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INTERNATIONAL ATOMIC ENERGY AGENCY
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SUMMER COLLEGE IN CONDENSED MATTER ON
" STATISTICAL PHYSICS OF FRUSTRATED SYSTEMS "

(28 July - 15 August 1997)

" Spin-glass mean field theory and replicas" (PART II)

presented by:

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Deficiencies of replica-symmetric ansatz ?

- Hint #1 : Ising s.g. $\rightarrow$ Entropy $S \geq 0$.

  RS ansatz $\rightarrow$ $S(T=0) \leq 0$!

- Test # 2 : Stability analysis. (Almeida + Thouless 78)

  $\overline{\ln Z}$ involves $\exp(-N\,\Phi(\{m^\alpha\},\{q^{\alpha\beta}\}))$

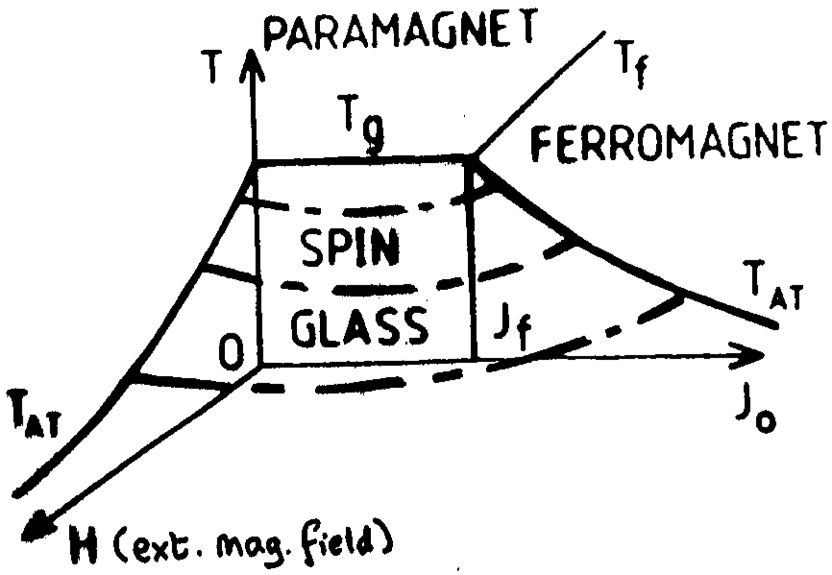
  ? Fluctuation analysis

$$m^\alpha = m + \varepsilon^\alpha$$
$$q^{(\alpha\beta)} = q + \eta^{(\alpha\beta)}$$

- expand Φ to second order
- Look at normal modes of h.o.
- Require stability

- Result:
  - RS ok for m
  - Not everywhere for q
  - Limit of stability $\rightarrow$ AT line.

de Almeida - Thouless surface for SK spinglass



...g. 9. De Almeida Thouless surface (indicated by chain-hatching) for the limit of stability of the ...plica-symmetric Ansatz for mean-field theory for a random-bond Ising model. The Ansatz is stable on the side of the surface closer to the origin. The phase line beneath the AT surface is calculated within the Parisi Ansatz.

[Re]plica symmetry breaking (RSB) ansätze

One-step RSB.

A 'natural' suggestion

- Divide n replicas into n/m groups of m replicas
- Choose

$$q^{\alpha\beta} = q_1 \;;\; \alpha,\beta \text{ in same group}$$

$$\phantom{q^{\alpha\beta} =} q_0 \;;\; \alpha,\beta \text{ in different groups}$$

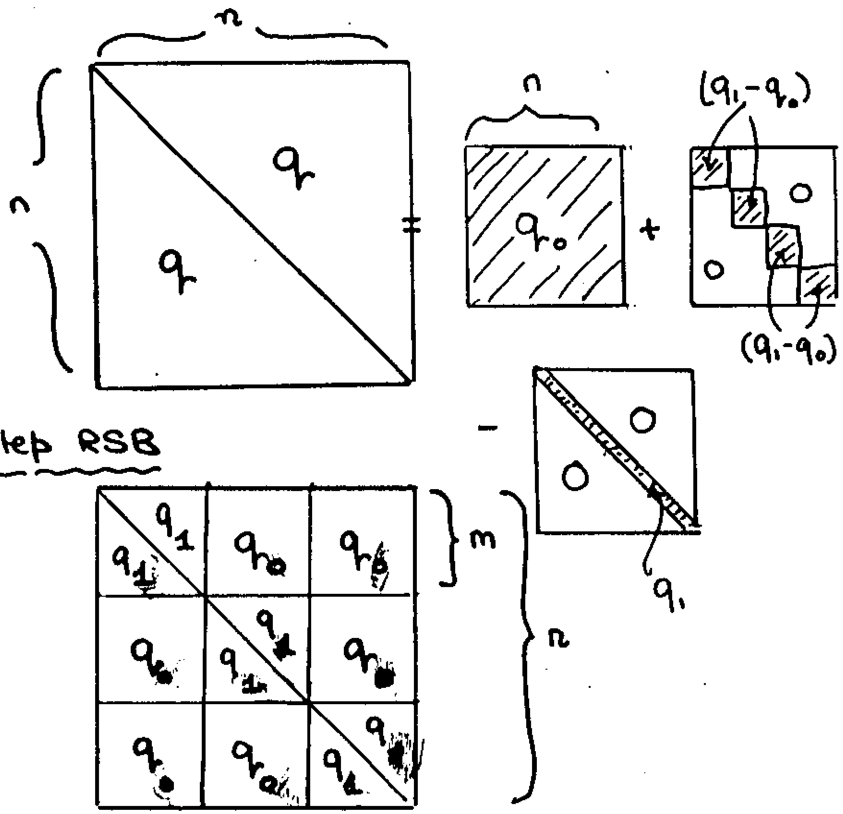
- Extremize Φ w.r.t. q_0, q_1, m

→ Improvement

- Less negative $S(T=0)$
- Less unstable normal modes ω_{AT}
- Quite good numerical agreement with simulations for g.s. energy
  - But still not perfect for SK model. (ok for some others)

$q^{\alpha\beta}$ matrix

- Replica-symmetric (RS)
- One step RSB



$$= -\frac{\beta}{2}\Big\{ q_0\Big(\sum_\alpha \sigma^\alpha\Big)^2 + (q_1-q_0)\sum_{\substack{\text{diag}\\\text{blocks}}}\Big(\sum_{\alpha\subset\text{block}} \sigma^\alpha\Big)^2 - n q_1 \Big\}$$

Partially

Explicit analysis for one-step RSB

$$q^{\alpha\beta} = q_1 \quad ; \quad \alpha,\beta \quad \text{same diag block}$$

$$= q_0 \quad ; \quad \alpha,\beta \quad \text{diff diag block}$$

$$= 0 \quad ; \quad \alpha=\beta.$$

$$n\left\{\begin{array}{|c|c|c|c|}\hline q_1 & q_0 & q_0 & q_0 \\ \hline & q_1 & q_0 & q_0 \\ \hline & & q_1 & q_0 \\ \hline & & & q_1 \\ \hline \end{array}\right. \;\}m$$

- Relevant non-trivial term in $Z_{eff}(q)$

$$\tilde{Z}(q) = \sum_{\substack{\sigma \\ \text{one-site}}} \exp(-\beta \tilde{H}(q,\sigma))$$

$$H(q,\sigma) = -\beta \sum_{1\le\alpha<\beta\le n} q^{\alpha\beta}\sigma^{\alpha}\sigma^{\beta}$$

- Gaussian transform for each quadratic form

$$\rightarrow \int D_{q_0}(z) \prod_{k=1}^{n/m} \int D_{(q_1-q_0)}(y_k)$$

$$\sum_{\{\sigma\}} \exp(\beta[z\sum\sigma^{\alpha} + \sum_{k=1}^{n/m} y_k (\sum_{\text{a blocks}} \sigma^{\alpha})])$$

where $D_q(z) = \frac{1}{(2\pi q)^{1/2}} \exp(-z^2/2q)\,dz$

$$= \int D_{q_0}(z)\left[\int D_{(q_1-q_0)}(y)\,(2\cosh(\beta[z+y]))^m\right]^{n/m}$$

Note: This has still to go into

$$F = -kT \lim_{N\to\infty} \lim_{n\to 0} \frac{1}{n}\left\{\int \prod \tilde{d}q^{\alpha\beta}\,(\tilde{Z}(q))^N - 1\right\}$$

$$(\tilde{Z}(q))^N \sim \exp(N \ln \tilde{Z}(q))$$

gives extremal dominance → mean field soln.

Extremize wrt. q_0, q_1, m.

(Not do this in detail here)

Full Parisi RSB

Philosophy

- Need to break replica-symmetry
- One step insufficient
  - → Two steps : better but still not enough (for SK)
- Need continuum of breakings / 'fractal' hierarchy.

Parisi scheme

- Hierarchy of steps

(1) Sub-divide $n \times n$ into $m_1 \times m_1$ blocks

Take $q^{\alpha\beta} = q_1$ in off-diag blocks

(2) Sub-divide $m_1 \times m_1$ diagonal blocks into $m_2 \times m_2$ sub-blocks.

Take $q^{\alpha\beta} = q_2$ in off-diagonal sub-blocks

(3) Continue sub-division in this way ad infinitum

→ • infinite regression of sub-divisions of diagonal blocks : $n \geq m_1 \geq m_2 \cdots \geq 1$

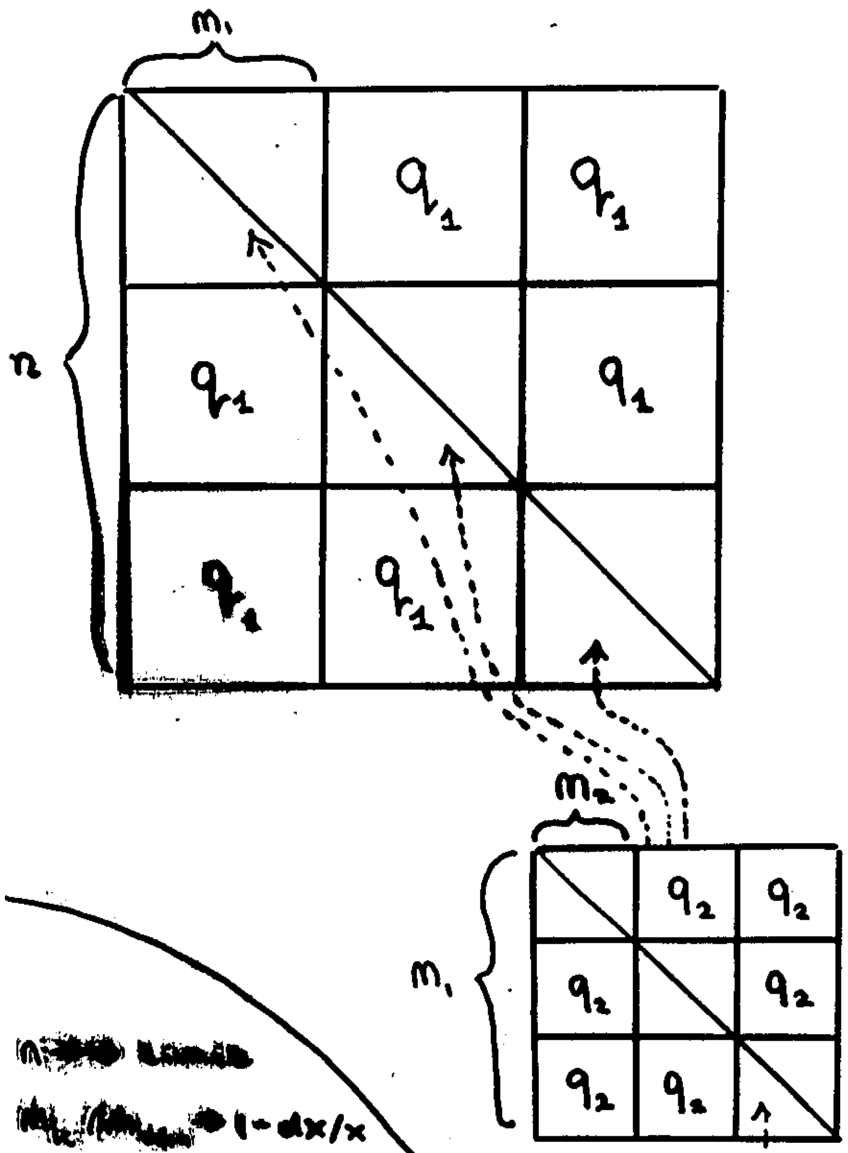
• corresponding $q_1, q_2 \cdots\cdots$

- Continuum limit + $n \to 0$

$$m_k / m_{k+1} \to 1 - dx/x$$

$$q_k \to q(x) \quad ; 0 \leq x \leq 1$$

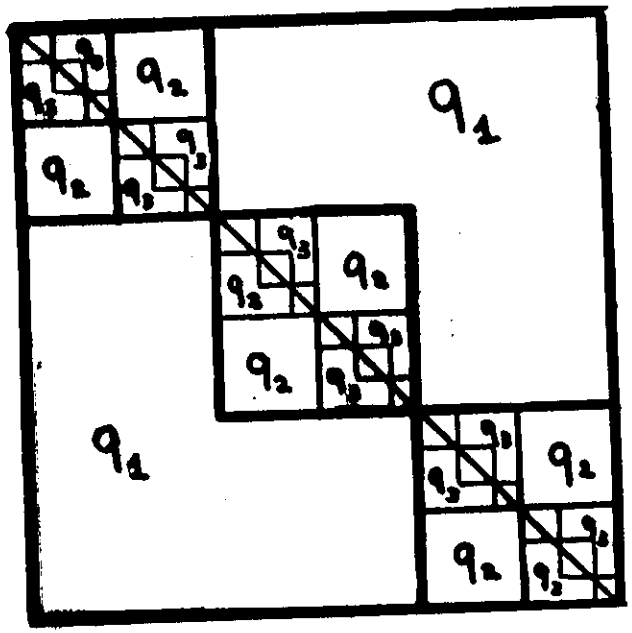
- Treat $q(x)$ as variational OP to take extremum of $\Phi[\{q(x)\}]$



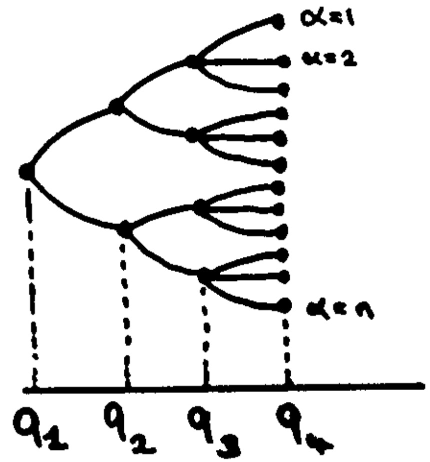
sub-divide
diagonal blocks
ad infinitum.

[redacted]
[redacted] $1-dx/x$
[redacted] $0 \le x \le 1$

Parisi matrix: first few stages



cf. evolutionary tree



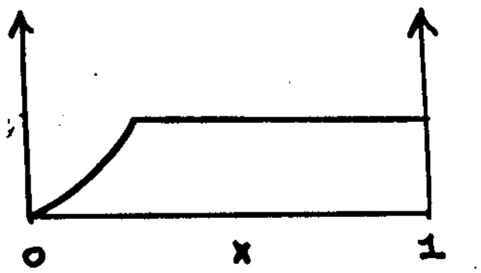
q given by nearest
common ancestor

Extremize w.r.t. $q(x)$; $0 \leq x \leq 1$: $\lim_{n \to 0}$

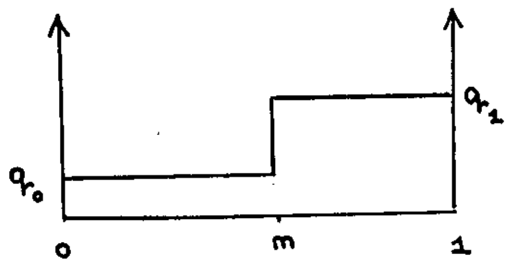
RS region $\quad q(x) = q$; all x

RSB region $\quad q(x)$ has structure

eg. $q(x)$



One-step RSB.



Interpretation of Replica-Symmetry Breaking

- **Overlap**

(i) Between two microstates s, s'

$$q^{ss'} = N^{-1} \sum_i \sigma_i^{s} \sigma_i^{s'}$$

→ Distribution of overlaps

$$\tilde{P}(q) = \sum_{s,s'} \underset{\substack{\uparrow \\ \text{microstate prob.}}}{p_s}\, p_{s'}\, \delta(q - q^{ss'})$$

eqm: $\quad p_s = \exp(-\beta H_s) / Z$

Replica theory + Parisi ansatz

$$\rightarrow \quad \overline{\tilde{P}(q)} = \int_0^1 \delta(q - \underset{\substack{\uparrow \\ \text{Parisi OP.}}}{q(x)})\, dx = dx/dq$$

$\therefore \left(\frac{dx}{dq}\right)$ gives average overlap distn

A further connection

Overlap

(a) between 2 macrostates S, S'

$$q^{SS'} = N^{-1} \sum_i m_i^S m_i^{S'}$$

Thermodynamic average of σ_i in macrostate S

→ Macrostate overlap dist^n

$$P(q) = \sum_{SS'} P_S P_{S'} \delta(q - q^{SS'})$$

prob of macrostate $P_S \sim \exp(-\beta F_S)$

Conceptual link

Can show $P(q) = \tilde{P}(q)$: eq^m.

$$\therefore \frac{dx}{dq} = \overline{P(q)}$$

Orientation

- **Conventional Ising ferromagnet.**

→ 2 thermodynamic states: mag $\pm m$

$$\therefore P(q) = \frac{1}{2}\left[\delta(q - m^2) + \delta(q + m^2)\right]$$

$\underline{S} = \underline{S}'$

$\underline{S} = \overline{\underline{S}}'$ (inver

Eliminated by infinitesmal fiel

One peak, one state for $q > 0$.

- **Replica-symmetric 'spin glass'**

$q(x) = \text{constant}$ → $\frac{dx}{dq} = \delta\text{-fn}$

→ one peak in $\overline{P}(q)$

→ one thermodyn state.

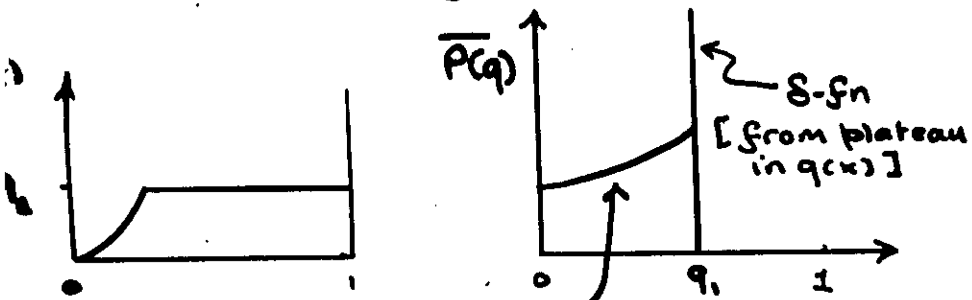
Replica-symm broken spin-glass

$q(x)$ has structure $\rightarrow \frac{dx}{dq} \neq \delta$-fn

$\rightarrow \overline{P(q)}$ has structure

$\rightarrow$ many non-equivalent thermodynamic states

$\rightarrow$ multiple non-equivalent attractors in rugged energy landscape.



varying overlaps from non-equiv states $q^{SS'}$; $S \neq S'$

corresponds to self-overlap q^{SS}; same for all S

$$\boxed{\text{RSB} \rightarrow \text{Many non-equiv. thermodyn. states}}$$

Order parameters

Observables

- Suppose many non-equiv macrostates

Prob: $P(\alpha) = \dfrac{\exp(-\beta F(\alpha))}{\sum_\alpha \exp(-\beta F(\alpha))}$

macrostate label.

) $q_{EA} = \lim_{t\to\infty} \lim_{N\to\infty} \overline{\langle S_i(t_0) S_i(t+t_0)\rangle}$

gives high barriers between macrostates. System cannot get over

$\rightarrow \overline{\sum_\alpha P_\alpha (M_\alpha)^2} = q(1)$ in Parisi language

) Gibbs average

$q = \overline{\langle\sigma_i\rangle^2} = \overline{m_i^2} = \overline{\left(\sum_\alpha P_\alpha m_i^\alpha\right)^2}$

$= \overline{\sum_{\alpha\beta} P_\alpha P_\beta M_\alpha M_\beta}$

$= \int_0^1 q(x)\,dx$ in Parisi language.

Interpretation of other thermodynamic measures

- Average over all Gibbs phase space

  Use full $P(q)$. (A)

- Short-time average

  System stays in one macrostate.
  Use only peak of $P(q)$ (B)

- Field cooled

  Use (A) : eg $\chi_{FC} = T^{-1}\{1 - \overline{\langle\sigma\rangle^2}\}$

  $= T^{-1}\int dx\,(1 - q(x))$

- Zero field cooled

  Use (B) : eg $\chi_{ZFC} = T^{-1}(1 - q_{max}(T))$.

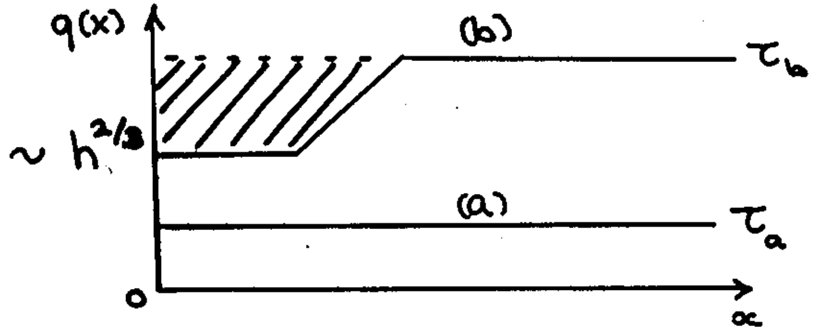
  $= T^{-1}(1 - q(1))$

  But strictly true dynamics is more subtle.

System with external field

$$H \to H - h\sum_i \sigma_i$$

Parisi theory prediction



(a) $T > T_{AT}$: AT ≡ de Almeida-Thouless

(b) $T < T_{AT}$

$\tau = T_g - T$

ie (a) Replica-symmetric : $\chi_{FC} = \chi_{ZFC}$

(b) Replica-symm broken : $\chi_{FC} = \chi_{ZFC} + \Delta$

Hatched area → anomaly Δ.

$$\Delta = (kT)^{-1}\left\{q(1) - \int_0^1 q(x)\,dx\right\}$$

Susceptibilities of Ising spin glass in field H 44

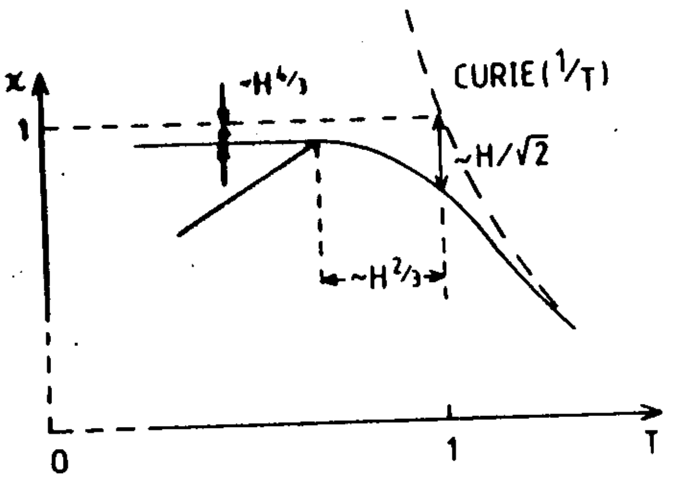


Fig. 12. Susceptibility of an Ising spin glass in an applied field H, as predicted by Parisi's mean-field theory. The upper curve shows the full Gibbs average, obtained from the full $q(x)$ and interpreted as the field-cooled (FC) susceptibility. The lower curve shows the result of restricting to one thermodynamic state, as obtained from $q(1)$ and interpreted as the zero-field-cooled susceptibility.

Upper $\chi = T^{-1}(1-\int q(x))$: FC

Lower $\chi = T^{-1}(1-q(1))$: ZFC

Ultrametricity

Distances : if $d_{ab} \geq d_{bc} \geq d_{ca}$

then $d_{ab} = d_{bc}$ (ultrametric).

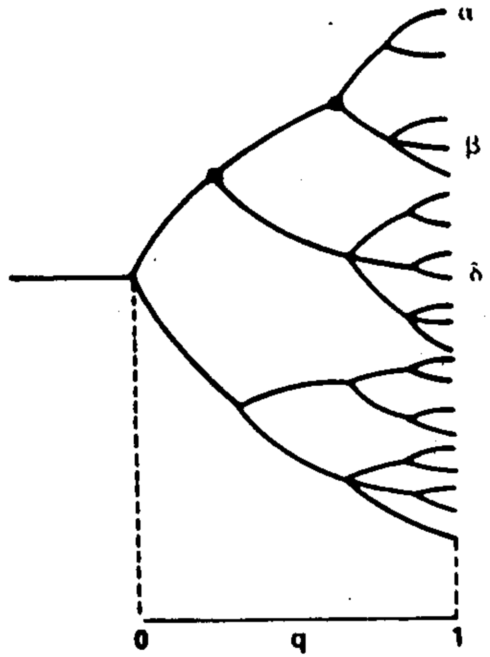


Fig. 13. An evolutionary tree, illustrating the occurrence of ultrametricity. If the overlap between two final states is measured by the degree of evolution of their nearest common ancestor, as shown, then for any group of three final states, the two smallest overlaps are equal.

Overlaps (complements of separation)

If $q^{ab} \leq q^{bc} \leq q^{ca}$

then $q^{ab} = q^{bc}$ (two smallest overlaps are equal).

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Self-averaging

$$\lim_{N\to\infty} \left\{ \frac{\{\overline{A^2} - (\bar{A})^2\}^{1/2}}{\bar{A}} \right\} = 0$$

for observable of positive expec^n val.

Parisi theory →

- self-averaging for normal observables (energy, free energy, ..)
- but dist^n of overlaps is non-self-averaging.

cf. sample-to-sample variations in other problems: proteins, evolution...

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Often need extra manipulations to get convenient forms

① Introduction of representations of unity to move quantities to more convenient locations

eg. $f(x) = \int dy\, \delta(x-y) f(y)$

now in δ fn

Often convenient if $f(y)$ higher than linear.

② Exponentiation of δ-fns

eg $\delta(x-y) = \int \frac{dz}{(2\pi)} \exp(iz(x-y))$

now x enters linearly in exponenti

③ Inversion of complete square in exponential

$$\exp(\lambda a^2) = (2\pi)^{-1/2} \int dy \exp\left(-\frac{y^2}{2} + (2\lambda)^{1/2} a y\right)$$

Useful if x above $\sim a^2$ and we want $\exp(\ldots a)$

linear.

As well as

④ Extremal dominance

$$\int d\underline{x}\, d\underline{y} \dots d\underline{z}\ \exp(-N\Phi(\underline{x}, \underline{y}, \underline{z} \dots))$$

↑

May have replica labels.

③ Replica-symmetric ansatz

⑥ RSB / Parisi

Simulations — Monte-Carlo

- Subtleties of detail
  - equilibration (esp Bhatt + Young)
  - finite size

→ phase transitions / crit exponents
- simple exptl. observables $E, m \dots$
- subtle theor. observables $q, P(q) \dots$

- Give just a hint here.

(i) $q = N^{-1} \sum_i \langle \sigma_i \rangle^2$

$$\langle \sigma_i \rangle = \tau^{-1} \int_{t_0}^{t_0+\tau} dt\, \sigma_i(t)$$

t_0 long enough to have equilibrated (but ....*)

τ Large enough for system to have explored adequate phase space but not to experience global inversion

(ii) Upper + lower bounding methods
Look for convergence to indicate measure of relevant initial transient / 'eq^n' time.

(Bhatt + Young)

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* In fact systems may never truly equilibrate — see Cugliandolo

Test for equilibration

① $q^{u}(t) = N^{-1} \sum_i \sigma_i(t+t_0)\,\sigma_i(t_0)$

$t_0 >$ eq^n time

determined 'a posteriori'

$t=0 \rightarrow q^{u}(0) = 1$

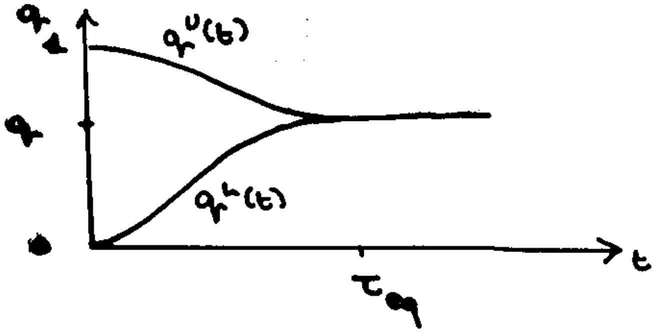
$t \rightarrow \tau_{eq} \rightarrow q^{u}(t) \rightarrow q$

② $q^{L}(t) = N^{-1} \sum_i \sigma_i^{A}(t+t_0)\,\sigma_i^{B}(t+t_0)$

2 real replicas A, B.

$t=0 \rightarrow q^{L}(0) = 0$

$t \rightarrow \tau_{eq} \rightarrow q^{L}(t) \rightarrow q$



Extra sophistication

Study $q^{(2)} = N^{-1} \sum_{ij} \langle \sigma_i \sigma_j \rangle^2$

(Invariant under global spin-flip)

$\rightarrow$ sim^r $q^{U(2)}, q^{L(2)}$ analysis

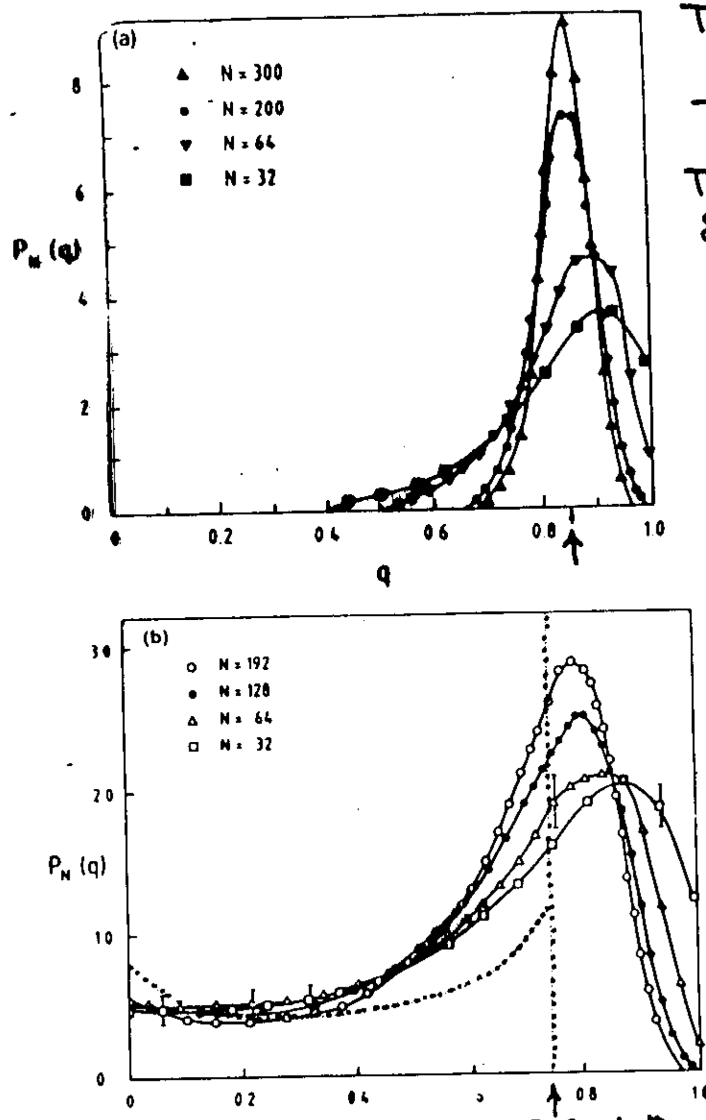
[Ref: Bhatt + Young : PRL 54, 924 ('85)
PR B37, 5606 ('88)
+ several brief reviews]

Simulation of P(q)

- Run $m \geq 2$ copies of system
  identical quenched disorder
  independent evolution (eg M.C.)
- Equilibrate first : $t_0 > \tau_{eq}$
  zeroth order step determines this
- Cross-correlate replicas
  Plot histogram of instantaneous overlaps $q^{ab} = N^{-1} \sum_i \sigma_i^a \sigma_i^b$
  Run for $t > \tau_{eq}$ after initial (ignored) t_0

$\rightarrow P(q)$

Simulations of $\overline{P}(q)$ for SK model



$T < T_g$; $h \neq 0$

$T > T_{AT}$

Theory predicts δ-fn at arrow in limit $N \to \infty$

$T < T_g$

$H = 0$.

Theor pred^n shown dotted.

δ-fn in theory

Fig. 14. (a) Monte Carlo simulations of $[P_N(q)]$ for the SK model for several sizes at $T = 0.4J$, $H = 1.2J$, which is above the AT line. The results are consistent with a Gaussian distribution centred on the value of q given by replica mean-field theory (indicated by the arrow) and of width scaling as $N^{-1/2}$; from: Young 1985. (b) Monte Carlo simulations of $[P_N(q)]$ for the SK model for several sizes at $T = 0.4$, $H = 0$, which is below the AT line. The dotted line is an approximate solution of Parisi's equations and consists of a delta function of width $\frac{1}{2}$ at $q = 0.744$ and a continuum with finite weight down to $q = 0$; from: Young 1983.

A.P. Young.

Extn → self-averaging ?, ultrametricity ?

Self-averaging :

Look at different instances of $\{J\}$

→ different $P(q)$ → non-self-averaging

Ultrametricity

Consider triples of replicas / copies

Look at peaks → q^{12}, q^{23}, q^{31}

Order $q_1 \geq q_2 \geq q_3$

Look at distn $P(\delta q)$; $\delta q = q_2 - q_3$

Ultrametricity indicated by sharpening towards $\delta(\delta q)$ as volume/size increased

For more details see : Caracciolo et. al., J. Physique **51**, 1877 (1990)

Bhatt + Young : J. Magn. Mag. Mat. **54-57**, 191 (1986)

Alternative approach : Bansoar, S + Sourlas ; J. Phys A **20**, L1 (1987) for Ultrametricity

Ultrametricity test   SK model

Dist^n of diff between 2 smallest overlaps in set of 3 (from 3 states)

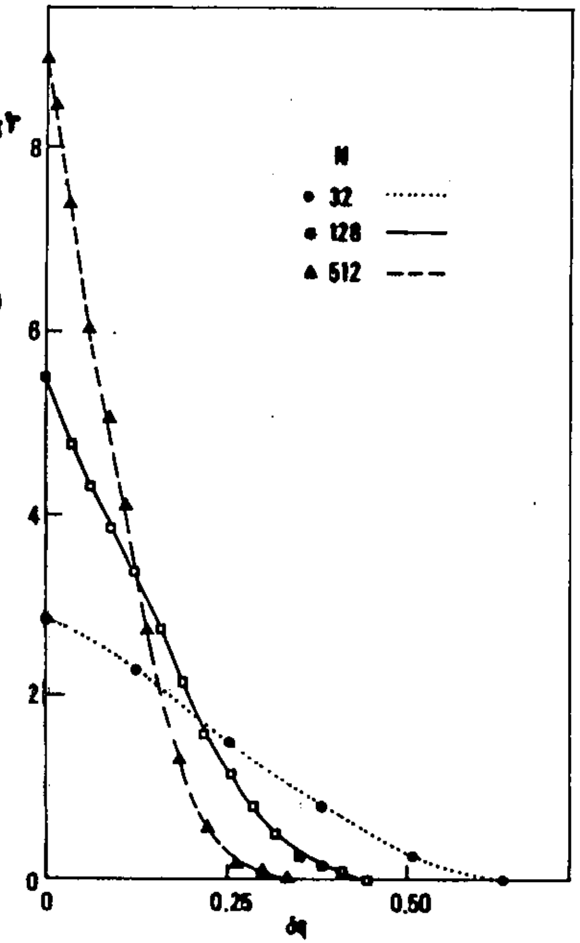
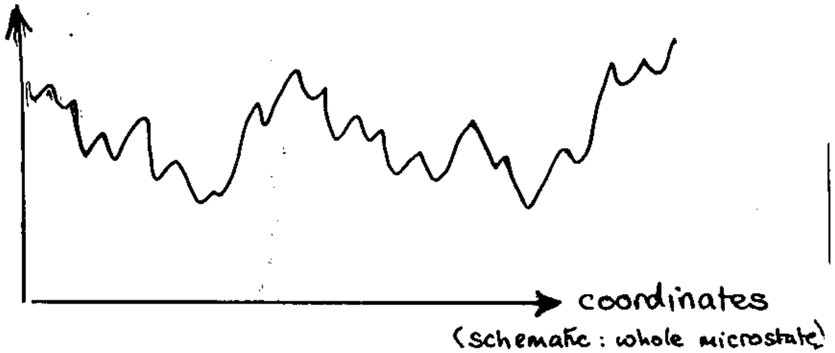


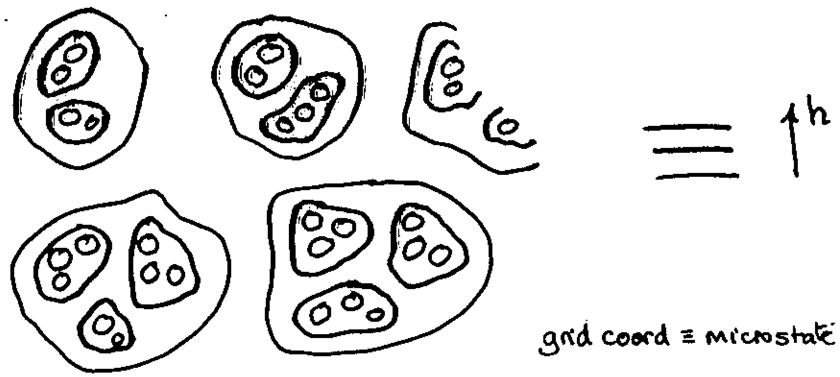
Fig. IV.2 Probability distribution of the difference between the two smaller overlaps ($\delta q = q_{mid} - q_{min}$) for a fixed larger overlap value $q = 0.5$. The temperature is $0.6\,T_{c}$. In the $N \to \infty$ limit, the distribution is expected to become a δ function at the origin (from Ref. 9).

Bhatt + Young

→ Complex / rugged landscape



or as contour map



→ hierarchical, rugged,

difficult to find minima, barriers,

metastability, slow stochastic dynamics

+ complex / chaotic evolution with control parameters (eg temp., field).

→ Paradigm for complex disordered systems.

Continuous + Discontinuous RSB

One-step No ext field.

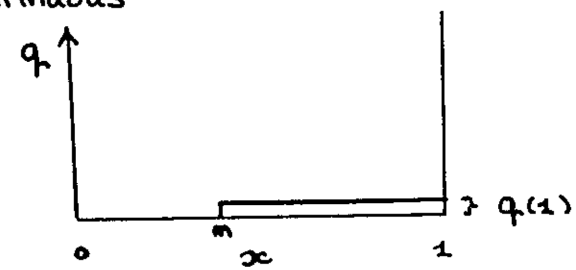
(i) $T > T_c$: $q(x) = 0$; all x, $0 \le x \le 1$

(ii) $T < T_c$: $q(x) \neq 0$

2 scenarios ($T < T_c$)

- Continuous

eg Potts s.g. $p < p_c = 4$

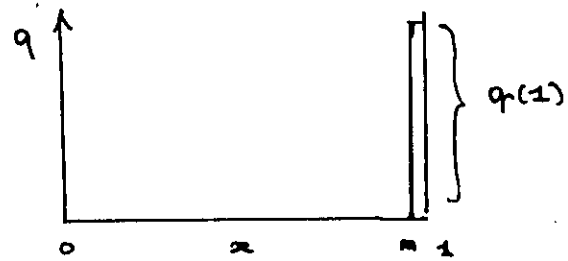


$q(1) \to 0$ as $T \to T_c^-$
$(1-m) \neq 0$ at transition

- Discontinuous

eg Potts s.g. $p > p_c$

$p > 2$-spin spherical model



$q(1)$ finite at transition
$(1-m) \to 0$ as $T \to T_c^-$

Mean field theory for SK without first disorder averaging

Thouless, Anderson, Palmer (TAP)

$$H = -\sum_{(ij)} J_{ij}\,\sigma_i\sigma_j \;-\; \sum_i h_i\sigma_i$$

- Usual m.f.t. (allowing for site variation)

$$m_i = \tanh\Big(\beta\big(\sum_j J_{ij}\, m_j + h_i\big)\Big)$$

Might think: solve self-consistently for $\{m_i\}$

But misses self-corr^n.

- Need to take out effect of site i on its own effective field h_i

TAP: $$m_i = \tanh\Big(\beta\big(\sum_j J_{ij} m_j + h_i - \beta\sum_j J_{ij}^2(1-m_j^2)\, m_i\big)\Big)$$

Needed because $\overline{J_{ij}^2} \sim O(N^{-1})$

as well as $\overline{J_{ij}} \sim O(N^{-1})$

$\rightarrow$ $$m_i = \tanh\big(\beta \sum_j J_{ij} m_j + h_i - \beta J^2(1-q)\, m_i\big)\big)$$

- Extn: Cavity method

  $\rightarrow$ Alternative derivation of replica results

  (see Mezard, Parisi, Virasoro)

A divergent susceptibility at phase transition?

cf. Staggered susceptibility for antiferromag.

Spin-glass susceptibility

- Need some corr^n fn which goes long-range at transition
- $\langle \sigma_i \sigma_j \rangle$ goes non-zero at transition but sign can be $\pm$
- $\langle \sigma_i \sigma_j \rangle^2$ becomes always +ve.

$$\therefore \quad \chi_{SG} = N^{-1} \sum_{ij} \overline{\chi_{ij}^2}$$

$$= N^{-1} \beta^2 \sum_{ij} \overline{\left\{ \left(\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \right)^2 \right\}}$$

DIVERGES AT T_g!

- Related to non-linear susceptibility

$$m = \chi h - \chi_{nl}^{(3)} h^3 + \ldots$$

$T > T_g$: $\quad \chi_{nl}^{(3)} = \beta \left(\chi_{SG} - \frac{2}{3} \beta^2 \right)$

Another perspective on RSB

Orientation: Conventional ferromagnet

$$H = - \sum J_{ij} \sigma_i \sigma_j$$

ferro

$\rightarrow$ 2 low temperature states $m = \pm \tilde{m}$

Spin up/down degen^{te}

? Break symmetry

Add small field conj to order parameter

$$\Delta H = - h \sum_i \sigma_i$$

favours m in dir^n of h.

Free energy $F(h)$: non-analytic at $h \to 0$

Magnetization $m(h)$: discontinuous at $h \to 0$

Spin glass . RSB

Conjugate field to OP ?

Couple replicas to one another.

$$H = H(\{\sigma^1\}) + H(\{\sigma^2\}) - \epsilon \sum_i \sigma_i^1 \sigma_i^2$$

'real' replicas

With RSB $F(\epsilon)$ is non-analytic as $\epsilon \to 0$

$q(\epsilon)$ discontinuous as $\epsilon \to 0$

aracciolo et al. 3-D S.G.

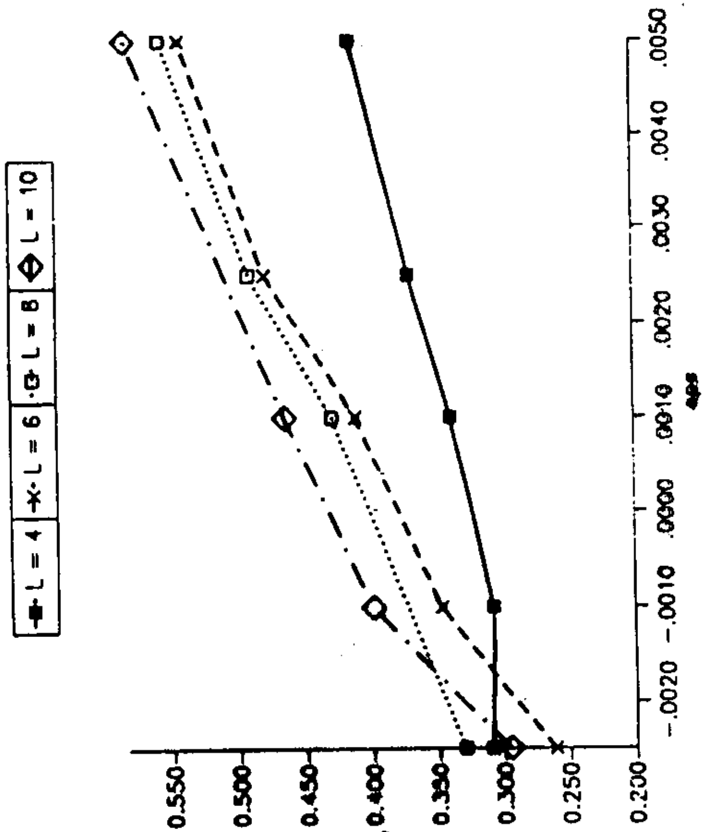


Fig. 15. — q(ε) as function of ε for L = 4, 6, 8, and L = 10 and T = 1.

OSR. $q(\epsilon)$ discontinuous as $\epsilon \to 0$

Hard optimization

- Minimize cost fn.

$$C(\{\underline{x}\}\{\underline{y}\})$$

$\{\underline{y}\}$ ← variables (many: too many to try all options)

$\{\underline{x}\}$ ↑ fixed parameters ↑ possibly randomly chosen from some distn.

(1) • $C\{\underline{y}\}$ smooth.

Minimize by gradient descent

(2) • $C\{\underline{y}\}$ rugged

Difficult due to hills + ridges obscuring true minimum

→ Simulated annealing (eg Metropolis) to probabalistically climb hills.

$$\Delta C \sim kT_A$$

T_A ↑ artificial annealing temp.

T_A reducing → 0.

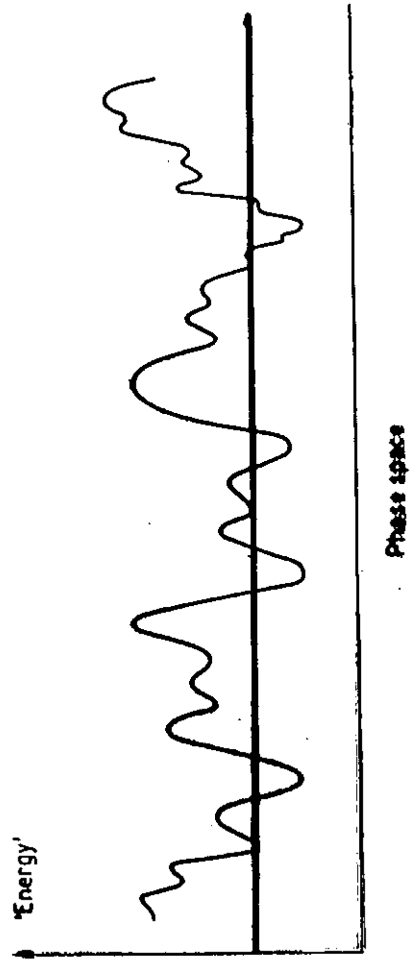


FIG. 4.

Analytic analysis of typical consequence of SA

(or analytic optimization)

- Cost fn → 'Hamiltonian'
- Annealing temp → 'Temperature'

→ 'Free energy' $\quad F = -kT_A \ln Z$

'partition fn'

$$Z = \operatorname*{Tr}_{\{y\}} \exp\left(-\frac{C(\{x\},\{y\})}{T_A}\right)$$

$$\text{Minimal cost} = \lim_{T_A \to 0} F(T_A)$$

- Typical case: average over $\{x\}$ with $P(\{x\})$
  - Carry out formally by replica th.
  - Solve + take limit $T_A \to 0$

Macrodynamics of SK spin glass away from eq^M

Coolen + DS.

- System

$$H = -\sum_{i<j} J_{ij}\sigma_i\sigma_j$$

$$J_{ij} = \frac{J_0}{N} + \frac{J}{\sqrt{N}} z_{ij} \; ; \; \overline{z_{ij}} = 0 , \; \overline{z_{ij}^2} = 1$$

- Macroparameters (minimal set)

$$m(\underline{\sigma}) = N^{-1}\sum_i \sigma_i \qquad \text{(magnetization)}$$

$$r(\underline{\sigma}) = N^{-3/2}\sum_{i<j}\sigma_i z_{ij}\sigma_j \qquad \text{(random contrib to energy)}$$

- Flow eqns (using self-averaging + equipartitioning ansätze)

$$\frac{dm}{dt} = \int dz\, D_{m,r}(z)\, f(m,r,z;T)$$

known

$$\frac{dr}{dt} = \int dz\, D_{m,r}(z)\, g(m,r,z;T)$$

'noise'-distribution

---

* Sim^r analysis for neural networks

Finite-dimensional spin glasses

$$H = -\sum_{(ij)} J_{ij}\,\sigma_i\sigma_j$$

(ij): nearest neighbour

J_{ij}: random ; Gaussian or ± 1.

Monte Carlo simulations. Bhatt + Young.

Basic idea: study OP q and look for transition as fn of T

: need finite-size analysis.

? **How to simulate q**

- 2 replicas of system. (same $\{J_{ij}\}$)
- Initialize random
- Evolve (MC) independently at same temperature T (eg. Metropolis)
- Run for longish time τ to equilibrate
- Average over further long time

$$q(t) = N^{-1}\sum_i \sigma_i^1(t)\,\sigma_i^2(t)$$

$\rightarrow$ measure of $\quad q = N^{-1}\sum_i \langle\sigma_i\rangle^2$

- Finite-size effects blur sharpness of q

$$P(q) = L^{\beta/\nu}\,\tilde{P}\left(q L^{\beta/\nu}, L^{1/\nu} t\right) \quad + \text{less singular corr}$$

$t = (T - T_c)/T_c$

- Useful measure is Binder ratio.

$$g = \frac{1}{2}\left\{3 - \frac{\langle q^4\rangle}{\langle q^2\rangle^2}\right\}$$

$\langle\ \rangle$ = both thermal average + disorder average.

High T : $P(q) \sim$ Gaussian $\rightarrow g = 0$

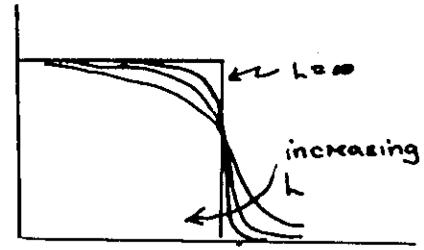
Low T : $P(q) \sim$ delta fn $\rightarrow g = 1$
if unique state

Infinite system $\rightarrow$ step at $T = T_c$

Finite system $\rightarrow$ rounding.

g dimensionless $\rightarrow$ fn of $L/\xi \sim L t^{\nu}$

$$\therefore\ g = \tilde{g}\left(L^{1/\nu} t\right)$$



3-d short-range spin glass

Kawashima + Young: Phys. Rev. B53, R484 ('96)

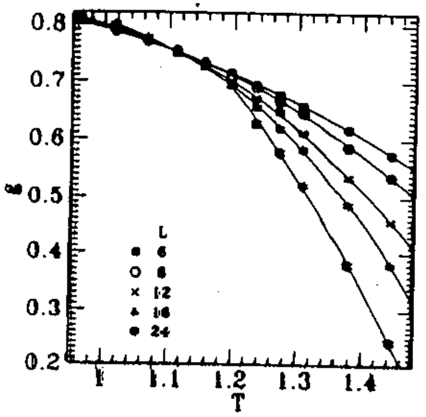


FIG. 1. Results for the Binder ratio g, defined in Eq. (3), for different sizes and temperatures. The lines are smooth curves through the data and are only intended as guides to the eye.

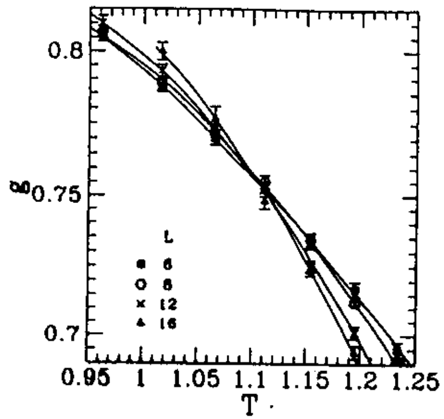


FIG. 2. An enlarged view of the data in Fig. 1 in the crucial region where the curves come together.

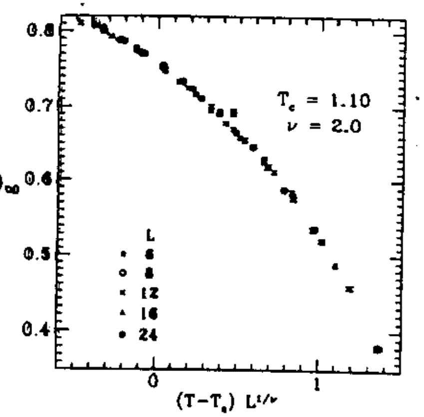


FIG. 3. A scaling plot for g according to the form in Eq. (4).

$$g = \tilde{g}\left(L^{1/\nu}(T-T_c)\right)$$

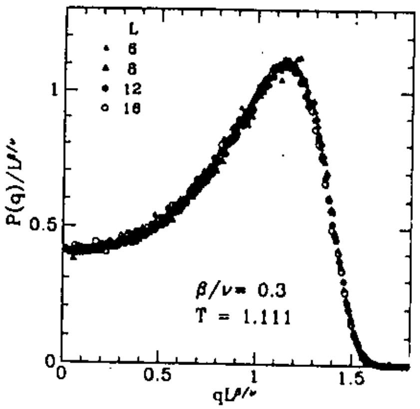


FIG. 4. A scaling plot for $P(q)$ at $T = 1.1113$ (which is close to the the critical point) according to the form in Eq. (7). According to Eq. (8), the value $\beta/\nu = 0.3$ corresponds to $\eta = -0.4$.

$$P(q) = L^{\beta/\nu}\tilde{P}\left(L^{\beta/\nu}q,\ L^{1/\nu}(T-T_c)\right)$$

Self-induced quenched disorder

(model glass transition)

Bouchaud + Mezard: J. Physique I 4, 1109 ('94)

Problem: Periodic Hamiltonian $\rightarrow$ Glassy state.

eg. Bernasconi model

$$H = \frac{J_0^2}{2N}\sum_{k=1}^{N-1}\left\{\sum_{i=1}^{N-k}\sigma_i\sigma_{i+k}\right\}^2$$

spin corr^{n}s

$$= \frac{J_0}{2N}\sum_{k=1}^{N-1} R_k^2$$

NB. Difficult because frustrated

Simulations:
- Monte Carlo (T)
- Reduce T
- Freezes to glassy state as T is reduced beyond critical temp.

Origin of glassy behaviour: effect of corr^{ns} between distant spins

$k \to \infty$ as $N \to \infty$

Bouchaud-Mézard: Replace Hamiltonian by randomly connected one

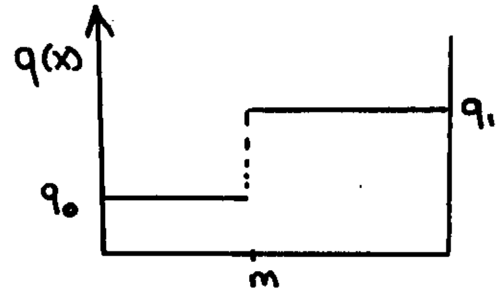
Original: $$H = \frac{J_0^2}{2N} \sum_{k=1}^{N-1} \left\{ \sum_{i=1}^{N-k} \sigma_i \sigma_{i+k} \right\}^2$$

New: $$H = \frac{1}{2N} \sum_{k=1}^{N-1} \left\{ \sum_i^N \sum_j^N J_{ij}^{(k)} \sigma_i \sigma_j \right\}^2$$

Random connectivity matrices, indep for each k, each element equal to J_0 with prob $(N-k)/N^2$, zero otherwise.

Now 'solve' using replicas.

One-step RSB



Find: transition at T_g

$T > T_g$: $q(x) = 0$

$T < T_g$: $q_0 = 0$, q_1 close to 1

$m \sim T/T_g$

ie first order transition to state with glassy behaviour (non-equiv thermodyn. state)
(reminiscent of REM).

My own excursion into glassy behaviour from periodic Hamiltonian } Josephson junction array in field (long-range int^{ns}) { P.R.L. **75**, 713 (95)

Return briefly to Bouchaud-Mézard study of Bernasconi model → s.g.

$$H = \frac{J_0}{2N} \sum_{k=1}^{N-1} \left(\sum_{i=1}^{N-k} \sigma_i \sigma_{i+k} \right)^2$$

$$\equiv \sum_{ij} \frac{J_{ij}}{2\sqrt{N}} \sigma_i \sigma_j$$

with $J_{ij} \equiv \frac{J_0}{\sqrt{N}} \sum_k \sigma_{i+k} \sigma_{j+k}$

ie equiv to SK Hamiltonian with couplings determined by spins themselves

Slow dynamics → couplings effectively in 'spin glass' phase self-consistent random quenched var.

cf. Coolen, Penney, S study of coupled spins + interactions (in s.g. region).

Dynamics of mean-field models (outline)

(de Dominicis + Peliti) (Sompolinsky + Zippelius) (Kirkpatrick + Thirumalai) (Kurchan + Cugliandolo)

Langevin dynamics

$$\frac{\partial \phi_i}{\partial t} = -\frac{\partial H}{\partial \phi_i} + \eta_i(t)$$

noise : white, variance T

→ Generating fn.

$$\tilde{Z} = \int \mathcal{D}\phi \prod_{it} \delta\left(\frac{\partial \phi_i}{\partial t} + \frac{\partial H}{\partial \phi_i} - \eta_i(t) \right) \mathcal{J}$$

$\mathcal{D}\phi = \prod_{it} d\phi_i(t)$

Can add $\sum \lambda_i \phi_i$ to H → generating facility

Jacobian

- Replace δ-fn by exponentiated form

$$\tilde{Z} = \int \mathcal{D}\phi \, \mathcal{D}\hat{\phi} \exp\left(i \int dt \sum_i \hat{\phi}_i(t) \left(\frac{\partial \phi_i}{\partial t} + \frac{\partial H}{\partial \phi_i} - \eta_i(t) \right) \right) \mathcal{J}.$$

- Average over noise η

$$\rightarrow Z = \int \mathcal{D}\phi \, \mathcal{D}\hat{\phi} \exp\left\{ i \sum \hat{\phi}_i(t) \left[\frac{\partial \phi_i}{\partial t} + \frac{\partial H}{\partial \phi_i} + T \hat{\phi}_i(t) \right. \right.$$

- Note: normalization s.t. $Z(\lambda=0) = 0$.

Z ... disorder

eg $H = -\sum_{ij} J_{ij} \phi_i \phi_j$

+ local term to keep $|\phi| \sim 1$ for Ising.

$$\frac{\partial H}{\partial \phi_i} = -\sum J_{ij} \phi_j$$

$$\sum_i \int dt \, \hat{\phi}_i(t) \frac{\partial H}{\partial \phi_i(t)} = -\sum_{ij} \int dt \, J_{ij} \hat{\phi}_i(t) \phi_j(t)$$

Symmetrize

$$= -\sum_{(ij)} \int dt \, J_{ij} \left(\hat{\phi}_i(t) \phi_j(t) + \hat{\phi}_j(t) \phi_i(t) \right)$$

Average over J_{ij}

$$\rightarrow \overline{Z} \sim \int \mathcal{D}\phi \, \mathcal{D}\hat{\phi} \, \exp\Big\{ \sum_i \int dt \left(\hat{\phi}_i(t) \dot{\phi}_i(t) + \hat{\phi}_i(t) \hat{\phi}_i(t) \right)$$

$$- \frac{\beta^2 J^2}{N} \int dt \, dt' \sum_{ij} \left(\hat{\phi}_i(t) \phi_j(t) + \hat{\phi}_j(t) \phi_i(t) \right) \left(\hat{\phi}_i(t') \phi_j(t') + \phi_j(t') \phi_i(t') \right) \Big\}$$

rescaled to give more usual.

Ignoring Jacobian + local term.

$\rightarrow$ single site via

$$\int \mathcal{D}C \prod_{tt'} \delta\left(C(tt') - N^{-1} \sum_i \hat{\phi}_i(t) \phi_i(t') \right) = 1$$

$$\int \mathcal{D}D \, \Pi \, \delta\left(D(tt') - N^{-1} \sum_i \phi_i(t) \phi_i(t') \right) = 1$$

$$\int \mathcal{D}\hat{D} \, \Pi \, \delta\left(\hat{D}(tt') - N^{-1} \sum_i \hat{\phi}_i(t) \hat{\phi}_i(t') \right) = 1$$

etc.

or by (completing square)$^{-1}$

$$\int \mathcal{D}D \, \mathcal{D}\hat{D} \, \mathcal{D}C \; \exp\left(\hat{D}D + CC \right)$$

$$\int \mathcal{D}\phi \, \mathcal{D}\hat{\phi} \, \exp\left(\hat{\phi}\dot{\phi} + \hat{\phi}\hat{\phi} + D\hat{\phi}\hat{\phi} + \hat{D}\phi\phi + C\hat{\phi}\phi \right)$$

shorthand

$$\int dt \, dt' \, C(tt') \sum_i \phi_i(t) \hat{\phi}_i(t')$$

Now single-site $\sum_i \rightarrow N$

$\rightarrow$ extremal dominance.

$\rightarrow$ Self-consistent eqns for $C, D, \hat{D}$.

$\rightarrow$ Mean field dynamics.

Cases

(i) $\lim_{N \to \infty} \lim_{t, t' \to \infty}$ → mean-field thermodyn

~ Parisi

(ii) $\lim_{t, t' \to \infty} \lim_{N \to \infty}$ → new phenomena. aging etc. Cugliandolo

---

Causality → $\hat{D} = 0$

Some recent work. (Ioffe + DS)

Escape from one metastable state to another for N large but finite

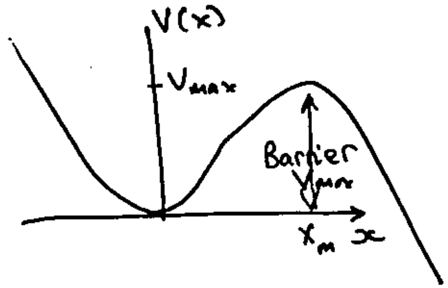
Need to allow for $\hat{D} \neq 0$

Analagous to going over barrier – rare fluctuation
cf. Bouchaud. Instanton.

SK model dynamics over barriers → too involved for here.

∴ Just idea on toy model.

Particle in potential $V(x) = v x^2 (1-x)$



Dynamics

$$\frac{dx}{dt} = -\frac{dV}{dx} + \xi(t) \qquad \langle \xi(t)\xi(t')\rangle = 2T\delta(t-t')$$

Generating fnl

$$Z = \int \mathcal{D}x\, \mathcal{D}\hat{x}\, \exp(-A(x,\hat{x}))\, J(x))$$

$$A(x,\hat{x}) = \int \left[i\hat{x}\left(\frac{dx}{dt} + \frac{dV}{dx}\right) + T\hat{x}^2 \right] dt$$

$$J(x) = \mathrm{Det}\left(\frac{d}{dt} + \frac{d^2(\beta V)}{dx^2}\right)$$ ← ignore (to get results only to exp. accuracy)

Low T → saddle pt dominance.

Vary action w.r.t. $x, \hat{x}$

$$\left.\begin{aligned} \frac{d\hat{x}}{dt} - \frac{d^2V}{dx^2}\hat{x} &= 0 \\ \frac{dx}{dt} + \frac{dU}{dx} &= 2iT\hat{x} \end{aligned}\right\} \rightarrow 2 \text{ solns}$$

1) Standard soln.

$$\hat{x} = 0 \quad , \quad \frac{dx}{dt} = -\frac{dV}{dx}$$

particle slides down potential practically unaffected by thermal noise.

$\rightarrow A = 0$

(ii) Non-standard (instanton) soln

$$iT\hat{x} = \frac{dx}{dt} \quad , \quad \frac{dx}{dt} = \frac{dV}{dx}$$

particle goes up hill

(iii) Escape path = soln (ii) up to V_{max}, x_{max}
+ soln (i) for $x > x_{max}$.

$\rightarrow$ Action $A = \beta V_{max}$

Probability $\exp(-\beta V_{max})$. $\rightarrow$ usual escape prob.

Ioffe + DS

Extn to SK model.

Allow for solns $\hat{D} \neq 0$ (self-consistently)

$\rightarrow$ prob of transitions between metastable states.

Small reduced temp τ $\rightarrow$ barriers $\sim N\tau^6$.