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IX TRIESTE WORKSHOP ON
OPEN PROBLEMS IN
STRONGLY CORRELATED SYSTEMS

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MULTIFRACTALITY AT THE ANDERSON TRANSITION: A DRIVING FORCE OR JUST A COINCIDENCE?

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These are preliminary lecture notes, intended only for distribution to participants.

Multifractality at the Anderson transition: a driving force or just a coincidence?

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Outline:

① Two criteria of localization.
Multifractality: a nontrivial distribution of $|\psi(r_0)|^2$ in disordered samples

② Multifractality within the nonlinear σ model "old" and "new" approaches.

③. Do multifractality & localization always come together?

④ "Multifractal" behaviour at the transition: some illustrations.

Collaboration and numerous discussions with:

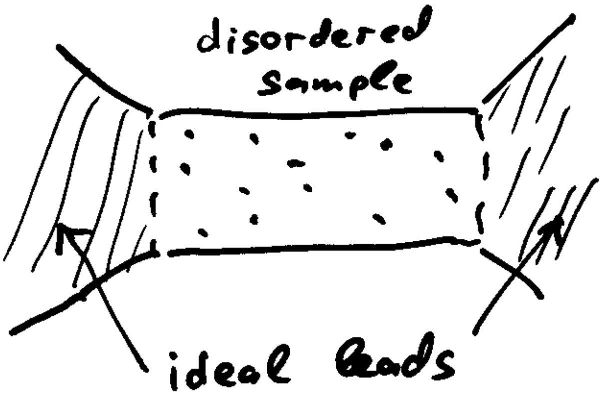
V. Kravtsov
B. Altshuler
J. Chalker
R. Smith
V. Yudson
F. Wegner

Numerous (and heated) discussions:

V. Falko
K. Efetov
B. Muzykantskii
D. Khmelnitskii
A. Mirlin

Two criteria for localization

I. Behaviour of conductance

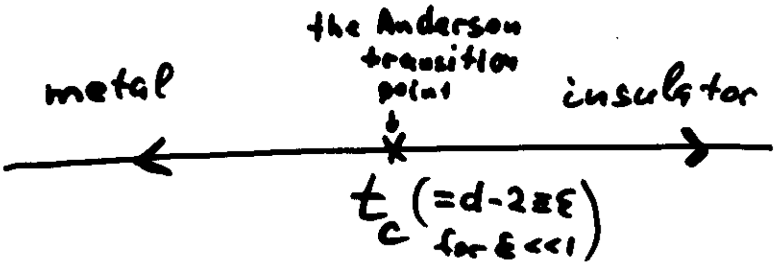


$$G \equiv \frac{1}{R} = \frac{I}{U}$$ in the two-lead device

$$g \sim G/(e^2/\hbar)$$

dimensionless conductance

$$\beta(g) \equiv \frac{d\ln g}{d\ln L} = \begin{cases} d-2-\dfrac{1}{g}+O(g^{-4}) & (B=0) \\[2ex] d-2-\dfrac{2}{g^2}+O(g^{-4}) & (B\neq 0) \end{cases}$$



The Ioffe-Regel criterion:

$$\ell \lesssim \lambda_F \Leftrightarrow g \lesssim 1$$

Two criteria for localization

II. The "inverse participation ratio" "$\langle|\psi(r)|^4\rangle$"

As $\int|\psi|^2 d^d r = 1$, $\underline{\langle|\psi|^2\rangle = L^{-d}}$ in a metal

$\langle\ldots\rangle$ - average over disorder

$$\therefore \left\langle \int |\psi^4| d^d r \right\rangle \sim \langle|\psi^4|\rangle L^d \propto \begin{cases} L^{-d}, & \text{metal} \\ \text{const}, & \text{insulator} \end{cases}$$

at the Fermi level

$\lim_{L\to\infty} |\psi^4(r)| > 0$: Gor'kov - Berezinskii (1969) criterion for localization

Are the two criteria equivalent ?

Inverse participation numbers (IPN)

more information on the structure of wave functions

$$P_n = \frac{1}{\nu}\left\langle \sum_\alpha |\psi_\alpha(\vec{r}_0)|^{2n} \delta(\varepsilon_\alpha - E)\right\rangle = \int P^n f(P)\, dP$$

distribution of local WF amplitudes

$$f(P) = \frac{1}{\nu}\left\langle \sum_\alpha \delta\left(P - |\psi_\alpha(\vec{r}_0)|^2\right)\delta(\varepsilon_\alpha - E)\right\rangle$$

$\nu = \frac{1}{\Delta L^d}$ is the average DOS

Δ – mean level spacing

Obviously,

$$P_n \propto \begin{cases} L^{-d(n-1)} & \text{metal} \\ \text{const} & \text{insulator} \end{cases}$$

no new information as compared to P_2

However, at the transition point (mobility edge)

$$\boxed{P_n \propto L^{-d^*(n)\,(n-1)}}$$

where $d^*(n) \neq d$, and in some range $d^*(n) \propto d - tn$

Such a behaviour is called multifractal

A lot of numerical data shows this behaviour

First: Soukoulis & Economou, 1984)

Analytical results:

- Wegner (1980)
  interpretation by Castellani, Peliti (1986)
- LDOS: Altshuler, Kravtsov, Lerner (1986, 19…
  Lerner (1988, 1…
- A new twist: Khmelnitskii, Muzykantski… (1995, 1996, …
  Efetov, Falko (1995)
  Mirlin (1996)
  many others 96+ …

Otherwise, moments of LDOS:

$$\nu(r) = \sum_\alpha |\Psi_\alpha(r)|^2 \delta(\varepsilon_\alpha - \varepsilon_F)$$

This is more convenient for an open sample (when individual levels are broadened)

$$f(\nu) = \left\langle \sum_\alpha \delta\left(\nu - |\psi_\alpha(r)|^2 \delta(\varepsilon_\alpha - \varepsilon_F)\right) \right\rangle$$

The result in a 2D "metal": (AKL 1986, 1991)

$$f(\nu) = \int f_{norm}(\nu - \nu_1)\, f_{tail}(\nu_1)\, d\nu_1$$

$$f_{tail} = A\, e^{-\frac{1}{4u} \ln^2 \alpha \delta \nu} \qquad \left(\text{for } \begin{matrix} B=0 \\ \beta=1 \end{matrix}\right)$$

$$u = \ln \frac{\sigma_0}{\sigma} \underset{g_0 \gg 1}{\approx} g_0^{-1} \ln \frac{L}{\ell}$$

$$\underbrace{g^{-1} u^2}_{\sim 1} + \underbrace{g^{-4} u^4}_{\ll 1}$$

Nonlinear σ model: a framework for calculating either g (and $\langle g^n \rangle$), or IPN:

$$F[\gamma] = \frac{\pi\nu}{4} \int d^d r \; \mathrm{Tr}\left[2D(\nabla Q)^2 + \gamma \Lambda Q\right]$$

$$\frac{1}{t} \equiv \frac{\pi\nu D}{8} = 16\pi \underset{\substack{\uparrow \\ \text{dimensionless} \\ \text{conductance}}}{g} \cdot L^{2-d} \; ;$$ t is the coupling const.

$$\boxed{Q(r) = U^+(r) \Lambda U(r)}, \qquad \Lambda = \mathrm{diag}(I, -I)$$

The RG describes behaviour of g at $d = 2+\varepsilon$ (or $d=2$).

The same functional may be used for calculating IPN or the LDOS moments

E.g., $$\langle \nu^n_{(r)} \rangle \propto \frac{\delta^n}{\delta\gamma(r)^n} \ln \int \mathcal{D}Q \, e^{-F[\gamma]}$$

It gives $$\langle \nu^n \rangle \sim \frac{1}{g^{2n-2}} \nu_0^n$$ - a very regular behaviour

Similarly,

$$f(P) \propto e^{-PL^d}, \qquad P_n \sim L^{-d(n-1)}$$

However, additional contributions to $\langle \nu^n \rangle$ (or IPN) are made by

Wegner 1980

Altshuler Kravtsov Lerner 1986 1991

$$\boxed{F_{add} = \int \sum_{k=2}^{\infty} \gamma_K \, \mathrm{Tr}(\Lambda Q)^K \, d^d r}$$

These composite operators arise in a microscopic derivation of NLσM from $H = H_0 + V(r)$)

random potential

The RG in the one-loop order:

$$\gamma_K \propto \gamma_K(0)\, e^{u K(K-1)}$$

$$u \equiv \ln \frac{\sigma_0}{\sigma} = \begin{cases} \ln\left[1 - g_0^{-1} \ln \frac{L}{\ell}\right]^{-1} & d=2 \\ \varepsilon \ln \frac{L}{\ell}, & d = 2+\varepsilon,\ L < \xi \\ \varepsilon \ln \frac{\xi}{\ell}, & d = 2+\varepsilon,\ L > \xi \end{cases}$$

bare — physical — conductivity

In the metallic region at $d=2$ it leads to

$$\langle \nu^n \rangle \propto P_n \propto \left(\frac{\ell}{L}\right)^{d(n-1)} e^{u n(n-1)}$$

$$\to L^{-d^*(n)(n-1)}$$

with $\boxed{d^*(n) = 2 - \frac{n}{\beta g}}$

$\beta = 1$ when $B = 0$

$\beta = 2$ when $B \neq 0$ (more than one flux through the system)

The composite operators do not affect g.
Another set of additional vertices might do:

$$F_{grad} = \int d^d r \sum_{\kappa=4}^{\infty} z_\kappa \text{“} Tr(\nabla Q)^{e \kappa} \text{”}$$

Ignoring the tensor stucture of the vertices

$$\dim Z_\kappa = d(\kappa-1) - g^{-1}\kappa(\kappa-1)$$

one-loop order

$$+ O(g^{-4}\kappa^4)$$

??

Kravtsov, Lerner, Yudson 198[illegible]
Lerner & Wegner 19[illegible]
Wegner 1992,9[illegible]
Castilla & Chakravar[illegible] 1993

These operators contribute directly to $\langle g^n \rangle$
and indirectly to IPN
(but their contribution is dominant)

Altshuler, Kravt[illegible] & Lerner 1986, 19[illegible]

There is no perturbative contributions to $\langle g \rangle$
but nonperturbative ones are not forbidden by symmetry

Wegner's conjecture (1991):
either a dangerous behaviour is overturned in higher loop
or there might exist a new fixed point

A new approach:

saddle-point approximation within the σ model

Khmelnitskii & Muzykantskii 1995, 96

Falko, Efetov 1995, 96

Mirlin 1996

+ . . . .

For the distribution of IPN:

$$f(P) \propto \lim_{\gamma\to 0} \int DQ\, \delta\left(P - \frac{\pi\nu\gamma}{2}\underline{\mathrm{Str}}\,\Lambda Q\right) e^{-F(Q)}\,\mathrm{Str}\,\Lambda Q((\ldots$$

the "metallic result", $f(P) \propto e^{-PL^d}$

may be obtained in the zero-d limit (Efetov & Prigodin, 1994)

$$\boxed{Q(r_0) \equiv Q_0}$$

observation point

The idea: to go beyond the trivial limit, one takes into account inhomogeneous fluctuations of the field $Q(r_0)$ near Q_0.

Parametrization:

$$Q = V^{-1} \Lambda V \; ; \qquad V = \underbrace{\begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix}}_{\text{anticommuting variables}} \exp \underbrace{\begin{bmatrix} 0 & -iu_2 \frac{\hat{\theta}}{2} \bar{v}_2 \\ -iv_2 \frac{\hat{\theta}}{2} \bar{u}_2 & 0 \end{bmatrix}}_{\text{commuting variables}}$$

$$\hat{\theta} = \begin{pmatrix} \theta \tau_0 & 0 \\ 0 & i\theta_1 \tau_0 \end{pmatrix} , \quad \begin{matrix} 0 < \theta < \pi \\ 0 < \theta_1 < \infty \end{matrix}$$ "non-compact variable"

In the limit $\gamma \to 0$ only the noncompact variables are essential.

To integrate over zero modes, one use the decomposition

$$Q(r) = V_0^{-1} \tilde{Q}(r) V_0 \, , \qquad \tilde{Q}(r) = \tilde{V}_0^{-1} \Lambda \tilde{V}$$

$$Q(r_0) = \Lambda$$

The limit $\gamma \to 0$ makes the integration over V_0 exact.

After the exact integration over "zero modes",

$$f(\mathcal{P}) = \frac{d^2 \Phi(\mathcal{P})}{d\mathcal{P}^2} ,$$

$$\Phi(\mathcal{P}) = \int_{Q(r_0)=\Lambda} e^{-F} \mathcal{D}Q$$

$$F[Q,\mathcal{P}] = \int d^d r \; \mathrm{Str}\left[\frac{1}{t}(\nabla Q)^2 - \frac{\mathcal{P}}{4} \Lambda Q \Pi \right]$$

the projection onto the noncompact sector

Apart from the inhomogeneous "boundary condition" $Q(r_0)=\Lambda$, this looks like the standard NLσM.

However, for $\mathcal{P} \neq 0$, the standard $Q=\Lambda$ does not minimize F.

An optimal configuration turns out to be inhomogeneous, and involves only <u>one</u> (out of at least 8) of the parameters representing Q: a so-called non-compact angle θ_1.

The saddle-point equation has the Liouville form

$$\boxed{\nabla^2 \theta(r) = -\frac{2\pi P}{g} e^{-\theta(r)}}$$

B.C. : $\theta(r_0) = 0$ at the "observation point"

$(\vec{n}\nabla)\,\theta(r) = 0$ at the surface of a sample

The optimal free energy:

$$\boxed{F_0 = \int d^d r \left\{ \frac{2\beta}{t} (\nabla\theta)^2 + P e^{-\theta} \right\}}$$

contains only the one-component scalar field

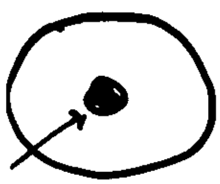
$\underline{\underline{d=2}}$:

Approximate solution (the exact one is known but unnecessary)

$$\theta^{(0)}(r) = 2\mu \ln \frac{r}{l}$$

the solution to the laplas eq. satisfying $\theta(r_0)=0$ for $r_0 \leq l$

; μ can be found from the first iteration + B.C. at the surface



sample geometry

the central lead of size $\sim l^2$

For "large" P, $\quad P L^d \gg \frac{g}{\ln L/l}$,

$$\mu \approx \frac{1}{2} \frac{\ln[g^{-1} P L^d]}{\ln L/l}$$

Thus the optimal free energy is

$$F_0 \simeq \beta g_0 \left\{ \mu + \mu^2 \ln\frac{L}{\ell} \right\} \underset{\text{large } \mathcal{P}}{\approx} \frac{\beta g_0}{4} \frac{\ln^2 T}{\ln L/\ell}$$

$$T \equiv \frac{\mathcal{P} L^d \ln L/\ell}{g_0}$$

This leads immedeatly to the IPN distribution function for large $\mathcal{P}$

$$f(\mathcal{P}) \simeq A e^{-F_0} = A e^{-\frac{\beta g_0}{4 \ln L/\ell} \cdot \ln^2 T}$$

The preexponential factor A is contributed by the θ_1- fluctuations near the saddle point.

For small $\mathcal{P}$, $F_0 \approx \beta L^d \mathcal{P}$ which leads to the exponential (Porter - Thomas) distribution.

The IPN are directly obtained as

$$\mathcal{P}_n \propto \left(\frac{\ell}{L}\right)^{-d^*(n)\cdot(n-1)}, \quad d^*(n) = 2 - \frac{n}{\beta g}$$

the only case to be considered by the saddle-point method

For $g_0 \gg 1$ (neglecting the weak localization effects) this is in exact agreement with the results of the RG calculations of the extended σ model.

A few features of this solution

① Coincidence of the results of perturbative RG treatment
(which could be done with either replicas or SuSy)

and very non-perturbative results of the saddle-point methods
(which could be used only with SuSy)

is quite surprising.
<u>No additional vertices were required in SuSy</u>

②. Higher gradients indicate an instability of a spatially homogeneous Q-configuration. The inhomogeneous saddle point could be a correct vacuum for constructing RG flows

③ The tails of the distribution function describe rare events (untypically large $|\psi^2(r)|$). But do they correspond to the prelocalized states ?

Two different scales (for $B \neq 0$)

Localization at $d=2$:

$$\frac{d \ln g}{d \ln L} = -\frac{2}{g^2}$$

$\therefore$ If $g = g_0 \gg 1$ at scale of l_{el},

then g falls off to $g \sim 1$

(the Mott-Ioffe-Regel limit)

at

$$L \sim l_0 \, e^{\frac{g_0^2}{2}}$$

On the other hand, mesoscopic effects (and multifractality) set in at scale

$$L \sim l_0 \, e^{g_0}$$

The two scales merge only when disorder is strong ($g_0 \sim 1$)

An example of the model with multifractality but without localization

$$\dot{\vec{r}} = \vec{\xi}(t) + \vec{V}(\vec{r})$$

random noise — quenched (potential) disorder

RW in quenched disorder

$$\Updownarrow$$

$$\partial_t \rho_{\vec{r}} = \sum_{r'} \left(W_{rr'} \rho_{r'} - W_{r'r} \rho_r \right) \quad \text{with } W_{rr'} \neq W_{r'r}$$

(eg, due to the presense of charged impurities)

The long-wave limit of the both models above is described by the field theory:

D. Fisher et al, 1985
Aronovitz & Nelson, 1985
Kravtsov, Lerner, Yudson, 1985
Bouchaud, Georges et al, 1987, 1990
. . . .

$$S = \int d^d r\, \bar{\varphi}\,(i\omega + D\partial^2)\,\varphi +$$

$$+ \frac{i\gamma}{2} \int d^d r\, d^d r'\, (\partial_\alpha \bar{\varphi}\varphi)_r\, F_{\alpha\beta}(r-r')\, (\partial_\beta \bar{\varphi}\varphi)_{r'}$$

Here D and γ arise from

$$\langle \xi_\alpha(t)\, \xi_\beta(t') \rangle = 2D\,\delta_{\alpha\beta}\,\delta(t-t')$$

$$\langle V_\alpha(r)\, V_\beta(r') \rangle = \gamma\, F_{\alpha\beta}(r-r')$$

For potential disorder, $F_{\alpha\beta}(\vec{q}) = q_\alpha q_\beta / q^2$.

$$\Gamma \equiv \frac{\gamma}{4\pi D^2}$$ is the coupling constant.

It appears that the model is "super-renormalizable":

$$\boxed{\frac{d \ln \Gamma}{d \ln L} = 0 \text{ in all orders of expansion in } \Gamma}$$

Kravtsov, L., Yudson 1986

Honkonen, Pismak, & Vasiliev 1988

Bouchaud et al 1990

Thus

$$D(t) \equiv \frac{\partial}{\partial t} \langle r^2(t) \rangle \underset{t \gg \tau}{\simeq} D_0 \left(\frac{t}{\tau}\right)^{-\frac{\Gamma}{2}}$$

This is "subdiffusion". Obviously, no localization.

Now note that D can fluctuate with disorder.

For the hopping model,

$$D(r) = \frac{1}{2} \sum_{r'} (\vec{r} - \vec{r}')^2 W_{rr'}$$

These fluctuations are naively irrelevant for a weak disorder ($\Gamma \ll 1$). However...:

The fluctuations of $D(r)$ lead to the appearance of additional operators in the field theory.

$$S_{add} = i \sum_{k=2}^{\infty} \Gamma_k \int d^d r \prod_{j=1}^{k} (\partial_{\alpha_j} \bar{\varphi} \partial_{\alpha_j} \varphi)$$

Y. Shapir, 199.
Lerner, 1993, 1995

RG eqs for Γ_k turn out to be identical to those for "additional charges" of the NLGM.
As a result:

$$\boxed{\dim \Gamma_k = d(k-1) - \Gamma k(k-1)}$$

This immediately leads to the multifractal dimensions coinciding with those in the quantum diffusion model (in the weak-disorder limit) with

$$g^{-1} \to \Gamma :$$

$$d^*(n) = 2 - \Gamma n \qquad \left(n < \frac{2}{\Gamma}\right).$$

Therefore, "multifractality" exists in the total absence of localization

Multifractality at the mobility edge: an illustration.

Return probability:

$$p(t) = \langle |\Psi(0,t)|^2 \rangle \qquad \Psi(\vec{r},t) = \ell^{\frac{d}{2}} \sum_{\alpha}^{N_0} \psi_\alpha^*(0)\psi_\alpha(r) e^{-i\varepsilon_\alpha t}$$

is clearly related to the 2nd multifractal dimensionality $d^*(2)$

$$p(t) \sim \ell_0^{-\frac{\eta}{2}} \left(\frac{\hbar\nu}{t}\right)^{1-\frac{\eta}{d}}$$

$$\eta \equiv d - d^*(2)$$

Chalker & Daniell 1988

Chalker 1990

. . . . .

This is a convenient object for numerics.

Note that

$$p(t) = P(0,t) \quad \text{where} \quad P(r,t) \propto e^{-\frac{r^2}{4D_{eff}t}}$$

$$\frac{1}{Dq^2 - i\omega} \xrightarrow[g \to g_c]{} \frac{1}{D_{eff}(q,\omega)\, q^2 - i\omega} \xrightarrow[\text{small } \omega]{} \frac{1}{Aq^{d-\eta} - i\omega}$$

D_{eff} and thus (possibly) g knows about MF.

Level rigidity

$$\Sigma_2 = \langle (\delta N)^2 \rangle - \langle N^2 \rangle = \begin{cases} \ln N & \text{meta} \\ N & \text{insulat.} \end{cases}$$

This is related to the two-level correlations

$$R_2(\omega) = \frac{1}{\nu^2} \langle \nu(\varepsilon)\, \nu(\varepsilon + \omega) \rangle - 1 \quad :$$

$$\Sigma_2 = \frac{1}{\Delta^2} \int_{-E}^{E} (E - |\omega|)\, R_2(\omega)\, d\omega$$

The compressibility :

$$\chi = \lim_{\langle N \rangle \to \infty} \Delta \cdot \lim_{L \to \infty} \frac{d\Sigma_2}{dE} = \int_{-\infty}^{\infty} R(\omega)\, d\omega \equiv \underset{t \to 0}{K(t)}$$

where the form-factor

$$K(t) = \int_{-\infty}^{\infty} e^{-i\varepsilon t} R(\varepsilon)\, d\varepsilon$$

χ is "universal" :

$$\chi = \begin{cases} 0, & \text{metal} \\ 1, & \text{insulator} \end{cases}$$

Relation to $p(t)$:

$$K(t) = \frac{1}{2} \frac{|t|\, p(t)}{\pi\hbar\nu + \int_{0^+}^{t} p(t)\,dt}$$

Chalker, Lerner, & Smith 1996

With the multifractal

$$p(t) \propto t^{-1+\eta/d},$$

we have

$$\boxed{\chi = \lim_{t\to 0} \lim_{L\to\infty} K(t) = \frac{\eta}{2d}}$$

Chalker, Kravtsov & Lerner, 1996

Intermediate between 0 and 1,

and entirely due to "multifractality".

OPEN QUESTIONS..?

One in the relation to this talk:

how to build an expansion near the new vacuum (a spatially inhomogeneous saddle point).

It might help in answering a question whether (and how) multifractality affects the Anderson transition, i.e. leads to some non-trivial behaviour even of the mean conductance.