small control property is automatically fulfilled.

applied with this $V(x)$, and we obtain a new stabilizing feedback with the same order of differentiability as f and g_1 (at least for $x \neq 0$).

We have already noticed that Artstein's theorem is limited to affine systems. However, the following extension holds (see the remark after Lemma 2.1 in [46]; see also [107]).

Theorem 21 *Consider a system of the form (4.1), where f is continuous and $f(0,0) = 0$. The following statements are equivalent.*

(i) There exists a discontinuous feedback which stabilizes the system in Filippov's sense and which fulfills the additional condition

$$\lim_{\delta \to 0} \operatorname{ess\,sup}_{\|x\|<\delta} \|u(x)\| = 0 \tag{4.7}$$

(ii) Tthere exists a smooth control Liapunov function for which the small control property holds.

4.2 Asymptotic controllability

In this section we assume that $f(x,u)$ is continuous with respect to the pair $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$, and Lipschitz continuous with respect to x (uniformly with respect to u). We assume also that $f(0,0) = 0$.

Definition 11 *System (4.1) is said to be* globally asymptotically controllable *at the origin (see [34]) if there exist $C_0 > 0$, $C > 0$ such that:*

(a) for each $x_0 \in \mathbb{R}^n$ there exists an admissible input $u_{x_0}(t) : [0, +\infty) \to \mathbb{R}^m$ such that the unique solution $\varphi(t; x_0, u_{x_0}(\cdot))$ is defined for all $t \geq 0$ and satisfies

$$\lim_{t \to +\infty} \varphi(t; x_0, u_{x_0}(\cdot)) = 0 \tag{4.8}$$

(b) for each $\varepsilon > 0$ it is possible to find $\eta > 0$ such that if $\|x_0\| < \eta$ then there exists an admissible input $u_{x_0}(t)$ such that (4.8) holds, and in addition

$$\|\varphi(t; x_0, u_{x_0}(\cdot))\| < \varepsilon \quad \text{for all} \quad t \geq 0 \tag{4.9}$$

(c) if in (b) the state x_0 satisfies also $\|x_0\| < C_0$, then the input $u_{x_0}(t)$ can be chosen in such a way

$$\|u_{x_0}(t)\| \leq C$$

for a.e. $t \geq 0$.

If (4.8) is required to hold only for each x_0 in some neighborhood of the origin, then we say that the system is locally asymptotically controllable.

The meaning of this definition is that the system is asymptotically driven toward zero by means of an open loop, bounded control which depends on the initial state.

It is clear that if (4.1) is stabilizable by means of a continuous feedback, then it is asymptotically controllable. The converse is true if the system is linear[2], but not in general. The classical counter-example is given by the nonholonomic integrator: it is possible to prove that the system is asymptotically controllable: however, we know that it does not pass Brockett's test, so that it is not continuously stabilizable. In fact, because of Ryan's extension of Brockett's test [118], it follows that large classes of asymptotically controllable systems can be stabilized not even by discontinuous feedback, at least as far

[2] For linear systems asymptotic controllability, stabilizability by continuous feedback and stabilizability by linear feedback are all equivalent: see [67].

as the solutions are intended in Filippov's sense. Important progress toward the solution of this problem has been recently made ([34], [1], [105]) by exploiting suitable extensions of the notion of control Liapunov function and/or new notions of solutions for discontinuous ordinary differential equations.

In order to give an idea of such developments, we start by a simple remark. It is clear that if an affine system without drift (like the nonholonomic integrator and the Artstein example (4.3)) admits a smooth control Liapunov function, then the small control property is automatically fulfilled. It follows from this simple remark and Theorem 20, that there exist no smooth control Liapunov functions for (4.2) and (4.3). Nevertheless, both systems are asymptotically controllable. This suggests the possibility of characterizing asymptotic controllability by some weaker notion of control Liapunov function.

Note that if the differentiability assumption about V is relaxed, then the monotonicity condition can be no more expressed in the form (4.4). In [125] (see also [128]) E. Sontag proved that if f is locally Lipschitz continuous with respect to both x, u, then the global asymptotic controllability is equivalent to the existence of a continuous global control Liapunov function. The monotonicity condition is expressed in [125] by means of Dini derivatives along the solutions (see Chapter 2). In [136] (see also [34]) it is pointed out that the same condition can be also expressed by means of contingent directional derivatives.

With these motivations, we propose a general definition.

Definition 12 *Let* V : $R^n \to \mathbb{R}$ *be continuous, positive definite and radially unbounded. Moreover, let* $D(x)$ *be a set valued map (*$D(x)$ *should be thought of as some generalized gradient of the map* V*). We say that* V *is a* (nonsmooth) global control Liapunov function (with respect to D) *if there exist two maps* $W : \mathbb{R}^n \to \mathbb{R}$ *and* $\sigma : [0, +\infty) \to [0, +\infty)$ *such that:*

1) W *is continuous, positive definite and radially unbounded*

2) σ *is increasing*

3) for each $x \in \mathbb{R}^n$ *and each* $p \in D(x)$ *there exists* $u_{x,p} \in \mathbb{R}^m$ *such that* $||u_{x,p}|| \leq \sigma(||x||)$ *and*

$$p[f(x) + G(x)u_{x,p} \leq -W(x) \ . \tag{4.10}$$

Problem 18 *There is another possible definition which does not make use of the map* σ*. There exist a continuous, positive definite and radially unbounded map* $W : \mathbb{R}^n \to \mathbb{R}$ *for which the following holds. For each compact set* $K \subset \mathbb{R}^n$ *there exists a compact set* $U \subset \mathbb{R}^m$ *such that for each* $x \in K$ *and each* $p \in D(x)$ *there exists* $u_{x,p} \in U$ *such that (4.10) holds. Prove that the two formulations are actually equivalent.*

Now, an improvement of the aforementioned Sontag's result can be stated in the following way ([105], [106], see also [132]).

Theorem 22 *Consider the system (4.1) and assume that* f *is locally Lipschitz continuous with respect to both* x, u*. Then, global asymptotic controllability is equivalent to the existence of a nonsmooth global control Liapunov function* $V(x)$ *(with respect to the proximal gradient* $\underline{\partial_P} V(x)$*). In addition,* $V(x)$ *can be taken locally Lipschitz continuous.*

Note that this result applies in particular to Artstein's example (4.3) (by the way, a locally Lipschitz continuous control Liapunov function for (4.3) is explicitly given in [132]).

We conclude this chapter by recalling the following stabilizability results.

Theorem 23 *([34]) Assume that the system (4.1) is globally asymptotically controllable. Then it can be stabilized by time-sampled discontinuous feedback*[3]*.*

[3] Roughly speaking, this means that the value of the feedback remains constant for a small interval of time

Theorem 24 *([107]) Assume that the system (4.1) admits a locally Lipschitz continuous, nonsmooth global control Liapunov function $V(x)$ (with respect to the Clarke gradient $\partial_C V(x)$). Then there exists also a <u>smooth</u> control Liapunov function, so that the system is actually stabilizable in Filippov sense.*

We emphasize that the tool used to express the monotonicity condition actually plays a crucial role.

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