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Return method: some applications to flow control

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Dedicated to Jean Lévine for his 50^{th} *birthday*

Abstract

Due to recent progress in advanced technologies in many fields of engineering sciences, applications of flow control are developing very quickly. In this paper we survey only a tiny and theoretical part of the recent results obtained in flow control, namely some results on the controllability and on the stabilizability of the equations of incompressible fluids which have been obtained by means of the return method.

1 Introduction

For finite dimensional system one knows many powerful sufficient conditions for local controllability of a nonlinear control system. This is not the case in infinite dimension, where, roughly speaking, the only known general result is that if the linearized control system at an equilibrium is controllable, then the nonlinear control system is locally controllable at the equilibrium. The return method, that we have introduced in [11] for a stabilisation problem in finite dimension and first used in infinite dimension for the controllability of the Euler equations in [13], allows in some cases to get the local controllability at the equilibrium of the nonlinear control system even if the linearized control system at the equilibrium is not controllable. The idea of the return consists in the following one. If one can find a trajectory of the nonlinear control system such that

- it starts and ends at the equilibrium,
- the linearized control system around this trajectory is controllable,

then, in general, the implicit function theorem allows to conclude that one can go from any state close to the equilibrium to any other state close to the equilibrium.

In this paper, we sketch some results in flow control which has been obtained by this method, namely

- Global controllability results of the Euler equations of incompressible fluids,

- Global controllability results for the Navier-Stokes equations of incompressible fluids,
- Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations
- Null global asymptotic stabilizability by means of explicit boundary feedback laws for the 2-D inviscid incompressible fluids on simply connected domains

2 Return method

In order to explain this method, let us just consider the problem of local controllability of a control system in finite dimension. So we consider the control system

$$\dot{x} = f(x, u),$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control ; we assume that f is of class C^∞ and satisfies

$$f(0,0) = 0.$$

There are various possible definitions of local controllability. Here we use the following one, called the small time local controllability,

Definition 1 *The control system $\dot{x} = f(x,u)$ is small time locally controllable if for every $T > 0$ there exist $\varepsilon > 0$ in $(0, +\infty)$ such that, for every $x_0 \in \mathbb{R}^n$ and $x_1 \in \mathbb{R}^n$ both of norm less than ε, there exists a bounded measurable function $u : [0,T] \to \mathbb{R}^m$ such that, if x is the (maximal) solution of $\dot{x} = f(x, u(t))$ which satisfies $x(0) = x_0$, then $x(T) = x_1$.*

One does not know any interesting necessary and sufficient condition for small time local controllability but there are many useful necessary conditions and sufficient conditions which have been found during the last thirty years. See for example the papers by A. Agrachev [2], R.M. Bianchini and G. Stefani [5, 6], H. Hermes [38], M. Kawski [43], H.J. Sussmann [70, 71], H.J. Sussmann and V. Jurdjevic [72], and A. Tret'yak [73]. Note that all these conditions rely on Lie bracket and that this geometric tool does not seem to give good results for distributed control systems - in this case x is an infinite dimensional space -. On the other hand for *linear* distributed control systems there are powerful methods to prove controllability - e.g. the H.U.M. method due to J.-L.Lions, see [52]. The return method consists in reducing the local controllability of a nonlinear control system to the existence of - suitable - periodic (or "almost periodic" -see below the cases of the Navier-Stokes control system and of the shallow water equations) trajectories and to the controllability of *linear* systems. The idea is the following one: assume that, for every positive real number T, there exists a measurable bounded function $\bar{u} : [0,T] \to \mathbb{R}^m$ such that, if we denote by $\bar{x}$ the (maximal) solution of $\dot{\bar{x}} = f(\bar{x}, \bar{u}(t))$, $\bar{x}(0) = 0$, then

$$\bar{x}(T) = 0, \tag{2.1}$$

$$\text{the linearized control system around } (\bar{x}, \bar{u}) \text{ is controllable on } [0,T]. \tag{2.2}$$

Then it follows easily from the inverse mapping theorem - see e.g. [65], Theorem 7 p. 126 - that $\dot{x} = f(x,u)$ is small time locally controllable. Let us recall that the linearized control system around $(\bar{x}, \bar{u})$ is the time-varying control system

$$\dot{y} = A(t)y + B(t)v, \tag{2.3}$$

where the state is $y \in \mathbb{R}^n$, the control is $v \in \mathbb{R}^m$ and $A(t) = (\partial f/\partial x)(\bar{x}(t), \bar{u}(t))$, $B(t) = (\partial f/\partial u)(\bar{x}(t), \bar{u}(t))$.

For the linear control system (2.3), controllability on $[0,T]$ means, by definition, that for every y_0 and y_1 in $\mathbb{R}^n$, there exists a bounded measurable function $v : [0,T] \to \mathbb{R}^m$ such that if $\dot{y} = A(t)y + B(t)v$ and $y(0) = y_0$, then $y(T) = y_1$. There is a well known Kalman-type sufficient condition for the controllability of (1.5) due to Silverman and Meadows [62] -see also [65, Prop. 3.5.16]-. This is the following one.

Proposition 2 *Assume that for some $\bar{t}$ in $[0,T]$*

$$\text{Span}\left\{\left(\frac{d}{dt} - A(t)\right)^i B(t)_{|t=\bar{t}}\, v;\ v \in \mathbb{R}^m,\ i \geq 0\right\} = \mathbb{R}^n, \tag{2.4}$$

then the linear control system (2.3) is controllable on $[0,T]$. Moreover if A and B are analytic on $[0,T]$ and if the linear control system (2.3) is controllable on $[0,T]$, then (2.4) holds for all $\bar{t}$ in $[0,T]$.

Note that if one takes $\bar{u} \equiv 0$, then the above method just gives the well known fact that if the time-invariant linear system

$$\dot{y} = \frac{\partial f}{\partial x}(0,0)y + \frac{\partial f}{\partial u}(0,0)v$$

is controllable, then the nonlinear control system $\dot{x} = f(x,u)$ is small time locally controllable. But it may happen that (2.2) does not hold for $\bar{u} \equiv 0$, but holds for other choices of $\bar{u}$. Let us give simple examples.

Example 3 We take $n = 2$, $m = 1$ and consider the control system

$$\dot{x}_1 = x_2^3,\ \dot{x}_2 = u.$$

Let us take $\bar{u} \equiv 0$; then $\bar{x} \equiv 0$ and the linearized control system around $(\bar{x}, \bar{u})$ is

$$\dot{y}_1 = 0,\ \dot{y}_2 = v.$$

which is clearly not controllable. Let us now take $\bar{u} \in C^\infty([0,T];\mathbb{R})$ such that

$$\int_0^{T/2} \bar{u}(t)dt = 0,$$
$$\bar{u}(T-t) = \bar{u}(t),\ \forall\, t \in [0,T].$$

Then one easily checks that

$$\bar{x}_2(T/2) = 0,$$
$$\bar{x}_2(T-t) = -\bar{x}_2(t),\ \forall t \in [0,T],$$
$$\bar{x}_1(T-t) = \bar{x}_1(t),\ \forall t \in [0,T].$$

In particular, we have

$$\bar{x}_1(T) = 0\,,\ \bar{x}_2(T) = 0.$$

The linearized control system around $(\bar{x}, \bar{u})$ is

$$\dot{y}_1 = 3\bar{x}_2^2(t) y_2\,,\ \dot{y}_2 = v.$$

Hence

$$A(t) = \begin{pmatrix} 0 & 3\bar{x}_2(t)^2 \\ 0 & 0 \end{pmatrix},\ B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and one easily sees that (2.4) holds if and only if

$$\exists\, i \in \mathbb{N} \text{ such that } \frac{d^i \bar{x}_2}{dt^i}(\bar{t}) \neq 0. \tag{2.5}$$

Note that (2.5) holds for at least a $\bar{t}$ in $[0,T]$ if (and only if) $u \not\equiv 0$. So (2.2) holds if (and only if) $u \not\equiv 0$.

Example 4 *We take $n = 3$, $m = 2$ and the control system is*

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = x_1 u_2 - x_2 u_1. \tag{2.6}$$

Again one can check that the linearized control system around $(\bar{x}, \bar{u})$ is controllable on $[0,T]$ if and only if $\bar{u} \not\equiv 0$. Note that, for the control system (2.6), it is easy to achieve the "return condition" (2.1). Indeed, if

$$\bar{u}(T-t) = -\bar{u}(t),\ \forall\, t \in [0,T],$$

then

$$\bar{x}(T-t) = \bar{x}(t),\ \forall\, t \in [0,T]$$

and, in particular,

$$\bar{x}(T) = \bar{x}(0) = 0.$$

Example 5 Let us now give an example, which has some relations with the 1-D tank studied in section 4, where the return method can be used to get large time local controllability. For this example the control system is

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1 x_2, \tag{2.7}$$

where the state is $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and the state is $u \in \mathbb{R}$. Let us first point out that that this control system is not small time locally controllable. Indeed if $(x, u) : [0, T] \to \mathbb{R}^4 \times \mathbb{R}$ is a trajectory of the control system (2.7) such that $x(0) = 0$ then

$$x_3(T) = \int_0^T x_1^2(t) \cos(T - t)dt, \tag{2.8}$$

$$x_4(T) = x_1^2(T) - \int_0^T x_1^2(t) \sin(T - t)dt \tag{2.9}$$

In particular if $x_1(T) = 0$ and $T \leqslant \pi$ then $x_4(T) \leqslant 0$ with equality if and only if $x \equiv 0$. So, if for $T > 0$ we denote by $\mathcal{P}(T)$ the following controllability property

$\mathcal{P}(T)$ There exists $\varepsilon > 0$ in $(0, +\infty)$ such that, for every $x_0 \in \mathbb{R}^n$ and $x_1 \in \mathbb{R}^n$ both of norm less than ε, there exists a bounded measurable function $u : [0, T] \to \mathbb{R}$ such that, if x is the (maximal) solution of (2.7) which satisfies $x(0) = x_0$, then $x(T) = x_1$,

then, for every $T \in (0, \pi]$, $\mathcal{P}(T)$ is false. Let us show how the return method can be used to prove that

$$\mathcal{P}(T) \text{ holds for every } T \in (\pi, +\infty). \tag{2.10}$$

Let $T > \pi$. Let

$$\eta = \frac{1}{2}\text{Min } (T - \pi, \pi).$$

Let $\bar{x}_1 : [0, T] \to \mathbb{R}$ be a function of class C^∞ such that

$$\bar{x}_1(t) = 0\, \forall t \in [\eta, \pi] \cap [\pi + \eta, T], \tag{2.11}$$

$$\bar{x}_1(t + \pi) = x_1(t)\, \forall t \in [0, \eta]. \tag{2.12}$$

Let $\bar{x}_2 : [0, T] \to \mathbb{R}$ and $\bar{u} : [0, T] \to \mathbb{R}$ be such that

$$\bar{x}_2 = \dot{\bar{x}}_1,\ \bar{u} = \dot{\bar{x}}_2 + \bar{x}_1 + \bar{u},$$

In particular

$$\bar{x}_2(t) = 0\, \forall t \in [\eta, \pi] \cap [\pi + \eta, T], \tag{2.13}$$

$$\bar{x}_2(t + \pi) = x_2(t)\, \forall t \in [0, \eta]. \tag{2.14}$$

Let $\bar{x}_3 : [0, T] \to \mathbb{R}$ and $\bar{x}_4 : [0, T] \to \mathbb{R}$ be defined by

$$\dot{\bar{x}}_3 = \bar{x}_4,\ \dot{\bar{x}}_4 = -\bar{x}_3 + 2\bar{x}_1\bar{x}_2, \tag{2.15}$$

$$\bar{x}_3(0) = 0,\ \bar{x}_4(0) = 0. \tag{2.16}$$

So $(\bar{x}, \bar{u})$ is a trajectory of the control system (2.7). Then, using (2.8), (2.9), (2.11), (2.13), (2.12), (2.14), one sees that

$$\bar{x}(T) = 0.$$

If $\bar{x}_1 \equiv 0$, $(\bar{x}, \bar{u}) \equiv 0$ and the linearized control system around $(\bar{x}, \bar{u})$ is not controllable. But, as one easily checks using the Kalman-type sufficient condition for the controllability of linear time-varying control sytem due to Silverman and Meadows (Proposition 2), if $\bar{x}_1 \not\equiv 0$ then the linearized control system around $(\bar{x}, \bar{u})$ is controllable. This shows (2.10).

One may wonder if the local controllability of $\dot{x} = f(x,u)$ implies the existence of u in $C^\infty([0,T];\mathbb{R}^m)$ such that (2.1) and (2.2) hold. It has been proved to be true by Sontag in [64]. Let us also remark that the above examples suggest that for many choices of $\bar{u}$ then (2.2) holds. This in fact holds in general. More precisely let us assume that

$$\left\{h(0); h \in \text{Lie}\left\{\frac{\partial f}{\partial u^\alpha}(\cdot,0),\ \alpha \in \mathbb{N}^m\right\}\right\} = \mathbb{R}^n, \tag{2.17}$$

where Lie $\mathcal{F}$ denotes the Lie algebra generated by the vector fields in $\mathcal{F}$; then for generic u in $C^\infty([0,T];\mathbb{R}^m)$ (2.2) holds; this is proved in [12], and in [67] if f is analytic. Let us recall that by a theorem due to Sussmann and Jurdjevic [72], (2.17) is a necessary condition for local controllability if f is analytic.

The return method does not seem to give any new interesting controllability result if x lies in a finite dimensional space ; in particular

- The small time local controllability in Example 3 follows from the Hermes condition [38, 71],
- The small time local controllability in Example 4 follows from Rashevski-Chow's theorem [59, 9],
- The large time local controllability (more precisely (2.10)) follows from a general result obtained by R. Bianchini about unilateral variations in [7] (one considers the trajectory $(x,u) \equiv (0,0)$).

But it gives some new results for the controllability of distributed control system as we are now to show in the following sections.

3 Controllability of the Euler and Navier-Stokes equations

Let us introduce some notations. Let $l \in \{2,3\}$ and let Ω be a bounded nonempty connected open subset of $\mathbb{R}^l$ of class C^∞. Let Γ_0 be an open subset of $\Gamma := \partial\Omega$ and let Ω_0 be an open subset of Ω. We assume that

$$\Gamma_0 \cup \Omega_0 \neq \emptyset. \tag{3.1}$$

The set Γ_0 is the part of the boundary and Ω_0 is the part of the domain Ω on which the control acts. The fluid that we consider is incompressible so that the velocity field y satisfies

$$\text{div } y = 0.$$

On the part of the boundary $\Gamma\backslash\Gamma_0$ where there is no control the fluid does not cross the boundary: it satisfies

$$y.n = 0 \text{ on } \Gamma\backslash\Gamma_0, \tag{3.2}$$

where n denotes the outward unit normal vector field on Γ. When the fluid is viscous it satifies on $\Gamma\backslash\Gamma_0$, besides (3.2), some extra conditions which will be specified later on. For the moment being, let us just call by **BC** all the boundary conditions (*including* (3.2)) satisfied by the fluid on $\Gamma\backslash\Gamma_0$.

Let us introduce the following definition.

Definition 6 *A trajectory of the Navier-Stokes control system (resp. Euler control system) on the interval of time* $[0,T]$ *is an application* $y : \overline{\Omega} \times [0,T] \to \mathbb{R}^l$ *of class* C^∞ *such that, for some function* $p : \overline{\Omega} \times [0,T] \to \mathbb{R}$ *of class* C^∞,

$$\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y + \nabla p = 0 \textit{ in } (\overline{\Omega}\backslash\Omega_0) \times [0,T], \tag{3.3}$$

$$(\textit{resp. } \frac{\partial y}{\partial t} + (y\cdot\nabla)y + \nabla p = 0 \textit{ in } (\overline{\Omega}\backslash\Omega_0) \times [0,T]) \tag{3.4}$$

$$\textit{div } y = 0 \textit{ in } \overline{\Omega} \times [0,T], \tag{3.5}$$

$$y(\cdot,t) \textit{ satisfies the boundary conditions BC on } \Gamma\backslash\Gamma_0, \forall t \in [0,T]. \tag{3.6}$$

The real number $\nu > 0$ appearing in (3.3) is the viscosity. J.–L. Lions' problem of controllability is the following one: let $T > 0$, let y_0 and y_1 in $C^\infty(\overline{\Omega};\mathbb{R}^l)$ be such that

$$\text{div } y_0 = 0 \text{ in } \overline{\Omega}, \tag{3.7}$$

$$\text{div } y_1 = 0 \text{ in } \overline{\Omega}, \tag{3.8}$$

$$y_0 \text{ satisfies the boundary conditions BC on } \Gamma\backslash\Gamma_0, \tag{3.9}$$

$$y_1 \text{ satisfies the boundary conditions BC on } \Gamma\backslash\Gamma_0, \tag{3.10}$$

does there exist a trajectory y of the Navier-Stokes or the Euler control system such that

$$y(\cdot,0) = y_0 \text{ in } \overline{\Omega}, \tag{3.11}$$

and, for an appropriate topology –see [53, 54]–,

$$y(\cdot,T) \text{ is ``close'' to } y_1 \text{ in } \overline{\Omega}? \tag{3.12}$$

That is to say, starting with the initial data y_0 for the velocity field, we ask whether there are trajectories of the control system considered (Navier-Stokes if $\nu > 0$, Euler if $\nu = 0$) which, at a fixed time T, are arbitrarly close to the given velocity field y_1. If this problem has always a solution one says that the control system considered is approximately controllable.

Note that (3.3), (3.5), (3.6) and (3.11) have many solutions. In order to have uniqueness one needs to add extra conditions. These extra conditions are the controls.

In the case of the Euler control system one can even require instead of (3.12) the stronger condition

$$y(\cdot, T) = y_1 \text{ in } \overline{\Omega}. \tag{3.13}$$

If y still exists with this stronger condition, one says that the Euler control system is exactly controllable. Of course, due to the smoothing of the Navier-Stokes equations, one cannot expect to have (3.13) instead of (3.12) for general y_1. We will see in subsection 3.2 a way to replace (3.13) in order to recover a natural definition of (exact) controllability of the Navier-Stokes condition.

This section is organized as follows

- In subsection 3.1 we consider the case of the Euler control system,
- In subsection 3.2 we consider the case of the Navier-Stokes control system.

3.1 Controllability of the Euler equations

In this section the boundary conditions BC in (3.6), (3.9), and (3.10) are respectively

$$y(x,t).n(x) = 0, \ \forall (x,t) \in (\Gamma \backslash \Gamma_0) \times [0,T], \tag{3.14}$$

$$y_0(x).n(x) = 0, \quad \forall x \in \Gamma \backslash \Gamma_0, \tag{3.15}$$

$$y_1(x).n(x) = 0, \quad \forall x \in \Gamma \backslash \Gamma_0. \tag{3.16}$$

For simplicity we assume that

$$\Omega_0 = \emptyset,$$

i.e. we study the case of boundary control (see [14] when $\Omega_0 \neq \emptyset$ and $l = 2$). In that case a control is given by $y.n$ on Γ_0 with $\int_{\Gamma_0} y.n = 0$ and by curl y if $l = 2$ and (curl y)$.n$ if $l = 3$ at the points of $\Gamma_0 \times [0,T]$ where $y.n < 0$: these boundary conditions, (3.14), and the initial condition (3.11) imply the uniqueness of the solution to the Euler equations (3.4) -up to an arbitrary function of t which may be added to p-; see also [44] for the existence of solution.

Let us first point out that in order to have (exact) controllability one needs that

$$\Gamma_0 \text{ intersects every connected component of } \Gamma. \tag{3.17}$$

Indeed, let $\mathcal{C}$ be a connected component of Γ which does not intersect Γ_0 and assume that, for some smooth Jordan curve γ_0 on $\mathcal{C}$ (if $l = 2$ one takes $\gamma_0 = \mathcal{C}$),

$$\int_{\gamma_0} y_0.ds \neq 0, \tag{3.18}$$

but that

$$y_1(x) = 0, \forall x \in \mathcal{C}. \tag{3.19}$$

Then there is no solution to our problem, that is there is no $y \in \mathcal{C}^\infty(\overline{\Omega} \times [0,T]; \mathbb{R}^2)$ and $p \in \mathcal{C}^\infty(\overline{\Omega} \times [0,T]; \mathbb{R})$ such that (3.5), (3.4), (3.11), (3.13), and (3.14) hold. Indeed, if such a solution (y,p) exists, then, by Kelvin's law,

$$\int_{\gamma(t)} y(\cdot,t).ds = \int_{\gamma_0} y_0.ds (\in \mathbb{R}), \tag{3.20}$$

where $\gamma(t)$ is the Jordan curve obtained, at time t, from the points of the fluids which at time 0 where on γ_0; in other words $\gamma(t)$ is the image of γ_0 by the flow map associated to the time-varying vector field y. But (3.13), (3.18), (3.19) and (3.20) are in contradiction.

Conversely, if (3.17) holds, then the Euler control system is exactly controllable:

Theorem 7 *Assume that Γ_0 intersects every connected component of $\partial\Omega$. Then the Euler control system is exactly controllable.*

Theorem 7 has been proved in

- [13] when Ω is simply-connected and $l = 2$,
- [14] when Ω is multi-connected and $l = 2$,
- [34] when Ω is contractible and $l = 3$,
- [35] when Ω is not contractible and $l = 3$.

The strategy of the proof of Theorem 7 relies on the "return method" Applied to the controllability of the Euler control system the return method consists in looking for $(\bar{y}, \bar{p})$ such that (3.5), (3.4), (3.11), (3.13) hold, with $y = \bar{y}, p = \bar{p}$, $y_0 = y_1 = 0$ and such that the linearized control system around around the trajectory $\bar{y}$ is controllable under the assumptions of Theorem 7. With such a $(\bar{y}, \bar{p})$ one may hope that there exists (y,p) -close to $(\bar{y}, \bar{p})$- satisfying the required conditions, at least if y_0 and y_1 are "small". Finally, by using some scaling argument, one can deduce from the existence of (y,p) when y_0 and y_1 are "small" the existence of (y,p) even if y_0 and y_1 are not "small".

Let us emphasize that one cannot take $(\bar{y}, \bar{p}) = (0,0)$. Indeed, with such a choice of $(\bar{y}, \bar{p})$, (3.5), (3.4), (3.11), (3.13) hold, with $y = \bar{y}, p = \bar{p}$, $y_0 = y_1 = 0$, but the linearized control system around $\bar{y} = 0$ is not at all controllable. Indeed the linearized control system around $\bar{y} = 0$ is

$$\text{div } z = 0 \text{ in } \overline{\Omega} \times [0,T], \tag{3.21}$$

$$\frac{\partial z}{\partial t} + \nabla \pi = 0 \text{ in } \overline{\Omega} \times [0,T], \tag{3.22}$$

$$z(x,t).n(x) = 0, \forall (x,t) \in (\Gamma \backslash \Gamma_0) \times [0,T].$$

Taking the curl of (3.22), one gets

$$\frac{\partial \text{curl } z}{\partial t} = 0,$$

which clearly shows that the linearized control system is not controllable. So one needs to consider other $(\bar{y}, \bar{p})$. Let us briefly explain how one constructs "good" $(\bar{y}, \bar{p})$ when $l = 2$ and Ω is simply connected. In such a case one easily checks the existence of a harmonic function θ in $C^\infty(\overline{\Omega})$ such that

$$\nabla\theta(x) \neq 0, \forall x \in \overline{\Omega},$$

$$\frac{\partial\theta}{\partial n} = 0 \text{ on } \Gamma \backslash \Gamma_0.$$

Let $\alpha \in C^\infty(0, T)$ vanishing 0 and T. Let

$$(\bar{y}, \bar{p})(x, t) = (\alpha(t)\nabla\theta(x), -\alpha'(t)\theta(x) - \frac{1}{2}\alpha^2(t) \mid \nabla\theta(x) \mid^2).$$

Then (3.5), (3.4), (3.11), (3.13) hold, with $y = \bar{y}, p = \bar{p}$, $y_0 = y_1 = 0$. Moreover, using arguments relying on an extension method analogous to the one introduced by D.L. Russell in [60], one can see that the linearized control system around $\bar{y}$ is controllable.

When Γ_0 does not intersect all the connected components of Γ_0 one can get, if $l = 2$, approximate controllability and even exact controllability outside every arbitrarily small neighborhood of the union Γ^* of the connected components of Γ which does not intersect Γ_0. More precisely, one has

Theorem 8 *[14]. Assume that $l = 2$. There exists a constant c_0 depending only on Ω such that, for every Γ_0 as above, every $T > 0$, every $\varepsilon > 0$, and every y_0, y_1 in $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying (3.7), (3.8), (3.15) and (3.16), there exists a trajectory y of the Euler control system on $[0, T]$ satisfying (3.11) such that*

$$y(x, T) = y_1(x), \ \forall x \in \overline{\Omega} \text{ such that } dist(x, \Gamma^*) \geqslant \varepsilon, \tag{3.23}$$

$$|y(\cdot, T)|_{L^\infty} \leqslant c_0(|y_0|_{L^2} + |y_1|_{L^2} + |curl y_0|_{L^\infty} + |curl y_1|_{L^\infty}). \tag{3.24}$$

In (3.23), $\text{dist}(x, \Gamma^*)$ denotes the distance of x to Γ^*, i.e.

$$\text{dist } (x, \Gamma^*) = \text{ Min } \{|x - x^*|; x^* \in \Gamma^*\}. \tag{3.25}$$

We use the convention dist $(x, \emptyset) = +\infty$ and so Theorem 8 implies Theorem 7. In (3.24) $| \ |_{L^r}$, for $r \in [1, +\infty]$, denotes the L^r–norm on Ω. Let us point out that, y_0, y_1, and T as in Theorem 8 being given, it follows from (3.23) and (3.24) that, for every r in $[1, +\infty)$,

$$\lim_{\varepsilon \to 0^+} |y(0, T) - y_1|_{L^r} = 0; \tag{3.26}$$

that is Theorem 8 implies approximate controllability in the L^r-space for every r in $[1, +\infty)$. Let us notice that, if $\Gamma^* \neq \emptyset$, then, again by Kelvin's law, approximate controllability for the L^∞-norm does not hold. More precisely let us consider the case $l = 2$ and let us denote by

$\Gamma_1^*, \ldots, \Gamma_k^*$ the connected components of Γ which does not meet Γ_0. Let y_0, y_1 in $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying (3.7), (3.8), (3.15) and (3.16). Assume that for some $i \in \{1, \ldots, k\}$

$$\int_{\Gamma_i^*} y_0.ds \neq \int_{\Gamma_i^*} y_1.ds$$

Then for $\varepsilon > 0$ small enough there is no trajectory y of the Euler control system on $[0, T]$ satisfying (3.11) such that

$$|y(\cdot, T) - y_1|_{L^\infty} \leq \varepsilon \tag{3.27}$$

One may wonder if, on the contrary one assumes that

$$\int_{\Gamma_i^*} y_0.ds = \int_{\Gamma_i^*} y_1.ds, \ \forall i \in \{1, \ldots, k\}. \tag{3.28}$$

Then O. Glass has proved that one has approximate controllability in L^∞ and even in the Sobolev spaces $W^{1,p}$ for every $p \in [1, +\infty)$. Indeed he has proved in [36]

Theorem 9 *Assume that $l = 2$. For every $T > 0$, and every y_0, y_1 in $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying (3.7), (3.8), (3.15), (3.16) and (3.28), there exists a sequence $(y^k)_{k \in \mathbb{N}}$ of trajectories of the Euler control system on $[0, T]$ satisfying (3.11) such that*

$$y^k(x, T) = y_1(x), \ \forall x \in \overline{\Omega} \text{ such that } dist(x, \Gamma^*) \geqslant 1/k, \ \forall k \in \mathbb{N}, \tag{3.29}$$

$$y^k(\cdot, T) \to y_1 \text{ in } W^{1,p}(\Omega) \text{ as } k \to +\infty, \ \forall p \in [1, +\infty). \tag{3.30}$$

Again the convergence in (3.30) is optimal: since the vorticity curl y is conserved along the trajectories of the vector field y one cannot have the convergence in $W^{1,\infty}$. In order to have convergence in $W^{1,\infty}$ one needs to add a relation between curl y_0 and curl y_1 on the Γ_i for $i \in \{1, \ldots, l\}$. In this direction O. Glass has proved in [36]

Theorem 10 *Assume that $l = 2$. Let $T > 0$, and let y_0, y_1 in $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ be such that (3.7), (3.8), (3.15), (3.16) and (3.28) hold. Assume that, for every $i \in \{1, \ldots, l\}$, there exits a diffeomorphism D_i of Γ_i^* preserving the orientation such that*

$$curl\ y_1 = (curl\ y_0) \circ D_i.$$

Then there exists a sequence (y^n) of trajectories of the Euler control system on $[0, T]$ satisfying (3.11), (3.29) and

$$y^k(\cdot, T) \to y_1 \text{ in } W^{2,p}(\Omega) \text{ as } k \to +\infty, \ \forall p \in [1, +\infty). \tag{3.31}$$

Again, one cannot expect a convergence in $W^{2,\infty}$ without an extra assumption on y_0 and y_1 -see [36]-.

3.2 Controllability of the Navier-Stokes equations

In this section $\nu > 0$. We now need to specify the boundary conditions BC. Three types of conditions are considered

- Stokes boundary condition,
- Navier boundary condition,
- curl condition.

The Stokes boundary condition is the well known no-slip boundary condition

$$y = 0 \text{ on } \Gamma \backslash \Gamma_0, \tag{3.32}$$

which of course implies (3.2).

The Navier boundary condition [57] imposes, condition (3.2), which is always assumed, and

$$\overline{\sigma} y.\tau + (1-\overline{\sigma}) n^i \left(\frac{\partial y^i}{\partial x^j} + \frac{\partial y^j}{\partial x^i} \right) \tau^j = 0 \text{ on } \Gamma \backslash \Gamma_0, \tag{3.33}$$

where $\overline{\sigma}$ is a constant in $[0,1)$, $n = (n^1, \dots, n^l)$ and $\tau = (\tau^1, \dots, \tau^l)$ is any tangent vector field on the boundary Γ. In (3.33) we also have used the usual summation convention. Note that the Stokes boundary condition (3.32) corresponds to the case $\overline{\sigma} = 1$, which we will not include in the Navier boundary condition considered here. The boundary condition (3.33) with $\overline{\sigma} = 0$ corresponds to the case where there the fluid slips on the wall without friction. It is the appropriate physical model for some flow problems; see [33] for example. The case $\overline{\sigma} \in (0,1)$ corresponds to a case where there the fluid slips on the wall with friction; it is also used in models of turbulence with rough walls; see, e.g., [49]. Note that in [10] F. Coron has derived rigorously the Navier boundary condition (3.33) from the boundary condition at the kinetic level (Boltzmann equation) for compressible fluids. Let us also recall that C. Bardos, F. Golse, and D. Levermore have derived in [4] the incompressible Navier-Stokes equations from a Boltzmann equation.

Let us point out that, using (3.2), one sees that, if $l = 2$ and if τ is the unit tangent vector field on $\partial\Omega$ such that (τ, n) is a direct basis of $\mathbb{R}^2$, (3.33) is equivalent to

$$\sigma y.\tau + \text{curl } y = 0 \text{ on } \Gamma \backslash \Gamma_0$$

with $\sigma \in C^\infty(\Gamma; \mathbb{R})$ defined by

$$\sigma(x) = \frac{2(1-\overline{\sigma})\kappa(x) - \overline{\sigma}}{1-\overline{\sigma}}, \forall x \in \Gamma, \tag{3.34}$$

where κ is the curvature of Γ defined through the relation $\frac{\partial n}{\partial \tau} = \kappa \tau$. In fact we will not use this particular character of (3.34) in our considerations; Theorem 14 below holds for every $\sigma \in C^\infty(\Gamma; \mathbb{R})$.

Finally the curl condition is considered in dimension 2 ($l = 2$). This condition is, condition (3.2) which is always assumed, and

$$\text{curl } y = 0 \text{ on } \Gamma \backslash \Gamma_0. \tag{3.35}$$

It corresponds to the case $\sigma = 0$ in (3.34).

As mentionned in the introduction, due to smoothing property of the Navier-Stokes equation, one cannot expect to get (3.13), at least for general y_1. For these equations, the good notion for exact controllability is not passing from a given state (y_0) to another given state (y_1). As proposed by A. Fursikov and O. Yu Imanuvilov in [28, 29], the good definition for exact controllability is passing from a given state (y_0) to a given *trajectory* ($\hat{y}_1$). This leads to the following, still open, problem of exact controllability of the Navier-Stokes equation with the Stokes, or Navier, or curl condition.

Open Problem 11 *Let $T > 0$. Let $\hat{y}_1$ be a trajectory of the Navier-Stokes control system on $[0, T]$. Let $y_0 \in C^\infty(\overline{\Omega}; \mathbb{R}^l)$ satisfying (3.7) and (3.9). Does there exist a trajectory y of the Navier-Stokes control system on $[0, T]$ such that*

$$y(x, 0) = y_0(x), \ \forall x \in \overline{\Omega}, \tag{3.36}$$

$$y(x, T) = \hat{y}_1(x, T), \ \forall x \in \overline{\Omega}? \tag{3.37}$$

Let us point out that the (global) approximate controllability of the Navier-Stokes control system is also an open problem. Related to the open problem 11 one knows two types of results

- local results,
- global results,

which we briefly describe in the next subsections

3.2.1 Local results

These results do not rely on the return method, but on the HUM and difficult Carleman's ineaqualities. Let us introduce the following definition.

Definition 12 *The Navier-Stokes control system is locally (for the Sobolev H^1 − norm) exactly controllable along the trajectory $\hat{y}_1$ on $[0, T]$ of the Navier-Stokes control system if there exists $\epsilon > 0$ such that, for every $y_0 \in C^\infty(\overline{\Omega}; \mathbb{R}^l)$ satisfying (3.7), (3.9) and*

$$| y_0 - \hat{y}_1(\cdot, 0) |_{H^1(\Omega)} < \epsilon,$$

there exists a trajectory y of the Navier-Stokes control system on $[0, T]$ satisfying (3.36) and (3.37).

Then one has the following results.

Theorem 13 *The Navier-Stokes control system is locally exactly controllable* *(i) along every trajectory for the curl condition or the Navier boundary condition in dimension 2 (l=2),*

(ii) along every trajectory if $\Gamma_0 = \Gamma$,

(iii) along every stationary trajectory with compact support for the Stokes condition.

Case (i) has been obtained by A.V. Fursikov and O. Yu Imanuvilov in [29, 30]. Case (ii) has been obtained by A.V. Fursikov in [27]. Case (iii) has been obtained by O. Yu Imanuvilov in [41] and [42].

3.2.2 Global results

Let $d \in \mathcal{C}^0(\overline{\Omega}; \mathbb{R})$ be defined by

$$d(x) = \text{ dist } (x, \Gamma) = \text{ Min } \{|x - x'|; x' \in \Gamma\}.$$

In [15] the following theorem is proved

Theorem 14 *Let $T > 0$, let y_0 and y_1 in $\mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^2)$ be such that (3.7) and (3.8) hold. Let us also assume that y_0 and y_1 satisfies the Navier boundary condition (3.33). Then there exists a sequence $(y^k; k \in \mathbb{N})$ of trajectories of the Navier-Stokes control system on $[0, T]$ with the Navier boundary condition (3.33) such that, as $k \to +\infty$,*

$$\int_\Omega d^\mu |y^k(\cdot, T) - y_1| \to 0, \ \forall \mu > 0, \tag{3.38}$$

$$|y^k(\cdot, T) - y_1|_{W^{-1,\infty}(\Omega)} \to 0, \tag{3.39}$$

and, for all compact K included in $\Omega \cup \Gamma_0$,

$$|y^k(\cdot, T) - y_1|_{L^\infty(K)} + |curl\ y^k(\cdot, T) - curl\ y_1|_{L^\infty(K)} \to 0. \tag{3.40}$$

In this theorem $W^{-1,\infty}(\Omega)$ denotes the usual Sobolev space of first derivatives of functions in $L^\infty(\Omega)$ and $|\ |_{W^{-1,\infty}(\Omega)}$ one of it's usual norms, for example the norm given in [1, Section 3.10].

As in the proof of the controllability of the 2-D Euler equations of incompressible inviscid fluids (see section 3.1), one uses the return method. Let us recall that it consists in looking for a trajectory of the Navier-Stokes control system $\bar{y}$ such that

$$\bar{y}(\cdot, 0) = \bar{y}(\cdot, T) = 0 \text{ in } \overline{\Omega}, \tag{3.41}$$

and such that the linearized control system around the trajectory $\bar{y}$ has a controllability in a "good" sense. With such a $\bar{y}$ one may hope that there exists y – close to $\bar{y}$ – satisfying the required conditions, at least if y_0 and y_1 are "small". Note that the linearized control system around $\bar{y}$ is

$$\frac{\partial z}{\partial t} - \nu \Delta z + (\bar{y} \cdot \nabla) z + (z \cdot \nabla) \bar{y} + \nabla \pi = 0 \text{ in } (\overline{\Omega} \backslash \Omega_0) \times [0, T], \tag{3.42}$$

$$\mathrm{div} z = 0 \text{ in } \overline{\Omega} \times [0,T], \tag{3.43}$$

$$z.n = 0 \text{ on } (\Gamma \backslash \Gamma_0) \times [0,T], \tag{3.44}$$

$$\sigma z.\tau + \mathrm{curl}\ z = 0 \text{ on } (\Gamma \backslash \Gamma_0) \times [0,T]. \tag{3.45}$$

In [29, 30] A. Fursikov and O. Immanuvilov have proved that this linear control system is controllable (see also [51] for the approximate controllability). Of course it is tempting to consider the case $\bar{y} = 0$. Unfortunately, it is not clear how to deduce from the controllability of the linear system (3.42) with $\bar{y} = 0$, the existence of a trajectory y of the Navier-Stokes control system (with the Navier boundary condition) satisfying (3.11) and (3.12) if y_0 and y_1 are not small. For this reason, one does not use $\bar{y} = 0$, but $\bar{y}$ similar to the one constructed in [14] to prove the controllability of the 2-D Euler equations of incompressible inviscid fluids; these $\bar{y}$ are chosen to be "large" so that, in some sense, "Δ" is small compared to "$(\bar{y} \cdot \nabla) + (\ \cdot \nabla)\bar{y}$".

Remark 15 *In fact with the $\bar{y}$ we use, one does not have (3.41): we have only the weaker property*

$$\bar{y}(\cdot, 0) = 0, \ \bar{y}(\cdot, T) \textit{ is "close" to } 0 \textit{ in } \overline{\Omega}. \tag{3.46}$$

But the controllability of the linearized control system around $\bar{y}$ is strong enough to take care of the fact that $\bar{y}(\cdot, T)$ is not equal to 0 but only close to 0.

Note that (3.38), (3.39), and (3.40) are not strong enough to imply

$$|y^k(\cdot, T) - y_1|_{L^2(\Omega)} \to 0, \tag{3.47}$$

i.e. to get the approximate controllability in L^2 of the Navier-Stokes control system. But, in the special case where $\Gamma_0 = \Gamma$, (3.38), (3.39), and (3.40) are strong enough to imply (3.47). Moreover, gluing together the proofs of Theorem 13 and of (ii) of Theorem 14, one gets

Theorem 16 *[19] The open problem 11 has a positive answer when $\Gamma_0 = \Gamma$ and $l = 2$.*

This result has been recently generalized by A. Fursikov and O. Immanuvilov in [32] to the case $l = 3$ Let us also mention that, in [24], C. Fabre has obtained, in every dimension, an approximate controllability result of two natural "cut off" Navier-Stokes equations. Her proof relies on a general method introduced by E. Zuazua in [74] to prove approximate controllability of semilinear wave equations. This general method is based on H.U.M. (Hilbert Uniqueness Method, due to J.-L. Lions [52]) and on a fixed point technique; see also [25] where C. Fabre, J.-P. Puel and E. Zuazua use this method to prove approximate controllability of semilinear heat equations.

Remark 17 *It is usually accepted that the viscous Burgers equation provides a realistic simplification of the Navier-Stokes system in fluid Mechanics. But J.I. Diaz has proved in [22] that the viscous Burgers equation is not approximately controllable; see also [28]. For the nonviscous Burgers equation, results have been obtained by F. Ancona and A. Marson in [3] and by Th. Horsin in [40].*

4 Local controllability of a 1-D tank containing a fluid modeled by the shallowm water equations

In this section, we consider a 1-D tank containing an inviscid incompressible irrotational fluid. The tank is subject to one-dimensional horizontal moves. We assume that the horizontal acceleration of the tank is small compared to the gravity constant and that the height of the fluid is small compared to the length of the tank. This motivates the use of the shallow water equations to describe the motion of the fluid; see e.g. [21, Sec. 4.2]. Hence the dynamics equations considered are -see [23]-

$$H_t\,(t,x) + (Hv)_x\,(t,x) = 0, \tag{4.1}$$

$$v_t\,(t,x) + \left(gH + \frac{v^2}{2}\right)_x (t,x) = -u\,(t)\,, \tag{4.2}$$

$$v(t,0) = v(t,L) = 0, \tag{4.3}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}\,(t) = u\,(t)\,, \tag{4.4}$$

$$\frac{\mathrm{d}D}{\mathrm{d}t}\,(t) = s\,(t)\,, \tag{4.5}$$

where (see figure 1),

- L is the length of the 1-D tank,
- $H\,(t,x)$ is the height of the fluid at time t and for $x \in [0,L]$,
- $v\,(t,x)$ is the horizontal water velocity of the fluid *in a referential attached to the tank* at time t and for $x \in [0,L]$ (in the shallow water model, all the points on the same vertical have the same horizontal velocity),
- $u\,(t)$ is the horizontal acceleration of the tank in the absolute referential,
- g is the gravity constant,
- s is the horizontal velocity of the tank,
- D is the horizontal displacement of the tank.

This is a control system, denoted Σ, where

- the state is $Y = (H, v, s, D)$,
- the control is $u \in \mathbb{R}$.

Our goal is to study the local controllability of this control system Σ around the equilibrium point

$$(Y_e, u_e) := ((H_e, 0, 0, 0), 0).$$

This problem has been raised by F. Dubois, N. Petit and P. Rouchon in [23].

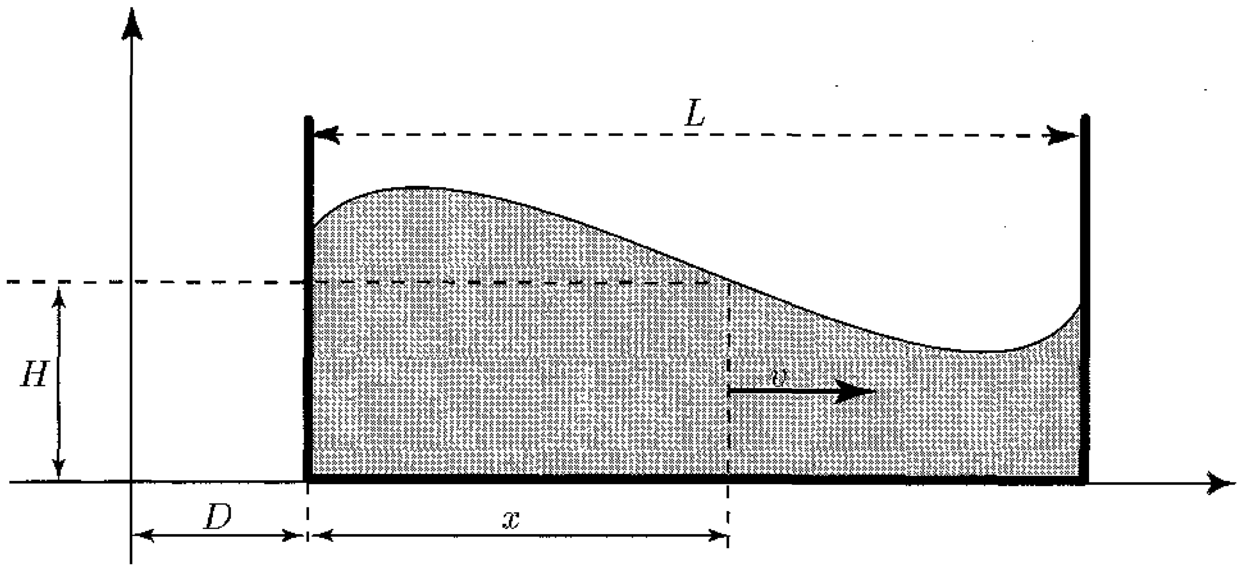

Figure 1: Fluid in the 1-D tank

Of course, the total mass of the fluid is conserved so that, for every solution of (4.1) to (4.3),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H(t,x)\,\mathrm{d}x = 0. \tag{4.6}$$

(One gets (4.6) by integrating (4.1) on $[0, L]$ and by using (4.3) together with an integration by parts.) Moreover, if H and v are of class C^1, it follows from (4.2) and (4.3) that

$$H_x(t,0) = H_x(t,L) \ \ (= -u(t)/g). \tag{4.7}$$

Therefore we introduce the vectorial space E of functions $Y = (H, v, s, D) \in C^1([0, L]) \times C^1([0, L]) \times \mathbb{R} \times \mathbb{R}$ such that

$$H_x(0) = H_x(L), \tag{4.8}$$

$$v(0) = v(L) = 0, \tag{4.9}$$

and consider the affine subspace $\mathcal{Y} \subset E$ of $Y = (H, v, s, D) \in E$ satisfying

$$\int_0^L H(x)\mathrm{d}x = LH_e. \tag{4.10}$$

The vectorial space E is equipped with the usual norm

$$|Y| := |H|_1 + |v|_1 + |s| + |D|,$$

where, for $w \in \mathcal{C}^1([0,L])$,

$$|w|_1 := \text{Max}\{|w(x)| + |w_x(x)|;\ x \in [0,L]\}.$$

With these notations, we can define a trajectory of the control system Σ.

Definition 18 *Let T_1 and T_2 be two real numbers satisfying $T_1 \leqslant T_2$. A function $(Y,u) = ((H,v,s,d),u) : [T_1,T_2] \to \mathcal{Y} \times \mathbb{R}$ is a trajectory of the control system Σ if*

(i) the functions H and v are of class C^1 on $[T_1,T_2] \times [0,L]$,

(ii) the functions s and D are of class C^1 on $[T_1,T_2]$ and the function u is continuous on $[0,T]$,

(iii) the equations (4.1) to (4.5) hold for every $(t,x) \in [T_1,T_2] \times [0,L]$.

Our main result states that the control system Σ is locally controllable around the equilibrium point (Y_e, u_e). More precisely, one has the following theorem.

Theorem 19 *There exists $T > 0$, $C_0 > 0$ and $\eta > 0$ such that, for every $Y_0 = (H_0, v_0, s_0, D_0) \in \mathcal{Y}^2$, and for every $Y_1 = (H_1, v_1, s_1, D_1) \in \mathcal{Y}^2$ such that*

$$|H_0 - H_e|_1 + |v_0|_1 < \eta,\ |H_1 - H_e|_1 + |v_1|_1 < \eta,\ |s_1 - s_0| + |D_1 - s_0 T - D_0| < \eta,$$

there exists a trajectory $(Y,u) : [0,T] \to \mathcal{Y} \times \mathbb{R},\ t \mapsto ((H(t), v(t), s(t), (t)), u(t))$ of the control system Σ such that

$$Y(0) = Y_0 \text{ and } Y(T) = Y_1, \tag{4.11}$$

and, for every $t \in [0,T]$,

$$\begin{aligned} &|H(t) - H_e|_1 + |v(t)|_1 + |u(t)| < \\ &\quad C_0 \left(\sqrt{|H_0 - H_e|_1 + |v_0|_1 + |H_1 - H_e|_1 + |v_1|_1} + |s_1 - s_0| + |D_1 - s_0 T - D_0| \right). \end{aligned} \tag{4.12}$$

As a corollary of this theorem, any steady state $Y_1 = (0,0,0,D_1)$ can be reached from any other steady state $Y_0 = (0,0,0,D_0)$. More precisely, one has the following corollary.

Corollary 20 *Let T, C_0 and η be as in Theorem 19. Let D_0 and D_1 be two real numbers and let $\eta_1 \in (0,\eta]$. Then, there exists a trajectory $(Y,u) : [0, T(|D_1 - D_0| + \eta_1)/\eta_1] \to \mathcal{Y} \times \mathbb{R},\ t \mapsto ((H(t), v(t), s(t), (t)), u(t))$ of the control system Σ such that*

$$Y(0) = (0,0,0,D_0) \text{ and } Y(T(|D_1 - D_0| + \eta_1)/\eta_1) = (0,0,0,D_1), \tag{4.13}$$

$$|H(t) - H_e|_1 + |v(t)|_1 + |u(t)| < C_0 \eta_1\ \forall t \in [0, T(|D_1 - D_0| + \eta_1)/\eta_1]. \tag{4.14}$$

Let us give the main steps of the proof of Theorem 19. Let us first point out that by scaling arguments one can assume without loss of generality that

$$L = g = H_e = 1. \tag{4.15}$$

Indeed, if we let

$$H^*(t,x) := \frac{1}{H_e} H\left(\frac{t}{L\sqrt{H_e g}}, \frac{x}{L}\right),\ v^*(t,x) := \frac{1}{\sqrt{H_e g}} v\left(\frac{t}{L\sqrt{H_e g}}, \frac{x}{L}\right),$$
$$u^*(t) := \frac{1}{LH_e g} u\left(\frac{t}{L\sqrt{H_e g}}\right),\ s^*(t) := \frac{1}{\sqrt{H_e g}} s\left(\frac{t}{L\sqrt{H_e g}}\right),\ D^*(t) := LD\left(\frac{t}{L\sqrt{H_e g}}\right),$$

with $x \in [0,1]$, then equations (4.1) to (4.5) are equivalent to

$$\begin{aligned}
&H_t^*(t,x) + (H^* v^*)_x(t,x) = 0,\\
&v_t^*(t,x) + \left(H^* + \frac{v^{*2}}{2}\right)_x(t,x) = -u^*(t),\\
&v^*(t,0) = v^*(t,1) = 0,\\
&\frac{\mathrm{d}s^*}{\mathrm{d}t}(t) = u^*(t),\\
&\frac{\mathrm{d}D^*}{\mathrm{d}t}(t) = s^*(t).
\end{aligned}$$

From now on, we always assume that we have (4.15). Since $(Y,u) = ((H,v,s,D),u)$ is a trajectory of the control system Σ if and only if $((H,v,s-a,D-at-b),u)$ is a trajectory of the control system Σ, we may assume without loss of generality that $s_0 = D_0 = 0$.

The proof of Theorem 19 relies again on the return method. So one looks for a trajectory $(\bar{Y},\bar{u}) : [0,T] \to \mathcal{Y} \times \mathbb{R}$ of the control system Σ satisfying

$$\bar{Y}(0) = \bar{Y}(T) = Y_e, \tag{4.16}$$

$$\text{the linearized control system around } (\bar{Y},\bar{u}) \text{ is controllable.} \tag{4.17}$$

Let us point out that, as already noticed by F. Dubois, N. Petit and P. Rouchon in [23], property (4.17) does not hold for the natural trajectory $(\bar{Y},\bar{u}) = (Y_e,u_e)$. Indeed the linearized control system around (Y_e,u_e) is

$$(\Sigma_0)\begin{cases} h_t + v_x = 0,\\ v_t + h_x = -u(t),\\ v(t,0) = v(t,1) = 0,\\ \frac{\mathrm{d}s}{\mathrm{d}t}(t) = u(t),\\ \frac{\mathrm{d}D}{\mathrm{d}t}(t) = s(t), \end{cases} \tag{4.18}$$

where the state is $(h,v,s,D) \in \mathcal{Y}_0$, with

$$\mathcal{Y}_0 := \left\{(h,v,s,D) \in E;\ \int_0^L h\mathrm{d}x = 0\right\},$$

and the control is $u \in \mathbb{R}$. But (4.18) implies that, if

$$h(0,1-x) = -h(0,x) \text{ and } v(0,1-x) = v(0,x) \ \forall x \in [0,1],$$

then

$$h(t,1-x) = -h(0,x) \text{ and } v(t,1-x) = v(0,x) \ \forall x \in [0,1], \ \forall t.$$

Remark 21 *Even if the control system (4.23) is not controllable, F. Dubois, N. Petit and P. Rouchon have proved in [23] that one can move from any steady state $(h_0, v_0, s_0, D_0) = (0,0,s_0,D_0)$ to any steady state $(h_1, v_1, s_1, D_1) = (0,0,s_1,D_1)$ for this control system (see also [58] when the tank has a non-straight bottom). This does not imply that the same property also holds for the nonlinear control system Σ, but it follows from Corollary 20, that this property indeed also holds for the nonlinear control system Σ. Moreover the fact that this motion is possible for the control system (4.23) explains why, in the right hand side of (4.12), one has $|s_1 - s_0| + |D_1 - s_0 T - D_0|$ and not $(|s_1 - s_0| + |D_1 - s_0 T - D_0|)^{1/2}$.*

As in [11, 13, 14, 19, 32, 34, 35, 66] one has to look for more complicated trajectories $(\bar{Y}, \bar{u})$ in order to have (4.17). In fact, as in [15], one can require instead of (4.16), the weaker property

$$\bar{Y}(0) = Y_e \text{ and } \bar{Y}(T) \text{ is close to } Y_e \tag{4.19}$$

and hope that, as it happens for the Navier-Stokes control system -see above and [15]-, the controllability around $(\bar{Y}, \bar{u})$ will be strong enough to tackle the problem that $\bar{Y}(T)$ is not Y_e but only close to Y_e. Moreover, since as it is proved in [23], one can move for the linear control system Σ_0, from $Y_e = (0,0,0,0)$ to $(0,0,s_1,D_1)$, it is natural to try not to "return" to Y_e, but requires instead (4.19) the property

$$\bar{Y}(0) = Y_e \text{ and } \bar{Y}(T) \text{ is close to } (0,0,s_1,D_1). \tag{4.20}$$

In order to use this method, one first needs to have trajectories of the control system Σ such that the linearized control system around these trajectories are controllable. Let us give an example of a family of such trajectories. Let us fix a positive real number T^* in $(2,+\infty)$. For $\gamma \in (0,1]$ and $(a,b) \in \mathbb{R}^2$, let us define $(Y^{\gamma,a,b}, u^\gamma) : [0,T^*] \to \mathcal{Y} \times \mathbb{R}$ by

$$Y^{\gamma,a,b}(t,x) := \left(1 + \gamma\left(\frac{1}{2} - x\right), 0, \gamma t + a, \gamma\frac{t^2}{2} + at + b\right) \ \forall t \in [0,T^*], \ \forall x \in [0,1], \tag{4.21}$$

$$u^\gamma(t) := \gamma \ \forall t \in [0,T^*]. \tag{4.22}$$

Then, $(Y^{\gamma,a,b}, u^\gamma)$ is a trajectory of the control system Σ. The linearized control system around this trajectory is the following control system

$$(\Sigma_\gamma) \begin{cases} h_t + \left(\left(1 + \gamma\left(\frac{1}{2} - x\right)\right) v\right)_x = 0, \\ v_t + h_x = -u(t), \\ v(t,0) = v(t,1) = 0, \\ \frac{ds}{dt}(t) = u(t), \\ \frac{dD}{dt}(t) = s(t) \end{cases} \tag{4.23}$$

where the state is $(h, v, s, D) \in \mathcal{Y}_0$ and the control is $u \in \mathbb{R}$. This linear control system Σ_γ is controllable if $\gamma > 0$ is small enough (see [18] for a proof). Unfortunately the controllability of Σ_γ does not seem to imply directly the local controllability of the control system Σ around the trajectory $(Y^{\gamma,a,b}, u^\gamma)$. Indeed the map from $\mathcal{Y} \times C^0([0,T])$ into $\mathcal{Y}$ which associates to any initial data $Y_0 = (H_0, v_0, s_0, D_0) \in \mathcal{Y}$ and to any $u \in C^0([0,T])$ such that

$$H_{0x}(0) = H_{0x}(1) = -u(0)$$

the state $Y(T) \in \mathcal{Y}$, where $Y = (H, v, s, D) : [0,T] \to \mathcal{Y}$ satisfies (4.1) to (4.5) and $Y(0) = Y_0$ is well-defined and continuous on a small open neighborhood of $(Y_e, 0)$ (see e.g. [50]) but is not of class C^1 on this neighborhood. So one cannot use the classical inverse function theorem to get the desired local controllability. To take care of this problem, one adapts the usual iterative scheme used to prove the existence of solutions to hyperbolic systems (see e.g. [20, p. 476-478], [39, p. 54-55], [50, p. 96-107], [55, p. 35-43] or [61, p. 106-116] -see also [13, 14, 19, 32, 34, 35] for the Euler and the Navier control system for incompressible fluids): one uses the following inductive procedure $(h^n, v^n, s^n, D^n, u^n) \mapsto (h^{n+1}, v^{n+1}, s^{n+1}, D^{n+1}, u^{n+1})$ so that

$$h_t^{n+1} + v^n h_x^{n+1} + \left(1 + \gamma\left(\frac{1}{2} - x\right) + h^n\right) v_x^{n+1} - \gamma v^{n+1} = 0 \tag{4.24}$$

$$v_t^{n+1} + h_x^{n+1} + v^n v_x^{n+1} = -u^{n+1}(t) \tag{4.25}$$

$$v^{n+1}(t,0) = v^{n+1}(t,L) = 0, \tag{4.26}$$

$$\frac{\mathrm{d}s^{n+1}}{\mathrm{d}t}(t) = u^{n+1}(t), \tag{4.27}$$

$$\frac{\mathrm{d}D^{n+1}}{\mathrm{d}t}(t) = s^{n+1}(t), \tag{4.28}$$

and $(h^{n+1}, v^{n+1}, s^{n+1}, D^{n+1}, u^{n+1})$ has the required value for $t = 0$ and for $t = T^*$. Unfortunately we have only been able to prove that the control system (4.24)-(4.28), where the state is $(h^{n+1}, v^{n+1}, s^{n+1}, D^{n+1})$ and the control is u^{n+1}, is controllable under a special assumption on (h^n, v^n), see [18]. Hence one has to insure that, at each iterative step, (h^n, v^n) satisfies this condition, which turns out to be possible. So one gets the following proposition, which is proved in [18].

Proposition 22 *There exist $C_1 > 0$, $\mu > 0$ and $\gamma_0 \in (0,1]$ such that, for every $\gamma \in [0, \gamma_0]$, for every $(a,b) \in \mathbb{R}^2$ and for every $(Y_0, Y_1) \in \mathcal{Y}^2$ satisfying*

$$|Y_0 - Y^{\gamma,a,b}(0)| \leqslant \mu\gamma^2 \text{ and } |Y_1 - Y^{\gamma,a,b}(T^*)| \leqslant \mu\gamma^2,$$

there exists a trajectory $(Y, u) : [0, T^*] \to \mathcal{Y} \times \mathbb{R}$ of the control system Σ such that

$$Y(0) = Y_0 \text{ and } Y(T^*) = Y_1,$$

$$\left|Y(t) - Y^{\gamma,a,b}(t)\right| + |u(t)| \leqslant C_1\gamma \;\forall t \in [0, T^*]. \tag{4.29}$$

One now needs to construct, for every given $\gamma > 0$ small enough, trajectories $(Y, u) : [0, T^0] \to \mathcal{Y} \times \mathbb{R}$ of the control system Σ satisfying

$$Y(0) = (1,0,0,0) \text{ and } |Y(T^0) - Y^{\gamma,a,b}(0)| \leqslant \mu\gamma^2, \tag{4.30}$$

and trajectories $(Y,u):\ [T^0+T^*, T^0+T^*+T^1] \to \mathcal{Y}\times\mathbb{R}$ of the control system Σ such that

$$Y\left(T^0+T^1+T^*\right) = (1,0,s_1,D_1) \text{ and } \left|Y\left(T^0+T^*\right) - Y^{\gamma,a,b}\left(T^*\right)\right| \leqslant \mu\gamma^2, \tag{4.31}$$

for suitable choice of $(a,b)\in\mathbb{R}^2$, $T^0>0$, $T^1>0$. Let us first point out that it follows from [23] that one knows explicit trajectories $(Y^l,u^l):\ [0,T^0]\to\mathcal{Y}\times\mathbb{R}$ of *the linearized control system around* $(0,0)$ (i.e. the control system Σ_0) satisfying $Y^l(0)=0$ and $Y^l(T^0)=Y^{\gamma,a,b}(0)$. (In fact F. Dubois, N. Petit and P. Rouchon have proved in [23] that the linear control system Σ_0 is flat -a notion introduced by M. Fliess, J. Lévine, P. Martin and P. Rouchon in [26]-. They have given a complete explicit parametrization of the trajectories of Σ_0 by means of an arbitrary function and a 1-periodic function.) Then, the idea is that, if one moves "slowly", the same control u^l gives a trajectory $(Y,u):\ [0,T^0]\to\mathcal{Y}\times\mathbb{R}$ of the control system Σ such that (4.30) holds. More precisely, let $f_0\in C^4([0,4])$ be such that

$$f_0 = 0 \text{ in } [0,1/2]\cup[3,4], \tag{4.32}$$

$$f_0(t) = s/2\ \forall t\in[1,3/2], \tag{4.33}$$

$$\int_0^4 f_0(t_1)\mathrm{d}t_1 = 0. \tag{4.34}$$

Similarly, let $f_1\in C^4([0,4])$ and $f_2\in C^4([0,4])$ be such that

$$f_1 = 0 \text{ in } [0,1/2]\cup[1,3/2] \text{ and } f_1 = 1/2 \text{ in } [3,4], \tag{4.35}$$

$$\int_0^3 f_1(t_1)\mathrm{d}t_1 = 0, \tag{4.36}$$

$$f_2 = 0 \text{ in } [0,1/2]\cup[1,3/2]\cup[3,4], \tag{4.37}$$

$$\int_0^4 f_2(t_1)\mathrm{d}t_1 = 1/2. \tag{4.38}$$

Let

$$\mathbb{D} := \left\{\left(\bar{s},\bar{D}\right)\in\mathbb{R}^2;\ |\bar{s}|\leqslant 1,\ \left|\bar{D}\right|\leqslant 1\right\}.$$

For $\left(\bar{s},\bar{D}\right)\in\mathbb{D}$, let $f_{\bar{s},\bar{D}}\in C^4([0,4])$ be defined by

$$f_{\bar{s},\bar{D}} := f_0 + \bar{s}f_1 + \bar{D}f_2. \tag{4.39}$$

For $\epsilon\in(0,1/2]$ and for $\gamma\in\mathbb{R}$, let $u^{\epsilon,\gamma}_{\bar{s},\bar{D}}:\ [0,3/\epsilon]\to\mathbb{R}$ be defined by

$$u^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) := \gamma f'_{\bar{s},\bar{D}}(\epsilon t) + \gamma f'_{\bar{s},\bar{D}}(\epsilon(t+1)). \tag{4.40}$$

Let $\left(h^{\epsilon,\gamma}_{\bar{s},\bar{D}}, v^{\epsilon,\gamma}_{\bar{s},\bar{D}}, s^{\epsilon,\gamma}_{\bar{s},\bar{D}}, D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right):\ [0,3/\epsilon]\to C^1([0,1])\times C^1([0,1])\times\mathbb{R}\times\mathbb{R}$ be such that (4.18) holds for $(h,v,s,D) = \left(h^{\epsilon,\gamma}_{\bar{s},\bar{D}}, v^{\epsilon,\gamma}_{\bar{s},\bar{D}}, s^{\epsilon,\gamma}_{\bar{s},\bar{D}}, D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right)$, $u=u^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ and

$$\left(h^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0,\cdot), v^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0,\cdot), s^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0), D^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0)\right) = (0,0,0,0).$$

From [23] one gets that

$$h^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t,x) = -\frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+x)) + \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+1-x)), \tag{4.41}$$

$$v^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t,x) = \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+x)) + \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+1-x)) - \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon t) - \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+1)), \tag{4.42}$$

$$s^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) = \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon t) + \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+1)), \tag{4.43}$$

$$D^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) = \frac{\gamma}{\epsilon^2} F_{\bar{s},\bar{D}}(\epsilon t) + \frac{\gamma}{\epsilon^2} F_{\bar{s},\bar{D}}(\epsilon(t+1)), \tag{4.44}$$

with

$$F_{\bar{s},\bar{D}}(t) := \int_0^t f_{\bar{s},\bar{D}}(t_1)\mathrm{d}t_1.$$

In particular, using also (4.32) to (4.38), one gets

$$h^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+t,x\right) = \gamma\left(\frac{1}{2}-x\right) \text{ and } v^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+t,x\right) = 0 \ \forall t \in \left[0,\frac{1-2\epsilon}{2\epsilon}\right], \ \forall x \in [0,1], \tag{4.45}$$

$$s^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+t\right) = \frac{\gamma}{\epsilon} + \frac{\gamma}{2} + \gamma t \ \forall t \in \left[0,\frac{1-2\epsilon}{2\epsilon}\right], \tag{4.46}$$

$$D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+t\right) = D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) + \left(\frac{\gamma}{\epsilon}+\frac{\gamma}{2}\right)t + \frac{\gamma}{2}t^2 \ \forall t \in \left[0,\frac{1-2\epsilon}{2\epsilon}\right], \tag{4.47}$$

$$h^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon},x\right) = 0 \text{ and } v^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon},x\right) = 0 \ \forall x \in [0,1], \tag{4.48}$$

$$s^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon}\right) = \frac{\gamma}{\epsilon}\bar{s} \text{ and } D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon}\right) = \frac{\gamma}{2\epsilon}\bar{s} + \frac{\gamma}{\epsilon^2}\bar{D}. \tag{4.49}$$

Let $H^{\epsilon,\gamma}_{\bar{s},\bar{D}} = 1 + h^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ and $Y^{\epsilon,\gamma}_{\bar{s},\bar{D}} = \left(H^{\epsilon,\gamma}_{\bar{s},\bar{D}}, v^{\epsilon,\gamma}_{\bar{s},\bar{D}}, s^{\epsilon,\gamma}_{\bar{s},\bar{D}}, D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right)$. Consider

$$a_{\epsilon,\gamma} := \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(1) + \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(1+\epsilon) = \frac{\gamma}{\epsilon} + \frac{\gamma}{2}, \ b^{\bar{s},\bar{D}}_{\epsilon,\gamma} := \frac{\gamma}{\epsilon^2} F_{\bar{s},\bar{D}}(1) + \frac{\gamma}{\epsilon^2} F_{\bar{s},\bar{D}}(1+\epsilon) = D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right).$$

Using (4.21), (4.45), (4.46) and (4.47), one has

$$Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) = Y^{\gamma,a_{\epsilon,\gamma},b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}(0,\cdot), \tag{4.50}$$

and, if $\epsilon \in (0, 1/(2(T^*+1))]$,

$$Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+T^*\right) = Y^{\gamma,a_{\epsilon,\gamma},b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}(T^*). \tag{4.51}$$

The next proposition, which is proved in [18], shows that one can achieve (4.30) with $u = u^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ for suitable choices of T^0, ϵ and γ.

Proposition 23 *There exists a constant $C_2 > 2$ such that, for every $\epsilon \in (0, 1/C_2]$, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$ and for every $\gamma \in [0, \epsilon/C_2]$, there exists one and only one map $\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}} : [0, 1/\epsilon] \to \mathcal{Y}$ satisfying the two following conditions*

$$(\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}, u^{\epsilon,\gamma}_{\bar{s},\bar{D}}) \text{ is a trajectory of the control system } \Sigma \quad (\text{on } [0, 1/\epsilon]),$$
$$\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0) = (1, 0, 0, 0)\,,$$

and this unique map $\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ verifies

$$\left|\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) - Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t)\right| \leqslant C_2\epsilon\gamma^2 \ \forall t \in [0, 1/\epsilon]. \tag{4.52}$$

In particular, by (4.45),

$$\left|\tilde{v}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right)\right|_1 + \left|\tilde{h}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) - \gamma\left(\frac{1}{2} - x\right)\right|_1 \leqslant C_2\epsilon\gamma^2. \tag{4.53}$$

Similarly, one has the following proposition, which shows that (4.31) is achieved with $u = u^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ for suitable choices of T^1, ϵ and γ.

Proposition 24 *There exists a constant $C_3 > 2(T^* + 1)$ such that, for every $\epsilon \in (0, 1/C_3]$, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$, and for every $\gamma \in [0, \epsilon/C_3]$, there exists one and only one map $\hat{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}$: $[(1/\epsilon) + T^*, 3/\epsilon] \to \mathcal{Y}$ satisfying the two following conditions*

$$\left(\hat{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}, u^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right) \text{ is a trajectory of the control system } \Sigma \quad (\text{on } [(1/\epsilon) + T^*, 3/\epsilon]),$$
$$\hat{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon}\right) = \left(1, 0, \frac{\gamma}{\epsilon}\bar{s}, \frac{\gamma}{2\epsilon}\bar{s} + \frac{\gamma}{\epsilon^2}\bar{D}\right) = Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon}\right),$$

and this unique map $\hat{Y}^{\epsilon,\gamma}$ verifies

$$\left|\hat{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) - Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t)\right| \leqslant C_3\epsilon\gamma^2 \ \forall t \in [(1/\epsilon) + T^*, 3/\epsilon]. \tag{4.54}$$

In particular, by (4.45),

$$\left|\hat{v}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right)\right|_1 + \left|\hat{h}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) - \gamma\left(\frac{1}{2} - x\right)\right|_1 \leqslant C_2\epsilon\gamma^2. \tag{4.55}$$

Let us choose

$$\epsilon := \text{ Min}\left(\frac{1}{C_2}, \frac{1}{C_3}, \frac{\mu}{2C_2}, \frac{\mu}{2C_3}\right) \leqslant \frac{1}{2}. \tag{4.56}$$

Let us point out that there exists $C_4 > 0$ such that, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$ and for every $\gamma \in [-\epsilon, \epsilon]$,

$$\left|H^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right|_{C^2([0,3/\epsilon]\times[0,1])} + \left|v^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right|_{C^2([0,3/\epsilon]\times[0,1])} \leqslant C_4, \tag{4.57}$$

which, with straightforward estimates, leads to the next proposition, whose proof is omitted.

Proposition 25 *There exists $C_5 > 0$ such that, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$, for every $Y_0 = (H_0, v_0, s_0, D_0) \in \mathcal{Y}$ with*

$$|Y_0 - Y_e| \leqslant \frac{1}{C_5},\ s_0 = 0,\ D_0 = 0$$

and for every $\gamma \in [0, \epsilon/C_2]$, there exists one and only one $Y : [0, 1/\epsilon] \to \mathcal{Y}$ such that

$$(Y, u^{\varepsilon,\gamma}_{\bar{s},\bar{D}} - H_{0x}(0)) \text{ is a trajectory of the control system } \Sigma,$$
$$Y(0) = Y_0,$$

and this unique map Y satisfies

$$\left|Y(t) - \tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t)\right| \leqslant C_5 |Y_0 - Y_e|,\ \forall t \in [0, 1/\epsilon].$$

Similarly, (4.57) leads to the following proposition.

Proposition 26 *There exists $C_6 > 0$ such that, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$, for every $\gamma \in [0, \epsilon/C_3]$, and for every $Y_1 = (H_1, v_1, s_1, D_1) \in \mathcal{Y}$ such that*

$$\left|Y_1 - \left(1, 0, \frac{\gamma}{\epsilon}\bar{s}, \frac{\gamma}{2\epsilon}\bar{s} + \frac{\gamma}{\epsilon^2}\bar{D}\right)\right| \leqslant \frac{1}{C_6},\ s_1 = \frac{\gamma}{\epsilon}\bar{s},\ D_1 = \frac{\gamma}{2\epsilon}\bar{s} + \frac{\gamma}{\epsilon^2}\bar{D}$$

there exists one and only one $Y : [(1/\epsilon) + T^*, 3/\epsilon] \to \mathcal{Y}$ such that

$$(Y, u^{\varepsilon,\gamma}_{\bar{s},\bar{D}} - H_{1x}(0)) \text{ is a trajectory of the control system } \Sigma$$
$$Y(3/\epsilon) = Y_1,$$

and this unique map Y satisfies

$$\left|Y(t) - \hat{Y}^{\varepsilon,\gamma}_{\bar{s},\bar{D}}(t)\right| \leqslant C_6 \left|Y_1 - Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}(3/\epsilon)\right|,\ \forall t \in [(1/\epsilon) + T^*, 3/\epsilon].$$

Finally define

$$T := \frac{3}{\epsilon}, \tag{4.58}$$

$$\eta := \operatorname{Min}\left(\frac{\epsilon^2\mu}{2C_5(C_3^2 + C_2^2)}, \frac{\epsilon^2\mu}{2C_6(C_3^2 + C_2^2)}, \frac{\epsilon}{C_2}, \frac{\epsilon}{C_3}, \frac{1}{C_5}, \frac{1}{C_6}, \frac{\gamma_0^2\mu}{2C_5}, \frac{\gamma_0^2\mu}{2C_6}, \gamma_0\right). \tag{4.59}$$

We want to check that Theorem 19 holds with these constants for a large enough C_0. Let $Y_0 = (H_0, v_0, 0, 0) \in \mathcal{Y}$ and $Y_1 = (H_1, v_1, s_1, D_1) \in \mathcal{Y}$ be such that

$$|H_0 - 1|_1 + |v_0|_1 \leqslant \eta,\ |H_1 - 1|_1 + |v_1|_1 \leqslant \eta,\ |s_1| + |D_1| \leqslant \eta. \tag{4.60}$$

Let

$$\gamma := \operatorname{Max}\left(\sqrt{\frac{2C_5}{\mu}}\sqrt{|H_0 - 1|_1 + |v_0|_1}, \sqrt{\frac{2C_6}{\mu}}\sqrt{|H_1 - 1|_1 + |v_1|_1}, |s_1| + |D_1|\right), \tag{4.61}$$

$$\bar{s} := \frac{\epsilon}{\gamma}s_1,\ \bar{D} := \frac{\epsilon^2}{\gamma}\left(D_1 - \frac{s_1}{2}\right), \tag{4.62}$$

so that, thanks to (4.49),

$$s^{\epsilon,\gamma}_{\bar{s},\bar{D}} = s_1, \; D^{\epsilon,\gamma}_{\bar{s},\bar{D}} = D_1. \tag{4.63}$$

Note that, by (4.56), (4.60), (4.61) and (4.62),

$$(\bar{s}, \bar{D}) \in \mathbb{D}. \tag{4.64}$$

By (4.59), (4.60) and (4.61), we obtain that

$$\gamma \in \left[0, \text{Min}\ \left(\frac{\epsilon}{C_2}, \frac{\epsilon}{C_3}\right)\right]. \tag{4.65}$$

Then, by Proposition 25, (4.59), (4.60) and (4.65), there exists a function $Y^0 = (H^0, v^0, s^0, D^0): \; [0, 1/\epsilon] \to \mathcal{Y}$ such that

$$(Y^0, u^{\epsilon,\gamma}_{\bar{s},\bar{D}} - H_{0x}(0)) \text{ is a trajectory of the control system } \Sigma \text{ on } [0, 1/\epsilon], \tag{4.66}$$

$$Y^0(0) = Y_0, \tag{4.67}$$

$$\left|Y^0(t) - \tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t)\right| \leqslant C_5|Y_0 - Y_e| \; \forall t \in [0, 1/\epsilon]. \tag{4.68}$$

By (4.61) and (4.68),

$$\left|Y^0\left(\frac{1}{\epsilon}\right) - \tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right)\right| \leqslant \frac{\mu\gamma^2}{2}. \tag{4.69}$$

By Proposition 23, (4.56) and (4.65),

$$\left|\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) - Y^{\gamma, a_{\epsilon,\gamma}, b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}(0)\right| \leqslant C_2\epsilon\gamma^2 \leqslant \frac{\mu\gamma^2}{2},$$

which, with (4.69), gives

$$\left|Y^0\left(\frac{1}{\epsilon}\right) - Y^{\gamma, a_{\epsilon,\gamma}, b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}\right| \leqslant \mu\gamma^2. \tag{4.70}$$

Similarly, by Propositions 24 and 26, (4.56), (4.58), (4.59), (4.60), (4.61), (4.63) and (4.65), there exists $Y^1 = (H^1, v^1, s^1, D^1): \; [(1/\epsilon) + T^*, T] \to \mathcal{Y}$ such that

$$(Y^1, u^{\epsilon,\gamma}_{\bar{s},\bar{D}} - H_{1x}(0)) \text{ is a trajectory of the control system } \Sigma \text{ on } [(1/\epsilon) + T^*, T], \tag{4.71}$$

$$Y^1(T) = Y_1, \tag{4.72}$$

$$\left|Y^1(t) - \tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t)\right| \leqslant C_6\left|Y_1 - (1, 0, s_1, D_1)\right| \; \forall t \in [(1/\epsilon) + T^*, T], \tag{4.73}$$

$$\left|Y^1\left(\frac{1}{\epsilon} + T^*\right) - Y^{\gamma, a_{\epsilon,\gamma}, b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}(T^*)\right| \leqslant \mu\gamma^2. \tag{4.74}$$

By (4.59), (4.60) and (4.61),

$$\gamma \leqslant \gamma_0. \tag{4.75}$$

From Proposition 22, (4.70), (4.74) and (4.75), there exists a trajectory $(Y^*, u^*) : \; [0, T^*] \to \mathcal{Y}$ of the control system Σ satisfying

$$Y^*(0) = Y^0\left(\frac{1}{\epsilon}\right), \tag{4.76}$$

$$\left|Y^*(t) - Y^{\gamma, a_{\epsilon,\gamma}, b_{\epsilon,\gamma}^{\bar{s},\bar{D}}}(t)\right| \leqslant C_1\mu\gamma \;\forall t \in [0, T^*], \tag{4.77}$$

$$Y^*(T^*) = Y^1\left(\frac{1}{\epsilon} + T^*\right). \tag{4.78}$$

The map $(Y, u) : \; [0, T] \to \mathcal{Y}$ defined by

$$(Y(t), u(t)) = (Y^0(t), u_{\bar{s},\bar{D}}^{\epsilon,\gamma}(t) - H_{0x}(0)) \;\forall t \in [0, 1/\epsilon],$$
$$(Y(t), u(t)) = (Y^*(t - (1/\epsilon)), u^*(t - (1/\epsilon))) \;\forall t \in [1/\epsilon, (1/\epsilon) + T^*],$$
$$(Y(t), u(t)) = (Y^1(t), u_{\bar{s},\bar{D}}^{\epsilon,\gamma}(t) - H_{1x}(0)) \;\forall t \in [(1/\epsilon) + T^*, T],$$

is a trajectory of the control system Σ which, by (4.67) and (4.72), satisfies (4.11). Finally the existence of $C_0 > 0$ such that (4.12) holds follows from the construction of (Y, u), (4.7), (4.29), (4.41) to (4.44), (4.52), (4.54), (4.60), (4.61), (4.68), (4.73) and (4.77).

5 Null asymptotic stabilizability of the 2-D Euler control system

In subsection 3.1 we have considered the problem of the controllability of the Euler control system of incompressible inviscid fluid in a bounded domain. In particular we have seen that, if the controls act on an arbitrarily small open subset of the boundary which meets every connected component of this boundary, then the Euler equation are exactly controllable.

For linear control systems, the exact controllability implies the asymptotic stabilizability by means of feedback laws. This is well known for linear control systems of finite dimension and, by M. Slemrod [63], J.–L. Lions [52], I. Lasiecka-R. Triggiani [48] and V. Komornik [46], it also holds in infinite dimension in very general cases. But, as pointed out by H.J. Sussmann in [69], by E.D. Sontag and H.J. Sussmann in [68], and by R.W. Brockett in [8], this is no longer true for *nonlinear* control systems, even of finite dimension. For example (see [8]) the nonlinear control system (2.6) is globally controllable but $0 \in \mathbb{R}^3$ cannot be, even locally, asymptotically stabilized by means of feedback laws. Let us also notice that, as in this counter-example, the linearized control system of the Euler equation around the origin is not controllable.

Therefore it is natural to ask what is the situation for the asymptotic stabilizability of the origin for the 2-D Euler equation of incompressible inviscid fluid in a bounded domain when the controls act on an arbitrarily small open subset of the boundary which meets every connected component of this boundary. In this section we are going to see that the null global asymptotic stabilizability by means of feedback laws holds if the domain is simply connected.

Let Ω be a nonempty bounded connected and simply connected subset of $\mathbb{R}^2$ of class $\mathcal{C}^\infty$ and let Γ_0 be a non empty open subset of the boundary $\partial\Omega$ of Ω. This set Γ_0 is the location of the control. Let y be the velocity field of the inviscid fluid contained in Ω. We assume that the fluid is incompressible, so that

$$\text{div } y = 0. \tag{5.1}$$

Since Ω is simply connected, y is completely characterized by $\omega := \text{curl } y$ and $y.n$ on $\partial\Omega$ where n denotes the unit outward normal to $\partial\Omega$. For the problem of controllability, one does not really need to specify the control and the state: one considers the "Euler control system" as an under-determined system by requiring $y.n = 0$ on $\partial\Omega \setminus \Gamma_0$ instead of $y.n = 0$ on $\partial\Omega$ as for the uncontrolled usual Euler equation. For the stabilization problem, one needs to specify more precisely the control and the state. In this paper the state is ω. For the control there are at least two natural possibilities

(a) The control is $y.n$ on Γ_0 and the time derivative $\partial\omega/\partial t$ of the vorticity at the points of Γ_0 where $y.n < 0$, i.e. at the points where the fluid enters into the domain Ω,

(b) The control is $y.n$ on Γ_0 and the vorticity ω at the points where $y.n < 0$.

Let us point out that, by (5.1), in both cases $y.n$ has to satisfy $\int_{\partial\Omega} y.n = 0$. In this paper we study only case (a); for case (b), see [16].

Let us give stabilizing feedback laws. Let $g \in \mathcal{C}^\infty(\partial\Omega)$ be such that

$$\text{Support } g \subset \Gamma_0, \tag{5.2}$$

$$\Gamma_0^+ := \{g > 0\} \text{ and } \Gamma_0^- := \{g < 0\} \text{ are connected,} \tag{5.3}$$

$$g \neq 0, \tag{5.4}$$

$$\overline{\Gamma_0^+} \cap \overline{\Gamma_0^-} = \emptyset, \tag{5.5}$$

$$\int_{\partial\Omega} g = 0. \tag{5.6}$$

For every $f \in \mathcal{C}^0(\overline{\Omega})$, we denote

$$|f|_0 = \text{Max } \{|f(x)|\,; x \in \overline{\Omega}\}.$$

Our stabilizing feedback laws are

$$y.n = M\,|\omega|_0\, g \text{ on } \Gamma_0,$$

$$\frac{\partial\omega}{\partial t} = -M\,|\omega|_0\, \omega \text{ on } \Gamma_0^- \text{ if } |\omega|_0 \neq 0,$$

where $M > 0$ is large enough. With these feedback laws, a function $\omega : I \times \overline{\Omega} \to \mathbb{R}$, where I

is an interval, is a solution of the closed loop system Σ if

$$\frac{\partial \omega}{\partial t} + \text{ div } (\omega y) = 0 \text{ in } \overset{\circ}{I} \times \Omega, \tag{5.7}$$

$$\text{div } y = 0 \text{ in } \overset{\circ}{I} \times \Omega, \tag{5.8}$$

$$\text{curl } y = \omega \text{ in } \overset{\circ}{I} \times \Omega, \tag{5.9}$$

$$y(t).n = M \left|\omega(t)\right|_0 g \text{ on } \partial\Omega, \forall t \in I, \tag{5.10}$$

$$\frac{\partial \omega}{\partial t} = -M \left|\omega(t)\right|_0 \omega \text{ on } \{t; \omega(t) \neq 0\} \times \Gamma_0^-. \tag{5.11}$$

where, for $t \in \Omega$, $\omega(t) : \overline{\Omega} \to \mathbb{R}$ and $y(t) : \overline{\Omega} \to \mathbb{R}^2$ are defined by requiring $\omega(t)(x) = \omega(t,x)$ and $y(t)(x) = y(t,x), \forall x \in \overline{\Omega}$. More precisely, the definition of a solution of system Σ is

Definition 27 *Let I be an interval. A function $\omega : I \to C^0(\overline{\Omega})$ is a solution of system Σ if*

(i) $\omega \in C^0(I; C^0(\overline{\Omega}))(\cong C^0(I \times \overline{\Omega}))$,

(ii) For $y \in C^0(I \times \overline{\Omega}; \mathbb{R}^2)$ defined by requiring (5.8) and (5.9) in the sense of distributions and (5.10), one has (5.7) in the sense of distributions,

(iii) In the sense of distributions on the open manifold $\{t \in \overset{\circ}{I}; \omega(t) \neq 0\} \times \Gamma_0^-$ one has $\partial\omega/\partial t = -M \left|\omega(t)\right|_0 \omega$.

Our first theorem says that, for M large enough, the Cauchy problem for system Σ has at least one solution defined on $[0, +\infty)$ for every initial data in $C^0(\overline{\Omega})$. More precisely one has

Theorem 28 *There exists $M_0 > 0$ such that, for every $M \geqslant M_0$, the following two properties hold*

(i) For every $\omega_0 \in C^0(\overline{\Omega})$, there exists a solution of system Σ defined on $[0, +\infty)$ such that $\omega(0) = \omega_0$,

(ii) Any maximal solution of system Σ defined at time 0 is defined on $[0, +\infty)$ (at least).

Remark 29 *a. In this theorem, property (i) is in fact implied by property (ii) and Zorn's lemma. We state (i) in order to emphasize the <u>existence</u> of a solution to the Cauchy problem for system Σ. b. We do not know if the solution to the Cauchy problem is unique for positive time. (For negative time, one does not have uniqueness since there are solutions ω of system Σ defined on $[0, +\infty)$ such that $\omega(0) \neq 0$ and $\omega(T) = 0$ for $T \in [0, +\infty)$ large enough.) But let us emphasize that, already for control system in finite dimension, one considers feedback laws which are merely continuous; with these feedback laws, the Cauchy problem for the closed loop system may have many solutions. It turns out that this lack of uniqueness is not a real problem. Indeed, in finite dimension at least, if a point is asymptotically stable for a continuous vector field, then there exists, as in the case of regular vector fields, a (smooth) strict Lyapounov function. This result is due to Kurzweil [47].It is tempting to conjecture* *that a similar result hold in infinite dimension under reasonable assumptions. The existence of this Lyapounov function insures some robustness to perturbations. This is precisely this robustness which makes the interest of feedback laws compared to open loop controls. We will see that, for our feedback laws, there exists also a strict Lyapounov –see Proposition 33 below– and therefore our feedback laws provide some kind of robustness.*

Our next theorem shows that, at least for M large enough, our feedback laws globally and strongly asymptotically stabilize the origin in $C^0(\overline{\Omega})$ for system Σ.

Theorem 30 *There exists a positive constant $M_1 \geqslant M_0$ such that, for every $\varepsilon \in (0,1]$, every $M \geqslant M_1/\varepsilon$ and every maximal solution ω of system Σ defined at time 0,*

$$|\omega(t)|_0 \leqslant \ Min\ \{|\omega(0)|_0, \frac{\varepsilon}{t}\}, \forall t > 0. \tag{5.12}$$

Remark 31 *Due to the term $|\omega(t)|_0$ appearing in (5.10) and in (5.11) our feedback laws do not depend only on the value of ω on Γ_0. Let us point out that there is no asymptotically stabilizing feedback law depending only on the value of ω on Γ_0 such that the origin is asymptotically stable for the closed loop system. In fact, given a nonempty open subset Ω_0 of Ω, there is no feedback law which does not depend on the values of ω on Ω_0. This phenomenon is due to the existence of "phantom vortices": there are smooth stationary solutions $\bar{y} : \overline{\Omega} \to \mathbb{R}^2$ of the 2-D Euler equations such that Support $\bar{y} \subset \Omega_0$ and $\bar{\omega} :=$ curl $\bar{y} \neq 0$; see, e.g., [56]. Then $\omega(t) = \bar{\omega}$ is a solution of the closed loop system if the feedback law does not depend on the values of ω on Ω_0 –and vanishes for $\omega = 0$.*

Remark 32 *Let us emphasize that (5.12) implies that*

$$|\omega(t)|_0 \leqslant \varepsilon, \forall t \geqslant 1, \tag{5.13}$$

for every maximal solution ω of system Σ defined at time 0 (whatever is $\omega(0)$). It would be intreresting to know if one could have a similar result for the 2-D Navier-Stokes equations of viscous incompressible flows, that is if, given $\varepsilon > 0$, does there exist a feedback law such that (5.13) holds for every solution of the closed loop Navier-Stokes control system? Note that $y = 0$ on Γ_0 is a feedback which leads to asymptotic stabilization of the null solution of the Navier-Stokes control system. But this feedback does not have the required property. One may ask a similar question for the Burgers control system; for the null asymptotic stabilization of this control system, see the paper [45] by M. Krstić and the references therein.

The detailed proofs of Theorem 28 and of Theorem 30 are given in [16]. Let us just mention that Theorem 30 is proved by giving an explicit Lyapounov function. Let us give this Lyapounov function. Let $V : C^0(\overline{\Omega}) \to [0, +\infty)$ be defined by

$$V(\omega) = |\omega \exp(-\theta)|_0,$$

where $\theta \in C^\infty(\overline{\Omega})$ satisfies

$$\Delta\theta = 0 \text{ in } \overline{\Omega}, \tag{5.14}$$

$$\frac{\partial \theta}{\partial n} = g \text{ on } \partial\Omega. \tag{5.15}$$

(Let us point out that the existence of θ follows from (5.6).) Theorem 30 is an easy consequence of the following proposition.

Proposition 33 *There exists $M_2 \geqslant M_0$ and $\mu > 0$ such that, for every $M \geqslant M_2$ and every solution $\omega : [0,+\infty) \to C^0(\overline{\Omega})$ of system Σ, one has, for every $t \in [0,+\infty)$,*

$$[-\infty, 0] \ni \dot{V}(t) := \frac{d}{dt^+} V(\omega(t)) \leqslant -\mu M V^2(\omega(t)), \tag{5.16}$$

where $d/dt^+ V(\omega(t)) := \lim_{\varepsilon \to 0^+} (V(\omega(t+\varepsilon)) - V(\omega(t)))/\varepsilon$.

Let us end this section by some comments for the case where Ω is not simply connected In this case, in order to define the state, one adds to ω the real numbers $\lambda_1, \cdots, \lambda_g$ defined by

$$\lambda_i = \int y.\nabla^{\perp}\tau_i,$$

where, if one denotes by $\mathcal{C}_0, \mathcal{C}_1, \cdots, \mathcal{C}_g$ the connected components of Γ, the functions $\tau_i \in C^{\infty}(\overline{\Omega}), i \in \{1, \cdots, g\}$are defined by

$$\begin{aligned} \Delta\tau_i &= 0 \ , \\ \tau_i &= 0 \text{ on } \partial\Omega \setminus \mathcal{C}_i, \\ \tau_i &= 1 \text{ on } \mathcal{C}_i, \end{aligned}$$

and where $\nabla^{\perp}\tau_i$ denotes $\nabla\tau_i$ rotated by $\pi/2$. One has the following open problem

Open Problem 34 *Assume that $g \geq 1$ and that Γ_0 meets every connected component of Γ. Does there exist always a feedback law such that $0 \in C^0(\overline{\Omega}) \times \mathbb{R}^g$ is globally asymptotically stable for the closed loop system?*

Brockett's necessary condition [8] for the existence of asymptotically stabilizing feedback laws cannot be directly applied to our situation since our control system is of infinite dimension. But it leads to the following question.

Question *Assume that Γ_0 meets every connected component of Γ. Let $f \in C^{\infty}(\overline{\Omega})$. Does there exit $y \in C^{\infty}(\overline{\Omega}; \mathbb{R}^2)$ and $p \in C^{\infty}(\overline{\Omega})$ such that*

$$(y.\nabla)y + \nabla p = f \ \textit{in} \ \overline{\Omega} \ , \tag{5.17}$$

$$\text{div } y = 0 \ \textit{in} \ \overline{\Omega} \ , \tag{5.18}$$

$$y.n = 0 \ \textit{on} \ \Gamma \setminus \Gamma_0? \tag{5.19}$$

Let us point out that, by scaling arguments, one does not have to assume that f is "small" in this question. It turns out that the answer to this question is indeed positive. This has been proved in [17] if Ω is simply connected and by O. Glass in [37] for the general case.

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