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WINTER COLLEGE ON LASERS, ATOMIC AND MOLECULAR PHYSICS

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Quantum Mechanics and Interactions with Weak E.M. Waves

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from the lectures
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A Simple calculation of coherent states

Define the coherent states $|\alpha\rangle$ as:

$$a|\alpha\rangle = \alpha|\alpha\rangle \,, \quad [a, a^+] = 1 \qquad (1)$$

Also the $|n\rangle$ states are characterized by:

$$a^+a|n\rangle = n|n\rangle \,. \qquad (2)$$

Multiplying the equation (1) by $\langle n|$ we get:

$\langle n|a|\alpha\rangle = \alpha \langle n|\alpha\rangle$. (3) Now, from the relations:

$$\begin{aligned} a^+|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \langle n|a &= \langle n+1|\sqrt{n+1} \end{aligned} \,, \qquad (4)$$

and using the equation (3) we can write:

$$\sqrt{n+1}\,\langle n+1|\alpha\rangle = \alpha\langle n|\alpha\rangle \qquad (5).$$

Since the $|n\rangle$ states are complete, one can always write:

$$|\alpha\rangle = \sum_n c_n|n\rangle \qquad (6)$$

with $\quad c_n = \langle n|\alpha\rangle \,. \qquad (7)$

Equation (5) can be written as:

$$\sqrt{n+1}\,c_{n+1} = \alpha c_n \quad \text{or} \quad c_{n+1} = \frac{\alpha c_n}{\sqrt{n+1}} \qquad (8)$$

For example:

$$\begin{aligned} n=0 \qquad & c_1 = \alpha c_0/\sqrt{1} \\ n=1 \qquad & c_2 = \alpha c_1/\sqrt{2} = \alpha^2 c_0/\sqrt{1\cdot 2} \qquad (9) \\ \vdots \qquad & \\ n=n \qquad & c_n = \alpha^n c_0/\sqrt{n!} \end{aligned}$$

Now, from the normalization condition $\langle\alpha|\alpha\rangle = 1$ one can write:

$$1 = |c_0|^2 \sum_n \frac{|\alpha|^{2n}}{n!} \qquad (10)$$

or

$$c_0 = |c_0| = e^{-|\alpha|^2/2} \qquad (11)$$

↑(up to an arbitrary phase)

The probability of having n-photons in a coherent state is given by:

$$p_n = |c_n|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} = \frac{e^{-\bar{n}}(\bar{n})^n}{n!} \,, \qquad (12)$$

which is a Poisson distribution $[\,|\alpha|^2 = \bar{n}\,]$

Another way of seeing this is:

$$\bar{n} = \langle\alpha|a^+a|\alpha\rangle = |\alpha|^2\langle\alpha|\alpha\rangle = |\alpha|^2 \,. \qquad (13)$$

This can be generalized. If an expression is in the form of a normally ordered product, that is all creation operators to the left and annihilation operators to the right, then:

$$\langle\alpha|(a^+)^m(a)^n|\alpha\rangle = (\alpha^*)^m\alpha^n \qquad (14)$$

or the following substitution is valid:

$$a \longrightarrow \alpha$$
$$a^+ \longrightarrow \alpha^*$$

Let's calculate, now the dispersion of the number of photons.

The dispersion of an operator A is

$\sigma^2(A) = \langle A^2\rangle - \langle A\rangle^2$ or the root mean square deviation

$$\sigma(A) = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$$

So if $N = a^+ a$

$$\langle N^2\rangle_\alpha = \langle a^+ a a^+ a\rangle_\alpha \quad \text{and since} \quad [a, a^+] = 1$$

$$\langle N^2\rangle_\alpha = \langle a^+(1 + a^+ a)a\rangle_\alpha = \langle a^+ a\rangle_\alpha + \langle a^{+2} a^2\rangle_\alpha$$

$$\langle N^2\rangle_\alpha = |\alpha|^2 + |\alpha|^4 \qquad (15)$$

(This term originates from $[a, a^+] = 1$)

on the other hand

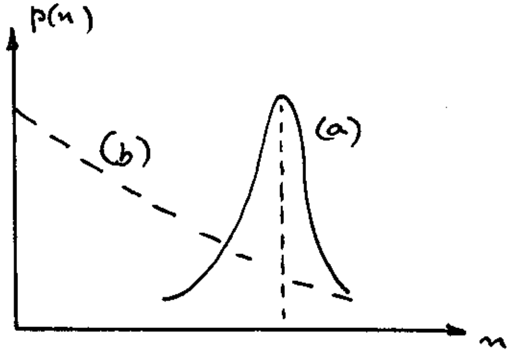
$$\langle N\rangle^2 = |\alpha|^4 \qquad (16)$$

therefore:

$$\sigma^2(N) = \langle N^2\rangle - \langle N\rangle^2 = \bar{n} \qquad (17)$$

The experimentally relevant quantity is:

$$\frac{\sigma(n)}{\bar{n}} = \frac{\sqrt{\bar{n}}}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} \xrightarrow[\text{as } \bar{n} \to \text{large}]{} 0$$



(Figure 1)

In figure 1, curve (a) represents the photon-statistics of a coherent state; curve (b) a chaotic state.

The probability p_n^{chaotic} is given by

$$p_n^{\text{chaotic}} = \left(\frac{\bar{n}}{1+\bar{n}}\right)^n \frac{1}{1+\bar{n}} \qquad (18)$$

(Bose Einstein Distribution)

The photon-statistics of a chaotic field is defined as follows, from maximum entropy requirements

$$p_n = \frac{e^{-\beta E_n}}{Z} = \frac{e^{-\beta(n\hbar\omega)}}{Z} \qquad (19)$$

where

$$Z = \sum_n e^{-\beta n\hbar\omega} = (1 - e^{-\beta\hbar\omega})^{-1} \qquad (20)$$

β is found as follows:

$$\langle E\rangle = \bar{n}\hbar\omega = \hbar\omega \sum_n n\, p(n) = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \qquad (21)$$

Hence for a thermal field in the classical limit $\hbar \to 0$

$$\langle E\rangle = kT = \frac{\hbar\omega}{1 + \beta\hbar\omega - 1} = \frac{1}{\beta} \Longrightarrow \beta = \frac{1}{kT} \qquad (22)$$

(22)

In general

$$e^{\beta\hbar\omega} - 1 = \frac{1}{\bar{n}} \quad \text{or} \quad e^{\beta\hbar\omega} = \frac{1+\bar{n}}{\bar{n}}$$

which leads to

$$\beta = \frac{1}{\hbar\omega} \ln\left(1 + \frac{1}{\bar{n}}\right) \qquad (23)$$

Replacing these expressions in p_n (eq. 19) one gets

$$p_n = \left(\frac{\bar{n}}{1+\bar{n}}\right)^n \frac{1}{1+\bar{n}} \qquad \text{(Bose-Einstein)} \quad (24)$$

which gives the probability statistics of a chaotic field in terms of the average photon number $\bar{n}$.

Also

$$\sigma^{\text{chaotic}}(n) = \sqrt{\bar{n}(\bar{n}+1)} \quad \text{for large } \bar{n} \qquad (25).$$

The main difference between the coherent and chaotic light is the following:

$$p(n) = \frac{e^{-\bar{n}} (\bar{n})^n}{n!}$$

$$\frac{\sigma(n)}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}$$

} coherent

For $\bar{n} >> 1$ $\quad \sigma(n)/\bar{n} \sim 1$

$$p(n) = \left(\frac{\bar{n}}{1+\bar{n}}\right)^n \frac{1}{1+\bar{n}}$$

} chaotic.

We will now show that a classical current produces a coherent state.

Theorem: Suppose that a system is represented by a Hamiltonian $H(a, a^+)$ that depends at most linearly with (a^+). In that case, if the initial state of the system is a coherent state, it will remain so at any time (or "coherent states are preserved")

Proof.-

If H is linear in a^+ $\quad \frac{\partial H}{\partial a^+} = F(a,t)$ independent of a^+

The Heisenberg Equation for a is:

$$\dot{a} = \frac{1}{i\hbar}[a, H] = \frac{1}{i\hbar}\frac{\partial H}{\partial a^+} \qquad (26)$$

So $\dot{a}$ depends on the annihilation operator only.

We can integrate equation (26) and obtain:

$$a_H(t) = F'(a,t) \quad . \quad (27)$$

Now, let the initial state be:

$$|\psi\rangle_0 = |\alpha_0\rangle \quad , \text{ then}$$

$$a_H(t)|\alpha_0\rangle = F'(\alpha_0, t)|\alpha_0\rangle \qquad (28)$$

(replacing $a \to \alpha_0$ as seen before)

Now

$$a_H(t) = U^{-1}(t)\, a\, U(t) \qquad \text{so}$$

$$U \;/\; U^{-1}(t)\, a\, U(t)|\alpha_0\rangle = F(\alpha_0, t)|\alpha_0\rangle$$

$$a\, U(t)|\alpha_0\rangle = F(\alpha_0, t)\, U(t)|\alpha_0\rangle$$

but

$$U(t)|\alpha_0\rangle \equiv U(t)|\psi_0\rangle = |\psi\rangle_t$$

and therefore

$$a|\psi\rangle_t = F(\alpha_0, t)|\psi\rangle_t \quad . \quad (29)$$

$|\psi\rangle_t$ is still an eigenstate of the annihilation operator, therefore a coherent state, thus proving the theorem.

Now, ~~the coupling~~ as an application, we study the coupling with a classical current. The coupling term of the Hamiltonian of the field with a classical current can be written as

$$H' = \int J(\vec{r},t)\cdot\vec{A}(\vec{r},t)\, d^3r \qquad (30)$$

where

$$A(\vec{r},t) = \sum_k{}' (a_k u_k + h.c) \quad . \quad (31)$$

We can also write:

$$J(\vec{r},t) = \sum_k{}' (J_k u_k + h.c) \qquad (32)$$

Therefore, since $\int u_k^* u_{k'}\, dV = \delta_{kk'}$

$$H' = g\sum_k{}' (a_k J_k^+ + a_k^+ J_k) \quad . \quad (33)$$

The entire Hamiltonian can be written as follows (take 1-mode for simplicity)

$$H = \hbar\omega a^+ a + g\,(a\,J^+(t) + a^+ J(t)) \tag{34}$$

[Linear in a^+ therefore the previous theorem applies]
The Heisenberg equation for a now is:

$$\dot a = \frac{1}{i\hbar}[a, H] = \underbrace{-i\omega a}_{\text{harmonic oscillator part}} + \underbrace{g\,J(t)}_{\text{a classical current acting as a forcing term in the harmonic oscillator}} \tag{35}$$

Integrating the equation (35), we get:

$$a_H(t) = a_0\, e^{-i\omega t} + \underbrace{g\int_0^t e^{-i\omega(t-t')} J(t')\, dt'}_{\equiv\, \alpha_1(t)\ \ (\text{c-number})} \tag{36}$$

or

$$a_H(t) = \alpha_0\, e^{-i\omega t} + \alpha_1(t) \tag{37}$$

For a free field ($\alpha_1 = 0$) and considering that
$\langle p \rangle \propto \mathrm{Im}\,\alpha$, $\langle q \rangle \propto \mathrm{Re}\,\alpha$
we have the classical oscillatory motion of the harmonic oscillator.
Suppose now that the initial state is the vacuum ($\alpha_0 = 0$) and there is no current.
Then we switch-on the classical current then $\alpha(t) = \alpha_1(t)$, the field will go to a coherent state and will have a Poisson distribution.
We can create a coherent state by:
a) A classical current to generate microwaves (antenna)
b) A Laser at optical frequencies.

Random phase coherent-field.
If the state is $|\alpha\rangle$, the density operator will be

$$\rho = |\alpha\rangle\langle\alpha| = e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n(\alpha^*)^m}{\sqrt{n!\,m!}}\,|n\rangle\langle m| \tag{38}$$

and is a pure state, so that $\rho^2 = \rho$.

Let us put $\alpha = |\alpha|\, e^{i\varphi}$, hence

$$\rho(\varphi, |\alpha|) = |\alpha\rangle\langle\alpha| = e^{-|\alpha|^2}\sum_{n,m}\frac{|\alpha|^n|\alpha|^m}{\sqrt{n!\,m!}}\, e^{i\varphi(n-m)}\,|n\rangle\langle m| \tag{39}$$

In reality, the absolute phase φ is not known. So we must average over φ. Define a new density operator

$$\bar\rho(|\alpha|) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \rho(\varphi, |\alpha|) = \sum_n p_n\,|n\rangle\langle n| \tag{40}$$

with
$$p_n = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{41}$$

Since
$$\frac{1}{2\pi}\int d\varphi\; e^{i\varphi(n-m)} = \delta_{n,m}. \tag{42}$$

The statistical operator $\bar\rho$ is now diagonal and represents a statistical mixture which does not contain phase information. Moreover $\mathrm{Tr}\,\bar\rho^2 < \mathrm{Tr}\,\bar\rho = 1$.

Weak Electromagnetic Interactions,
Perturbation Theory – Transition Probabilities.

We will assume a monochromatic field and study how it interacts with one atom.

$$H'(\text{dipole approximation}) = e\,\vec{r}\cdot\vec{E} \qquad (43)$$

$\vec{E}$: externally applied field

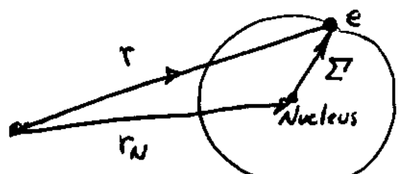
In NMR we have $\vec{\mu}\cdot\vec{B}$ where $\vec{\mu}$ is the magnetic dipole and B the magnetic field.

$e\vec{r} - \hat{\mu}$ electric dipole

We write the electric field as:

$$\vec{E} = (\vec{\sigma}\,E_0\,e^{i\omega_0 t} + c.c) \qquad (44)$$

In the electric-dipole approximation one assumes that the spatial dependence of E_0 plays no-role; that is (see figure 2)



$$e^{i\vec{k}\cdot\vec{r}} = e^{i\vec{k}\cdot\vec{r}_N}\, e^{i\vec{k}\cdot\vec{\Sigma}} \qquad (45)$$

but $e^{i\vec{k}\cdot\vec{r}_N}$ is a constant and

$$e^{i\vec{k}\cdot\vec{\Sigma}} \sim 1 \quad \text{if} \quad k\,r_B \ll 1$$

(r_B = Bohr's Radius)

or if $r_B \ll \lambda$ [Valid for instance at optical frequencies; not true for ~~[illegible]~~ very short wavelength]

$$H' = (e E_0\,\vec{r}\cdot\vec{\sigma}\,e^{i\omega_0 t} + c.c) \qquad (46)$$

Let: $\hat{\mu} \equiv e\,\vec{r}\cdot\sigma$ be the dipole moment operator in the direction of the field, then we can write:

$$H' = (\hat{\mu}\,E_0\,e^{i\omega_0 t} + c.c) \qquad (47)$$

Let's write Schrödinger's Equation:

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = (H_0 + H')\,|\psi\rangle \qquad (48)$$

where H_0 is the unperturbed Hamiltonian and let the $|n\rangle$ states be the eigenstates of H_0, namely:

$$H_0|n\rangle = E_n|n\rangle \qquad (49)$$

Then, we can always write

$$|\psi\rangle_t = \sum_n c_n(t)\,|n\rangle \qquad (50)$$

with $\quad c_n(t) = \langle n|\psi\rangle_t$

Therefore, Schrodinger's equation can be written as:

$$i\hbar\,\dot{c}_n(t) = E_n c_n(t) + \langle n|\,H'\,|\psi(t)\rangle$$

$$= E_n c_n + \sum_m \langle n|H'|m\rangle\, c_m(t)$$

$$i\hbar\,\dot{c}_n(t) = E_n c_n + \sum_m \langle n|H'|m\rangle\, c_m(t) \qquad (51)$$

Normally H' has ~~no~~ diagonal terms.

In case there are diagonal terms, one can always re-define the energy levels as:

$$E_n' = E_n + \langle n|H'|n\rangle \qquad (52)$$

Let's Write

$$H'_{nm} = \hat{\mu}_{nm}\, e^{i\omega_0 t} + c.c \qquad (53)$$

In order to eliminate the fast ~~to~~ time varying terms, we go to the interaction picture by using the following transformation:

$$c_n(t) = b_n(t)\, e^{-\frac{i}{\hbar}E_n t} \qquad (54)$$

Notice that if $H'=0$ from equation (51) we get: $c_n(t) = c_n(0)\, e^{-\frac{i}{\hbar}E_n t}$ and $b_n = \text{constant}$.

We now substitute c_n in equation (51) and obtain:

$$i\hbar\,\dot{b}_n(t) = \sum_m \langle n|H'|m\rangle\, b_m(t)\, e^{-i\omega_{nm}t} \qquad (55)$$

with $\quad \omega_{n,m} \equiv \dfrac{E_n - E_m}{\hbar} \qquad (56)$

This representation will help us to find the resonances.

$$H'_{nm} = \hat{\mu}_{n,m}\, E_0\,(e^{i\omega_0 t} + c.c)\,. \qquad (57)$$

(Assume E_0 [redacted] and μ_{nm} are real.

Then

$$i\hbar\,\dot{b}_n(t) = \sum_m E_0\,\hat{\mu}_{n,m}\, b_m \left[e^{i(\omega_0-\omega_{nm})t} + e^{-i(\omega_0+\omega_{nm})t}\right] \qquad (58)$$

and

$$p_n(t) = |b_n(t)|^2 = \text{probability of ocupation of the } n\text{th level.}$$

Resonances:

Atom

Incident light

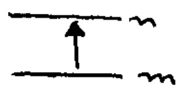
a) Case $\omega_{nm} = \omega_0 \geq 0$ (ω_0 positive)

In equation 58

$e^{i(\omega_0 - \omega_{nm})t} = 1$ Resonant term

$e^{-i(\omega_0 + \omega_{nm})t} \sim e^{-2i\omega_0 t} \longrightarrow$ irrelevant after a time $t > 1/\omega_0$ as compared to 1

$\omega_{nm} > 0$



$E_n > E_m$
(final) (initial)

This corresponds to a resonant absorption event.

b) Resonant Emission

$\omega_{nm} = -\omega_0$

$e^{i(\omega_0 - \omega_{nm})t}$ irrelevant

$e^{-i(\omega_0 + \omega_{nm})t} \approx 1$ resonant

$E_n - E_m < 0$ or $E_m > E_n$

(Emission)

Suppose now, quasi-resonance ~~case~~ ($\omega_0 \approx \omega_{nm}$)

1^{st} order perturbation treatment:

$$b_m(t) \xrightarrow[\text{approx.}]{1^{st}\text{ order}} b_m(0)$$

[To be replaced in the right hand side of equation (58)]

$$i\hbar \dot{b}_n = \sum_m E_0 \hat{\mu}_{nm} b_m(0) e^{+i(\omega_0 - \omega_{nm})t}$$

Defining $\Omega_{nm} = \dfrac{E_0 \hat{\mu}_{nm}}{\hbar}$ = Rabi flopping frequency

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If from the summation, we only select the quasi-resonant term and defining

$\delta = \omega_0 - \omega_{nm}$, we can write:

$$\dot{b}_n = \frac{E_0 \mu_{nm}}{\hbar} e^{-i\delta t} = \Omega e^{-i\delta t}$$

and finally get (integrating)

$$p_n(t) = |b_n(t)|^2 = \Omega^2 \left(\frac{\sin^2 \frac{\delta t}{2}}{\delta^2} \right) \quad (59)$$

This result breaks down at resonance

if $\delta = 0$

$p_n(t) = \Omega^2 t^2/4$ but since

$p_n \leq 1$, this is valid only if

$\Omega t << 1$ or $t << \frac{1}{\Omega}$

Since Ω is proportional to the field amplitude, this treatment is valid only for short times or very weak field.

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As we shall see next, equation (59) is correct at all times only if $\delta \neq 0$ and $\delta >> \Omega_R$

Exact Theory in the 2-level approximation.

When there is quasi-resonance and when μ_{21} is allowed by the selection rules, the transition probability to the other levels can be assumed small. Mathematically

$$|\psi\rangle = b_1(t)\,|1\rangle + b_2\,|2\rangle + \dots$$

$$b_n \simeq 0 \quad n \neq 1,2$$

What we did in the 1st order approximation was
$b_n(t) \longrightarrow b_n(0)$
now we remove this condition.
If we assume that we have only b_1 and b_2, we get 2 coupled equations

$$\dot{b}_1 = -i\Omega\, b_2(t)\, e^{i\delta t}$$
$$\dot{b}_2 = i\Omega\, b_1(t)\, e^{-i\delta t}$$

[Taking $b_1 = b_1(0)$ or $b_2 = b_2(0)$ in the right-hand side, we get back the 1st order approximation]
Notice that we have neglected the term $e^{i(\omega_0+\omega_{21})t}$ from the exact equation. This is the Rotating-Wave Approximation.
It is simple to verify that
$$\frac{d}{dt}\left[\,|b_1|^2 + |b_2|^2\,\right] = 0$$
Conservation of probability since $|b_1|^2 + |b_2|^2 = 1$

now, assume initially:
$b_1(0) = 1$
$b_2(0) = 0$
— 2
—•— 1 Initial

The solution for $b_2(t)$ is:

$$|b_2(t)|^2 = p_2(t) = \frac{4\Omega^2}{\delta^2+4\Omega^2}\sin^2\left(\frac{\sqrt{\delta^2+4\Omega^2}}{2}\,t\right)$$

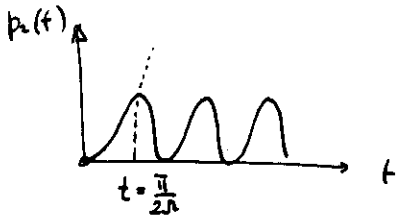
a) In the case of "out of resonance" or small field $\left[\frac{2\Omega}{\delta} << 1\right]$, one gets

$$p_2 \simeq \frac{4\Omega^2}{\delta^2}\sin^2 \delta t/2$$ which is the 1st order result.

(this result is valid for all times)

b) In the case of resonance or strong field $\delta << 2\Omega$

$$p_2(t) = \sin^2\Omega t\,, \quad p_1(t) = \cos^2\Omega t$$



The probability of occupation of the upper level becomes 1 at $t = \pi/2\Omega$

If $\Omega t << 1 \qquad p_2 \sim \Omega^2 t^2$ (as before)

The Feynman-Helworth Geometrical Representation.

In the 2-level approximation

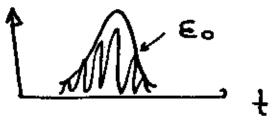
$$|\psi\rangle = c_1(t)\,|1\rangle + c_2(t)\,|2\rangle$$

and the Schrödinger Equation reduces to

$$i\hbar\,\dot{c}_1 = c_1E_1 + H'(t)\,c_2$$
$$i\hbar\,\dot{c}_2 = c_2E_2 + H'(t)\,c_2$$

where $H'(t) = \Omega\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$, $\Omega = \frac{E_0\mu}{\hbar}$

Note: The previous derivation doesnot assume that Ω or E_0 is a constant. As a matter of fact, for fast pulse phenomena, it will be a slowly varying envelope



Density Matrix:

$\rho = |\psi\rangle\langle\psi|$ for a pure state

using the expression for $|\psi\rangle$ we can write:

$$\rho = |c_1|^2\,|1\rangle\langle 1| + |c_2|^2\,|2\rangle\langle 2| + c_2^* c_1\,|1\rangle\langle 2| + c_1^* c_2\,|2\rangle\langle 1|$$

therefore, the matrix elements of ρ are:

$\rho_{11} = |c_1|^2 = p_1(t)$ = probability of occupying state #1

$\rho_{22} = |c_2|^2 = p_2(t)$ = " " " " #2

$\rho_{12} = c_2^* c_1 = \rho_{21}^*$

$\rho_{21} = c_1^* c_2$ $\qquad \rho_{21}^* = \rho_{12}$ (ρ is Hermitian)

So we can write

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \qquad \text{with: } \mathrm{Tr}\,\rho = 1$$

Note: the off diagonal elements of ρ are responsable for the polarization.

To see this, let's write $\mu = \begin{pmatrix} 0 & \mu_{12} \\ \mu_{12} & 0 \end{pmatrix}$

and $\langle\mu\rangle = \mathrm{Tr}(\rho\mu) = \rho_{12}\mu_{21} + \rho_{21}\mu_{12}$

as we can see if $\rho_{12} = 0$ there is zero expectation value for the polarization. In other words the atom must be in a superposition of the 2 states. Assuming $\mu = \mu_{12} = \mu_{21}$ = real

then $\langle\mu\rangle = 2\mu\,\mathrm{Re}\,\rho_{12}$

The system of equation for c_1 and c_2 can be written in term of the elements of ρ. One finds:

$$\dot\rho_{11} = -i(\rho_{11} - \rho_{21})\,H'(t)\,; \quad [\rho_{22} = 1 - \rho_{11}]$$

$$\dot\rho_{21} = -i\omega_{21}\rho_{21} + i\,H'(t)\cdot(\rho_{11} - \rho_{22})\,;$$

with $\omega_{21} = \frac{E_2 - E_1}{\hbar}$

For example, if we consider the case $H' = 0$ (no external field) $\langle\mu\rangle$ will have an oscillatory motion with frequency ω_{21}.

We will now perform a rotation to get rid of the fast time-dependence of H'.

$$\hat{\rho}_{21}(t) = \rho_{21}(t)\, e^{i\omega_0 t}$$

If we neglect the terms of the type $e^{2i\omega_0 t}$ (Rotating-Wave Approximation), we get:

$$\boxed{\begin{aligned} \dot{\tilde{\rho}}_{21} &= -i\,\delta\,\tilde{\rho}_{21} + i\,\Omega\,(\rho_{11}-\rho_{22}) \\ \dot{\rho}_{11} &= i\,(\tilde{\rho}_{12}-\tilde{\rho}_{21})\,\Omega \end{aligned}}$$

Let's define a vector (Bloch vector) whose components are:

$$\text{(pseudo-spin operator)}\left\{\begin{aligned} R_1 &= \rho_{12}+\rho_{21} = 2\,\mathrm{Re}\,\rho_{12} \\ R_2 &= i(\rho_{12}-\rho_{21}) = 2\,\mathrm{Im}\,\rho_{12} \\ R_3 &= \rho_{11}-\rho_{22} \end{aligned}\right.$$

with the conservation of probability condition $\rho_{11}+\rho_{22}=1$, we can write

$$\rho_{11} = \frac{1+R_3}{2}, \qquad \rho_{22} = \frac{1-R_3}{2}.$$

R_3 (if multiplied by the density of atoms) represents a population difference. If goes from 1 to -1

$R_3 = 1$ $\qquad\qquad$ $R_3 = -1$

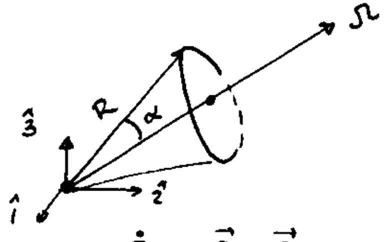
The equation of motion for ρ can be written as an equation for the $\vec{R}$ pseudo-vector as:

$$\dot{\vec{R}} = \vec{\Omega}\times\vec{R} \qquad \text{with } \vec{\Omega} = (0, \Omega_R, \delta)$$

Dinamically this corresponds to a precession of the $\vec{R}$ vector about $\vec{\Omega}$ with a frequency

$$|\vec{\Omega}| = \sqrt{\Omega_R^2+\delta^2}$$

To see this, we now prove that there are 2 constants of motion: $|R|$ and α (the angle between $\vec{R}$ and $\vec{\Omega}$)



Starting from $\dot{R} = \vec{\Omega}\times\vec{R}$

a) multiply by $\vec{R}\cdot$ and we get $\frac{d}{dt}|R|^2 = 0 \implies |R| = \text{constant}$

b) Multiply by $\Omega\cdot$ and we get (only if Ω is constant)
$$\frac{d}{dt}(\Omega\cdot R) = 0 \implies \alpha = \text{constant}$$

One of this quantities ($|R|$) is conserved independently whether Ω_R is a constant or not.

$$|R|^2 = R_1^2+R_2^2+R_3^2 = \text{constant}.$$

(this last statement is only modified if one introduces decay constants; to be seen later)

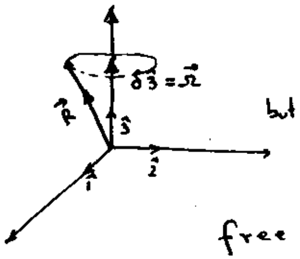
Explicitly, the equations for the components of $\vec{R}$ are:

$$\boxed{\begin{aligned} \dot{R}_1 &= \Omega_R R_3 - \delta R_2 \\ \dot{R}_2 &= \delta R_1 \\ \dot{R}_3 &= -\Omega_R R_1 \end{aligned}}$$

For Ω_R = constant the system of equations can be solved exactly giving $R_3 = \cos\sqrt{\Omega_R^2 + \delta^2}\, t$

Limiting Cases :

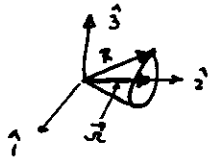
a) Free Case $\Omega_R = 0 \longrightarrow R_3$ is a constant



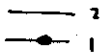
The energy is a constant but the R_1 and R_2 components are varying periodically in time. This is a free precession responsable for the "photon - echo".

b) Resonant Interaction $\delta = 0$

This corresponds in general to the following situation $\longrightarrow$



Assume that initially $R_1(0) = R_2(0) = 0$ (No polarization) which corresponds to the case



$\rho_{11}^0 = 1$
$\rho_{22}^0 = 0$

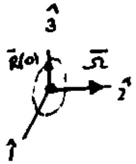
R_2 remains cero but:

$$\left.\begin{aligned} \dot{R}_1 &= \Omega R_3 \\ \dot{R}_3 &= -\Omega R_1 \end{aligned}\right\} \Rightarrow \ddot{R}_1 + \Omega^2 R_1 = 0$$

or

$$R_1 = \sin \Omega t$$
$$R_3 = \cos \Omega t$$

which corresponds to



Starting with cero polarization, we can create a maximum polarization for $\Omega t = \pi/2$. This is the superradiant state.

If $\Omega t = \pi$, $R_3 = -1$ and corresponds to a π pulse and a full inversion

full inversion, for a time $t = \pi/\Omega$, provided this time is shorter than the characteristic relaxation-time of the system.

The material of these lectures can be easily found in:

1) A. Messiah, Quantum Mechanics
North Holland, Amsterdam (1961)

2) R.J. Glauber, Optical coherence and quantum statistics, Lectures delivered at the Les Houches Summer School (1964) ed. by C. de Witt, A. Blandin and C. Cohen-Tannoudji, New York, Gordon and Breach (1965)

3) L. Allen and J.H. Eberly
Optical resonance and two level atoms
John Wiley, N.Y (1975)

4) M. Sargent III, M.O. Scully and W.E. Lamb Jr, Laser Physics (Addison-Wesley 1974)