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Summer School on Mathematical Control Theory

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Flatness based design
(Notes of the course)

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Flatness based design

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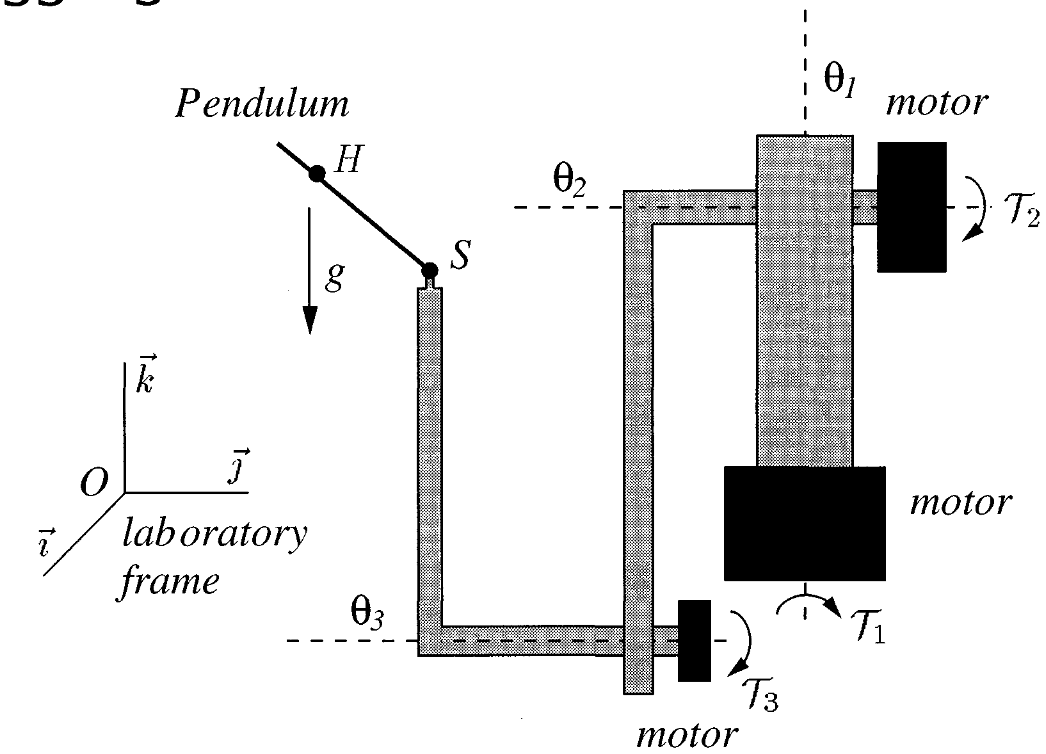
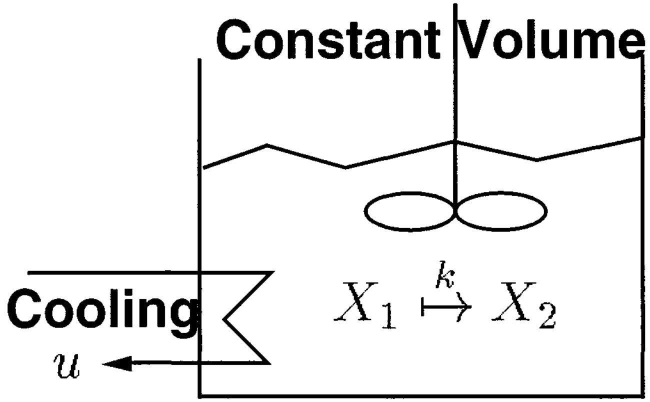
$2k\pi$ **the juggling robot**



5 degrees of freedom$(\theta_1, \theta_2, \theta_3)$ and the direction $\vec{SH}$. 3 motors.

A simple exothermic batch reactor.



Dynamics:

$$\frac{d}{dt}x_1 = -k_0 \exp(-E/R\theta)\ (x_1)^\alpha$$

$$\frac{d}{dt}\theta = k \exp(-E/R\theta)\ (x_1)^\alpha + u.$$

Modeling: mass and energy balance, Arhenius kinetics.

$$
\begin{aligned}
\frac{d}{dt}(V x_1) &= -V r(x,\theta) \\
\frac{d}{dt}(V x_2) &= V r(x,\theta) \\
\frac{d}{dt}(V \rho C_p \theta) &= V \Delta H r(x,\theta) + Q \\
r(x,\theta) &= k_0 \exp(-E/R\theta)\ (x_1)^\alpha
\end{aligned}
$$

Set $k = \frac{\Delta H}{\rho C_p}$ and $u = \frac{Q}{V \rho C_p}$ to obtain

$$
\begin{aligned}
\frac{d}{dt} x_1 &= -k_0 \exp(-E/R\theta)\ (x_1)^\alpha \\
\frac{d}{dt} \theta &= k \exp(-E/R\theta)\ (x_1)^\alpha + u.
\end{aligned}
$$

Explicit description of batch trajectories

Instead of fixing the initial condition $x_1(0) = x_1^0$, $\theta(0) = \theta^0$, the control $t \mapsto u(t)$ and integrating

$$\frac{d}{dt}x_1 = -k_0 \exp(-E/R\theta)\ (x_1)^\alpha$$
$$\frac{d}{dt}\theta = k \exp(-E/R\theta)\ (x_1)^\alpha + u(t)$$

take the system in the reverse way and assume that x_1 is a known time function

$$t \mapsto x_1 = y(t).$$

Then you bypass integration.

The inverse system has no dynamics

Set $x_1 = y(t)$ and compute θ and u knowing that

$$\frac{d}{dt}x_1 = -k_0 \exp(-E/R\theta)\ (x_1)^\alpha$$
$$\frac{d}{dt}\theta = k \exp(-E/R\theta)\ (x_1)^\alpha + u.$$

The mass conservation gives the temperature θ,

$$\exp(-E/R\theta) = -\frac{\frac{d}{dt}y}{k_0 y^\alpha} = \text{function of } (y, \dot{y}),$$

and energy balance gives exchanger duty u,

$$u = \frac{d}{dt}\theta + \frac{\frac{d}{dt}y}{k} = \text{function of } (y, \dot{y}, \ddot{y}).$$

Explicit description.

The system

$$\frac{d}{dt}x_1 = -k_0 \exp(-E/R\theta)\ (x_1)^\alpha, \qquad \frac{d}{dt}\theta = k \exp(-E/R\theta)\ (x_1)^\alpha + u$$

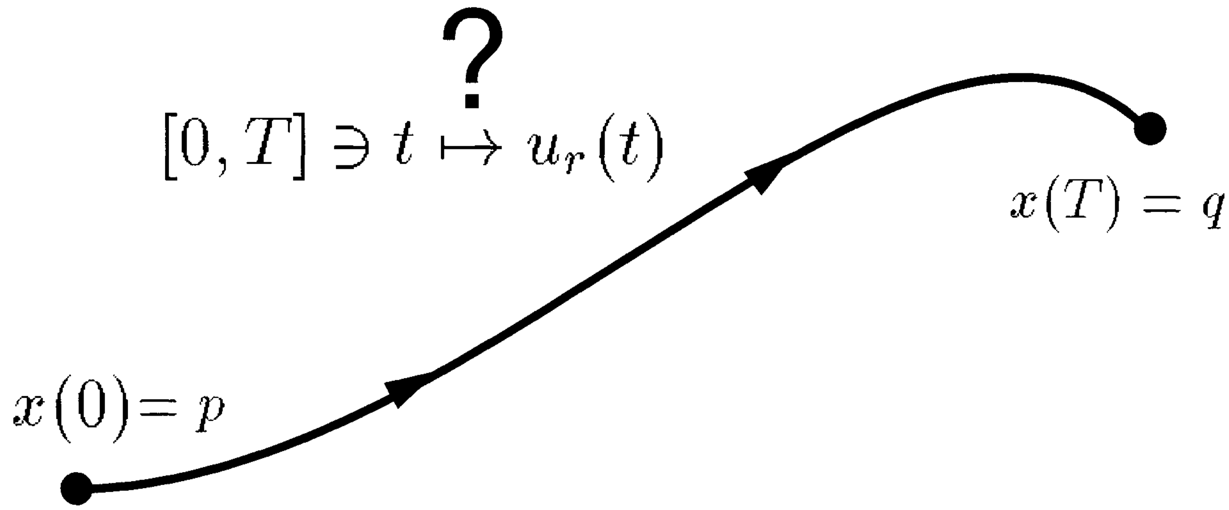
and the system

$$x_1 = y, \quad \theta = \text{function of } (y, \dot{y}), \quad u = \text{function of } (y, \dot{y}, \ddot{y})$$

represent the same object. It is just another presentation of the dynamics with an additional variable y and its derivatives. Dynamics admitting representation similar to the second system are called flat and the additional quantity y is then the flat output.

For the batch reactor, this is the simplest way to use the fact that the system is linearizable via static feedback and change of coordinates.

Motion planning: controllability.



Difficult problem because it requires, in general, the integration of

$$\frac{d}{dt}x = f(x, u(t)).$$

(for the batch reactor $x = (x_1, \theta)$).

Motion planning for the batch reactor

The initial condition

$$p = (x_1^0, \theta^0)$$

and final condition

$$q = (x_1^T, \theta^T)$$

provide initial and final positions and velocities for y:

$$y(0) = x_1^0 \qquad \frac{d}{dt}y(0) = -k_0 \exp(-E/R\theta^0)\ (x_1^0)^\alpha$$

$$y(T) = x_1^T \qquad \frac{d}{dt}y(T) = -k_0 \exp(-E/R\theta^T)\ (x_1^T)^\alpha$$

and in between $y(t)$ is free for $t > 0$ and $t < T$.

Motion planning for the batch reactor

Take $t \mapsto y^r(t)$ with such initial and final constraints. Compute u^r as

$$u^r = \text{function of } (y^r, \dot{y}^r, \ddot{y}^r)$$

Then the solution of the initial value problem

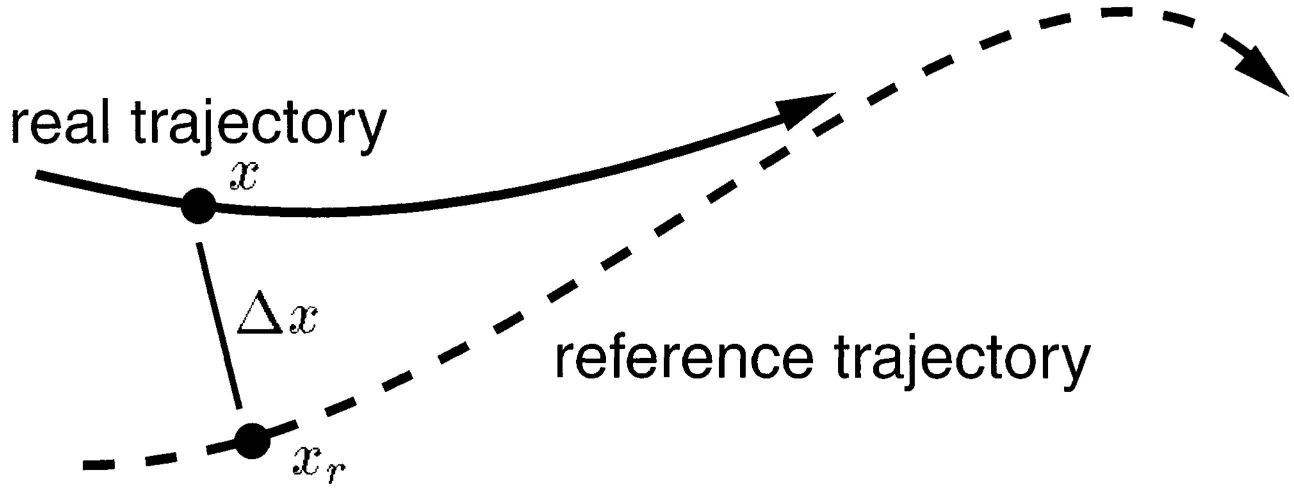
$$\frac{d}{dt}x_1 = -k_0 \exp(-E/R\theta)\ (x_1)^\alpha, \quad x_1(0) = x_1^0$$
$$\frac{d}{dt}\theta = k \exp(-E/R\theta)\ (x_1)^\alpha + u^r(t), \quad \theta(0) = \theta^0$$

is

$$x_1(t) = y^r(t), \quad \theta(t) = \text{function of } (y^r(t), \dot{y}^r(t))$$

and thus reaches $q = (x_1^T, \theta^T)$ at time T.

Tracking for $\frac{d}{dt}x = f(x,u)$: stabilization.



Compute Δu, $u = u_r + \Delta u$, such that $\Delta x = x - x_r$ tends to 0.

Tracking for the batch reactor

The reference trajectory defined via $t \mapsto y^r(t)$:

$x_1^r = y^r, \quad \theta^r = \text{function of } (y^r, \dot{y}^r), \quad u^r = \text{function of } (y^r, \dot{y}^r, \ddot{y}^r)$

The change of variable:

$$(x_1, \theta) \longleftrightarrow (y, \dot{y}).$$

The linearizing control:

$$u = \text{function of } (y, \dot{y}, v) = u^r + \Delta u$$

The stable closed-loop error dynamics:

$$\ddot{y} = v = \ddot{y}^r - 2\xi\omega_0(\dot{y} - \dot{y}^r) - \omega_0^2(y - y^r)$$

with $\xi > 0$ and $\omega_0 > 0$ design parameters.

Fully actuated mechanical systems

The computed torque method for

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] = \frac{\partial L}{\partial q} + M(q)u$$

consists in setting $t \mapsto q(t)$ to obtain u as a function of q, $\dot{q}$ and $\ddot{q}$.

(Fully actuated: $\dim q = \dim u$ and $M(q)$ invertible).

Two oscillators

Dynamics

$$\ddot{x}_1 = \omega_1^2(u - x_1), \quad \ddot{x}_2 = \omega_2^2(u - x_2).$$

Brunovsky output via u elimination:

$$\omega_2^2\ddot{x}_1 - \omega_1^2\ddot{x}_2 = \omega_1^2\omega_2^2(x_2 - x_1).$$

Controllable when $\omega_1 \neq \omega_2$ with Brunovsky (flat) output

$$y = \omega_2^2 x_1 - \omega_1^2 x_2.$$

For classical SISO system $z(s) = \frac{P(s)}{Q(s)}u(s)$ (z is the output here), then just put

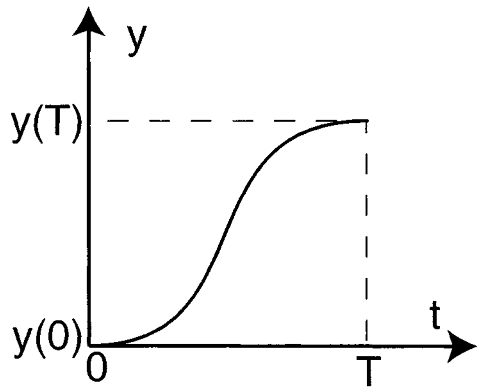
$$z(s) = P(s)y(s), \quad u(s) = Q(s)y(s).$$

Since P and Q have no common divisor, exist R and S such that $PR + QS = 1$, i.e., $y = Ry + Su$ is the "flat" output.

We have then

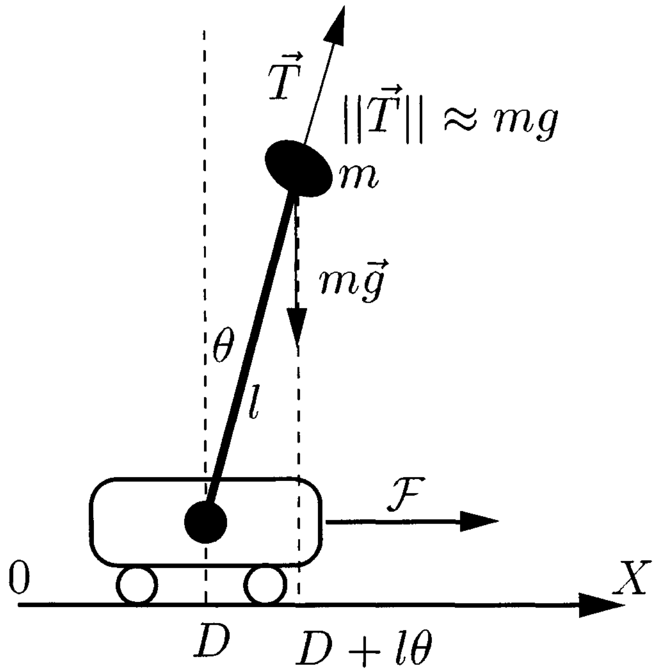
$$(x_1, x_2, u) = \text{linear combination of } (y, \ddot{y}, y^{(4)}).$$

Steady-state to steady-state steering via y with the following shape (take, e.g., a polynomial of degree 9):



$$\dot{y} = \ddot{y} = y^{(3)} = y^{(4)} = 0 \text{ for } t = 0$$
$$\dot{y} = \ddot{y} = y^{(3)} = y^{(4)} = 0 \text{ for } t = T$$

Inverted linearized pendulum



Under the small angle approximation:

$$\frac{d^2}{dt^2}(D + l\theta) = g\theta, \quad M\frac{d^2}{dt^2}D = -mg\theta + \mathcal{F}.$$

The Brunovsky output

The dynamics

$$\frac{d^2}{dt^2}(D + l\theta) = g\theta, \qquad M\frac{d^2}{dt^2}D = -mg\theta + \mathcal{F}$$

admits $y = D + l\theta$ as Brunovky (flat) output.

This comes from

$$\theta = \ddot{y}/g, \quad D = y - l\ddot{y}/g, \quad \mathcal{F} = m\ddot{y} + M(\ddot{y} - ly^{(4)}/g).$$

Up to static linear feedback and linear change of coordinates it reads

$$y^{(4)} = v.$$

Not very robust to control directly this fourth order system: the trolley dynamics is not well known (friction).

Hierarchical control

$$\frac{d^2}{dt^2}(y) = g(y-D)/l, \quad M\frac{d^2}{dt^2}D = -mg(y-D)/l + \mathcal{F}$$

Use the fact that $\mathcal{F}$ only appears in the trolley equation (strong structural and physical property) to dominate modelling uncertainties via a high gain loop on the trolley position.

High gain feedback with u as the set-point for the trolley position:

$$\mathcal{F} = -Mk_1\dot{D} - Mk_2(D-u)$$

with $k_1 > 10/\tau$, $k_2 > 10/\tau^2$ where $\tau = \sqrt{l/g}$ is the characteristic time of the pendulum (for this gain design, we assume that $m \leq M$).

The slow pendulum dynamics is close to:

$$\frac{d^2}{dt^2}(y) = g(y-u)/l$$

The well known slow pendulum dynamics

$$\frac{d^2}{dt^2}(y) = g(y-u)/l = \frac{y-u}{\tau^2}.$$

Take a reference trajectory (y_r, u_r)

$$\frac{d^2}{dt^2}(y_r) = \frac{y_r - u_r}{\tau^2}$$

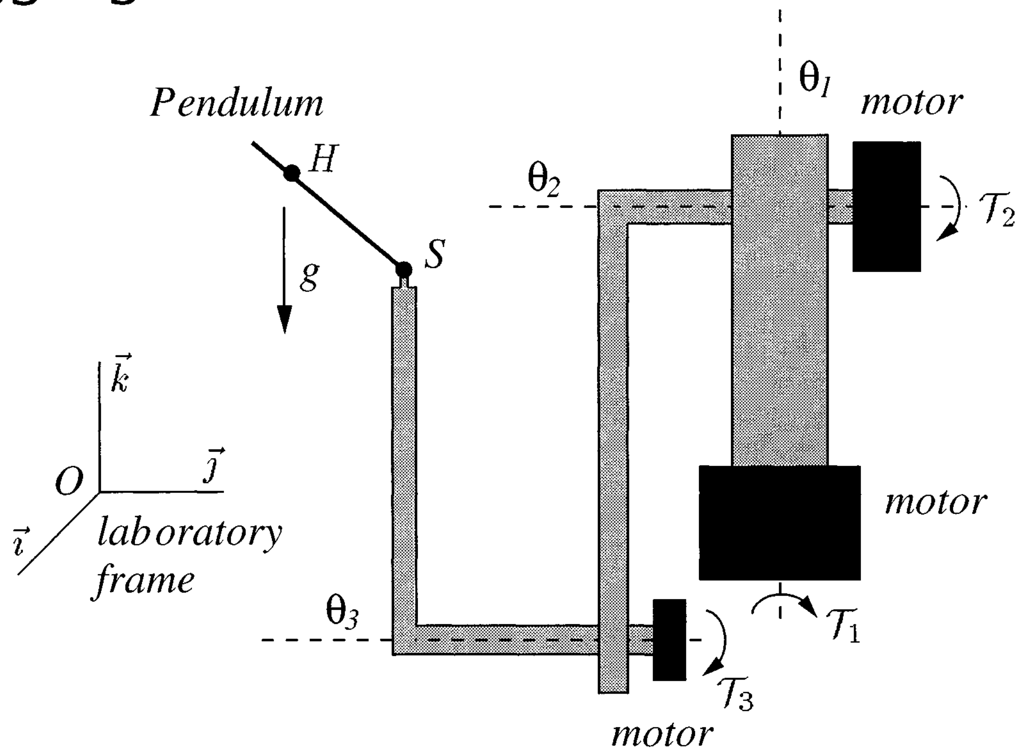
and set

$$u = y - \tau^2 \ddot{y}_r(t) + 2\tau(\dot{y} - \dot{y}_r(t)) + (y - y_r(t))$$

then the error dynamics $e = y - y_r$ satisfies the stable second order system

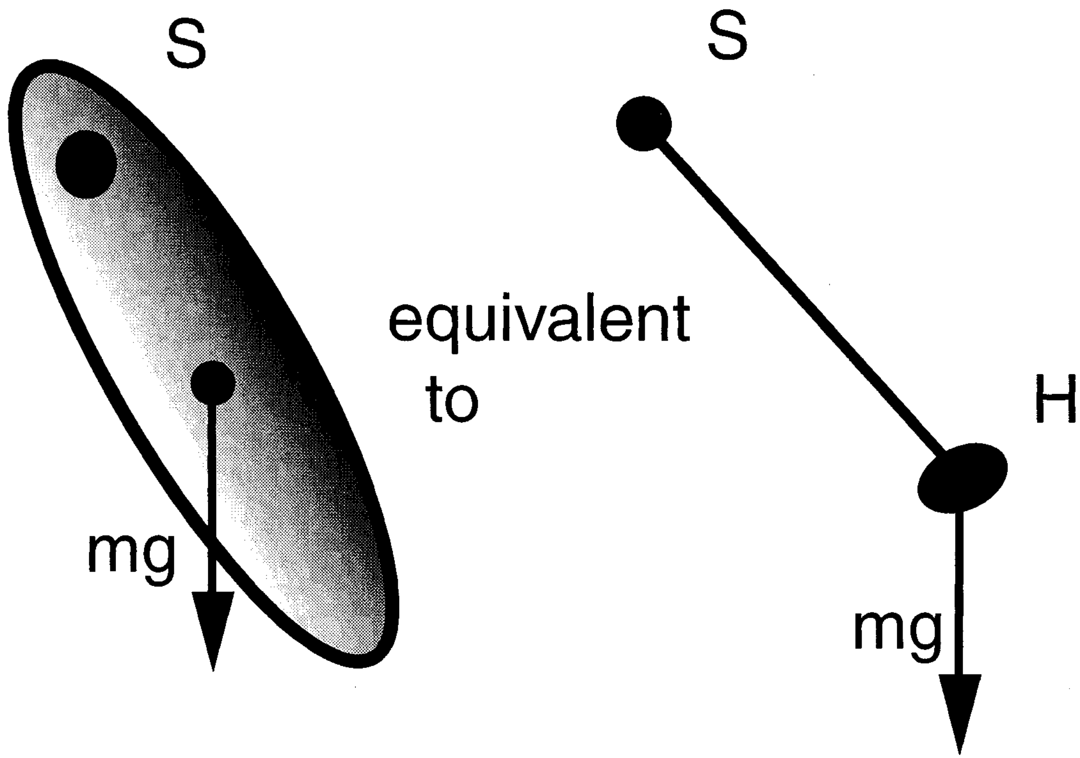
$$\ddot{e} = -\frac{2\dot{e}}{\tau} - \frac{e}{\tau^2}.$$

$2k\pi$ **the juggling robot**

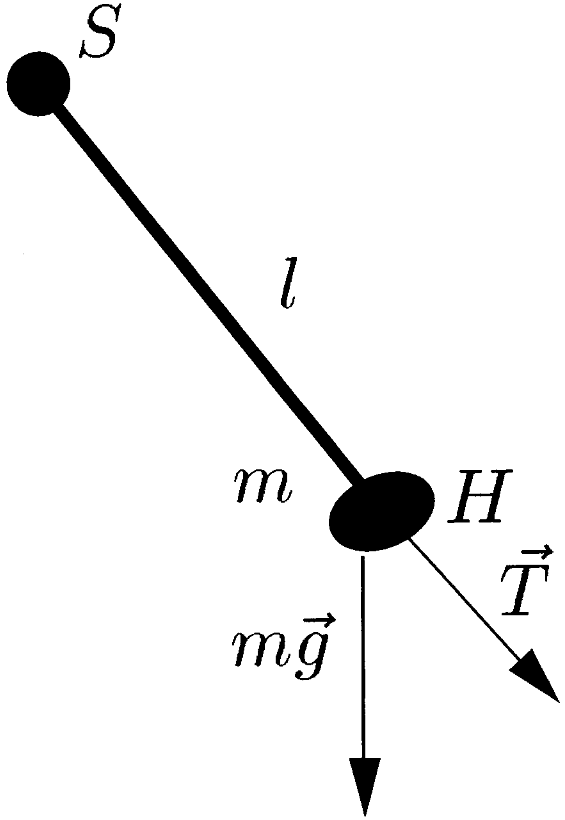


5 degrees of freedom$(\theta_1, \theta_2, \theta_3)$ and the direction $\vec{SH}$. 3 motors.

Huygens isochronous pendulum



The implicit model (S is the control)



Newton law

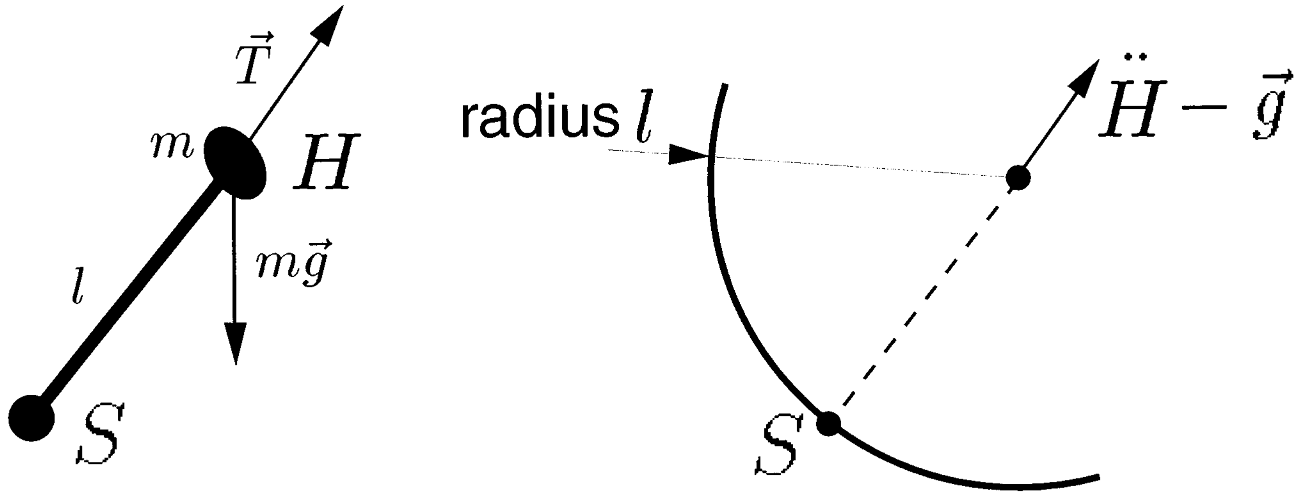
$$m\ddot{H} = \vec{T} + m\vec{g}$$

Constraints

$$\vec{T} // \overrightarrow{HS}$$

$$||\overrightarrow{HS}|| = l$$

H as flat output



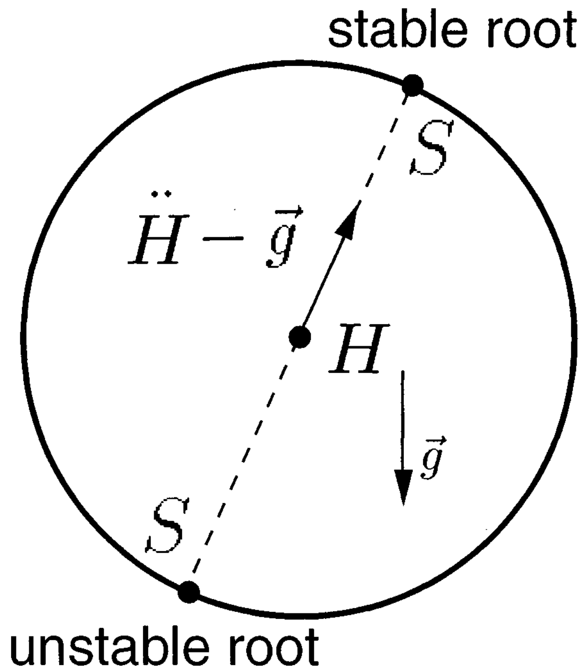
Since

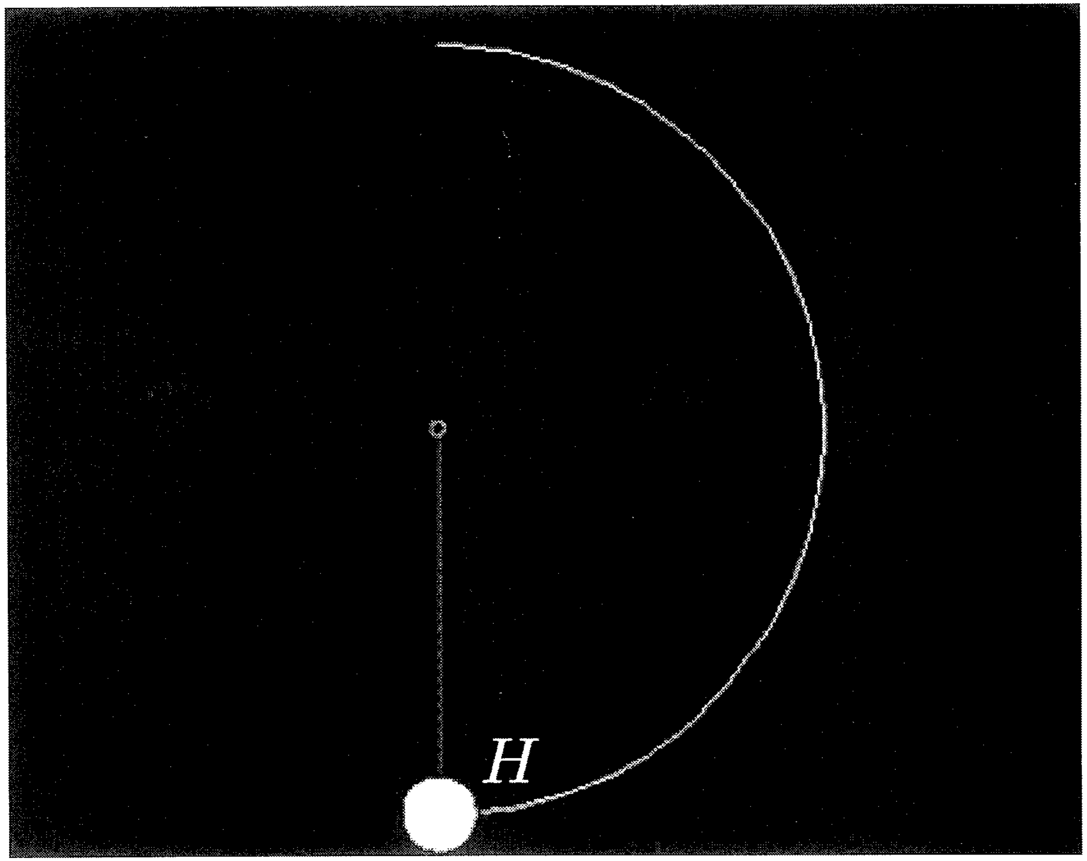
$$\vec{T}/m = \ddot{H} - \vec{g} \quad \text{and} \quad \vec{T} \; // \; \overrightarrow{HS}$$

we have S via

$$\overrightarrow{HS} \; // \; \ddot{H} - \vec{g} \quad \text{and} \quad HS = l.$$

Planning the inversion trajectory Any smooth trajectory connecting the stable to the unstable equilibrium is such that $\ddot{H}(t) = \vec{g}$ for at least one time t. During the motion there is a switch from the stable root to the unstable root (singularity crossing when $\ddot{H} = \vec{g}$)



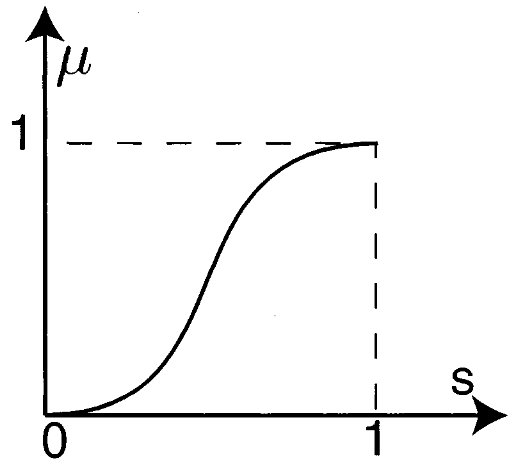

H

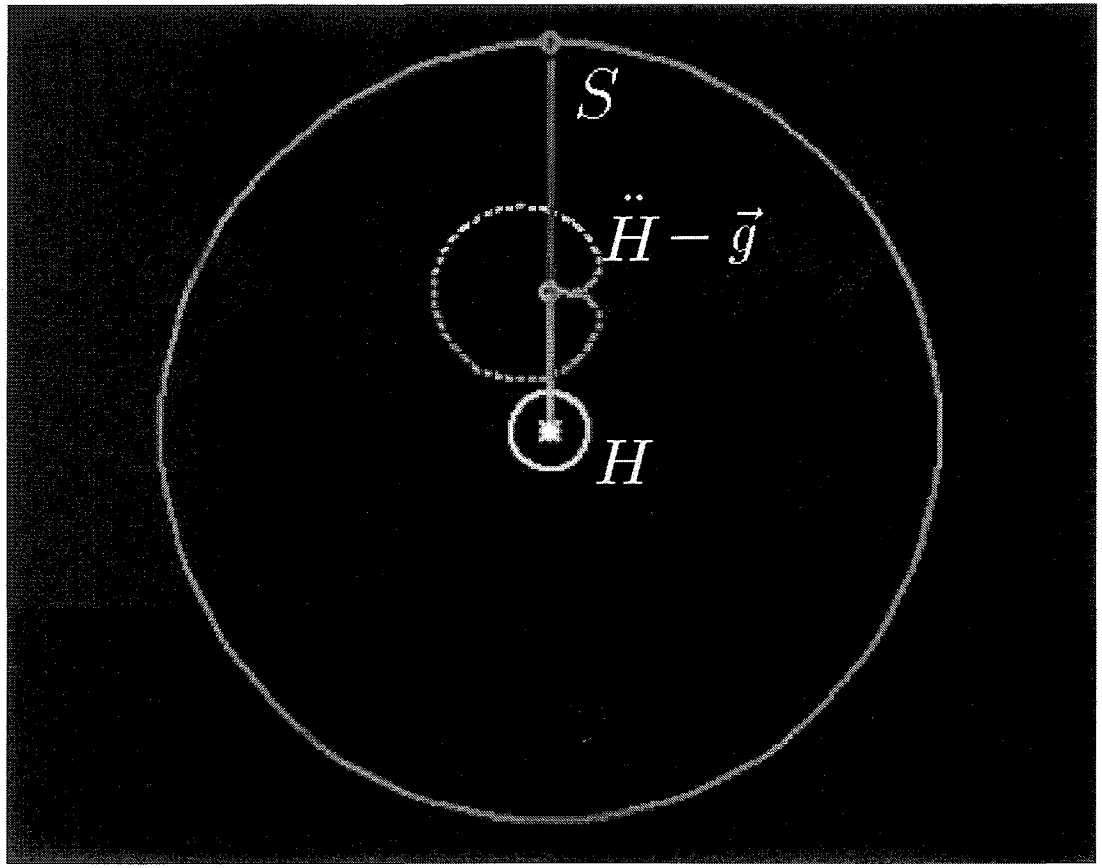
Crossing smoothly the singularity $\ddot{H} = \vec{g}$

The geometric path followed by H is a half-circle of radius $l$of center O:

$$H(t) = 0 + l \begin{bmatrix} \sin\theta(s) \\ -\cos\theta(s) \end{bmatrix} \text{ with } \theta(s) = \mu(s)\pi, \quad s = t/T \in [0, 1]$$

where T is the transition time and $\mu(s)$ a sigmoid function of the form:




S
$\ddot{H} - \vec{g}$
H

Time scaling and dilation of $\ddot{H} - \vec{g}$

Denote by $'$ derivation with respect to s. From

$$H(t) = 0 + l \begin{bmatrix} \sin\theta(s) \\ -\cos\theta(s) \end{bmatrix}, \quad \theta(s) = \mu(t/T)\pi$$

we have

$$\ddot{H} = H''/T^2.$$

Changing T to αT yields to a dilation of factor $1/\alpha^2$ of the closed geometric path described by $\ddot{H} - \vec{g}$ for $t \in [0, T]$ ($\ddot{H}(0) = \ddot{H}(T) = 0$), the dilation center being $-\vec{g}$.

The inversion time is obtained when this closed path passes through 0. This construction holds true for generic μ.

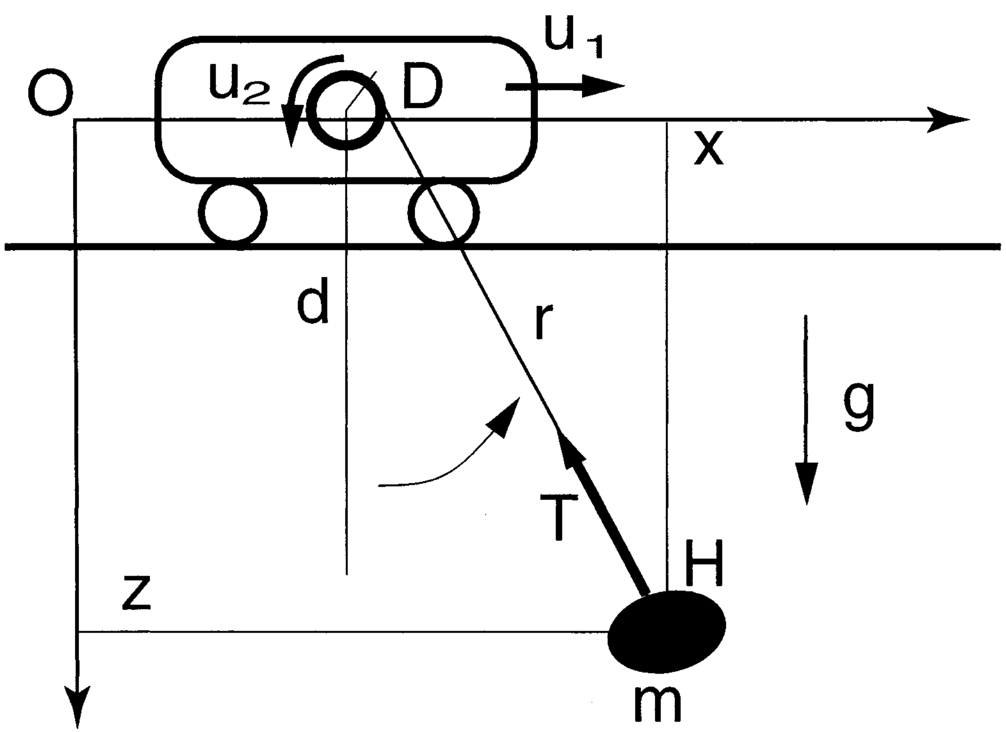
The crane

$$m\ddot{H} = \vec{T} + m\vec{g}$$

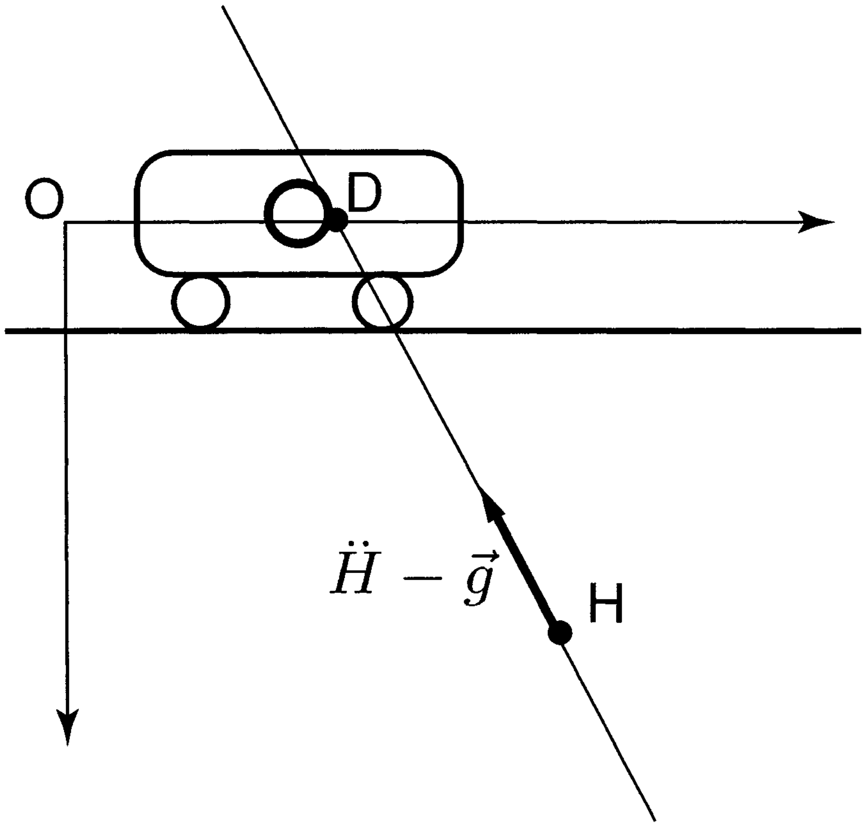
$$\vec{T} \,//\, \overrightarrow{HD}$$

$$HD = \mathsf{r}$$

$$\overrightarrow{0D} \cdot \vec{k} = 0$$



The geometric construction for the crane



Singularity when $\ddot{H} - \vec{g}$ is horizontal.

Flat systems (Fliess-et-al, 1992,. . . ,1999)

A basic definition extending remark of Isidori-Moog-DeLuca (CDC86) on dynamic feedback linearization (Charlet-Lévine-Marino (1989)):

$$\frac{d}{dt}x = f(x,u)$$

is flat, iff, exist $m = \dim(u)$ output functions $y = h(x,u,\ldots,u^{(p)})$, $\dim(h) = \dim(u)$, such that the inverse of $u \mapsto y$ has no dynamics, i.e.,

$$x = \Lambda\left(y, \dot{y}, \ldots, y^{(q)}\right), \quad u = \Upsilon\left(y, \dot{y}, \ldots, y^{(q+1)}\right).$$

Behind this: an equivalence relationship exchanging trajectories (absolute equivalence of Cartan and dynamic feedback: Shadwick (1990), Sluis (1992), Nieuwstadt-et-al (1994), . . .).

The equivalence relationship (Fliess-et-al 1992,. . . , 1999)

Elimination of u from the n state equations $\frac{d}{dt}x = f(x,u)$ provides an under-determinate system of $n-m$ equations with n unknowns

$$F\left(x, \frac{d}{dt}x\right) = 0.$$

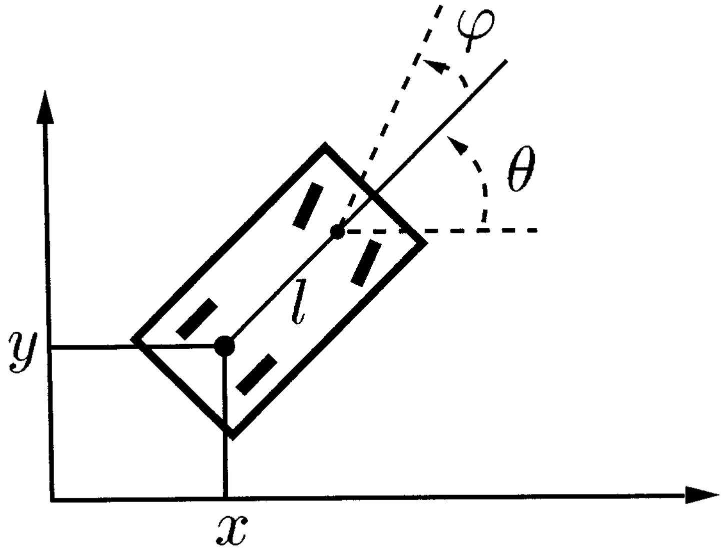
An **endogenous transformation** $x \mapsto z$ is defined by

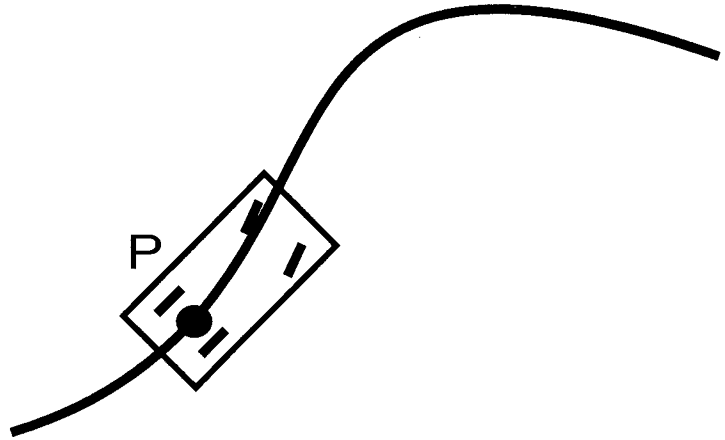
$$z = \Phi(x, \dot{x}, \dots, x^{(p)}), \quad x = \Psi(z, \dot{z}, \dots, z^{(q)})$$

(nonlinear analogue of uni-modular matrices).

Two systems are equivalents, iff, exists an endogenous transformation exchanging the trajectories or the equations. A system equivalent to the trivial equation $z_1 = 0$ with $z = (z_1, z_2)$ is flat with z_2 the flat output.

Single car





$$
\begin{cases}
\frac{d}{dt}x = v\cos\theta \\
\frac{d}{dt}y = v\sin\theta \\
\frac{d}{dt}\theta = \frac{v}{l}\tan\varphi = \omega
\end{cases}
$$

$$
\begin{cases}
v = \pm\|\frac{d}{dt}P\| \\
\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} = \frac{\frac{d}{dt}P}{v} \\
\tan\varphi = \frac{l\det(\dot{P},\ddot{P})}{v\sqrt{|v|}}
\end{cases}
$$

The time scaling symmetry

For any $T \mapsto \sigma(T)$, the transformation

$$t = \sigma(T), \quad (x, y, \theta) = (X, Y, \Theta), \qquad (v, \omega) = (V, \Omega)/\sigma'(t)$$

leave the equations

$$\frac{d}{dt}x = v \cos \theta, \quad \frac{d}{dt}y = v \sin \theta, \quad \frac{d}{dt}\theta = \omega$$

unchanged:

$$\frac{d}{dT}X = V \cos \Theta, \quad \frac{d}{dT}Y = V \sin \Theta, \quad \frac{d}{dT}\Theta = \Omega.$$

$SE(2)$ invariance

For any (a, b, α), the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X\cos\alpha - Y\sin\alpha + a \\ X\sin\alpha + Y\cos\alpha + b \end{bmatrix} \quad , \theta = \Theta - \alpha, \qquad (v, \omega) = (V, \Omega)$$

leave the equations

$$\frac{d}{dt}x = v\cos\theta, \quad \frac{d}{dt}y = v\sin\theta, \quad \frac{d}{dt}\theta = \omega$$

unchanged:

$$\frac{d}{dt}X = V\cos\Theta, \quad \frac{d}{dt}Y = V\sin\Theta, \quad \frac{d}{dt}\Theta = \Omega.$$

Control system with symmetries: a first definition.

The system

$$\frac{dx}{dt} = f(x,u)$$

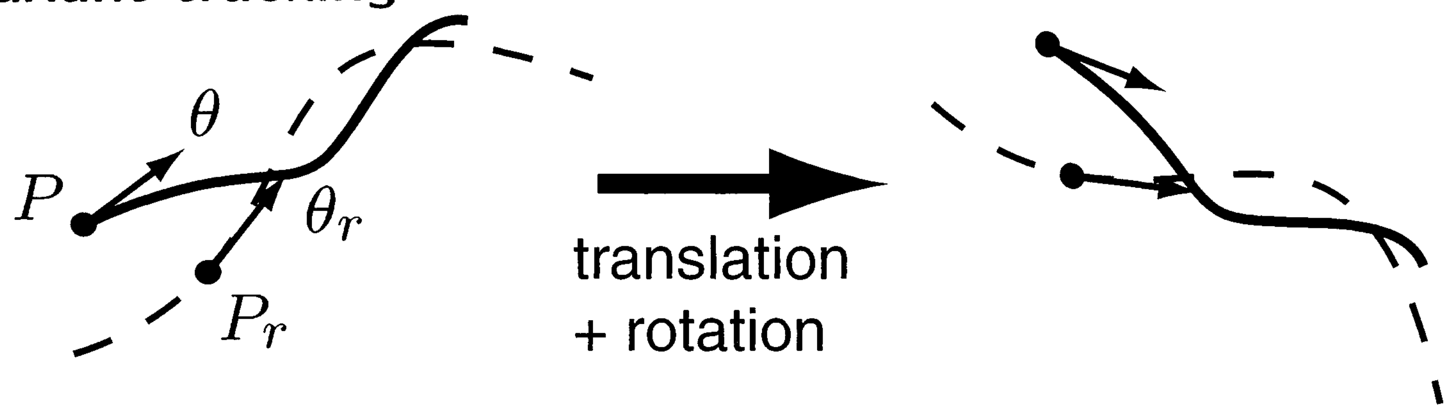
admitting a symmetry group G of transformations (any element of G is just a change of state-variables), iff, for any change of state variables $x = g(X)$, $g \in G$, exists a feedback

$$u = k(X,U)$$

such that with X, U the state equations remain unchanged

$$\frac{dX}{dt} = f(X,U).$$

Invariant tracking



Invariant tracking for the car: goal

Given the reference trajectory

$$t \mapsto s_r \mapsto P_r(s_r), \quad \theta_r(s_r), \quad v_r = \dot{s}_r, \quad \omega_r = \dot{s}_r \kappa_r(s_r)$$

and the state (P, θ)

Find an invariant controller

$$v = v_r + \dots, \quad \omega = \omega_r + \dots$$

Invariant tracking for the car: time-scaling

Set

$$v = \bar{v}\ \dot{s}_r, \quad \omega = \bar{\omega}\ \dot{s}_r$$

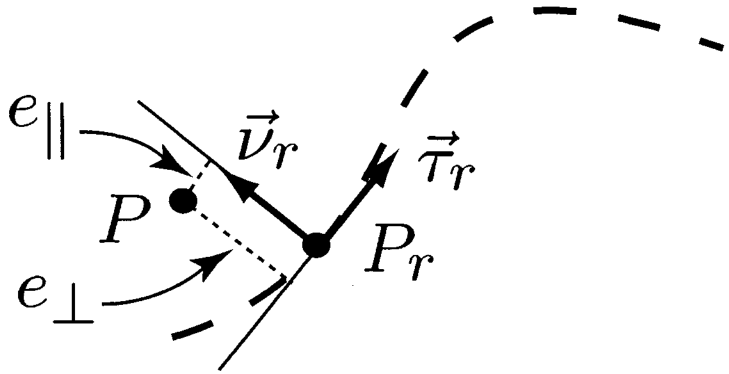
and denote by $'$ derivation versus s_r.

Equations remain unchanged

$$P' = \bar{v}\ \vec{\tau}, \quad \vec{\tau}' = \bar{\omega}\ \vec{\nu}$$

with $P = (x, y)$, $\vec{\tau} = (\cos\theta, \sin\theta)$ and $\vec{\nu} = (-\sin\theta, \cos\theta)$.

Invariant errors



Construct the decoupling and/or linearizing controller with the two following invariant errors

$$e_\parallel = (P - P_r) \cdot \vec{\tau}_r, \quad e_\perp = (P - P_r) \cdot \vec{\nu}_r.$$

Computations of $e_{\|}$ and $e_{\perp}$ derivatives

Since $e_{\|} = (P - P_r) \cdot \vec{\tau}_r$ and $e_{\perp} = (P - P_r) \cdot \vec{\nu}_r$ we have (remember that $' = d/ds_r$)

$$e'_{\|} = (P' - P'_r) \cdot \vec{\tau}_r + (P - P_r) \cdot \vec{\tau}'_r.$$

But $P' = \bar{v}\vec{\tau}$, $P'_r = \vec{\tau}_r$ and $\vec{\tau}'_r = \kappa_r \vec{\nu}_r$, thus

$$e'_{\|} = \bar{v}\vec{\tau} \cdot \vec{\tau}_r - 1 + \kappa_r (P - P_r) \cdot \vec{\nu}_r.$$

Similar computations for $e'_{\perp}$ yield:

$$e'_{\|} = \bar{v}\cos(\theta - \theta_r) - 1 + \kappa_r e_{\perp}, \quad e'_{\perp} = \bar{v}\sin(\theta - \theta_r) - \kappa_r e_{\|}.$$

Computations of $e_{\|}$ and $e_{\perp}$ second derivatives

Derivation of

$$e_{\|}' = \bar{v}\cos(\theta - \theta_r) - 1 + \kappa_r e_{\perp}, \qquad e_{\perp}' = \bar{v}\sin(\theta - \theta_r) - \kappa_r e_{\|}$$

with respect to s_r gives

$$\begin{aligned} e_{\|}'' = {} & \bar{v}'\cos(\theta - \theta_r) - \bar{\omega}\bar{v}\sin(\theta - \theta_r) \\ & + 2\kappa_r\bar{v}\sin(\theta - \theta_r) + \kappa_r' e_{\perp} - \kappa_r^2 e_{\|} \end{aligned}$$

$$\begin{aligned} e_{\perp}'' = {} & \bar{v}'\sin(\theta - \theta_r) + \bar{\omega}\bar{v}\cos(\theta - \theta_r) \\ & - 2\kappa_r\bar{v}\cos(\theta - \theta_r) - \kappa_r' e_{\|} + \kappa_r + \kappa_r^2 e_{\|}. \end{aligned}$$

The dynamics feedback in s_r time-scale

We have obtain

$$e''_{\parallel} = \bar{v}' \cos(\theta - \theta_r) - \bar{\omega}\bar{v} \sin(\theta - \theta_r) + W_{\parallel}$$
$$e''_{\perp} = \bar{v}' \sin(\theta - \theta_r) + \bar{\omega}\bar{v} \cos(\theta - \theta_r) + W_{\perp}$$

Choose $\bar{v}'$ and $\bar{\omega}$ such that

$$e''_{\parallel} = -\left(\frac{1}{L^1_{\parallel}} + \frac{1}{L^2_{\parallel}}\right) e'_{\parallel} - \left(\frac{1}{L^1_{\parallel} L^2_{\parallel}}\right) e_{\parallel}$$
$$e''_{\perp} = -\left(\frac{1}{L^1_{\perp}} + \frac{1}{L^2_{\perp}}\right) e'_{\perp} - \left(\frac{1}{L^1_{\perp} L^2_{\perp}}\right) e_{\perp}$$

Possible around a large domain around the reference trajectory since the determinant of the decoupling matrix is $\bar{v} \approx 1$.

The dynamics feedback in physical time-scale

In the s_r scale, we have the following dynamic feedback

$$\bar{v}' = \Phi(\bar{v}, P, P_r, \theta, \theta_r, \kappa_r, \kappa_r')$$
$$\bar{\omega} = \Psi(\bar{v}, P, P_r, \theta, \theta_r, \kappa_r, \kappa_r')$$

Since $' = d/ds_r = d/(\dot{s}_r dt)$ we have

$$\frac{d\bar{v}}{dt} = \Phi(\bar{v}, P, P_r, \theta, \theta_r, \kappa_r, \kappa_r') \ \dot{s}_r(t)$$
$$\bar{\omega} = \Psi(\bar{v}, P, P_r, \theta, \theta_r, \kappa_r, \kappa_r')$$

and the real control is

$$v = \bar{v}\dot{s}_r(t), \quad \tan\phi = \frac{l\bar{\omega}}{\bar{v}}$$

Nothing blows up when $\dot{s}_r(t)$ tends to 0: the controller is well defined around steady-state via a simple use of time-scaling symmetry.

Conversion into chained form destroys $SE(2)$ invariance

The car model

$$\frac{d}{dt}x = v\cos\theta, \quad \frac{d}{dt}y = v\sin\theta, \quad \frac{d}{dt}\theta = \frac{v}{l}\tan\varphi$$

can be transformed into chained form

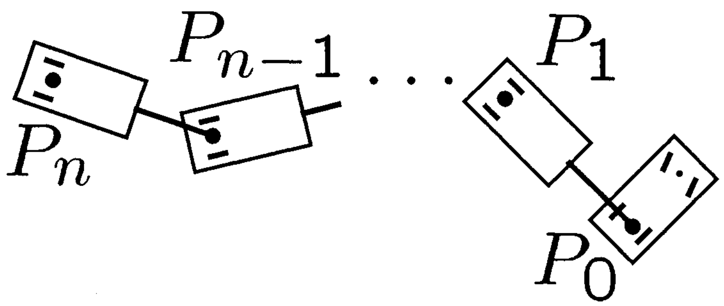
$$\frac{d}{dt}x_1 = u_1, \quad \frac{d}{dt}x_2 = u_2, \quad \frac{d}{dt}x_3 = x_2 u_1$$

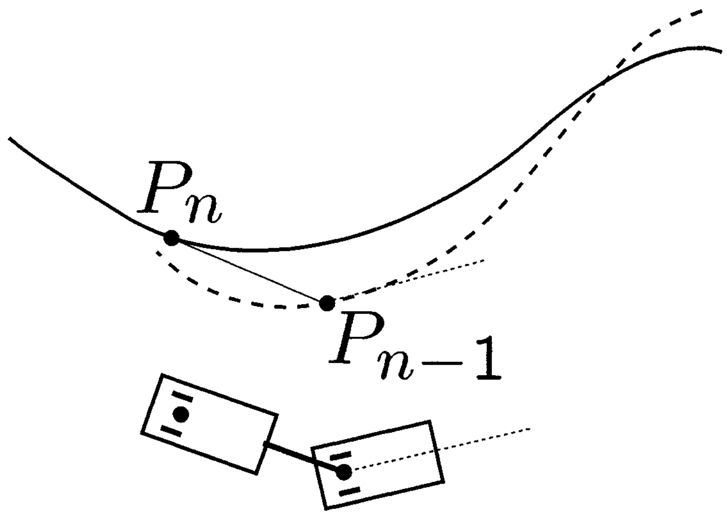
via change of coordinates and static feedback

$$x_1 = x, \quad x_2 = \frac{dy}{dx} = \tan\theta, \quad x_3 = y.$$

But the symmetries are not preserved in such coordinates: one privileges axis x versus axis y without any good reason. The behavior of the system seems to depend on the origin you take to measure the angle (artificial singularity when $\theta = \pm\pi/2$).

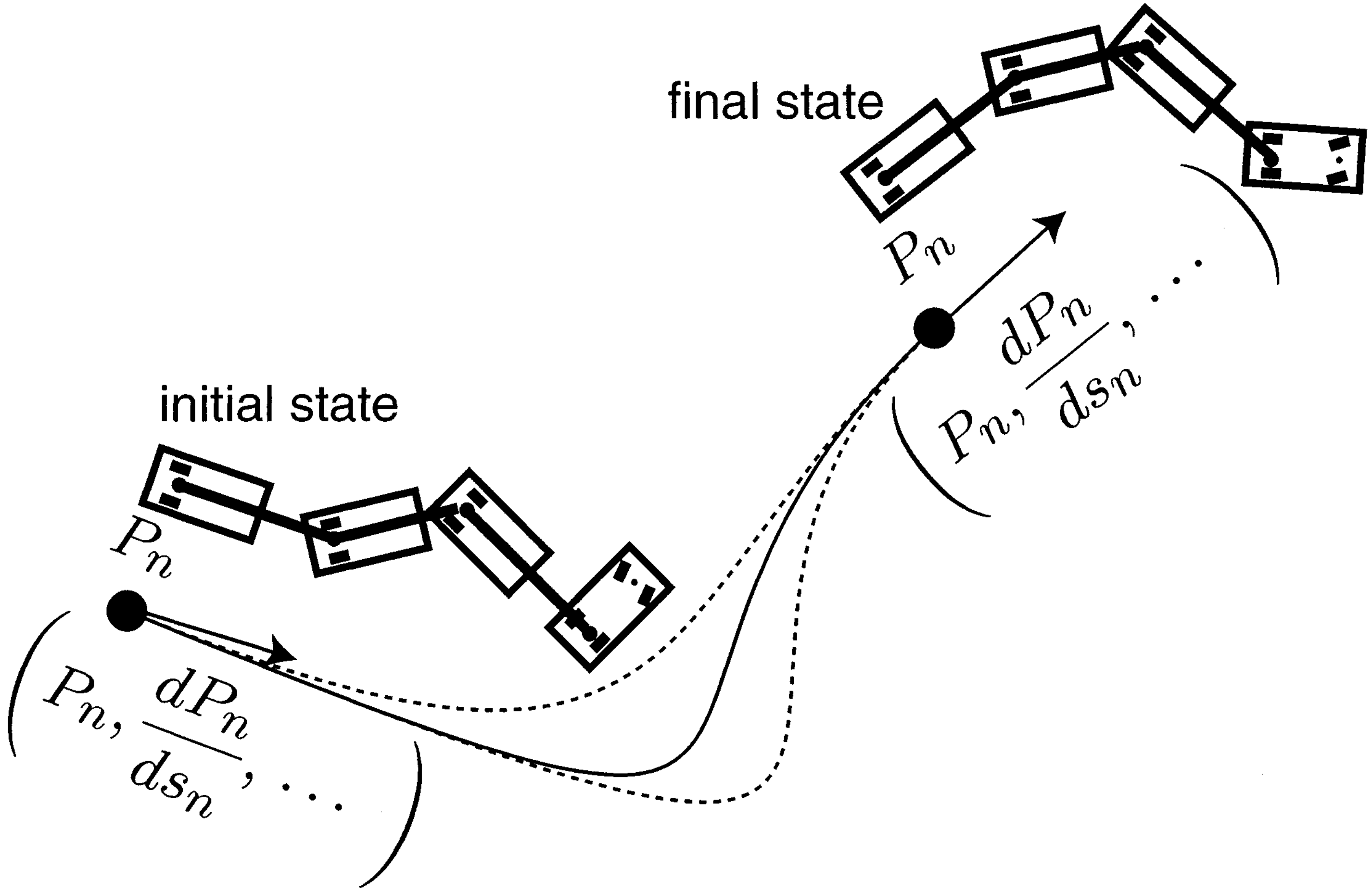
The standard n-trailers system



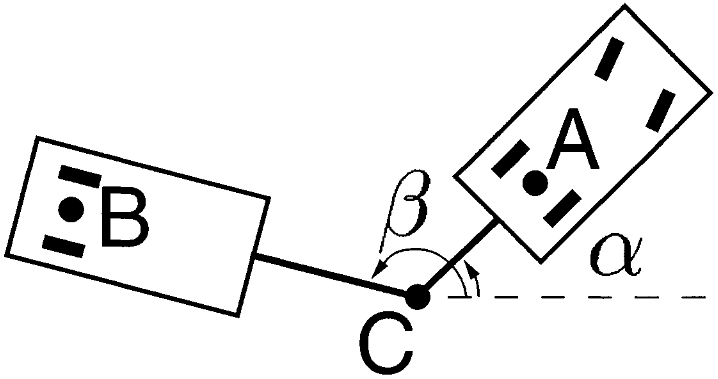


$$P_{n-1} = P_n + d_n \frac{dP_n}{ds_n}$$

Motion planning for the standard n trailers system

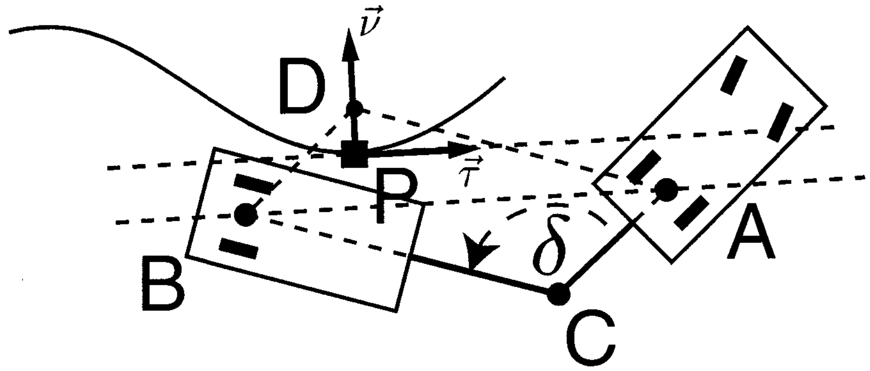


The general 1-trailer system (CDC93)



Rolling without slipping conditions ($A = (x, y)$, $u = (v, \varphi)$):

$$\begin{aligned}
\frac{d}{dt}x &= v \ \cos\alpha \\
\frac{d}{dt}y &= v \ \sin\alpha \\
\frac{d}{dt}\alpha &= \frac{v}{l}\tan\varphi \\
\frac{d}{dt}\beta &= \frac{v}{b}\left(\frac{a}{l}\tan\varphi\cos(\beta-\alpha)+\sin(\beta-\alpha)\right).
\end{aligned}$$



With $\delta = \widehat{BCA}$ we have

$$D = P - L(\delta)\vec{\nu} \quad \text{with} \quad L(\delta) = ab \int_0^{\pi+\delta} \frac{-\cos\sigma}{\sqrt{a^2+b^2+2ab\cos\sigma}}\, d\sigma$$

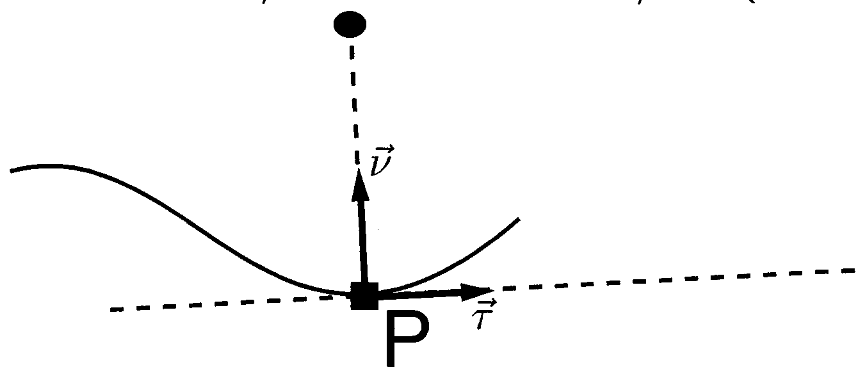
Curvature is given by

$$K(\delta) = \frac{\sin\delta}{\cos\delta\,\sqrt{a^2+b^2-2ab\cos\delta} - L(\delta)\,\sin\delta}$$

The geometric construction

Assume that $s \mapsto P(s)$ is known. Let us show how to deduce (A, B, α, β) the system configuration.

We know thus P, $\vec{\tau} = dP/ds$ and $\kappa = d\theta/ds$ (θ is the angle of $\vec{\tau}$:

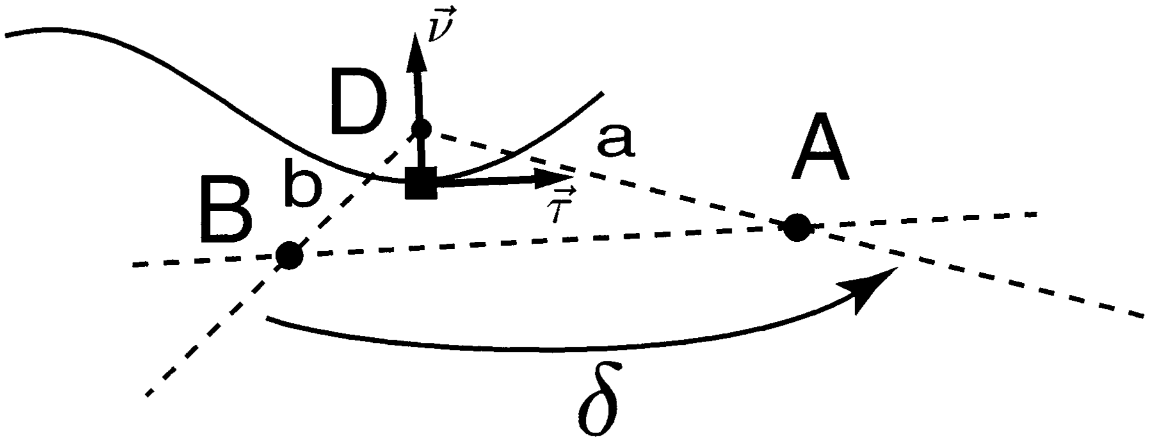


The geometric construction

From κ we deduce $\delta = \widehat{BCA}$ by inverting $\kappa = K(\delta)$.

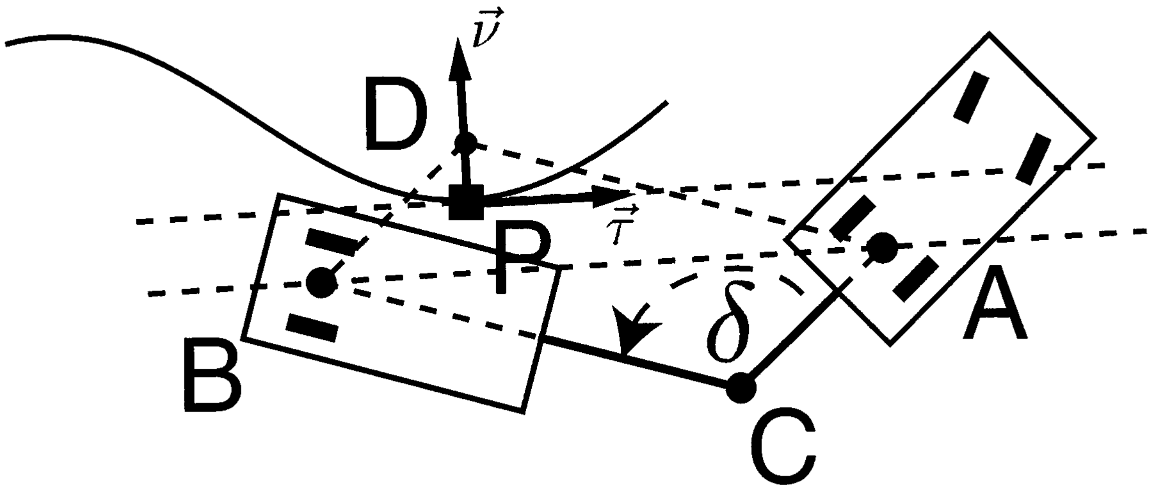
D is then known since $D = P - L(\delta)\vec{\nu}$.

Finally $\vec{\tau}$ is parallel to AB and $DB = a$ and $DA = b$.



The complete construction

One to one correspondence between P, $\vec{\tau}$ and κ and (A, α, β).



Differential forms: eliminate v from

$$\frac{d}{dt}x = v\ \cos\alpha, \quad \frac{d}{dt}y = v\ \sin\alpha, \quad \frac{d}{dt}\alpha = \frac{v}{l}\tan\varphi, \quad \frac{d}{dt}\beta = \ldots$$

to have 3 equations with 5 variables

$$\sin\alpha\ \tfrac{d}{dt}x - \cos\alpha\ \tfrac{d}{dt}y = 0$$

$$\tfrac{d}{dt}\alpha - \left(\frac{\tan\varphi\cos\alpha}{l}\right)\ \tfrac{d}{dt}x - \left(\frac{\tan\varphi\sin\alpha}{l}\right)\ \tfrac{d}{dt}y = 0$$

$$\tfrac{d}{dt}\beta \ldots$$

defining a module of differential forms, $I = \{\eta_1, \eta_2, \eta_3\}$

$$\eta_1 = \sin\alpha\ dx - \cos\alpha\ dy$$

$$\eta_2 = d\alpha - \left(\frac{\tan\varphi\cos\alpha}{l}\right)\ dx - \left(\frac{\tan\varphi\sin\alpha}{l}\right)\ dy$$

$$\eta_3 = d\beta - \ldots$$

Derived flag

Compute the sequence $I = I^{(0)} \supseteq I^{(1)} \supseteq I^{(2)} \ldots$ where

$$I^{(k+1)} = \{\eta \in I^{(k)} \mid d\eta = 0 \mod (I^{(k)})\}$$

and find that

$$\dim I^{(0)} = 3, \quad \dim I^{(1)} = 2, \quad \dim I^{(2)} = 1, \quad \dim I^{(3)} = 0.$$

The Cartesian coordinates (X, Y) of P are obtained via the Pfaff normal form of the differential form μ generating $I^{(2)}$

$$\mu = f(\alpha, \beta)\ dX + g(\alpha, \beta)\ dY.$$

(X, Y) is not unique; $SE(2)$ invariance simplifies computations.

Contact systems:

The driftless system $\frac{d}{dt}x = f_1(x)u_1 + f_2(x)u_2$ is also a Pfaffian system of codimension 2

$$\omega_i \equiv \sum_{j=1}^{n} a_i^j(x)\ dx_j = 0, \quad i = 1, \ldots, n-2.$$

Pfaffian systems equivalent via changes of x-coordinates to contact systems (related to chained-form, Murray-Sastry 1993)

$$dx_2 - x_3 dx_1 = 0, \quad dx_3 - x_4 dx_1 = 0, \quad \ldots dx_{n-1} - x_n dx_1 = 0$$

are mainly characterized by the derived flag (Weber(1898), Cartan(1916), Goursat (1923), Giaro-Kumpera-Ruiz(1978), Murray (1994), Pasillas-Respondek (2000), . . .).

Interest of contact systems (chained form):

$$dx_2 - x_3 dx_1 = 0, \quad dx_3 - x_4 dx_1 = 0, \quad \ldots dx_{n-1} - x_n dx_1 = 0$$

The general solution reads in terms of $z \mapsto w(z)$ and its derivatives,

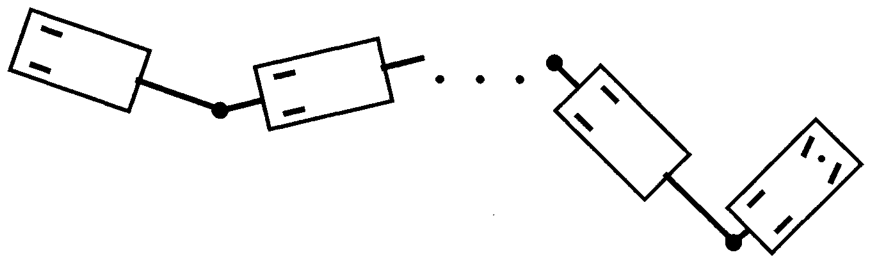
$$x_1 = z, \quad x_2 = w(z), \quad , x_3 = \frac{dw}{dz}, \quad \ldots \quad , x_n = \frac{d^{n-2}w}{dz^{n-2}}.$$

In this case, the general solution of $\frac{d}{dt}x = f_1(x)u_1 + f_2 u_2$ reads in terms of $t \mapsto z(t)$ any C^1 time function and any C^{n-2} function of z, $z \mapsto w(z)$. The quantities $x_1 = z(t)$ and $x_2 = w(z(t))$ play here a special role. We call them the flat output.

Flatness Characterization

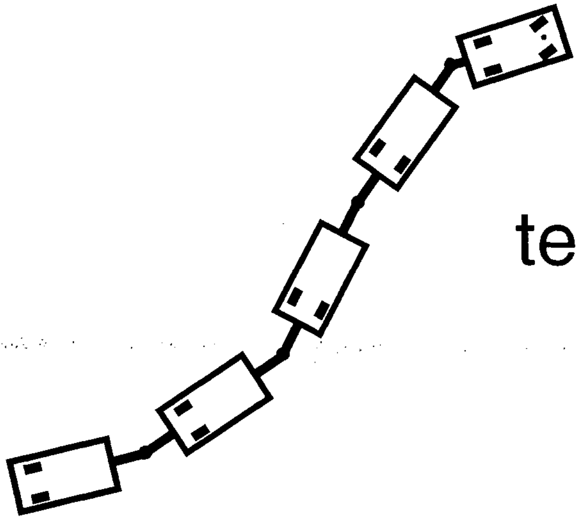
- Single input system.
- The ruled manifold criterion.
- Flatness and dynamic feedback linearization: endogenous feedback versus exogenous feedback.
- Driftless systems with two controls.

The general n-trailer system for $n \geq 2$ is not flat.

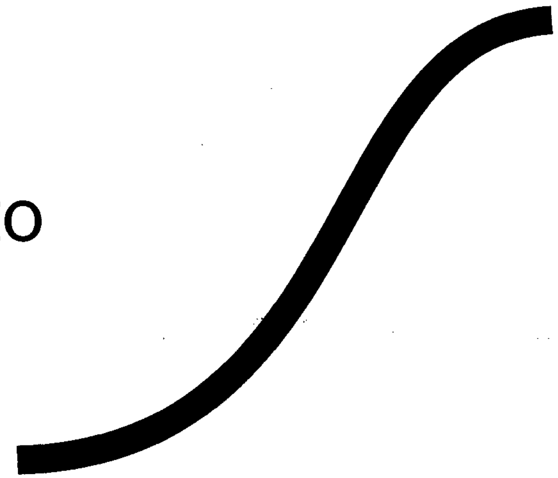


Proof: by pure chance, the characterization of codimension 2 contact systems is also a characterization of drifless flat systems (Martin-Rouchon 1994) (adding integrator, endogenous or exogenous or singular dynamic feedbacks are useless here).

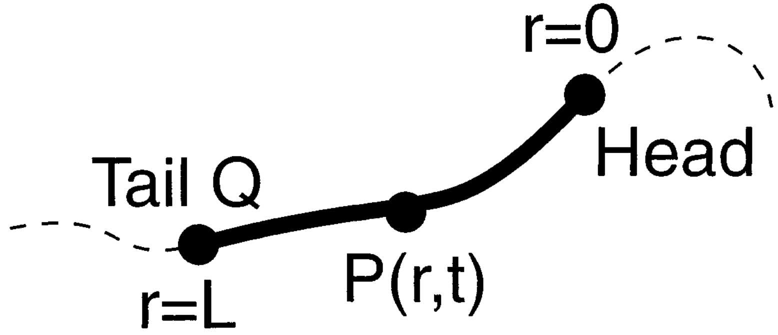
When the number n of trailers becomes large. . .



tends to



The nonholonomic snake: a trivial delay system.



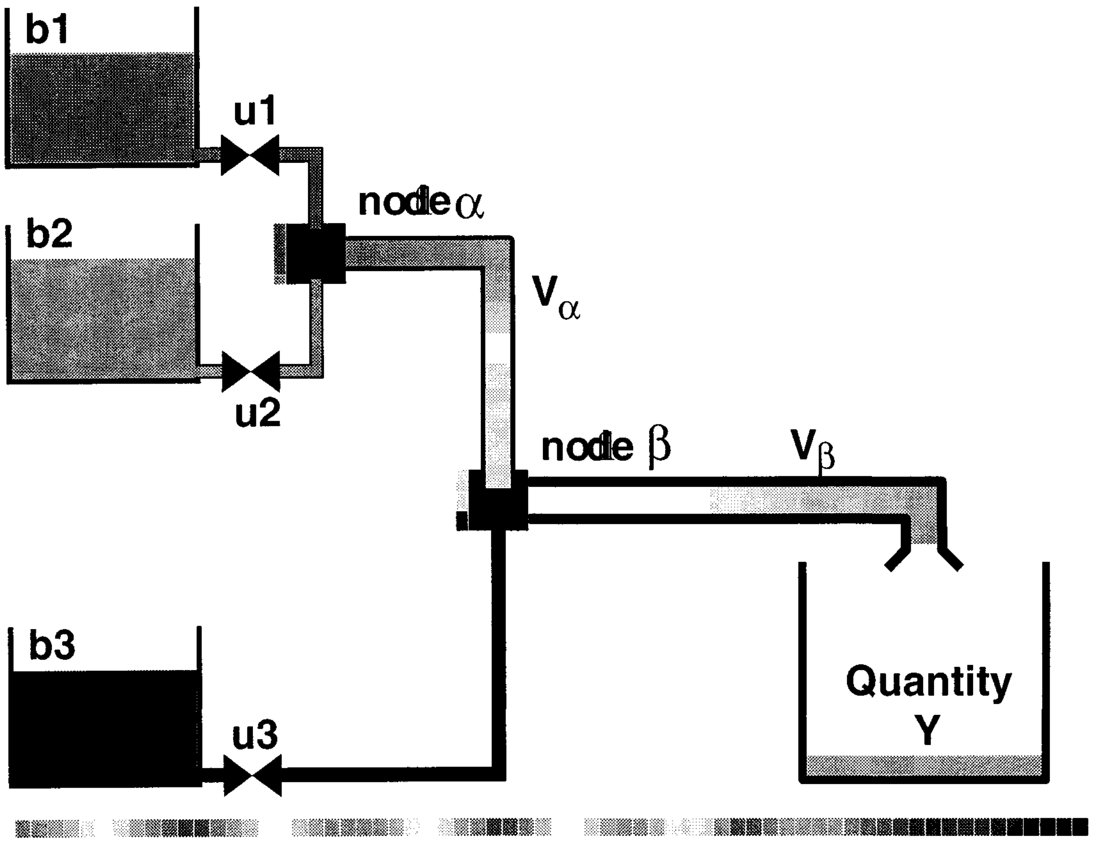
Implicit partial differential nonlinear system:

$$\left\| \frac{\partial P}{\partial r} \right\| = 1, \quad \frac{\partial P}{\partial r} \wedge \frac{\partial P}{\partial t} = 0.$$

General solution via $s \mapsto Q(s)$ arbitrary smooth:

$$P(r,t) = Q(s(t) + L - r) \equiv \sum_{k \geq 0} \frac{(L-r)^k}{k!} \frac{dQ^k}{ds^k}(s(t)).$$

Nonlinear mixing process: three tanks and two nodes.



The product tank quantities (Y_1, Y_2, Y_3) as flat output.

$$u_1(t) = \frac{Y_1' \circ \sigma_\alpha}{(Y_1' + Y_2') \circ \sigma_\alpha} \frac{(Y_1' + Y_2') \circ \sigma_\beta}{(Y_1' + Y_2' + Y_3') \circ \sigma_\beta} (Y_1' + Y_2' + Y_3') \circ \sigma \dot{\sigma}$$

$$u_2(t) = \frac{Y_2' \circ \sigma_\alpha}{(Y_1' + Y_2') \circ \sigma_\alpha} \frac{(Y_1' + Y_2') \circ \sigma_\beta}{(Y_1' + Y_2' + Y_3') \circ \sigma_\beta} (Y_1' + Y_2' + Y_3') \circ \sigma \dot{\sigma}$$

$$u_3(t) = (Y_1' + Y_2' + Y_3') \circ \sigma \dot{\sigma}(t) - u_1 - u_2$$

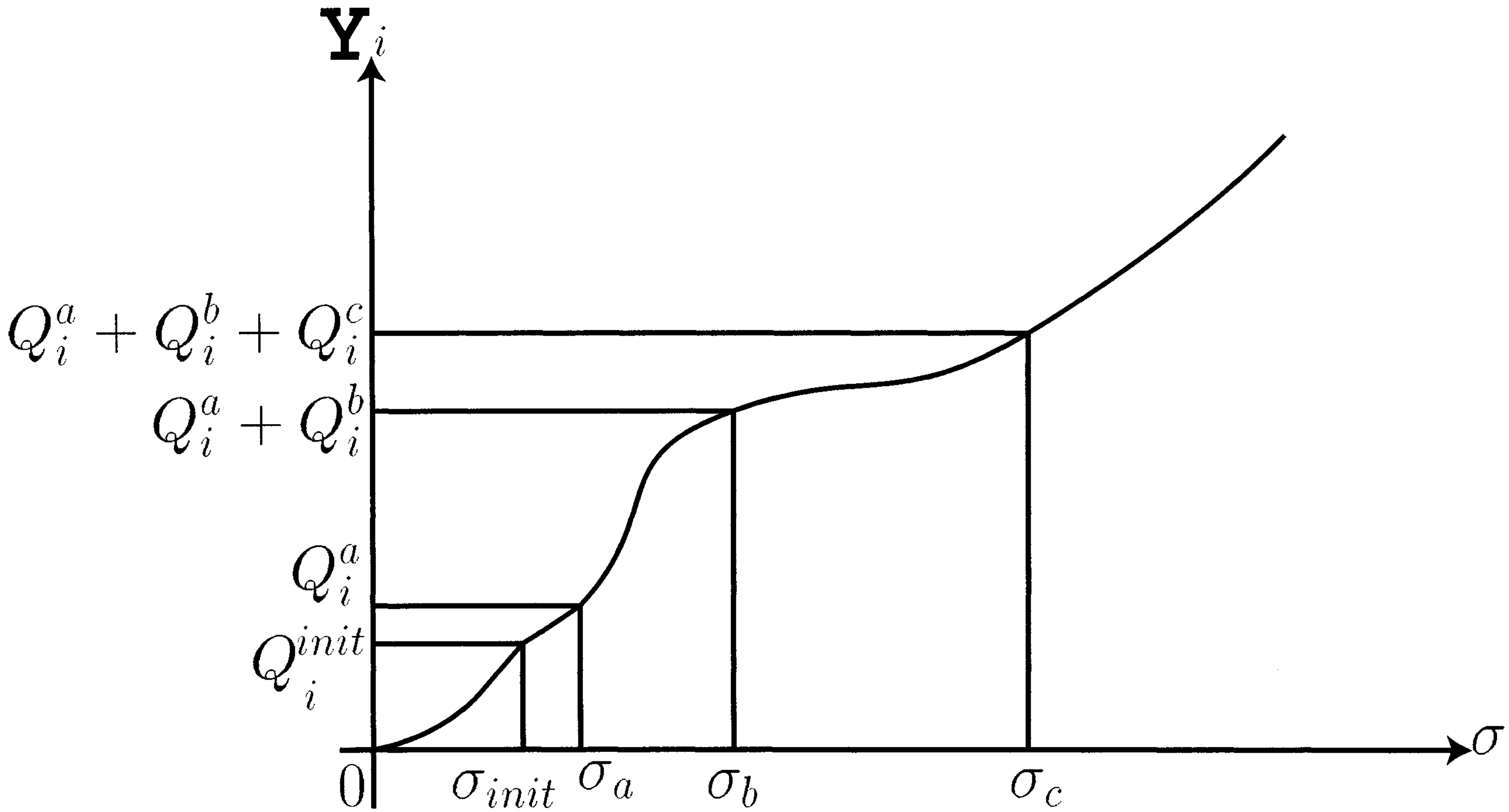
with

$$\sigma_\beta = (Y_1 + Y_2 + Y_3)^{-1} \circ (Y_1 + Y_2 + Y_3 + V_\beta)$$

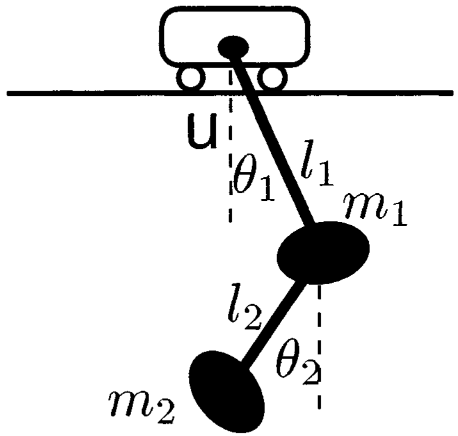
$$\sigma_\alpha = (Y_1 + Y_2)^{-1} \circ (Y_1 + Y_2 + V_\alpha) \circ \sigma_\beta$$

$' = \frac{d}{d\sigma}$, σ : arbitrary time function

Batch Scheduling



Two linearized pendulum in series

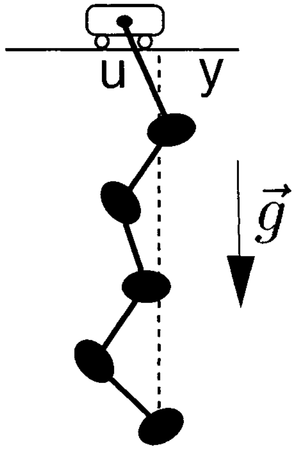


Flat output $y = u + l_1\theta_1 + l_2\theta_2$:

$$\theta_2 = -\frac{\ddot{y}}{g}, \quad \theta_1 = -\frac{m_1\overbrace{(y - l_2\theta_2)}^{\cdot\cdot}}{(m_1 + m_2)g} + \frac{m_2}{m_1 + m_2}\theta_2$$

and $u = y - l_1\theta_1 - l_2\theta_2$ is a linear combination of $(y, y^{(2)}, y^{(4)})$.

n **pendulum in series**

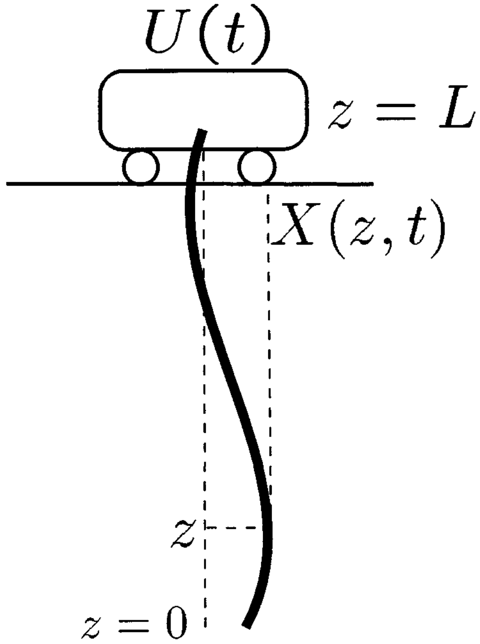


Flat output $y = u + l_1\theta_1 + \ldots + l_n\theta_n$:

$$u = y + a_1 y^{(2)} + a_2 y^{(4)} + \ldots + a_n y^{(2n)}.$$

When n tends to ∞ the system tends to a partial differential equation.

The heavy chain



$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

$$X(L,t) = U(t)$$

Flat output $y(t) = X(0,t)$ with

$$U(t) = \frac{1}{2\pi} \int_0^{2\pi} y\left(t - 2\sqrt{L/g}\ \sin\zeta\right)\, d\zeta$$

With the same flat output, for a discrete approximation (n pendulums in series, n large) we have

$$u(t) = y(t) + a_1 \ddot{y}(t) + a_2 y^{(4)}(t) + \ldots + a_n y^{(2n)}(t),$$

for a continuous approximation (the heavy chain) we have

$$U(t) = \frac{1}{2\pi} \quad \int_0^{2\pi} y\left(t + 2\sqrt{L/g}\ \sin\zeta\right)\ d\zeta.$$

Why? Because formally

$$y(t + 2\sqrt{L/g}\ \sin\zeta) = y(t) + \ldots + \frac{\left(2\sqrt{L/g}\ \sin\zeta\right)^n}{n!}\ y^{(n)}(t) + \ldots$$

But integral formula is preferable (divergence of the series. . .).

The general solution of the PDE

$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

is

$$X(z,t) = \frac{1}{2\pi} \int_0^{2\pi} y\left(t - 2\sqrt{z/g}\ \sin\zeta\right)\ d\zeta$$

where $t \mapsto y(t)$ is any time function.

Proof: replace $\frac{d}{dt}$ by s, the Laplace variable, to obtain a singular second order ODE in z with bounded solutions. Symbolic computations and operational calculus on

$$s^2 X = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right).$$

Symbolic computations in the Laplace domain

Thanks to $x = 2\sqrt{\frac{z}{g}}$, we get

$$x\frac{\partial^2 X}{\partial x^2}(x,t) + \frac{\partial X}{\partial x}(x,t) - x\frac{\partial^2 X}{\partial t^2}(x,t) = 0.$$

Use Laplace transform of X with respect to the variable t

$$x\frac{\partial^2 \widehat{X}}{\partial x^2}(x,s) + \frac{\partial \widehat{X}}{\partial x}(x,s) - xs^2\widehat{X}(x,s) = 0.$$

This is a the Bessel equation defining J_0 and Y_0:

$$\widehat{X}(z,s) = A(s)\ J_0(2\imath s\sqrt{z/g}) + B(s)\ Y_0(2\imath s\sqrt{z/g}).$$

Since we are looking for a bounded solution at $z = 0$ we have $B(s) = 0$ and (remember that $J_0(0) = 1$):

$$\widehat{X}(z,s) = J_0(2\imath s\sqrt{z/g})\widehat{X}(0,s).$$

$$\widehat{X}(z,s) = J_0(2\imath s\sqrt{z/g})\widehat{X}(0,s).$$

Using Poisson's integral representation of J_0

$$J_0(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \exp(\imath\zeta \sin\theta)\ d\theta, \qquad \zeta \in \mathbb{C}$$

we have

$$J_0(2\imath s\sqrt{x/g}) = \frac{1}{2\pi}\int_0^{2\pi} \exp(2s\sqrt{x/g}\sin\theta)\ d\theta.$$

In terms of Laplace transforms, this last expression is a combination of delay operators:

$$X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} y(t + 2\sqrt{z/g}\sin\theta)\ d\theta$$

with $y(t) = X(0,t)$.

Explicit parameterization of the heavy chain

The general solution of

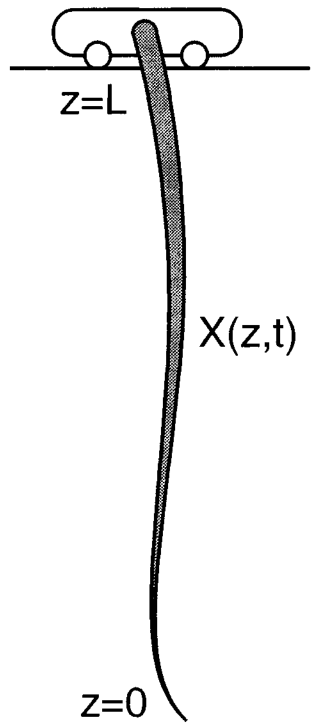
$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right), \qquad U(t) = X(L,t)$$

reads

$$X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} y(t + 2\sqrt{z/g}\sin\theta)\ d\theta$$

There is a one to one correspondence between the (smooth) solutions of the PDE and the (smooth) functions $t \mapsto y(t)$.

Heavy chain with a variable section



$$\begin{cases} \dfrac{\tau'(z)}{g} \quad \dfrac{\partial^2 X}{\partial t^2} = \dfrac{\partial}{\partial z}\left(\tau(z)\dfrac{\partial X}{\partial z}\right) \\ \\ X(L,t) = u(t) \end{cases}$$

The general solution of

$$\begin{cases} \dfrac{\tau'(z)}{g} \quad \dfrac{\partial^2 X}{\partial t^2} = \dfrac{\partial}{\partial z}\left(\tau(z)\dfrac{\partial X}{\partial z}\right) \\ \\ X(L,t) = u(t) \end{cases}$$

where $\tau(z) \geq 0$ is the tension in the rope, can be parameterized by an arbitrary time function $y(t)$, the position of the free end of the system $y = X(0,t)$, via delay and advance operators with **compact** support.

Sketch of the proof. Main difficulty: $\tau(0) = 0$. The bounded solution $B(z, s)$ of

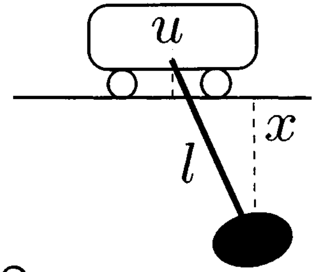
$$\frac{\partial}{\partial z}\left(\tau(z)\frac{\partial X}{\partial z}\right) = \frac{s^2\tau'(z)}{g}\ X$$

is an entire function of s, is of exponential type and

$$\mathbb{R} \ni \omega \mapsto B(z, \imath\omega)$$

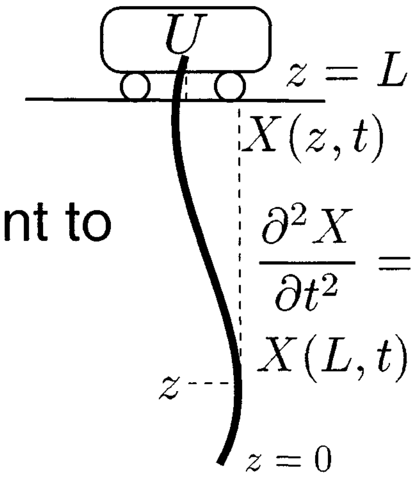
is L^2 modulo some J_0. By the Paley-Wiener theorem $B(z, s)$ can be described via

$$\int_a^b K(z, \zeta)\, \mathrm{exp}(s\zeta)\ d\zeta.$$



$$\frac{d^2x}{dt^2} = \frac{g}{l}(u - x)$$

is equivalent to



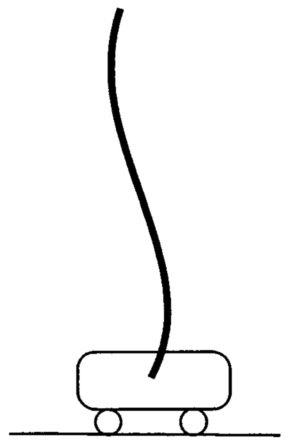
$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

$$X(L,t) = U$$

The following maps exchange the trajectories:

$$\begin{cases} x(t) = X(0,t) \\ u(t) = \frac{\partial^2 X}{\partial t^2}(0,t) \end{cases} \qquad \begin{cases} X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} x\left(t - 2\sqrt{z/g}\sin\zeta\right) d\zeta \\ U(t) = \frac{1}{2\pi}\int_0^{2\pi} x\left(t - 2\sqrt{L/g}\sin\zeta\right) d\zeta \end{cases}$$

The Indian rope.

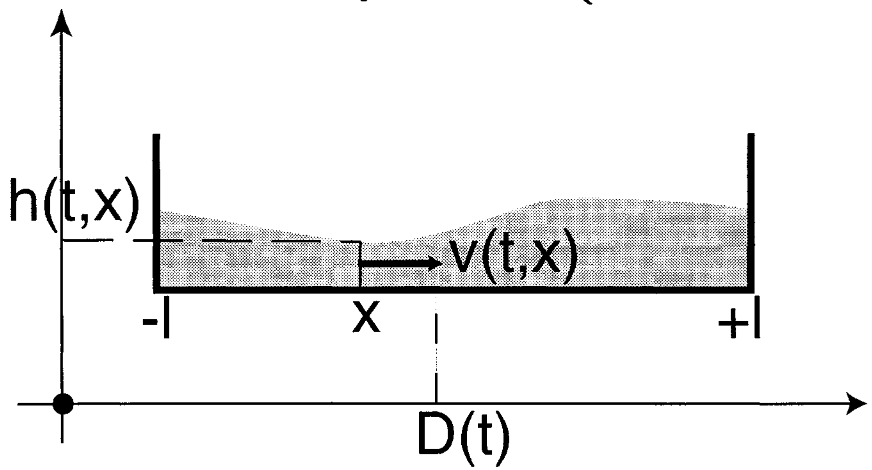


$$
\begin{aligned}
\frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right) + \frac{\partial^2 X}{\partial t^2} &= 0 \\
X(L,t) &= U(t)
\end{aligned}
$$

The equation becomes elliptic and the Cauchy problem is not well posed in the sense of Hadamard. Nevertheless **formulas are still valid with a complex time and y holomorphic**

$$X(z,t) = \frac{1}{2\pi} \int_0^{2\pi} y\left(t - (2\sqrt{z/g}\ \sin\zeta)\ \sqrt{-1}\right)\ d\zeta.$$

1D Tank: Saint-Venant equation (shallow water)

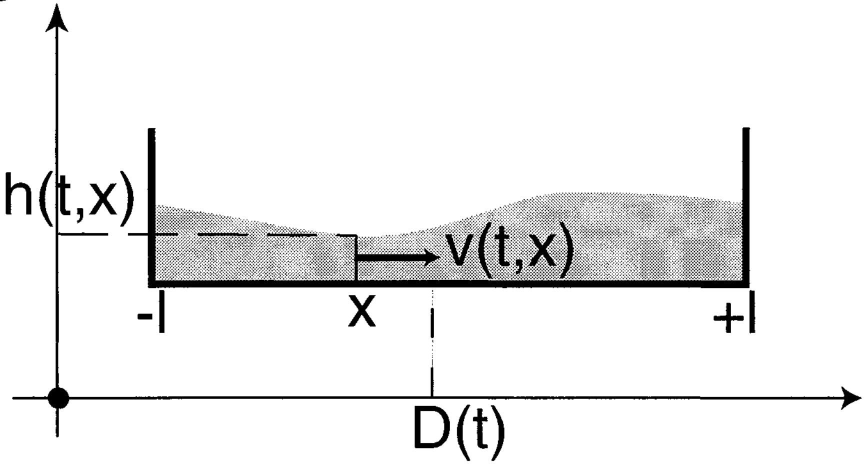


$$h_t + (hv)_x = 0, \qquad v_t + \ddot{D} + v v_x = -g h_x$$

with $v(t,-l) = v(t,l) = 0$.

"Steady-state controllable": Coron 2000.

1D tank: tangent linearization.



Assumptions: $h = \bar{h} + H,\ |H| \ll \bar{h};\ |\ddot{D}| \ll g,\ |v| \ll c = \sqrt{g\bar{h}}.$

$$\frac{\partial^2 H}{\partial t^2} = g\bar{h}\frac{\partial^2 H}{\partial x^2}, \qquad \frac{\partial H}{\partial x}(t,-l) = \frac{\partial H}{\partial x}(t,l) = -\frac{1}{g}\ \ddot{D}(t)$$

Non controllable system

Since $H = \phi(t + x/c) + \psi(t - x/c)$, with ϕ and ψ arbitrary, one gets

$$\begin{cases} \phi'(t+\Delta) - \psi'(t-\Delta) = -c\ddot{D}(t)/g \\ \phi'(t-\Delta) - \psi'(t+\Delta) = -c\ddot{D}(t)/g \end{cases}$$

with $2\Delta = l/c$. Elimination of D yields

$$\phi'(t+\Delta) + \psi'(t+\Delta) = \phi'(t-\Delta) + \psi'(t-\Delta).$$

So the quantity $\pi(t) = \phi(t) + \psi(t)$ satisfies an autonomous equation (torsion element of the underlying module)

$$\pi(t + 2\Delta) = \pi(t).$$

The system is not controllable.

Trajectories passing through a steady-state

Since $\pi(t) = \phi(t) + \psi(t) \equiv 0$ we have

$$\phi'(t+\Delta) + \phi'(t-\Delta) = -c\ddot{D}(t)/g$$

thus

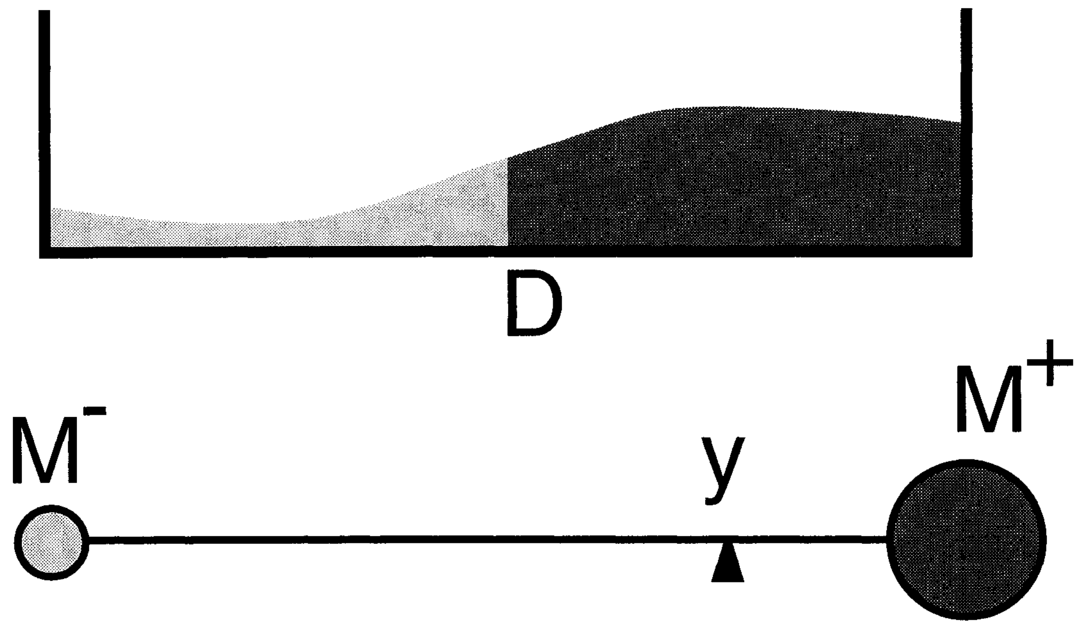
$$\phi(t) = -\left(\frac{c}{2g}\right) y'(t), \qquad D(t) = (y(t+\Delta) + y(t-\Delta))/2$$

and

$$\begin{cases} H(t,x) = \dfrac{1}{2}\sqrt{\dfrac{\bar{h}}{g}}\left[y'(t+x/c) - y'(t-x/c)\right] \\ D(t) = \dfrac{1}{2}\left[y(t+\Delta) + y(t-\Delta)\right] \end{cases}$$

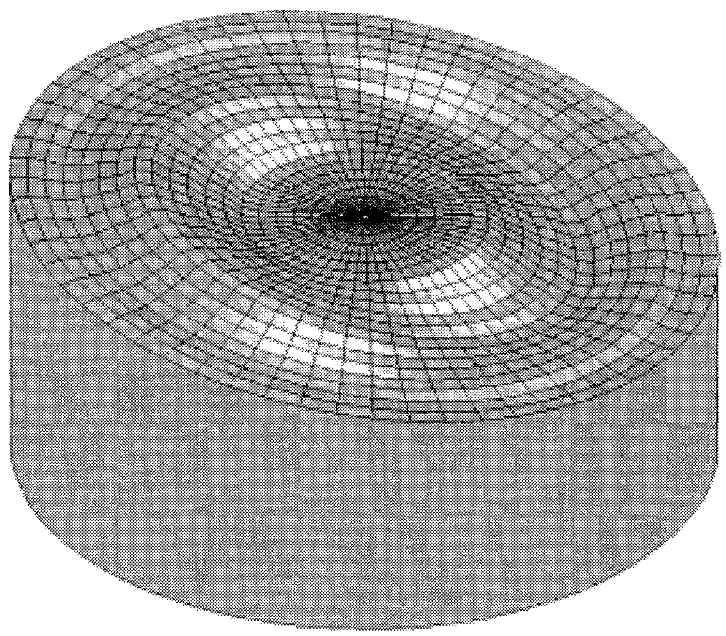
with $t \mapsto y(t)$ an arbitrary time function.

Physical interpretation of y



$$M^- = \int_{-l}^{0} h(t,x)\ dx \qquad M^+ = \int_{0}^{l} h(t,x)\ dx$$

The tumbler in movement: 2D cylindrical tank



Modelling the 2D tank

The liquid occupies a cylinder with vertical edges with the 2D domain Ω as horizontal section. The tangent linear equations are:

$$\frac{\partial^2 H}{\partial t^2} = g\bar{h}\Delta H \quad \text{in } \Omega$$

$$\nabla H \cdot \vec{n} = -\frac{\ddot{D}(t)}{g} \cdot \vec{n} \quad \text{on } \partial\Omega$$

with $D = (D_1, D_2)$, $\vec{n}$ the normal to $\partial\Omega$.

2D Tank, circular shape.

Steady-state motion planning results from a symbolic computations in polar coordinates:

$$
\begin{aligned}
H(t,x_1,x_2) = \frac{1}{\pi}\sqrt{\bar{h}/g}\int_0^{2\pi} \Bigg[\cos\alpha\ y_1'\left(t - \frac{x_1\cos\alpha + x_2\sin\alpha}{c}\right) \\
+ \sin\alpha\ y_2'\left(t - \frac{x_1\cos\alpha + x_2\sin\alpha}{c}\right)\Bigg]\ d\alpha \\
D_1(t) = \frac{1}{\pi}\int_0^{2\pi}\left[\cos^2\alpha\ y_1\left(t - \frac{l\cos\alpha}{c}\right)\right]\ d\alpha \\
D_2(t) = \frac{1}{\pi}\int_0^{2\pi}\left[\sin^2\alpha\ y_2\left(t - \frac{l\sin\alpha}{c}\right)\right]\ d\alpha
\end{aligned}
$$

with $t \mapsto y_1(t)$ and $t \mapsto y_2(t)$ as you want.

Open question

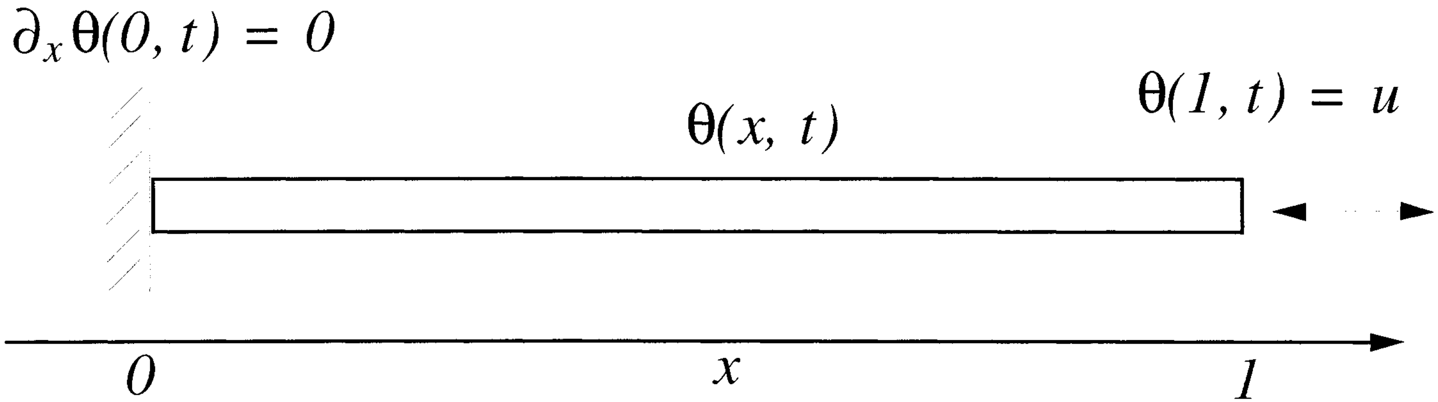
Under which conditions on Ω and is the 2D tank described by

$$\frac{\partial^2 H}{\partial t^2} = g\bar{h}\Delta H \quad \text{in } \Omega$$
$$\nabla H \cdot \vec{n} = -\frac{u}{g} \cdot \vec{n} \quad \text{on } \partial\Omega$$
$$\ddot{D}(t) = u$$

steady-state controllable ?

It is true for Ω a disk or a rectangle.

Heat equation



$$\partial_t \theta(x,t) = \partial_x^2 \theta(x,t), \quad x \in [0,1]$$
$$\partial_x \theta(0,t) = 0 \qquad \theta(1,t) = u(t).$$

Its general solution parameterized via $t \mapsto y(t) \in \mathbb{R}$, C^∞ ($y(t) := \theta(0,t)$)

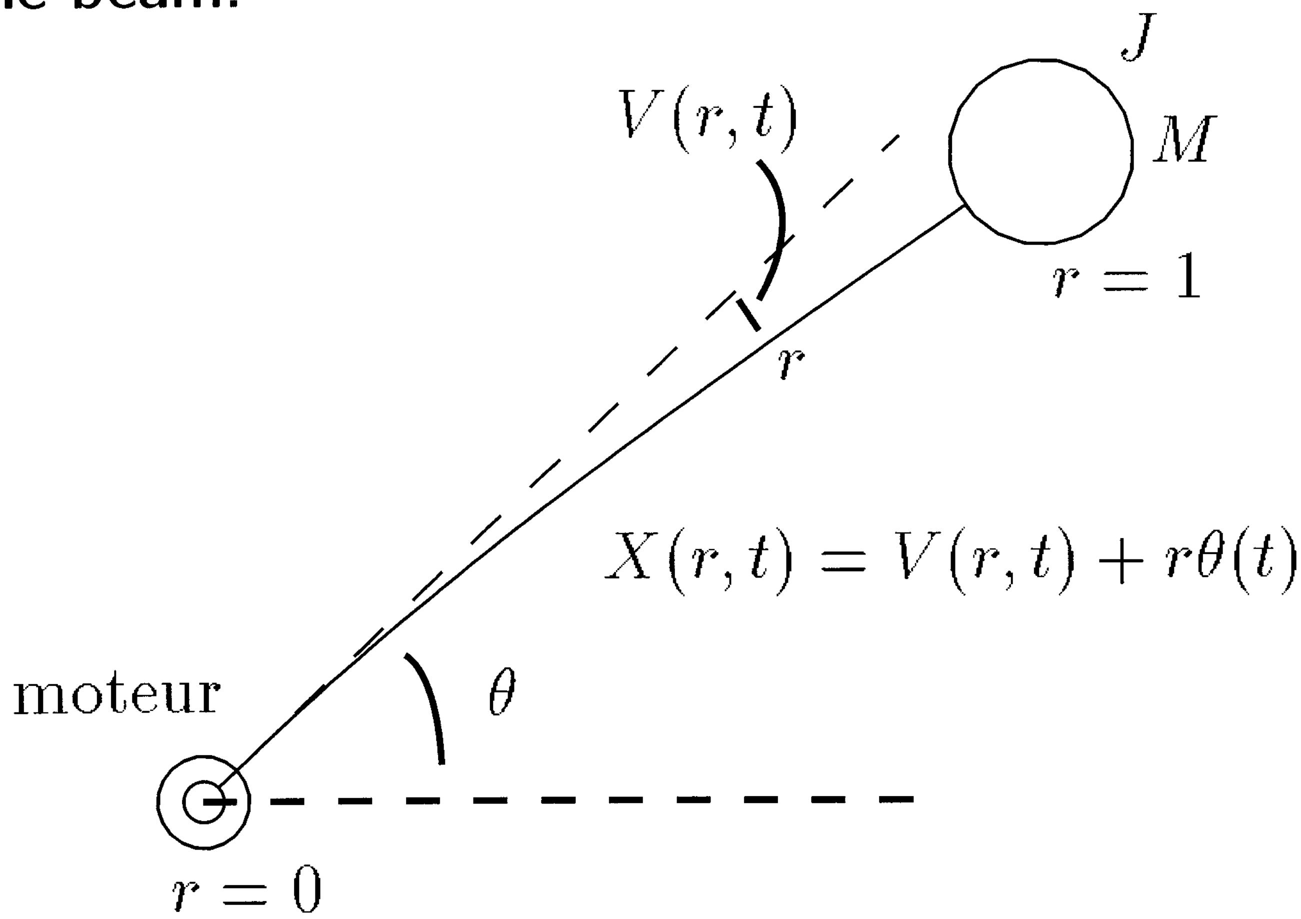
$$\theta(x,t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}\; x^{2i}$$

$$u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}.$$

Convergence when y is of Gevrey order $\sigma < 2$:

$$\exists\; K, M > 0, \quad \forall i \geq 0, \quad |y^{(i)}(t)| \leq M(Ki)^{\sigma i}.$$

Flexible beam.



Dynamics (Euler Bernoulli)

$$\partial_{tt}X = -\partial_{xxxx}X$$

$$X(0,t) = 0, \qquad \partial_x X(0,t) = \theta(t)$$
$$\ddot{\theta}(t) = u(t) + k\partial_{xx}X(0,t)$$

$$\partial_{xx}X(1,t) = -\lambda\partial_{ttx}X(1,t)$$
$$\partial_{xxx}X(1,t) = \mu\partial_{tt}X(1,t)$$

General solution via y C^∞ of Gevrey order ≤ 2:

$$X(x,t) = \sum_{n\geq 0} \frac{(-1)^n \; y^{(2n)}(t)}{(4n)!} P_n(x) + \sum_{n\geq 0} \frac{(-1)^n \; y^{(2n+2)}(t)}{(4n+4)!} Q_n(x)$$

with

$$P_n(x) = \frac{x^{4n+1}}{2(4n+1)} + \frac{(\Im - \Re)(1-x+\imath)^{4n+1}}{2(4n+1)} + \mu\Im(1-x+\imath)^{4n}$$

and

$$Q_n(x) = \frac{\lambda\mu(4n+4)(4n+3)(4n+2)}{2}\left((\Im - \Re)(1-x+\imath)^{4n+1} - x^{4n+1}\right) - \lambda(4n+4)(4n+3)\Re(1-x+\imath)^{4n+2}.$$

Reaction/diffusion systems

For

$$\begin{cases} \dfrac{\partial \theta}{\partial t} = \dfrac{\partial^2 \theta}{\partial x^2} + c(\theta) \\ \dfrac{\partial \theta}{\partial x}(0,t) = 0 \\ \theta(1,t) = u \end{cases}$$

the series can still be calculated. Convergence ?

Conclusion

Adding new quantities, the flat output here, is a powerful idea: constraint optimization and Lagrange multipliers, stabilization and Lyapounov function.

No algorithm to decide whether a system is flat or not: similar to Lyapounov functions or first integrals.

Importance of the physics: implicit description via differential/algebraic systems, symmetries.