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Introduction to General Circulation Models of the Atmosphere

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INTRODUCTION TO GENERAL CIRCULATION MODELS OF THE ATMOSPHERE

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1.GCMS: WHAT ARE THEY AND HOW ARE THEY USED?

Comprehensive mathematical representation of our current understanding of the physical processes responsible for the weather and climate. A tool to simulate the general circulation of the atmosphere, its evolution from a know initial state.

- Numerical Weather Prediction Models: Atmospheric GCMs.
Weather: detailed description of the evolution of the atmosphere over a few days. Initial condition problem. Models aimed at short time and high spatial resolution simulations.

- Climate Models: Atmospheric GCMs
Climate: Require long term atmospheric statistics, measures of the variability of meteorological parameters such as temperature and precipitation. Models aimed at long term and low spatial resolution integrations. Boundary value problem.

- Coupled Climate model: Atmospheric GCM coupled to ocean GCM. The focus is on prediction of the statistical properties of the climate system. Example: ENSO (El Nino Southern Oscillation), a seasonal forecast problem, where atmospheric variability associated with ocean variability is important.

- Climate System Models:
From the bottom of the ocean to the upper atmosphere.
Climate system: atmosphere, ocean, cryosphere (snow and ice) lithosphere (land), biosphere (natural and anthropogenic).

=> Models including both physical and bio-geo-chemical processes by means of coupling of a numbers of modules. Interactive models are those that allow for feedback between each other parts.

Questions that can be addressed by Climate System Models:

How and why the climate changed in the past?

What is the impact of the anthropogenic perturbations of the atmospheric composition on climate?

Climate System Models can be used for Climate projections:

Determination of the response of the atmospheric climate to changes in the external forcing such as increases in greenhouses gases, changes in aerosol distributions, variations in solar radiative fluxes, impacts of volcanos, etc.

2. BASIC EQUATIONS

A simulation with a GCM requires to compute the numerical solutions of a set of nonlinear partial differential equations based on the general principles of conservation of momentum, energy and mass.

These equations describe the physical processes that relate the temporal evolution of thermo-hydrodynamical quantities and are specialized to the problem of a thin layer of fluid on a rotating sphere => Primitive equations

External Forcing: Solar radiation.
(Orbit and inclination of rotation axis => Annual cycle)

Specified boundary and initial conditions.
(Distributions of lands and seas, land surface types: forest, grass, deserts,... affecting the heat, moisture and momentum exchanges between the surface and the atmosphere)

Primitive equations (Pexioto and Oort, 1992):

Equations of motion, conservation of mass for a continuous fluid, first law of thermodynamics, balance equation for water vapor (and cloud water and ice), and the equation of state.

Horizontal equations of motions in spherical coordinates:

$$\frac{du}{dt} = \left(f + u\frac{\tan\varphi}{a}\right)v - \frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial\lambda} + F_{\lambda}$$

$$\frac{dv}{dt} = -\left(f + u\frac{\tan\varphi}{a}\right)u - \frac{1}{\rho}\frac{\partial p}{\partial\varphi} + F_{\varphi}$$

where:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial z}$$

Hydrostatic equation:

$$\frac{\partial p}{\partial z} = -\rho g$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial \lambda}(\rho u) + \frac{\partial}{\partial \varphi}(\rho v \cos\varphi)\right] - \frac{\partial}{\partial z}(\rho w)$$

First law of thermodynamics (energy conservation):

$$C_p \frac{d}{dt}(\ln T) - R_d \frac{d}{dt}(\ln p) = \frac{Q}{T}$$

Equation of state for air (perfect gas):

$$p = \rho R_d T_v$$

Balance equation for water vapor:

$$\frac{dq}{dt} = (e - p) + D$$

Coordinates:

λ, longitude; φ, latitude; and z, geometrical height

Variables:

u, zonal wind; v, meridional wind; w, vertical velocity;
ρ, density of air; p, pressure; T, temperature;
[T_v, virtual temperature, = T[1+(Rv/Rd -1)q]

Note:

- The continuity equations is re-written in terms of the surface pressure (by vertical integration and by using the hydrostatic equation) that therefore substitute the density as a prognostic variable.

- A vertical coordinate that is function of pressure is generally used, most popular an hybrid coordinate, terrain following at the bottom and pure pressure above the tropopause.

f = *Coriolis parameter*
a = *radius of the Earth*
C_p = *atmospheric specific heat at constant pressure*
R_d = *gas constant for dry air*
R_v = *gas constant for water vapor*

FORCING and DISSIPATIVE TERMS (=> Parameterizations):

F_λ and F_φ: surface stress, wave stress, eddy and molecular diffusion;

Q: heating/cooling rates from radiative fluxes (solar and terrestrial); latent and convective heat; dissipation of kinetic energy

D: eddy and molecular diffusion.

e = evaporation; p = precipitation

Problems:

Equations:

- There are no analytical solutions => numerical inaccuracies Discrete resolution in space and time and parameterization of subgrid (unresolved) scale fluxes (turbulent motions, cloud processes, radiative processes, convective processes).

=> Some physical processes and feedback (cloud physics, for instance) involved are still poorly known;

Forcing, solar input:

- Radiative transfer schemes: computationally expensive.

- The distributions of short lived atmospheric constituents are not known (tropospheric ozone, aerosols, sources and sinks?)

- Variability in the solar flux.

Boundary and initial conditions:

- usually not sufficiently defined (incomplete, not accurate)

=> Differences between observed and simulated climate.

=> Simplification, specialization, selection of the natural processes represented in the model.

=> Variety of models.

What allowed (and is allowing) for the development of GCMs:

High speed computers, large data storage and handling facilities, global scale monitoring systems (satellites, radiosonde networks, surface observation networks).

3. NUMERICAL APPROACH TO THE PROBLEM

Discretization of the primitive equations, requirement: enough horizontal resolution to resolve synoptic eddies.

(i) Spectral transform method in the horizontal and finite difference in the vertical dimension. Finite series of spherical harmonic basis functions. Nonlinear terms and parametrized physics on associated Gaussian Grid (grid point space). FAST and ACCURATE at relatively low horizontal resolutions

Major problems: Distribution of tracers, treatment of mountains (ripples close to sharp changes in distribution).

(ii) Gridpoint models on a longitude latitude grid with polar filter (to overcome problems related to the convergence of the meridians): fully finite differences or finite volumes.

(iii) Geodesic (icosahedral - hexagonal grid) grids (renewed interest for high horizontal resolutions)

Parameterizations:

Shortwave and longwave radiative transfer:
Spectral range and resolution considered, so called line-by-line models cannot be used in GCMs. Distributions of gases, aerosols, clouds. Radiative properties of clouds, aerosols (microphysics)

Cumulus convection: vertical eddy transport of momentum, heat, moisture, cloud water and ice, tracers. Condensation and Evaporation.

Stratiform clouds and large scale condensation (prognostic).

Planetary boundary layer, turbulence transfer in the free atmosphere, kinetic energy dissipation.

Land-surface processes: surface - atmosphere exchanges.

Orographic gravity wave drag and momentum flux deposition due to a gravity wave spectrum.

Momentum deposition from upward propagating gravity waves:

(i) Reversal of the temperature gradient at the mesopause:
Mesospheric gravity wave breaking.
Downward control of the temperature distribution (Haynes et al.1991; McIntyre 1992; Garcia and Boville 1994); dynamical driving of the Brewer - Dobson circulation; i.e., the large scale circulation that transports chemical constituents and aerosols.

(ii) Middle Atmosphere tropical oscillations in zonal wind:
Semiannual oscillation at the stratopause and at the mesopause,
Quasi-biennial oscillation in the lower stratosphere.
Gravity waves forced by cumulus convection activity.

=> The middle atmosphere is mechanically driven (from below) and radiatively damped.

3. CURRENT RESEARCH TOPICS IN MIDDLE ATMOSPHERE MODELLING

- Extratropical dynamics [winter and spring seasons]:

(i) Relative importance of planetary and gravity wave driving for the residual mean meridional circulation.

(ii) Planetary - gravity wave interactions.

(iii) Improvements in planetary wave forcing (troposphere).

- Quasi-Biennial Oscillation:

(i) Vertical resolution and dissipation.

(ii) Role played by small scales of motion.

(iii) Role played by convection as a forcing mechanism of equatorial planetary waves and gravity waves.

- Parameterization of the momentum flux deposition due to a continuous spectrum of gravity waves.

References on the early development of general circulation models

Arakawa, A, 1966: Computational design for long term numerical integration of the equation of fluid motion: Two-dimensional incompressible flow. Part 1., J. Comp. Phys. 1, 119-143.

Arakawa, A and Lamb, 1977: Computational design of the basic dynamical procesess of the UCLA general circulation model. Methods in Comp. Phys. , 17, General circulation models of the atmosphere. Ed. J Chang, Academic Press, San Diego, California, USA.

Bourke W, 1972: An efficient, one level primitive equation spectral model, Mon. Wea. Rev. 100, 683-689.

Bourke W, 1974: A multi level spectral model. 1.Formulation and hemispheric integrations Mon. Wea. Rev., 102, 687-701.

Charney JG, R Fjortoft and J von Neumann, 1950: Numerical integration of the barotropic vorticity equation. Tellus, 2, 237-254.

Fels, S.B., J.D. Mahlman, M.D. Schwarzkopf, and R.W. Sinclair, Stratospheric sensitivity to perturbations in ozone and carbon dioxide: radiative and dynamical responses. J. Atmos. Sci., 37, 2265-2297, 1980.

Haltiner, GJ and RT Williams, 1980: Numerical prediction and dynamical meteorology. Wiley, New York, New York, USA.

Holton, JR, 1992: An introduction to dynamic meteorology. Academic Press, San Diego, California, USA.

Hoskins BJ, and AJ Simmons, 1975: A multi-layer spectral model and the semi-implicit method. Quart. J. Roy. Meteor. Soc., 101, 637-655.

Kasahara, A, 1974: Various vertical coodinate system used for numerical weather prediction, Mon. Wea. Rev., 102, 509-522.

Kurihara, Y, 1965: On the use of implicit and iterative methods for the time integration of the wave equation. Mon. Wea. Rev., 93, 33-46.

Machenhauer, B, 1979: The spectral method. In Numerical methods used in atmospheric models. GARP Publication series 17, 121-275, WMO, Geneva, Switzerland.

Orzag, SA, 1970: Transform method for the Calculation of vector coupled sums: Application to the spectral form of the vorticity equation, J. Atmos. Sci., 27, 890-895.

Pexioto JJ P and A H Oort, 1992: Physics of climate. American Institute of Physics, New York, New York, USA.

Phillips NA, 1957: A coordinate system having some special advantages for numerical forecasting. J Meteor. , 14, 184-185.

Richardson, LF, 1922: Weather prediction by mumerical process. Cambridge University Press, London, England. 1965: reprinted by Dover Pubblications, New York, New York, USA.

Robert, AJ, 1969: The integration of a spectral model of the atmosphere by the implicit method. Proceeding of WMO/ICSU, symposium on numerical weather prediction, Tokyo, 1968, S-VII, 19-24.

Robert, AJ, TL Yee and H Richie, 1985: A semi-lagrangian and semi-implicit numerical integration scheme for multilevel atmospheric models. Mon. Wea. Rev., 113, 388-394.

Simmons AJ, R Strüfing, 1981: An energy and angular momentum conserving finite difference schme, hybrid coordinates and medium range weather prediction. ECMWF Technical Report 28, 68 pp. Reading, England.

Sadourny, R, 1975: The dynamics of finite difference models of the shallow water equations. J. Amos. Sci., 32, 680-689.

Simmons AJ, BJ Hoskins and DM Burridge, 1978: Stability of the semi-implicit method of time integration. Mon. Wea. Rev., 106, 405-412.

Smagorinsky, J, 1963: General circulation experiments with the primitive equations. 1. The basic experiment. Mon. Wea. Rev., 91, 99-104.

Smagorinsky, J, S Manabe, and JL Halloway, 1965: Nemerical results from a nine level circulation model of the atmosphere. Mon. Wea. Rev. 93, 727-768.

Williamson DL, JT Kiehl, V Ramanathn, RE Dickinson, JJ Hack, 1987: Description of the NCAR CCM1, NCAR/TN-285+STR, Boulder, Colorado, USA.

Links related to GCM and climate research:

- On line documentation of the NCAR CCM3 model (Boulder, CO, USA):
http://www.cgd.ucar.edu/cms/ccm3/ccm3lsm_doc/

- Climate research at GFDL (Princeton, NJ, USA):
http://www.gfdl.noaa.gov/~kd/ClimateDynamics/climate.html

- General circulation model development at GISS (New York, NY, USA):
http://www.giss.nasa.gov/research/modeling/gcms.htm

- Model developments at Max Planck Institute (Hamburg, Germany):
http://www.mpimet.mpg.de/working_groups/wg2/index.html

- Hadley Centre for Climate Prediction and Research (Bracknell, UK):
http://www.meto.govt.uk/research/hadleycentre/

- Climate research at Laboratoire de Meteorologie Dynamics (Paris, France):
http://www.lmd.jussieu.fr/en/Climat/LMD_Climat_frame.html

- Atmospheric Model Intercomparison Project:
http://www-pcmdi.llnl.gov/amip/amiphome.html

NCAR/TN-285+STR
NCAR TECHNICAL NOTE

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Description of NCAR Community Climate Model (CCM1)

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2. CONTINUOUS GOVERNING EQUATIONS

The continuous equations are similar to those used by Bourke (1974) and Hoskins and Simmons (1975) and adopt the σ-vertical coordinate proposed by Phillips (1957).

a. Momentum Equations

The zonal and meridional components of the momentum equations may be written in σ coordinates as

$$\frac{\partial u}{\partial t} = \eta v - \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}(\Phi + E) - \frac{RT_v}{a\cos\phi}\frac{\partial}{\partial\lambda}\ell n p_s - \dot{\sigma}\frac{\partial u}{\partial\sigma} + F_{uV} + F_{uH}, \quad (2.a.1)$$

$$\frac{\partial v}{\partial t} = -\eta u - \frac{1}{a}\frac{\partial}{\partial\phi}(\Phi + E) - \frac{RT_v}{a}\frac{\partial}{\partial\phi}\ell n p_s - \dot{\sigma}\frac{\partial v}{\partial\sigma} + F_{vV} + F_{vH}, \quad (2.a.2)$$

where the absolute vorticity η, relative vorticity ζ, divergence δ, and kinetic energy E are given by

$$\eta = \zeta + f, \quad (2.a.3)$$

$$\zeta = \frac{1}{a\cos\phi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\phi}(u\cos\phi)\right], \quad (2.a.4)$$

$$\delta = \frac{1}{a\cos\phi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right], \quad (2.a.5)$$

$$E = \frac{1}{2}(u^2 + v^2). \quad (2.a.6)$$

Here f is the Coriolis parameter $(2\Omega\sin\phi)$, t is time, ϕ latitude, λ longitude, σ vertical coordinate (p/p_s), p_s surface pressure, $\dot{\sigma}$ vertical velocity in σ coordinates, u zonal wind component, v meridional wind component, Φ geopotential, a mean radius of the earth, R gas constant for dry air, and T_v virtual temperature. The virtual temperature is given by

$$T_v = \left[1 + \left(\frac{R_v}{R} - 1\right)q\right]T, \quad (2.a.7)$$

where T is temperature, q specific humidity, and R_v gas constant for water vapor. The vertical friction terms F_{uV}, F_{vV}, which include surface fluxes and vertical diffusion, and the horizontal diffusion terms F_{uH}, F_{vH} will be described shortly.

b. Vorticity and Divergence Equations

The momentum equations of the form (2.*a*.1) and (2.*a*.2) are not directly used by the model but rather, following Bourke (1974), their vorticity and divergence counterparts are obtained from (2.*a*.1) and (2.*a*.2) with the relations (2.*a*.3), (2.*a*.4), and (2.*a*.5). In addition, the variable

$$\mu = \sin\phi \tag{2.b.1}$$

is used for the meridional independent variable rather than latitude ϕ, and

$$U = u\cos\phi \tag{2.b.2}$$

$$V = v\cos\phi \tag{2.b.3}$$

are used when velocity components are needed. These forms are more convenient for the spectral representation. The equations for absolute vorticity and divergence can be written

$$\frac{\partial\eta}{\partial t} = \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}(N_v + \cos\phi F_{vV}) - \frac{1}{a}\frac{\partial}{\partial\mu}(N_u + \cos\phi F_{uV}) + F_{\eta H}, \tag{2.b.4}$$

$$\frac{\partial\delta}{\partial t} = \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}(N_u + \cos\phi F_{uV}) + \frac{1}{a}\frac{\partial}{\partial\mu}(N_v + \cos\phi F_{vV}) + F_{\delta H}$$
$$- \nabla^2(E + \Phi + RT_o \ell n p_s). \tag{2.b.5}$$

The spherical horizontal Laplacian operator is denoted ∇^2.

$$\nabla^2 = \frac{1}{a^2(1-\mu^2)}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{a^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}\right]. \tag{2.b.6}$$

The virtual temperature has been divided into two parts, one of which T_o is a function of σ only, in order to facilitate the incorporation of the semi–implicit time–integration scheme,

$$T_v'(\lambda,\mu,\sigma,t) = T_v(\lambda,\mu,\sigma,t) - T_o(\sigma). \tag{2.b.7}$$

The mean temperature T_o for the linearization associated with the semi-implicit scheme is specified *a priori*, usually to be 300°K (Simmons *et al.*, 1978). The nonlinear dynamical terms are

$$N_u = \eta V - RT'_v \frac{1}{a}\frac{\partial}{\partial\lambda}\ell n p_s - \dot{\sigma}\frac{\partial U}{\partial\sigma}, \tag{2.b.8}$$

$$N_v = -\eta U - RT'_v \frac{(1-\mu^2)}{a}\frac{\partial}{\partial\mu}\ell n p_s - \dot{\sigma}\frac{\partial V}{\partial\sigma}. \tag{2.b.9}$$

The horizontal diffusion terms $F_{\eta H}$ and $F_{\delta H}$ are formulated directly using η and δ rather than u and v. They are converted to the equivalent F_{uH} and F_{vH} forms for use in the frictional heating term in the thermodynamic equation.

c. Thermodynamic and Mixing–Ratio Equations

The thermodynamic equation is for the perturbation temperature T' calculated about the same mean T_o as used earlier for the virtual temperature.

$$\begin{aligned}\frac{\partial T'}{\partial t} = &-\frac{1}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}(UT') - \frac{1}{a}\frac{\partial}{\partial\mu}(VT') + T'\delta - \dot{\sigma}\frac{\partial T}{\partial\sigma} + \frac{RT_v}{C_p^*}\frac{\omega}{p} \\ &+ Q_S + Q_{\ell w} + F_{TV} + F_{TH} - \frac{1}{C_p^*}\left[u(F_{uV}+F_{uH}) + v(F_{vV}+F_{vH})\right],\end{aligned} \tag{2.c.1}$$

$$T'(\lambda,\mu,\sigma,t) = T(\lambda,\mu,\sigma,t) - T_o(\sigma), \tag{2.c.2}$$

$$C_p^* = \left[1 + \left(\frac{C_{p_v}}{C_p} - 1\right)q\right]C_p, \tag{2.c.3}$$

where C_p is the specific heat of dry air at constant pressure and C_{p_v} is the specific heat of water vapor at constant pressure. The vertical diffusion F_{TV}, which includes the sensible heat flux from the surface and the horizontal diffusion term F_{TH}, will be defined shortly. The pressure vertical velocity ω and the sigma coordinate vertical velocity $\dot{\sigma} = \frac{d\sigma}{dt}$ are given in the next section [(2.*d*.7) and (2.*d*.5)]. The solar atmospheric heating rate Q_S is given by (2.*g*.34) to (2.*g*.36) and the longwave atmospheric heating rate $Q_{\ell w}$ by (2.*g*.60).

The moisture forecast equation for the specific humidity q is

$$\frac{\partial q}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial}{\partial \lambda}(Uq) - \frac{1}{a}\frac{\partial}{\partial \mu}(Vq) + q\delta - \dot{\sigma}\frac{\partial q}{\partial \sigma} + S + F_{qV} + F_{qH}. \quad (2.c.4)$$

We shall discuss later the mathematical expressions for the source/sink term S for water vapor and the horizontal and vertical water–vapor diffusion terms F_{qH}, F_{qV}.

d. Vertical Velocities and Surface Pressure

The continuity equation in the σ–system is

$$\frac{\partial \ell n p_s}{\partial t} = -\mathbf{V} \cdot \nabla \ell n p_s - \delta - \frac{\partial \dot{\sigma}}{\partial \sigma}, \quad (2.d.1)$$

where $\mathbf{V}$ is the horizontal velocity vector with components (u, v) and

$$\mathbf{V} \cdot \nabla \ell n p_s = \frac{U}{a(1-\mu^2)}\frac{\partial \ell n p_s}{\partial \lambda} + \frac{V}{a}\frac{\partial \ell n p_s}{\partial \mu}. \quad (2.d.2)$$

The continuity equation (2.d.1) is not used directly but, when integrated in the vertical, gives equations for the surface–pressure tendency and for $\dot{\sigma}$. Integrating (2.d.1) from $\sigma = 0$ to $\sigma = 1$, with the boundary conditions

$$\dot{\sigma} = 0 \quad \text{at} \quad \sigma = 0 \quad \text{and} \quad 1, \quad (2.d.3)$$

gives the prognostic equation for surface pressure,

$$\frac{\partial \ell n p_s}{\partial t} = -\int_o^1 (\delta + \mathbf{V} \cdot \nabla \ell n p_s) d\sigma. \quad (2.d.4)$$

The diagnostic equation for the sigma vertical velocity $\dot{\sigma}$ is derived by integrating the continuity equation (2.d.1) vertically from the top of the atmosphere ($\sigma = 0$) to σ and substituting the surface–pressure–tendency equation (2.d.4),

$$\dot{\sigma} = \sigma \int_o^1 (\delta + \mathbf{V} \cdot \nabla \ell n p_s) d\sigma - \int_o^\sigma (\delta + \mathbf{V} \cdot \nabla \ell n p_s) d\sigma. \quad (2.d.5)$$

The pressure vertical velocity is obtained from (2.d.5) and (2.d.4) using

$$\frac{\omega}{p} = \frac{\dot{\sigma}}{\sigma} + \frac{d \ell n p_s}{dt}, \quad (2.d.6)$$

as

$$\frac{\omega}{p} = \mathbf{V} \cdot \nabla \ell n p_s - \frac{1}{\sigma} \int_o^{\sigma} (\delta + \mathbf{V} \cdot \nabla \ell n p_s) d\sigma. \tag{2.d.7}$$

e. Hydrostatic Equation and Equation of State

The hydrostatic equation is

$$\frac{\partial \Phi}{\partial \ell n \sigma} = -RT_v, \tag{2.e.1}$$

or in integral form,

$$\Phi = \Phi_s - \int_{\sigma=1}^{\sigma} RT_v d\ell n \sigma, \tag{2.e.2}$$

where Φ_s is the geopotential at the earth's surface.

The equation of state is that for a moist atmosphere,

$$p = \rho RT_v, \tag{2.e.3}$$

where ρ is the density and T_v was given earlier (2.a.7).