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INTERNATIONAL ATOMIC ENERGY AGENCY
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR/1003 - 13

SUMMER COLLEGE IN CONDENSED MATTER ON
" STATISTICAL PHYSICS OF FRUSTRATED SYSTEMS "

(28 July - 15 August 1997)

"Theoretical description of the out of equilibrium "
"Dynamics of glassy systems"

presented by:

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Theoretical Description of the Out of Equilibrium Dynamics of Glassy Systems.

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Contents:

Lecture I: Phenomenology.
- Experimental Observations
- Numerical Results
- Some Analytical Results.

Relevant Correlations.

Lecture II: (J. Kurchan)

Microscopic Dynamics
Equilibrium Theorems (time translation invariance & fluctuation-dissipation)
Bounds out of equilibrium.

Lecture III

General Properties of Corr & Resp Functions.
Cases: SG, Glasses, Domain Growth.

Lecture IV – V

MF Dynamics →
- Models
- Dyn eqs for disordered & non disordered non-linear syst
- Solution.
- cfr. with SG - exp.

Some Gral Refs.

- E. Vincent, J. Hammann, M. Ocio, JP Bouchaud, LFC
  To appear in Springer Verlag, E. Rubí Ed
  Proc from Sitges 96. cond-mat/9607224.
  Experiments
- JP Bouchaud, LFC, J. Kurchan, M. Mézard
  To appear in "SG & Random Fields", P Young Ed
  cond-mat/9702070
  General

GLASSY DYNAMICS

1 – EXPERIMENTS ≡ MOTIVATIONS

SPIN-GLASSES, POLYMER GLASSES, GELS, ETC.
RESPONSES, CORRELATIONS

'UNIVERSALITY'

MODELLING → SPIN-GLASSES, GLASSES

'MAIN' OBSERVATIONS (DATA)

2 – EQUILIBRIUM vs. NON EQUILIBRIUM

INVARIANCE UNDER TIME TRANSLATIONS (TTI)
FLUCTUATION-DISSIPATION THEOREM (FDT)

3 – GENERAL PROPERTIES OF CORRELATION AND RESPONSE FUNCTIONS.

4-5 – MF DYNAMICS FOR 'NON-TRIVIAL' MODELS

SOLUTION
GENERAL PREDICTIONS
'UNIVERSALITY' – SCALINGS – EXPERIMENTS.

GLASSES

- POLYMER GLASSES
- SPIN GLASSES
- STRUCTURAL GLASSES
- ORIENTATIONAL GLASSES
- VORTEX GLASSES

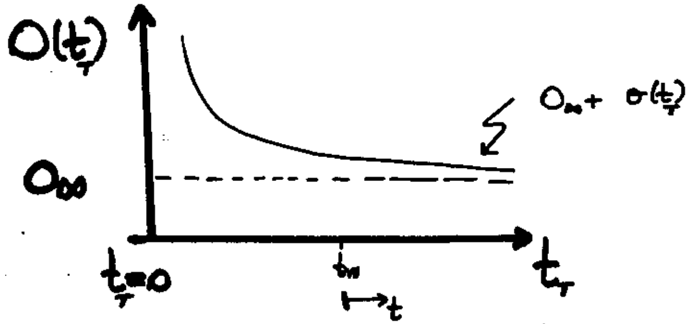
• • •

All rather different but

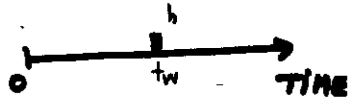
SIMILAR PHENOMENOLOGY

AT LOW TEMPERATURES

- ONE-TIME QUANTITIES eg. $\mathcal{E}(t_T)$, $m(t_T)$



- **APPEARS AS BEING EQUILIBRATED** for large times.



- TWO-TIME QUANTITIES eg.

'Local' Correlation $$C(t+t_w,t_w) \equiv \frac{1}{N}\sum_{i=1}^{N} \overline{\langle s_i(t+t_w)\, s_i(t_w)\rangle}$$

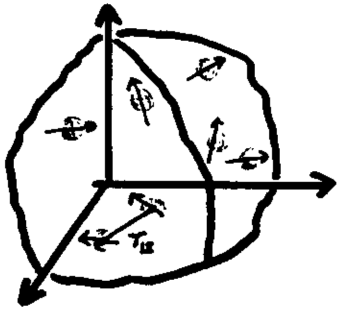
'Local' Response $$R(t+t_w,t_w) \equiv \frac{1}{N}\sum_{i=1}^{N} \overline{\frac{\delta \langle s_i(t+t_w)\rangle}{\delta h_i(t_w)}}\Bigg|_{h=0}$$

- **DEPEND ON BOTH (t_w, t)**
- **ARE NOT RELATED BY FDT**

$\boxed{\text{NO EQUILIBRIUM.}}$ I would talk about space for [a moment.

WHAT'S A SPIN GLASS? → MODELS

Magnetic Alloy, e.g. CuMn, AgMn



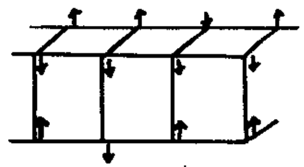
Mn Magnetic Impurities at Random Fixed Positions

Quenched Randomness

RKKY Interactions (Between Mn's)

$$J(\vec r_{12}) = J(|\vec r_{12}|) \simeq J_0\, \frac{\cos(2k_F|\vec r_{12}| + \phi_0)}{(k_F|\vec r_{12}|)^3}$$

Symmetric, Random (through $|\vec r_{12}|$), $\sim |\vec r_{12}|^{-3}$ Rapid decay



Mn : Ising Spins $s_i = \pm 1$

$J(|\vec r_{12}|)$: Gaussian Interactions

J_{ij}, $P[J]$ Gaussian

DISORDER

$$\boxed{H_J[\vec s] = \sum_{\langle ij\rangle} J_{ij}\, s_i s_j \longrightarrow \sum_{i\neq j} J_{ij}\, s_i s_j \quad \text{MF}}$$

3D EDWARDS-ANDERSON → SHERRINGTON-KIRKPATRICK

MODELS : MORE SIMPLIFICATIONS

$$H_J[\vec{s}] = \sum_{\langle ij \rangle} J_{ij}\, s_i s_j$$ 3DEA $\quad S_i = \pm 1$

$\downarrow$

MF $$H_J[\vec{s}] = \sum_{i \neq j} J_{ij}\, s_i s_j$$ SK No Geometry $S_i = \pm 1$

$\downarrow$

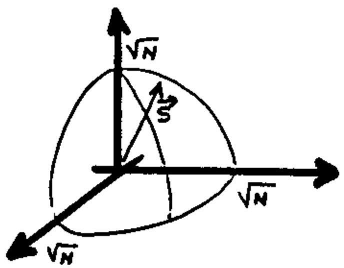
MF $$H_J[\vec{s}] = \sum_{i_1 \neq i_2 \cdots \neq i_p} J_{i_1 \cdots i_p}\, s_{i_1} \cdots s_{i_p}$$ p-spin Model $S_i = \pm 1$

DISORDER $P[J]$: $\begin{cases} \text{BIMODAL} \\ \text{GAUSSIAN } (p \geq 2) \quad \left(\dfrac{N^{p-1}}{\pi p!}\right)^{1/2} \exp\left[-\dfrac{N^{p-1} J^2_{i_1 \ldots i_p}}{p!}\right] \end{cases}$

SPINS . ISING : DIFFICULT!

: SPHERICAL

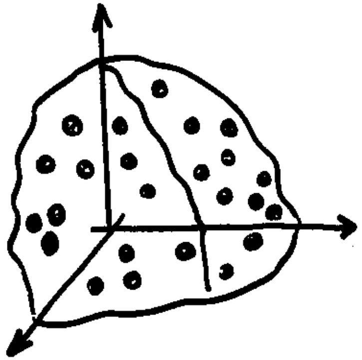
$\vec{S}$ ON A HYPERSPHERE (N-DIMENSIONAL)



Hope: $S_i = O(1), \forall i. \Rightarrow$ FINITE VALUE FOR S_i !

WHAT CAN FORM A 'STRUCTURAL' GLASS ?

$\longrightarrow$ MODELS



- 'PARTICLES' THAT 'MOVE' INSIDE THE SAMPLE

INTERACTIONS An example LÉNNARD-JONES

$$\boxed{V(\vec{r}_1, \vec{r}_2) = V(|\vec{r}_1 - \vec{r}_2|) = -\frac{C_1}{|\vec{r}_1 - \vec{r}_2|^6} + \frac{C_2}{|\vec{r}_1 - \vec{r}_2|^{12}}}$$

NO QUENCHED DISORDER

SIMPLIFIED MODELS

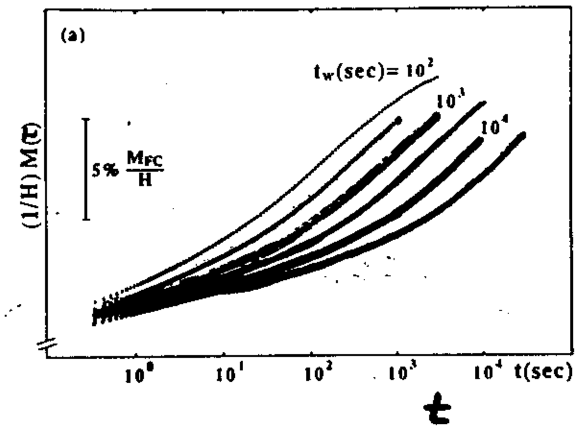
e.g. $$H[\vec{r}_a] = \sum_{a \neq b}^{N_{part}} \sum_{i=1}^{D} (r_i^a \cdot r_i^b)^p$$ or

$\sim$ p-spin model. Motivated by Spin-Glass Modelling.

AGING IN SPIN-GLASSES

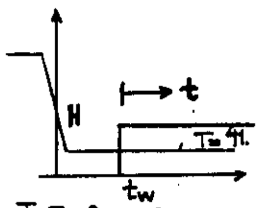
ZERO FIELD COOLING EXPERIMENT (ZFC)

$\frac{m(t)}{h}$



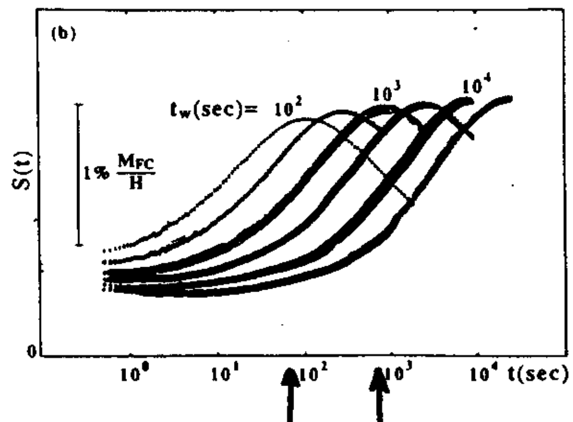
CuMn

(T_g = 45.3K)
h = 0.85



T: TEMPERATURE
h: MAGNETIC FIELD
$\longmapsto t$

$$S(t) = \frac{1}{h}\,\frac{\partial m(t)}{\partial \ln t} \quad (t = \tau \text{ HERE})$$



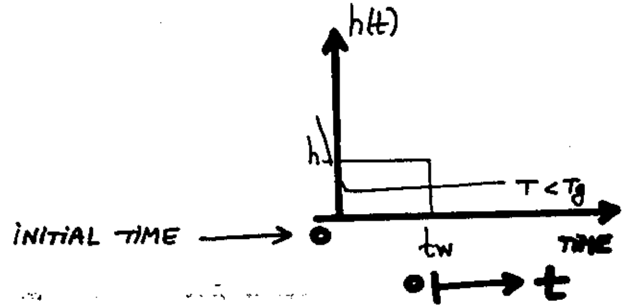
t_{w_1} = 100 s
t_{w_2} = 300 s
t_{w_3} = 1000 s
t_{w_4} = 3000 s
t_{w_5} = 10000 s
t_{w_6} = 30000 s

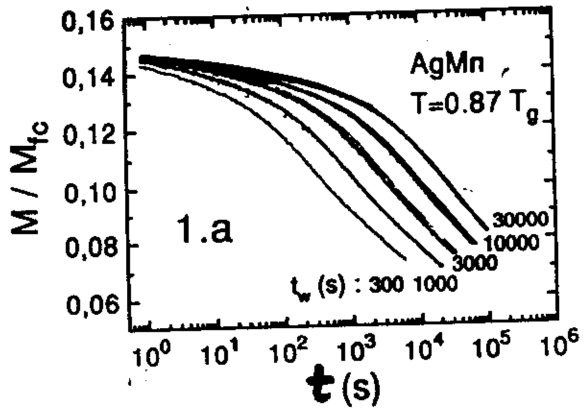
max S(t) AT $t \sim t_w$

L. SANDLUND, P. SVEDLINDH, P. NORDBLAD, O. BECKMAN ; PHYS. REV. LETT. 51 (83) 911.

THERMO-REMANENT MAGNETIZATIO

SACLAY-GROUP DATA





EXPLICIT t_w DEPENDENCE

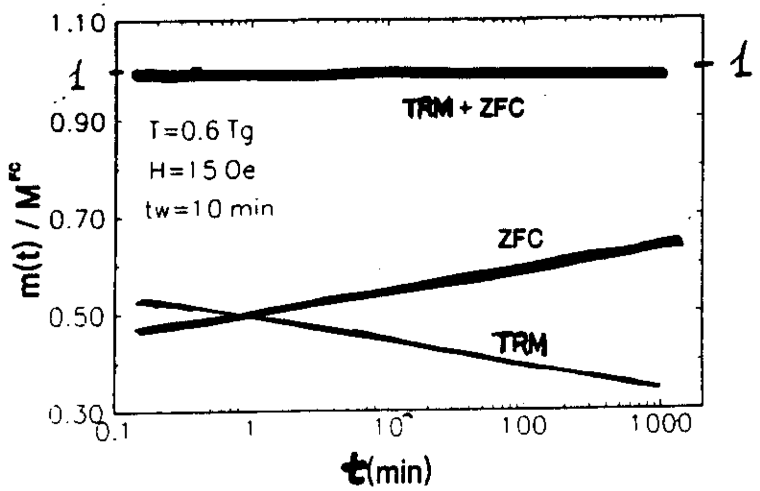
AGING

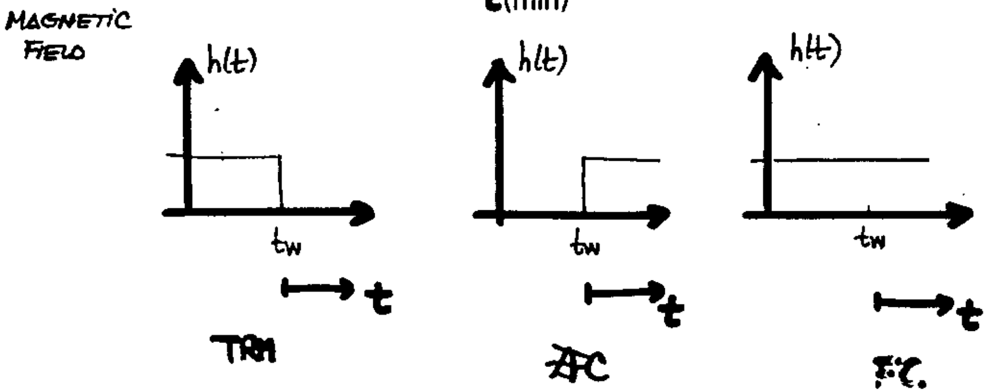
MECHANISM ?

SCALING ?

LINEAR RESPONSE THEORY

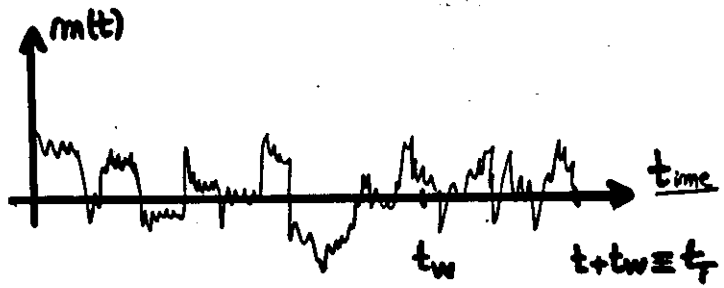
$$\frac{m^{TRM}}{m^{FC}} + \frac{m^{ZFC}}{m^{FC}} = 1$$





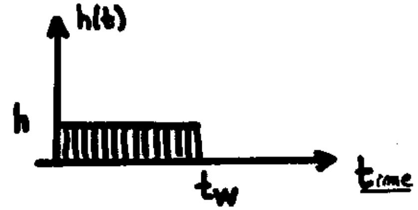
MOST PROBABLY $T_{\text{NON-LINEAR}}(h)$ FOR FINITE h

TRM – ZFC ⟷ RESPONSE



$$\left.\frac{\delta m(t_T)}{\delta h(t_w)}\right|_{h=0} \equiv R_m(t_T, t_w) \quad \text{RESPONSE}$$

IF WE NOW APPLY TRM

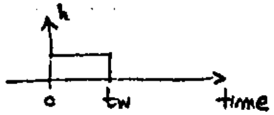


$$\frac{\Delta m(t_T, t_w)}{\Delta h} = \int_0^{t_w} ds \, \frac{\delta m(t_T)}{\delta h(s)} = \int_0^{t_w} ds \, R_m(t_T, s)$$

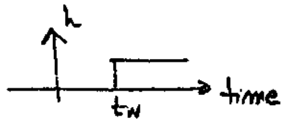
$$\boxed{m_{TRM}(t_T, t_w) = h \cdot \int_0^{t_w} ds \, R_m(t_T, s)}$$

LINEAR RESPONSE THEORY

$$m_{TRM}(t_T,t_W) \equiv h\int_0^{t_W} ds \; R(t_T,s)$$



$$m_{ZFC}(t_T,t_W) \equiv h\int_{t_W}^{t_T} ds \; R(t_T,s)$$



if $\boxed{R(t_T,s) = R(t_T-s)}$

$$m_{TRM}(t_T,t_W) = h\int_0^{t_T} ds\; R(t_T-s) - \underbrace{h\int_{t_W}^{t_T} ds\; R(t_T-s)}_{m_{ZFC}(t_T,t_W)}$$

$$m_{ZFC}(t_T,t_W) = h\int_{t_W}^{t} ds\; R(t_T-s)$$

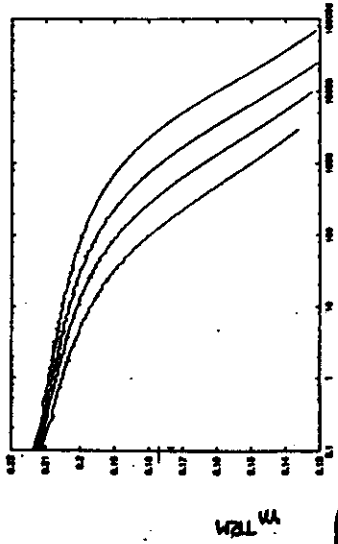
$$\boxed{m_{ZFC}(t_T,t_W) = h\int_0^{t} d\tau\; R(\tau) = m_{ZFC}(t \equiv t_T-t_W)}$$

$$m_{TRM}(t_T,t_W) = \underbrace{h\int_0^{t_T} d\tau\; R(\tau)}_{\text{CONST}} - h\int_0^{t} d\tau\; R(\tau)$$

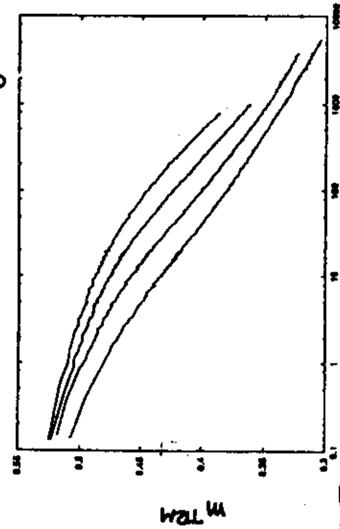
$$\boxed{m_{TRM}(t_T-t_W) = \text{CONST} - h\int_0^{t} d\tau\; R(\tau) = m_{TRM}(t)}$$

THIS PROOF CAN BE DONE BACKWARDS ⇒ $R(t_T-t_W) \Leftrightarrow m(t_T-\ldots$

THERMO REMANENT MAGNETIZATION (log-linear) PLOTS:

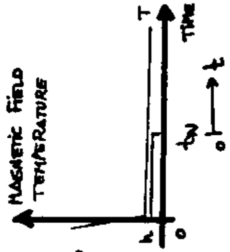


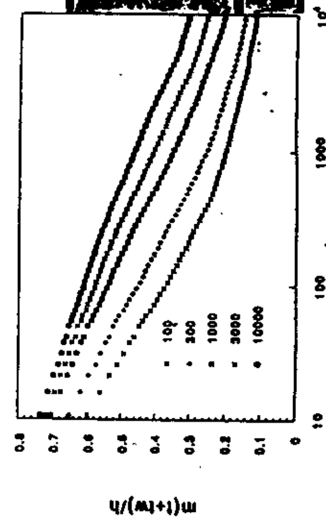
EXP Ag:Mn (Saclay-Data)



EXP Thiospinel (Saclay-Data)

3D 3D EA (H. Rieger)





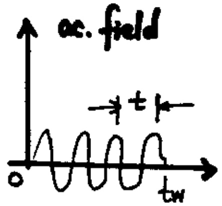
Linear Response

NF D-Hypercubic Cell (NF) ≃ SK

(L.F. Cugliandolo, J. Kurchan, F. Ritort)

OUT OF PHASE SUSCEPTIBILITY

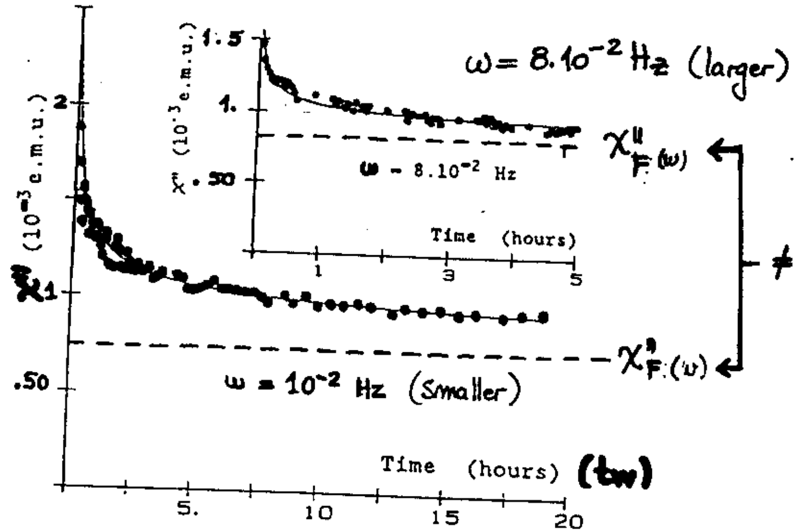
$T < T_g$: $\chi''(\omega, t_w)$



$\boxed{t \sim \frac{1}{\omega}}$

IN THESE PLOTS:

$\omega t_w \lesssim O(10^3)$



EXPERIMENTALLY OBSERVED $\left\{ 10^{-2}\,Hz \sim 1\,Hz \right.$ FREQUENCIES.

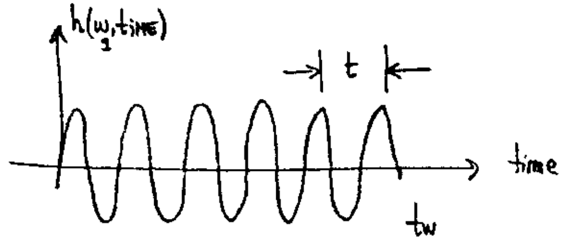
$\Longrightarrow$ t_w-DEPEND

AMPLITUDE OF

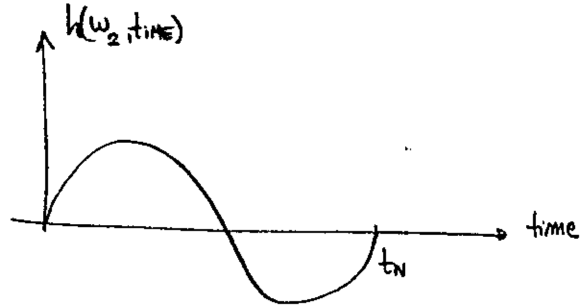
$$\chi''(\omega, t_w) \equiv \mathrm{Im} \int_0^{t_w} ds\, e^{i\omega s}\, \underbrace{R(t_w, s)}_{\text{RESPONSE FUNCTION}}$$

$\chi''(\omega, t_w) \sim$ DEPENDS UPON ω VS. t_w.

OUT OF PHASE SUSCEPTIBILITY



'large' $\omega_1 = \frac{2\pi}{t}$



'smallest' $\omega_2 = \frac{2\pi}{t_w}$

EXPERIMENTAL CONSTRAINT

$\boxed{\omega = \frac{2\pi}{t} \geq \frac{2\pi}{t_w}}$

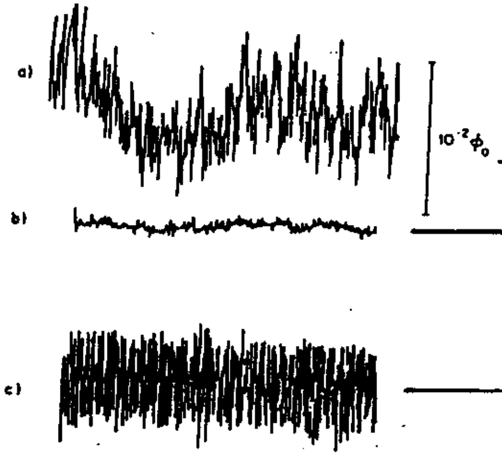
(INDEED, THERE MUST BE A FEW CYCLES IN SUCH A WAY TO HAVE A 'GOOD' 'MEASUREMENT')

THUS $\boxed{t \leq t_w}$

AND, IN SUSCEPTIBILITY MEASUREMENT ONE CANNOT REACH LARGER t's !

* OUT OF PHASE SUSC. MEASUREMENTS ARE BETTER BECAUSE h IS SMALLER BUT ARE WORSE BECAUSE ONE CANNOT REACH VERY LONG t's.

MAGNETIC NOISE IN SPIN-GLASSE



SPIN-GLASS
$CsNiFeF_6$
TIME-CORRELATIONS ?

→ EMPTY

→ COPPER (White Noise)

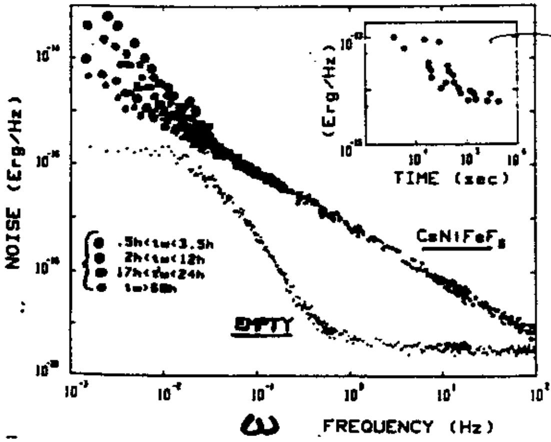
TTI ASSUMPTION ⇒
⇒ FOURIER TRANSFORM : $\boxed{C(t) \longrightarrow \tilde{C}(\omega)}$

M. Ocio, H. Bouchiat and P. Monod, J. Magn. Magn. Mat 54 (1986) 11.

BUT :

$\tilde{C}(\omega)$

$t_w \longrightarrow$



$f = 3.10^{-3}$ Hz

SPIN GLASS $T = 4.2$ K

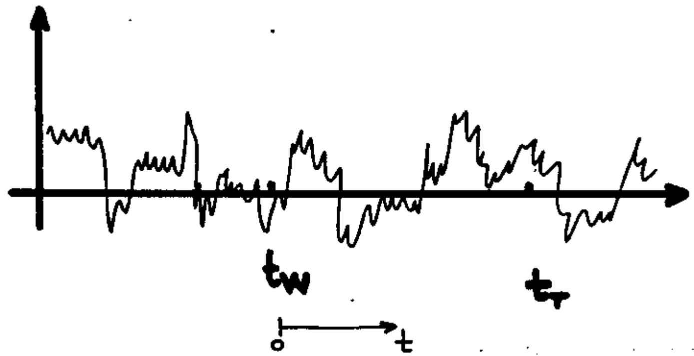
ω

$\longleftrightarrow$ 1/f NOISE (STATIONARY.)

$\longleftrightarrow$?

NON STATIONARY CONTRIBUTION : t_w-DEPENDENCE

CORRELATION FUNCTIONS



h=0

NO FIELD (PERTURBATION) APPLIED

$$\boxed{\frac{1}{N}\, m(t_r)\, m(t_w) \equiv C_m(t_r, t_w)}$$ CORRELATION

↑
NORMALIZATION

IN GLASSES: DENSITY - DENSITY CORRELATIONS

MEASUREMENT:

$\vec{r}$: Position in Space

$\rho(\vec{r},t)$: Time Dependent Density at $\vec{r}$

$$\rho(\vec{q},t) \equiv \int d^d r \; e^{i\vec{q}\cdot\vec{r}} \rho(\vec{r},t)$$

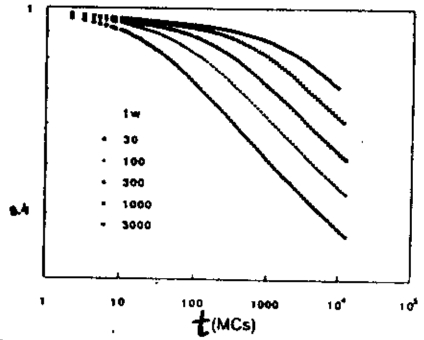
$$C_q(t_r, t_w) = \frac{\langle \rho^*(\vec{q},t_r)\, \rho(\vec{q},t_w) \rangle}{\langle |\rho(\vec{q},t_w)|^2 \rangle}$$ ← Normalization.

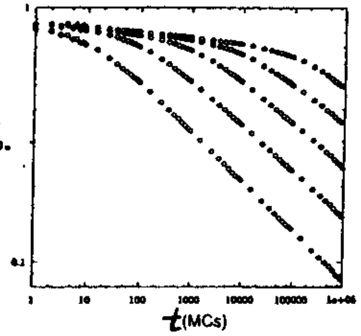
$\langle \cdot \rangle$ Average over dynamical histories (Numerics)

$\boxed{C(t+t_w, t_w) \quad \text{vs.} \quad t}$

log-log Plots:

$$C(t+t_w, t_w) = \frac{1}{N} \sum_{i=1}^{N} \overline{\langle S_i(t+t_w)\, S_i(t_w) \rangle}$$

a.


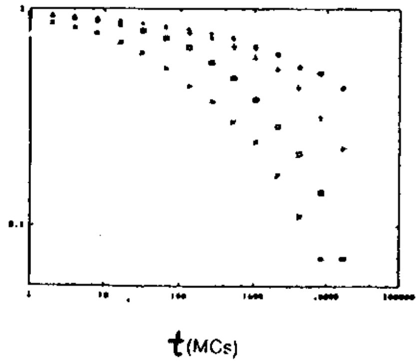
b.


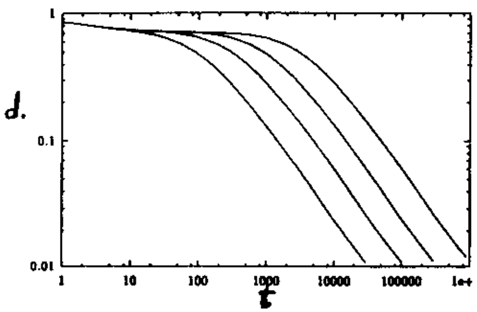
MF D-HYPERCUBIC CELL (SK)

(D=17) $t_w = 30, 10^2, 300, 10^3, 3000$

3D EA 3D

$t_w = 10, 10^2, 10^3, 10^4, 10^5$

c.


d.


MF P-SPIN (SPHERICAL)

(P=3) $t_w = 4, 64, 1024, 16.000$

MF P=2 SPHERICAL

(ANALYTICAL)

$t_w = 30, 100, 300, 1000.$

a. L.F. CUGLIANDOLO, J. KURCHAN, F. RITORT; PRB 49, 6331 (94).

b. H. RIEGER; J. PHYS A26, L615 (93).

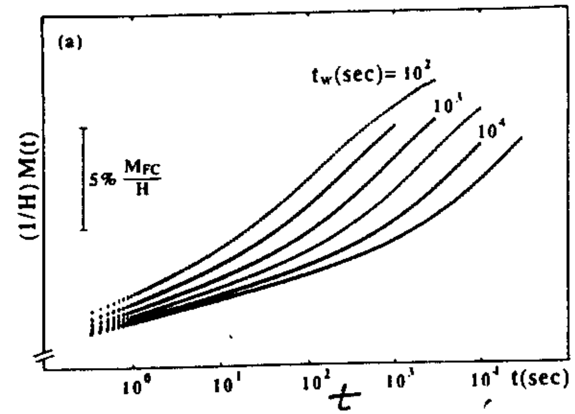
c. B. STROCKA, H. RIEGER, H. SCHRECKENBERG; UNPUBLISHED.

d. L. F. CUGLIANDOLO, D. S. DEAN; J. PHYS A28, 4213 (95).

$\boxed{\text{EXP ?}}$

PHYSICAL AGING $T<T_g$

Cu Mn SPIN-GLASS ZFC EXPERIMENT

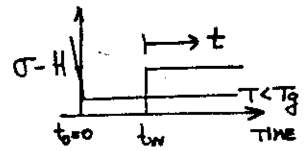


$T_g = 45.3$ K

$H = 0.8$ G

L. SANDLUND, P. SVEDLINDH, P. NORBLAD, O. BECKMAN, PHYS REV LETT 51 (83) 911.

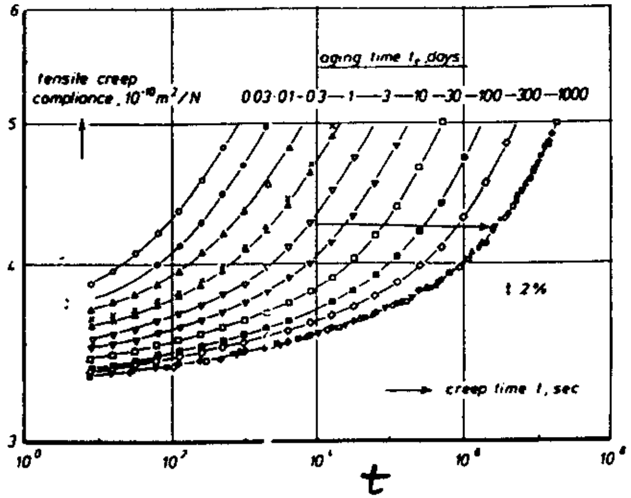
PVC POLYMER GLASS



H: MAGNETIC FIELD

σ: STRESS

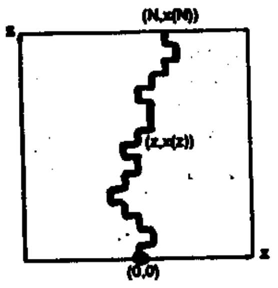
$\dot{\gamma} \to \infty$



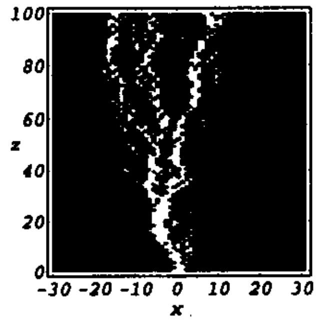
L. STRUICK, 'PHYSICAL AGING IN AMORPHOUS POLYMERS' (1978)

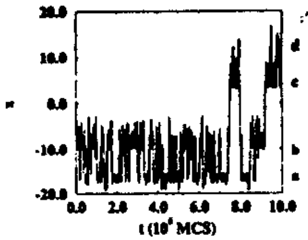
1+1 DIRECTED POLYMER

RANDOM ENVIRONMENT



↑ FIXED. END

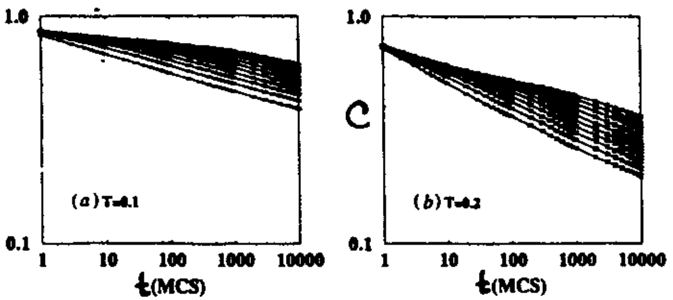




$$\mathcal{H} = \sum_{z=1}^{N} \mathbf{V}[z, x(z)]$$

WITH $V(z,x)$ TAKEN FROM A UNIFORM DIST $[-1,1]$.

Off-equilibrium dynamics of (1 + 1)-dimensional DPRM H Yoshino J. PHYS. A29, 1421 (1996).

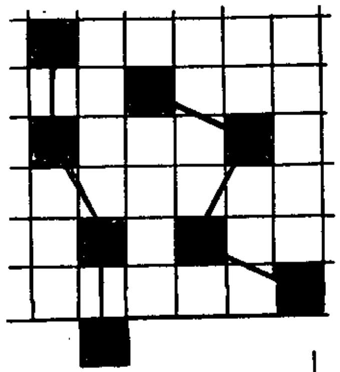


↕ DIFFERENT t_W's

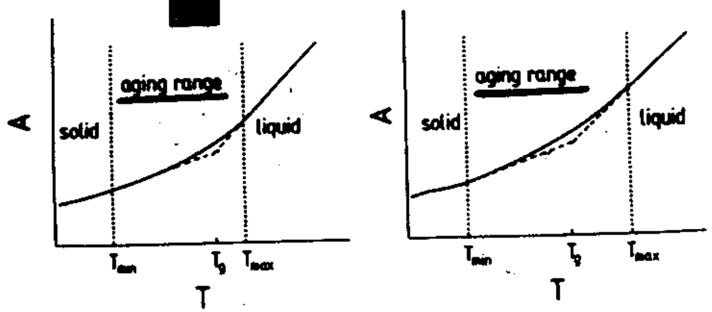
See Also A. BARRAT ... PHYS. REV E (97) 9701021/COND-MAT.

POLYMER GLASSES

LATTICE MODEL



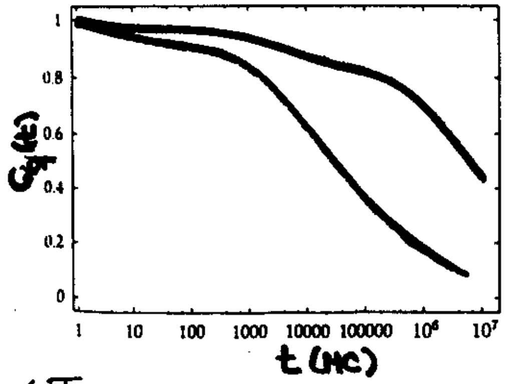
BINDER ET AL.



(a) experiment (b) simulation

MC DYNAMICS

J. BASCHNAGEL, PHYS. REV. 69, 135 (96).



$t_W = 4.10^6$ MC

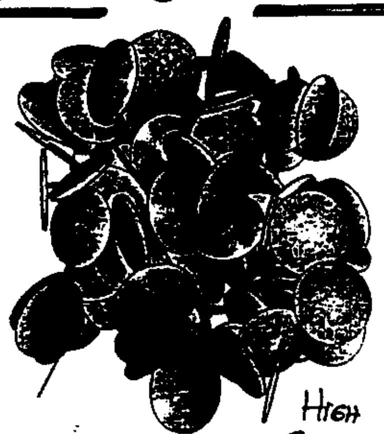
$t_W = 6.10^3$ MC

$T_{min} < T = 0.16 < T_{max}$

CLAY COLLOID SUSPENSIONS

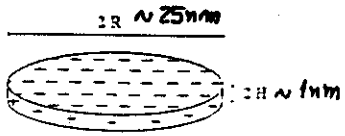
LANGEVIN DYNAMICS

'HOUSE OF CARDS'



HIGH CONCENTRATION

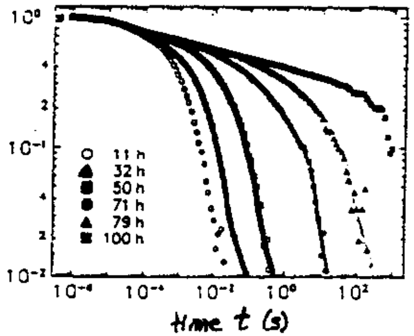
LOW CONCENTRATION



M. DIJKSTRA, J-P. HANSEN, P. A. MADDEN; PHYS. REV. LETT. 75, 223 (95)

W. KROON, G.H. WEGDAM, R. SPRICK; PHYS REV. E54 (96).

$C_q(t)$



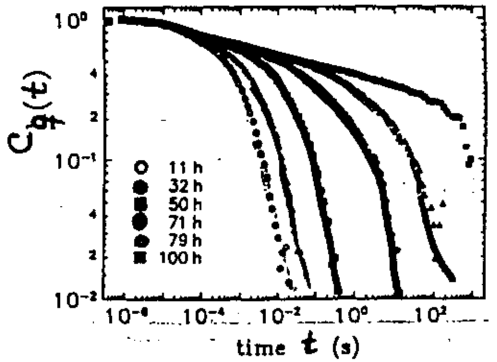
FOR A GIVEN q.

AGING AT LEAST UNTIL $\begin{cases} t_w \sim 100h \\ t \sim 1h \end{cases}$

CORRELATION FUNCTIONS

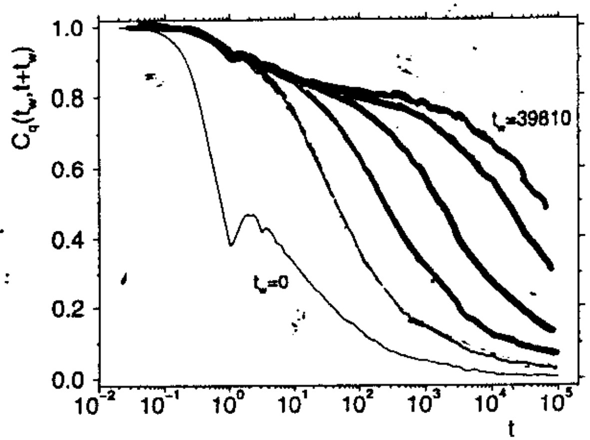
GLASSES

EXPERIMENT (GEL)



M. KROON, G.H. WEGDAM, R. SPRIK; PHYS. REV. E54, 1 (1996).

(MOLECULAR DYNAMICS) SIMULATION

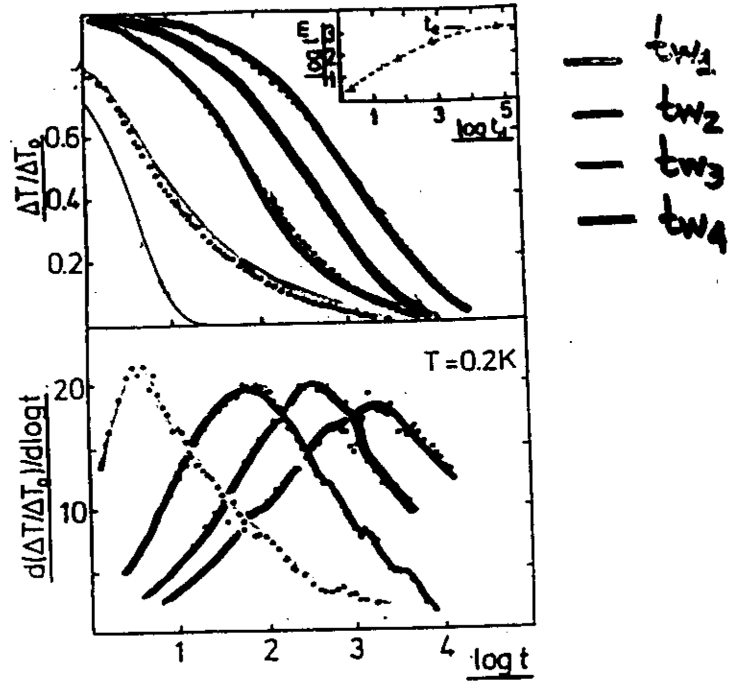


W. KOB, J.L. BARRAT PHYS. REV. LETT (1997).

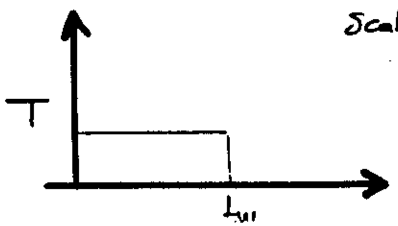
LÉNNARD-JONES

Also: G. PARISI (1997). MONTECARLO SIM

CHARGED DENSITY WAVES



K. BILJAKOVIC, J.C. LASJAUNIAS, P. MONCEAU, PRL (95).
K. BILJAKOVIC in 'Phase Transitions and Relaxation in Systems with Competing Energy Scales' Kluwer Ac. (1993).



TEMPERATURE JUMP.

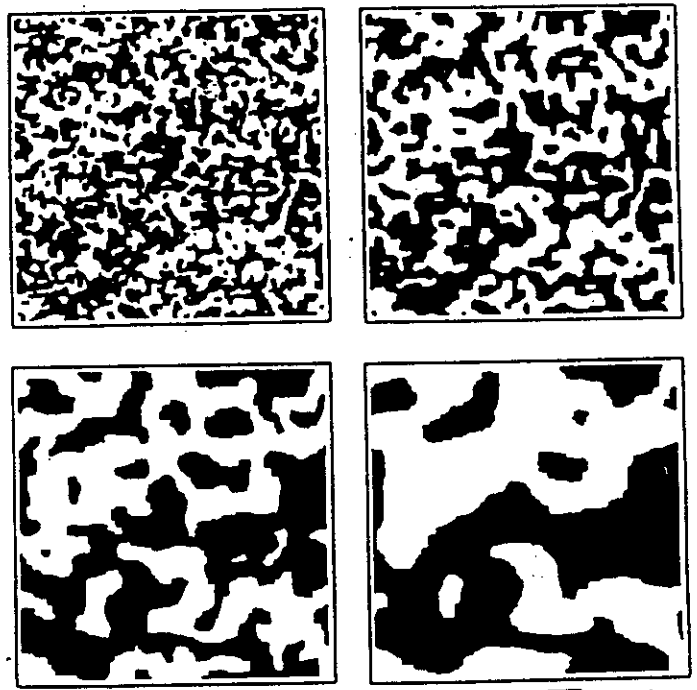
DOMAIN GROWTH

A.J. BRAY. (REVIEW). J. KISSNER

2D ISING MODEL (MC) T=0.

$t_T = 5$ $t_T = 1!$

$t_T = 60$ $t_T = 2$

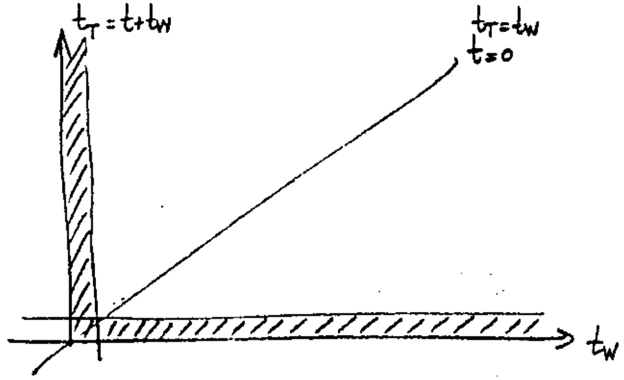


SCALING $\left.\begin{matrix} \\ C(r,t,t') \end{matrix}\right\} \rightarrow$ $\boxed{C(r/\ell(t)\,,\ r/\ell(t'))}$

$$C(r=0;t,t') = f\left(\frac{\ell(t')}{\ell(t)}\right)$$

WHAT SORT OF FUNCTIONS DO WE HAVE?

$\boxed{\text{TWO-TIME DEPENDENCES}}$



SCHEMATICALLY: 'FINITE TIMES,' ie $t_w \in [0, \delta]$
$t_T \in [0, \delta]$

WITH δ 'FINITE'

WE'LL ASSUME THAT 'GLASSY' SYSTEMS WE DEAL WITH FORGET WHAT HAPPENED TO THEM IN FINITE TIME INTERVALS.

- PROP I OF 'WEAK-LONG-TERM MEMORY.
  (SEE LATER!)

IF WE WANT TO STUDY THE ASYMPTOTIC BEHAVIOUR WE NEED TO TAKE THE LIMIT OF LARGE TIMES BUT WE HAVE TO EXPLICITATE HOW WE TAKE THIS LIMIT.

TIMES INVOLVED

MICROSCOPIC TIME τ_0 (1MC, 10^{-12} sec, τ_0 in LANGEVIN)

WAITING TIME t_w

'MEASURING' TIME $t_T = t + t_w$

EQUILIBRATION TIME t_{eq}

- IN GLASSES t_{eq} IS EXTREMELY LARGE.
- IN EXPERIMENTS IN SG PRESUMABLY NEVER REACHED.
- " MF SG t_{eq} IS BELIEVED TO BE $O(e^{N})$

THUS

$$\boxed{\tau_0 \ll (t_w, t_T) \ll t_{eq}}$$

IN THE CALCULATIONS, THE ASYMPTOTIC LIMIT MEANS

$$\boxed{\lim_{t_w, t_T \to \infty} \lim_{N \to \infty}}$$

WE STILL HAVE TO SAY HOW DO (t_w, t_T) GO TO ∞.

CORRELATIONS: DATA

$$C(t+t_w,t_w) = \frac{1}{N}\sum_{i=1}^{N} \overline{\langle S_i(t+t_w)\, S_i(t_w)\rangle}$$

VARIATION WITH RESPECT TO t FOR t_w FIXED.

$$\boxed{\bullet \quad \frac{\partial C(t+t_w,t_w)}{\partial t} \leq 0} \qquad 1.-$$

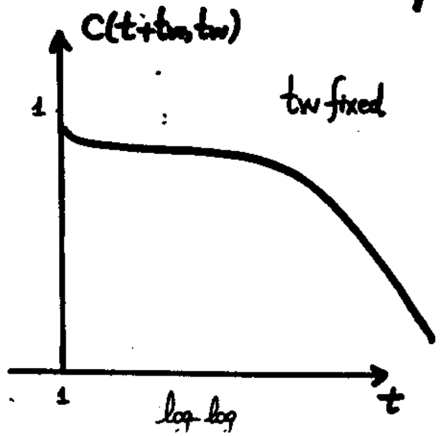
MONOTONOUSLY DECREASING

VARIATION WITH RESPECT TO t_w FOR $t+t_w$ FIXED.

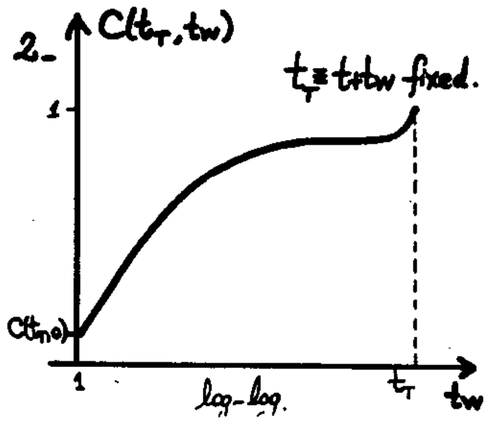
$$\boxed{\bullet \quad \left.\frac{\partial C(t+t_w,t_w)}{\partial t_w}\right|_{t+t_w \text{ fixed}} \geq 0} \qquad 2.-$$

MONOTONOUSLY INCREASING

1.-



2.-



CORRELATIONS DATA

$$C(t+t_w,t_w) = \langle \delta m(t+t_w)\ \delta m(t_w)\rangle$$
$$= \frac{1}{N}\sum_{i=1}^{N} \overline{\langle S_i(t+t_w)\, S_i(t_w)\rangle}$$

OBSERVATION: C DEPENDS ON (t,t_w) (BOTH!)

$$\boxed{\bullet \quad C(t+t_w,t_w) \neq C(t)} \qquad \forall (t,t_w)$$

NO INVARIANCE UNDER TIME-TRANSLATION

BUT

WHEN $t_w \gg t$ $\quad C(t+t_w,t_w) \sim C(t)$

$$\boxed{\begin{array}{l} \bullet \quad \lim_{t_w\to\infty} C(t+t_w,t_w) = C_F(t) \\ \bullet \quad \lim_{t\to\infty} C_F(t) = q_{EA} \end{array}}$$

(AGAIN: A 'TIMES-SECTOR' WHERE TTI HOLDS!)

WHEN $t \gg t_w$ $\quad C(t+t_w,t_w) \sim 0$

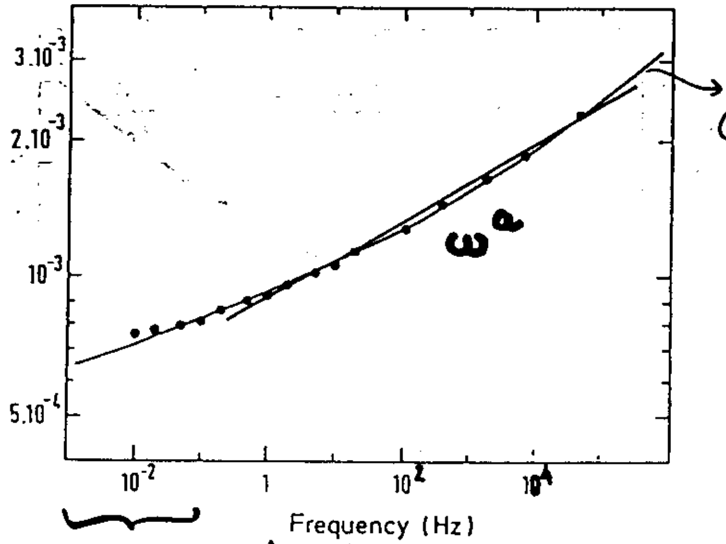
$$\boxed{\bullet \quad \lim_{t\to\infty} C(t+t_w,t_w) = 0}$$

OUT OF PHASE SUSCEPTIBILITY

$T < T_g$: $\chi''(\omega, t_w)$

ωt_w LARGE $\rightarrow$ $\boxed{\chi''_F(\omega)}$

'STATIONARY' PART : NO t_w-DEPENDENCE



$\underbrace{\qquad}$ $\underbrace{\qquad\qquad}$

HIGH FREQUENCIES

SMALLER FREQUENCIES

$\Downarrow$

OUT OF EQUILIBRIUM $\Rightarrow$ t_w-DEPENDENCE

$\left.\right\}$ $\chi''_F(\omega)$ IS NOT JUST A fn. OF ω !

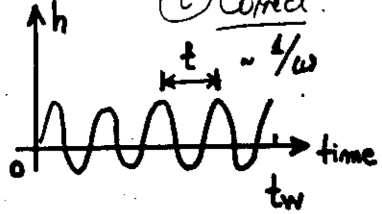
POWER LAW or LOGARITHMIC ? $\boxed{\chi''_F(\omega) \sim \omega^a}$

EXPONENT: $a(T)$: 0.01 – 0.1 ($T\uparrow$) (or $\log^\psi \omega$?)

OUT OF PHASE SUSCEPTIBILITY

$$\frac{\chi(\omega, t_w)}{h} = \int_0^{t_w} ds\, e^{i\omega s}\, R(t_w, s)$$

(t) Correct!



$\chi'(\omega, t_w) = \mathrm{Re}\,\chi(\omega, t_w)$ In phase Susceptibility

$\chi''(\omega, t_w) = \mathrm{Im}\,\chi(\omega, t_w)$ Out of phase Susceptibility.

Observation: $\chi''(\omega, t_w)$ Depends on both (ω, t_w).

$\boxed{\bullet\ \chi''(\omega, t_w) \neq \chi''(\omega)}$ $\forall (t, t_w)$

NO INVARIANCE UNDER TIME-TRANSLATIONS

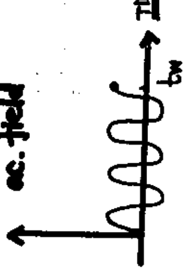
But if $\omega t_w \gg 1$ (large)

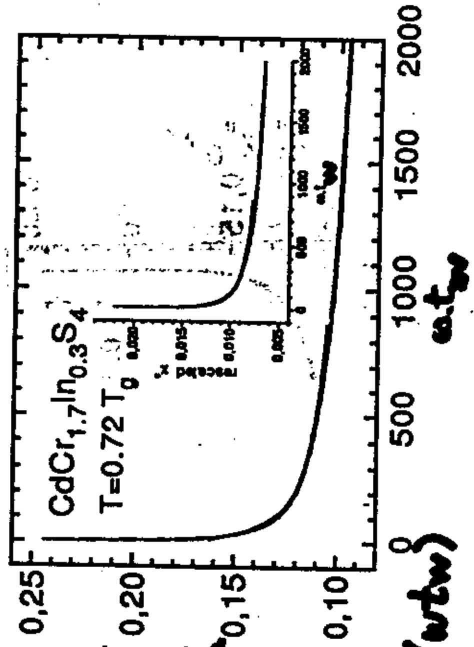
- $\lim_{\omega t_w \to \infty} \chi''(\omega, t_w) = \chi''_F(\omega) \sim \omega^{-a}$
- $\chi''(\omega, t_w) - \chi''_F(\omega) \sim (\omega t_w)^b$

$\hookrightarrow$ GENERAL $\forall (\omega, t_w)$

OUT OF PHASE SUSCEPTIBILITY

LOW AND HIGH FREQUENCIES DYNAMICAL SCALING





INSET: FULL SCALING

MAIN PLOT APPROX. FOR $\omega t_w > 1$

SCALING

$$\chi''(\omega, t_w) \sim \chi''_F(\omega) + f(\omega t_w)$$

$\omega = 0.01, 0.03, 0.1, 1.$ Hz.

$$\chi''(\omega, t_w) \sim ct. + \omega^a + ct. \cdot (\omega t_w)^{-b}$$

COND-MAT/96

E. VINCENT, J. HAMMANN, M. OCIO, JP BOUCHAUD, L.F. CUGLIANDOLO

SACLAY DATA

THERMOREMANENT MAGNETIZATION DATA

$$m^{TRM}(t+t_w, t_w) = h \cdot \int_0^{t_w} ds \ R(t+t_w, s)$$

OBSERVATION : m^{TRM} DEPENDS ON (t, t_w) BOTH!

$$\boxed{\bullet \quad R(t+t_w, t_w) \neq R(t)}$$

NO INVARIANCE UNDER TIME-TRANSLATION.

BUT:

WHEN $t_w \gg t$, $m^{TRM}(t+t_w, t_w) \sim m^{TRM}(t)$

$$\boxed{\bullet \quad \lim_{t_w \to \infty} R(t+t_w, t_w) = R_F(t)}$$

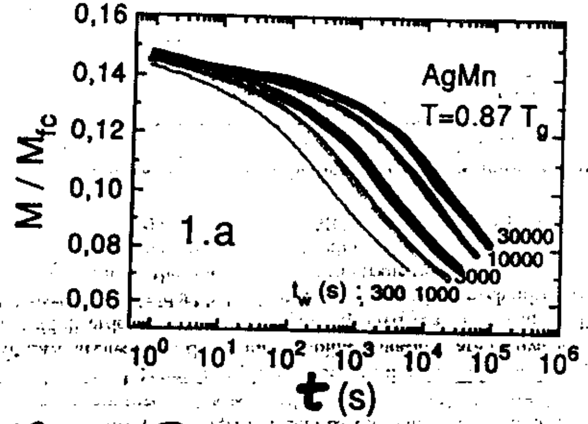
(DEFINES A 'TIMES-SECTOR' WHERE TTI HOLDS!)

WHEN $t \ggg t_w$ $m^{TRM}(t+t_w, t_w) \sim 0$

$$\boxed{\bullet \quad \lim_{t \to \infty} R(t+t_w, t_w) = 0}$$

NO REMANENT MAGNETIZATION

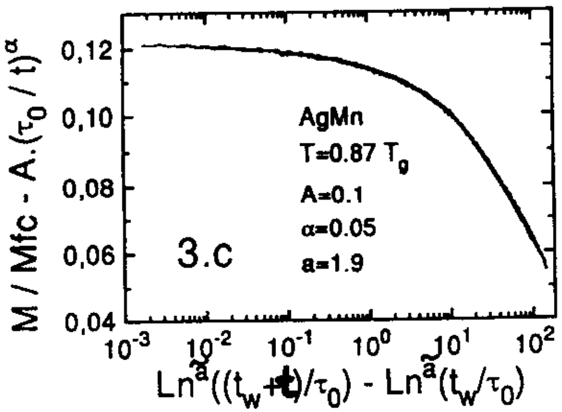
$$m^{TRM}(t+t_w, t_w) \sim m_F(t) + m_A\left(\frac{h(t_w/\tau_0)}{h(t+t_w/\tau_0)}\right)$$



ENHANCED POWER LAW:

$$h(t_w/\tau_0) = \exp \ln^a (t_w/\tau_0)$$

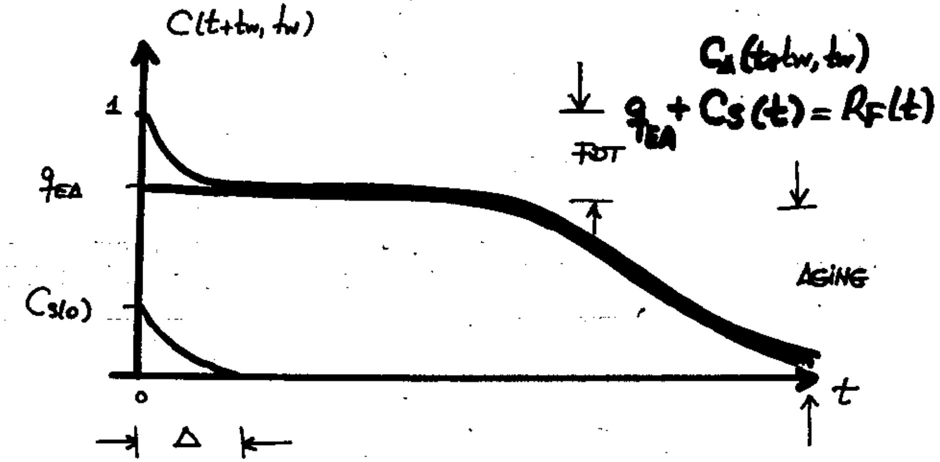
SCALING OF THE 5 CURVES



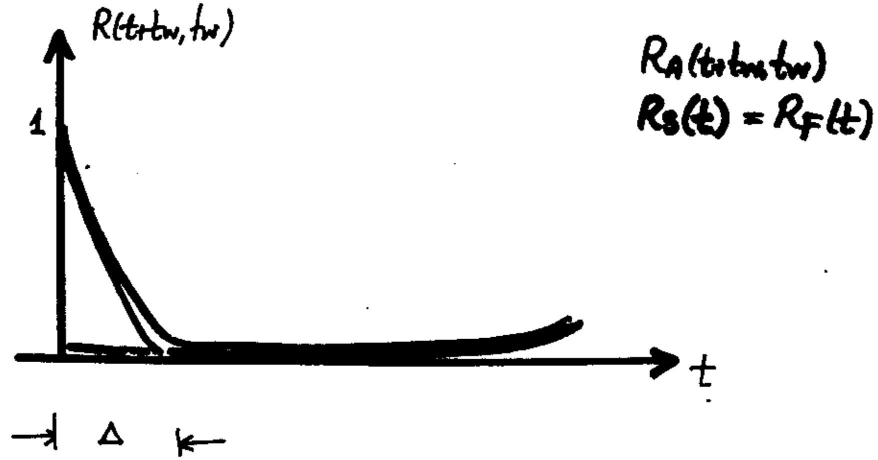
τ_0: MICROSCOPIC TIME SCALE.

SLOW DYNAMICS

CORRELATION FUNCTION $C(t+t_w, t_w)$



RESPONSE FUNCTION $R(t+t_w, t_w)$



SUMMARY

EXPERIMENTS

- SPIN-GLASSES
  - MAGN RELAX $h \neq 0$
  - OUT OF PH. SUSC. $h \neq 0$
  - MAGN NOISE $h = 0$
- COLLOIDS ⟶ DENS-DENS. CORR. $h =$
- POLYMER GLASSES → MECH. PROP. $f \neq 0$
- CHARGED DENSITY WAVES ⟶ CALORIMETRY

NUMERICS

- SPIN-GLASSES
  - MAGN. RELAX
  - AUTO-CORR
  - FINITE D
  - MF - MODELS
  - MC - DYNAMICS
- 'LENNARD-JONES' GLASSES
  - 'MAGN' RELAX
  - AUTO-CORR
  - MC-MD
- SINGLE POLYMER
  - RESPONSE
  - AUTO-CORR
  - MC
- POLYMER GLASSES AUTO-CORR
- LATTICE GASES, 'SELF-ORGANIZED CRITICALITY', ETC, ETC.

SCIENCE 267, 1924 (95).

FRONTIERS IN MATERIALS SCIENCE: ARTICLES

Physical Aging in Polymer Glasses

Ian M. Hodge

Physical aging refers to structural relaxation of the glassy state toward the metastable equilibrium amorphous state, and it is accompanied by changes in almost all physical properties. These changes, which must be taken into account in the design, manufacture, and use of glassy polymer materials and devices, present a daunting challenge to theorists.

— Let's define the $\boxed{\text{Microscopic Dynamics of the System}}$

τ_0: microsc. time scale

$\tau_0 \ll \tau_{exp}$: 'Experimental Time Scale' $\Longrightarrow$ $\boxed{\{J_{ij}\} \text{ fixed} \to \text{Quenched}}$

$\longrightarrow S_i(t)$ dynamical variables $i = 1, \dots, N$

What is their time evolution?

We want to choose an updating rule such that the system could eventually equilibrate, ie such that, for N finite and $t \to \infty$

$$\boxed{P(\vec{s}, t \to \infty) = \exp\left[-\beta H_J[\vec{s}]\right]} \quad (1)$$

Boltzmann's distribution $(\beta = 1/k_B T)$.

Since the system is in a thermal bath, we'll mimic the interaction between the magn moments and lattice vibrations (phonons) conducting electrons (in metallic SG) by a thermal noise $\xi_i(t)$ and 'transform' then the spins into stochastic variables $\boxed{S_i[\vec{\xi}]}$. ( )

Thus, we are led to consider a stochastic dynamics that leads to equilibrium at ∞-times. What ∞-times mean it's a difficult question that we'll answer later.

Eq. (1) means that, if the system is in a situation such that its conf. is weighted by the Boltzmann's weight then

$$\boxed{[f(s)]_\tau = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt\, f[\vec{s}(t)] = \langle f[\vec{s}] \rangle_{GB}} \quad (2)$$

The microscopic dynamics is taken to be Markovian meaning that each spin knows the conf of all other spins only one time step before. The spins do not have memory.

* Comment $\to$ Universality $\Rightarrow$ don't expect the macroscopic behaviour to depend strongly upon the dynamics

$\boxed{\text{Glauber Dynamics:}}$

The natural choice of dynamics for Ising spins is the one proposed by Glauber for the Ising Model. But, it is difficult to deal with it analytically in these problems.

$\boxed{\text{Monte Carlo:}}$ Just a computational way of implementing Glauber.

Soft spins $\longrightarrow$ $\boxed{\text{Langevin Dynamics}}$

Ising spins can be made continuous variables if we add a 'soft-spin' contraint

$$\boxed{H_{soft} = a \sum_i \left[(s_i^2 - 1)^2\right]} \qquad -\infty \leq s_i \leq \infty \quad (3)$$

to the energy, with a a parameter that one takes to ∞ at the end of the calculations.

Many of the common models deal directly with continuous spins, as is the case of the spherical models.

A well-known stochastic dynamics of continuous variables is provided by the Langevin Equation

$$\boxed{\Gamma^{-1}\, \partial_t s_i(t) = -\mu \frac{\delta H_J[\vec{s}]}{\delta s_i(t)} + \xi_i(t)} \quad (4)$$

with $\boxed{\langle \xi_i(t)\, \xi_j(t') \rangle = 2A\, \delta_{ij}\, \delta(t - t')}$ (5). white noise.

(In this expression one has neglected any 'inertia', any other parameter that could appear in front of the lhs can be reabsorbed in μ and A.)

[ONE CAN OBTAIN SUCH AN EQ. BY CONSIDERING THE INTERACTION OF SYSTEM VARIABLES S_i WITH BATH VARIABLES X_i REPRESENTED, e.g. BY HARMONIC OSCILLATORS, & BY INTEGRATING OUT THE BATH VARIABLES IN SUCH A WAY TO REPRESENT THE EFFECT OF THE BATH BY A RANDOM FORCE $\xi_i(t)$. SEE eg. Ford, Kac, Mazur. Feynman - Vernon].

- Solution to the L-eq

ONE HAS TO SOLVE Eq. (4) IN A PROBABILISTIC SENSE SINCE WE ONLY KNOW $P[\xi_i(t)]$. THE SOLUTION TO Eq. (4) READS

$$\vec{s}^{\,sol}_{\xi}(t) = \mathcal{F}[\vec{\xi}(\tau)] \qquad \text{with } t_0 \le \tau \le t \qquad (6)$$

AND HENCE ANY FUNCTIONAL OF $s_i(t)$, e.g. $\mathcal{F}[\vec{s}](t)$ MUST BE AVERAGED OVER VARIOUS THERMAL HISTORIES TO OBTAIN A 'DETERMINISTIC' RESULT:

$$\langle \mathcal{F}[\vec{s}](t) \rangle_{\xi} = \int \prod_t d\vec{\xi}(t)\, e^{-\int_0^t d\tau \frac{\xi^2(\tau)}{2A}}\; \mathcal{F}[\vec{s}^{\,sol}_{\xi}(t)] \qquad (7)$$

IN THE SG PROBLEM WE STILL HAVE A DEPENDENCE UPON THE $\boxed{\text{QUENCHED DISORDER } J_{ij}}$ TO TAKE INTO ACCOUNT. WE SHOULD ALSO TAKE THE AVERAGE OVER IT

$$\overline{\langle \mathcal{F}[\vec{s}](t) \rangle_{\xi}} = \underbrace{\int \prod_t d\vec{\xi}(t)\, e^{-\int_0^t d\tau \frac{\xi^2(\tau)}{2A}}}_{dP[\xi]}\; \mathcal{F}[\vec{s}^{\,sol}_{\xi}(t)] \qquad (8)$$

THIS AVERAGE MEANS

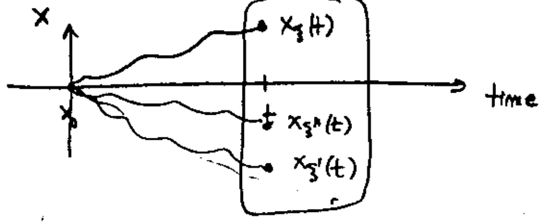
WE SOLVE THE L-EQ FOR A $\{\vec{\xi}(t)\}$ & GET $\vec{s}^{\,sol}_{\xi}(\tau)$

" " " " " ANOTHER $\{\vec{\xi}'(t)\}$ " " $\vec{s}_{\xi'}(\tau)$

ETC

EXAMPLE. JUST ONE VARIABLE

$$\dot{x} = -\mu \frac{\partial V}{\partial x} + \xi \quad \Rightarrow \quad x^{sol}_{\xi}(t) = \mathcal{F}[\xi](t)$$



AVERAGE OVER THESE VALUES.

CERTAINLY, ONE CAN FOLLOW THIS PROGRAM FOR THE SG-CASE & TRY TO FIND, e.g. DYN. EQS FOR $C_{tt'} = \langle x_t x_{t'} \rangle$ ETC. (SEE LECTURE IV).

OTHER WAY.

OBTAIN $\left\{ \begin{array}{l} \text{DYNAMICAL EQS. FOR CORR. \& RESPONSE} \\ \text{FUNCTIONS} \end{array} \right.$

if $N \to \infty$: CLOSE THE HIERARCHY & ONLY THE CORR & RESPONSE*: $\boxed{\{ C(t,t'),\ R(t,t') \} \text{ ENTER}}$

$\Rightarrow$ STUDY THE GENERAL PROPS OF $\{C, R\}$ TO USE THEM LATER

* TO BE DEF. LATER

NOTATION
HERE & IN THE REST OF III $\left\{ \begin{array}{l} t' = t_w \\ t = t_T = (t_w + \ldots \end{array} \right.$

$0 \longrightarrow t' \longrightarrow t \longrightarrow$ TIME

- FOKKER-PLANCK EQ

THE PROB. DIST OF $\vec{S}(t)$ INDUCES A TIME-DEP PROB. DIST OF $\vec{s}(t)$ THAT WE'LL CALL $\mathbb{P}(\vec{s},t)$. IF WE DEFINE *

$$\boxed{\mathbb{P}(\vec{s},t) = \int d\mathbb{P}[\vec{s}]\ \delta\left(\vec{s} - \vec{s}^{\,sol}_{\,\mathfrak{z}}(t)\right)} \qquad (9)$$

$$= \langle \delta\left(\vec{s} - \vec{s}^{\,sol}_{\,\mathfrak{z}}(t)\right)\rangle_{\mathfrak{z}}$$

THEN,

$$\boxed{\langle \mathcal{F}[\vec{s}](t)\rangle_{\mathfrak{z}} = \int d\vec{s}\ \ \mathbb{P}(\vec{s},t)\ \mathcal{F}[\vec{s}]} \qquad (10)$$

ONE CAN SHOW, THAT $\mathbb{P}(\vec{s},t)$ SATISFIES THE FOKKER-PLANCK EQ:

$$\boxed{\frac{\partial \mathbb{P}(\vec{s},t)}{\partial t} = \vec{\nabla}\left[\mathbb{P}(\vec{s},t)\ \mu \frac{\partial H_J}{\partial \vec{s}}\right] + A\ \nabla^2 \mathbb{P}(\vec{s},t)} \qquad (11)$$

WITH $\vec{\nabla} = \dfrac{\partial}{\partial \vec{s}}$

$\nabla^2 = \dfrac{\partial^2}{\partial s^2}$

*NB. $\int d\vec{s}\ \mathbb{P}(\vec{s},t) = \int d\vec{s}\int d\mathbb{P}[\vec{s}]\ \delta\left(\vec{s} - \vec{s}^{\,sol}_{\,\mathfrak{z}}(t)\right) = \int d\mathbb{P}[\vec{s}] = 1 \quad \forall t$

AND $\mathbb{P}(\vec{s},t) \geq 0 \quad \forall (\vec{s},t)$

THE TIME-DEPENDENT PROB. DENSITY $\mathbb{P}(\vec{s},t)$ SHOULD APPROACH $\mathbb{P}_{GB}(\vec{s})$ FOR LARGE TIMES, IF THE SYSTEM ARRIVES AT EQUILIBRIUM.

LET US CHECK WHEN

$\lim_{t\to\infty} \mathbb{P}(\vec{s},t) \overset{?}{=} \mathbb{P}_{GB}(\vec{s})$ IS A LARGE-TIME SOL TO FPEQ.

DEM

$$0 \overset{?}{=} \vec{\nabla}\left[e^{-\beta H_J[\vec{s}]}\ \mu \frac{\partial H_J}{\partial \vec{s}}\right] + A\nabla^2\left(e^{-\beta H_J[\vec{s}]}\right)$$

$$-\beta \frac{\partial H_J[\vec{s}]}{\partial \vec{s}}\ \mu \frac{\partial H_J[\vec{s}]}{\partial \vec{s}} + \mu \frac{\partial^2 H_J[\vec{s}]}{\partial s^2} +$$

$$+ A\left[-\beta \frac{\partial^2 H_J[\vec{s}]}{\partial s^2} + \beta^2 \frac{\partial H_J[\vec{s}]}{\partial \vec{s}}\frac{\partial H_J[\vec{s}]}{\partial \vec{s}}\right] \overset{?}{=} 0$$

$$\beta(A\beta - \mu)\left(\frac{\partial H_J[\vec{s}]}{\partial \vec{s}}\right)^2 + (\mu - A\beta)\left(\frac{\partial^2 H_J[\vec{s}]}{\partial s^2}\right) \overset{?}{=} 0$$

$$(\mu - A\beta)\left\{\frac{\partial^2 H_J[\vec{s}]}{\partial s^2} - \beta\left(\frac{\partial H_J[\vec{s}]}{\partial \vec{s}}\right)^2\right\} \overset{?}{=} 0$$

VALID $\forall \vec{s} \Rightarrow \quad \mu = A\beta \ \longrightarrow\ A = \frac{\mu}{\beta} = \mu k_B T$

$$\boxed{A = \mu k_B T} \qquad (12)$$

EINSTEIN'S RELATION OR 2nd FDT (KUBO)

Thus,

$$\text{L-EQ}\quad \begin{cases} \dot{s}_i(t) = -\mu \dfrac{\delta H}{\delta s_i(t)} + \xi_i(t) \\ \langle \xi_i(t)\, \xi_i(t') \rangle = 2\mu k_B T\, \delta(t-t') \end{cases}$$

F-P EQ

$$\frac{\partial P(\vec{s},t)}{\partial t} = \vec{\nabla}\left[P(\vec{s},t)\, \mu \frac{\partial H_J}{\partial \vec{s}} \right] + \mu k_B T\, \nabla^2 P(\vec{s},t)$$

USING THE RELATION BETWEEN FP & THE SCHRÖDINGER EQ THAT WE'LL EXPLAIN BELOW ONE CAN SHOW THAT $P_{GB}(\vec{s})$ IS THE UNIQUE SOL TO $\dfrac{\partial P(\vec{s},t)}{\partial t} = 0$.

THE FD-RELATION $(A = \mu k_B T)$ EXPRESSES THE FACT THAT THE BATH IS AT EQUILIBRIUM AT $t=0$ AND STAYS AT EQUILIBRIUM AT ALL SUBSEQUENT TIMES.

ANALOGY WITH THE SCHRÖDINGER EQ

$$\frac{\partial P(\vec{s},t)}{\partial t} = \vec{\nabla}\left[P(\vec{s},t)\, \mu \left(\frac{\partial H_J}{\partial \vec{s}}\right) \right] + \mu k_B T\, \nabla^2 P(\vec{s},t).$$

LET'S IDENTIFY $\boxed{-i\dfrac{\partial}{\partial \vec{s}} = \hat{\vec{p}}} \Leftrightarrow \boxed{\dfrac{\partial}{\partial \vec{s}} = i\hat{\vec{p}}}$ $\quad [\hat{\vec{p}}, \hat{\vec{s}}] = -i$

$\boxed{P(\vec{s},t) = |\phi(t)\rangle}$

IN THE SCHRÖDINGER'S REPRESENTATION WHERE STATES DEPEND UPON TIME BUT OPERATORS DON'T.

LET'S CALL $\boxed{\dfrac{\partial H_J}{\partial \vec{s}} = -\vec{F}_J[\vec{s}]}$ $\quad \left(\equiv \dfrac{-v[\vec{s}]}{\mu} \text{ DETERMINISTIC VELOCITY} \right)$

$$\frac{\partial P(\vec{s},t)}{\partial t} = -\vec{\nabla}\cdot\left[P(\vec{s},t)\, \mu \vec{F}_J[\vec{s}] \right] + \mu k_B T\, \nabla^2 P(\vec{s},t)$$

$$\frac{\partial}{\partial t}|\phi(t)\rangle = -i\hat{\vec{p}}\left[\mu \vec{F}_J[\hat{\vec{s}}]\, |\phi(t)\rangle \right] + \mu k_B T (i\hat{\vec{p}})^2 |\phi(t)\rangle$$

$$= -i\hat{\vec{p}}\left[\mu \vec{F}_J[\hat{\vec{s}}] + \mu k_B T(-i\hat{\vec{p}}) \right] |\phi(t)\rangle$$

$$\boxed{\frac{\partial}{\partial t}|\phi(t)\rangle = -H_{FP}[\hat{\vec{p}}, \hat{\vec{s}}]\, |\phi(t)\rangle} \quad (13)$$

$$\boxed{H_{FP}[\hat{\vec{p}}, \hat{\vec{s}}] = -\hat{\vec{p}}\left[-i\mu \vec{F}_J[\hat{\vec{s}}] - \mu k_B T \hat{\vec{p}} \right]} \quad (14)$$

(IF THERE'S A FIELD COUPLED TO $\hat{\vec{s}}$). $\quad \boxed{H^h_{FP}[\hat{\vec{p}}, \hat{\vec{s}}] = -\hat{\vec{p}}\left[-i\mu \vec{F}_J[\hat{\vec{s}}] - i\mu h - \mu k_B T \hat{\vec{p}} \right]}$

$$\left[\hat{\vec{p}}, B[\vec{s}]\right]\phi = -i\frac{\partial}{\partial\vec{s}}\left(B[\vec{s}]\,\phi\right) - B[\vec{s}]\left(-i\frac{\partial}{\partial\vec{s}}\phi\right)$$

$$= -i\left(\frac{\partial B}{\partial\vec{s}}\right)\phi - \cancel{iB[\vec{s}]\frac{\partial\phi}{\partial\vec{s}}} + \cancel{iB[\vec{s}]\frac{\partial\phi}{\partial\vec{s}}}$$

$$= -i\left(\frac{\partial B}{\partial\vec{s}}\right)\phi$$

$$\boxed{\left[\hat{\vec{p}}, B[\vec{s}]\right] = -i\,\frac{\partial B}{\partial\vec{s}}}$$

$$\left[\hat{\vec{p}}, B(\vec{s})\right] = \frac{\partial B}{\partial\vec{s}} \Rightarrow \left[B(\vec{s}), i\hat{\vec{p}}\right] = -\frac{\partial B}{\partial\vec{s}}$$

Mean Values in this Notation

$$\langle \mathcal{F}[\vec{s}](t)\rangle = \int_t \Pi d\vec{s}(t)\; \underbrace{1}\; \underbrace{\mathcal{F}[\vec{s}]}\; \underbrace{\mathcal{P}(\vec{s},t)}$$

$$= \langle - | \quad \mathcal{F}[\hat{\vec{s}}] \quad |\phi(t)\rangle$$

$$\boxed{\langle \mathcal{F}[\vec{s}](t)\rangle = \langle - |\, \mathcal{F}[\hat{\vec{s}}]\, |\phi(t)\rangle} \qquad (15)$$

Solution to eq FP-Schröd:

$$\boxed{|\phi(t)\rangle = e^{-H_{FP}\cdot t}\,|\mathrm{Init}\rangle} \qquad (16)$$

(if H_{FP} is indep of t, ie if $H_J[\vec{s}]$ is indep of t; otherwise, one has to include a time order.)

$$\langle \mathcal{F}[\vec{s}](t)\rangle = \langle - |\, \mathcal{F}[\hat{\vec{s}}]\, e^{-H_{FP}t}\,|\mathrm{Init}\rangle$$

Taking $\mathcal{F}[\vec{s}] = 1 \Rightarrow$ Eq. (15) implies

$$\boxed{\langle 1\rangle = 1 = \langle - |\phi(t)\rangle} \qquad \text{(Normalization)} \qquad (17)$$

$$\equiv \int_t \Pi d\vec{s}(t)\; \mathcal{P}(\vec{s},t) = 1$$

- Correlation Functions

$\langle A(t)\, B(t')\rangle$ $\qquad t > t' \geq t_0$

$$= \langle - |\, A\, e^{-H_{FP}(t-t')} B\, e^{-H_{FP} t'}\, |Init\rangle$$

In particular $\quad A \Rightarrow s_i(t)$
$\qquad\qquad\quad\;\; B \Rightarrow s_i(t')$

$$\langle s_i(t)\, s_i(t')\rangle = \langle - |\, s_i\, e^{-H_{FP}(t-t')}\, s_i\, e^{-H_{FP} t'}\, |Init\rangle$$

- If $|Init\rangle = |Eq\rangle \Rightarrow H_{FP}|Init\rangle = H_{FP}|Eq\rangle = 0.$

  $\Rightarrow e^{-H_{FP}t'}|Eq\rangle = |Eq\rangle$

$$\boxed{\langle A(t)\, B(t')\rangle = \langle - |\, A\, e^{-H_{FP}(t-t')} B\, |Eq\rangle = \mathcal{C}(t-t')} \quad (18)$$

And TTI holds

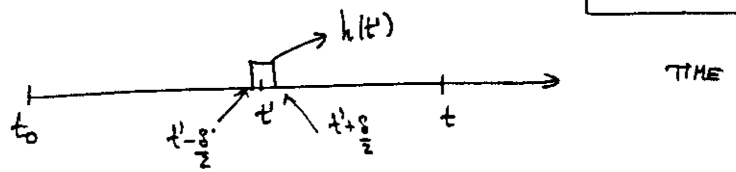
- If at $t' > t'' \geq t_0$: $|\phi(t'')\rangle = |Eq\rangle$ (ie even starting from a $|Init\rangle \neq |Eq\rangle$ if we can assume that at t'' Eq is established) then,

$$\boxed{\langle A(t)\, B(t')\rangle = \mathcal{C}(t-t')} \qquad (19)$$

Thus, time-translation invariance (TTI) is a direct consequence of $|\phi(t'')\rangle = |Eq\rangle$ $\forall$ operators A, B.

Response Function

$$\boxed{\begin{array}{l} H_J[\vec s] \to H_J[\vec s] - h\cdot \vec s \\ H_{FP}[\vec s] \to H_{FP}[\vec s] + i\mu\hat p \end{array}}$$



$$R(t,t') = \frac{\delta\langle A(t)\rangle}{\delta h(t')}\Big|_{h=0} \qquad \boxed{\text{Asuming } t>t'}$$

$$\langle A(t)\rangle = \langle -|\, A\, e^{-H_{FP}\, t}\, |Init\rangle$$

$$\langle A(t)\rangle_{h(t')} = \langle -|\, A\, e^{-H_{FP}(t-(t'+\frac{\delta}{2}))}\, e^{-H^h_{FP}[\cancel{t'}+\frac{\delta}{2})-(\cancel{t'}-\frac{\delta}{2})]}\, e^{-H_{FP}(t'-\frac{\delta}{2})}\, |Init\rangle$$

$$= \langle -|\, A\, e^{-H_{FP}[t-t'-\frac{\delta}{2}]}\, e^{-H^h_{FP}\,\delta}\, e^{-H_{FP}(t'-\frac{\delta}{2})}\, |Init\rangle$$

$$\frac{\delta\langle A(t)\rangle^h}{\delta h(t')}\Big|_{h=0} = \langle -|\, A\, e^{-H_{FP}[t-t'-\frac{\delta}{2}]} \left(-\frac{\delta H^h_{FP}}{\delta h}\right)\Big|_{h=0} \delta\, e^{-H_{FP}\,\delta}\, e^{-H_{FP}(}\, |Init\rangle$$

What's $\frac{\delta H^h_{FP}}{\delta h}\Big|_{h=0}$?

Using $+\frac{\delta H^h_{FP}}{\delta h} = i\mu\hat p$ and taking $\delta\to 0$.

$$\boxed{\frac{\delta\langle A(t)\rangle^h}{\delta h(t')}\Big|_{\substack{h=0\\ \delta=0}} = \langle -|\, A\, e^{-H_{FP}(t-t')}\,(i\mu\hat p)\, e^{-H_{FP}\,t'}\, |Init\rangle} \qquad (20)$$

$$R(t,t') = \frac{\delta \langle A(t)\rangle}{\delta h(t')}\Big|_{h=0} =$$

$$= \langle - | A\, e^{-H_{FP}(t-t')} (i\mu \hat{\vec{p}})\, e^{-H_{FP}t'} | Init\rangle$$

Causality

$\boxed{\text{If} \quad t<t'}$

$$R(t,t') = \langle - | i\mu \hat{\vec{p}}\; e^{-H_{FP}(t'-t)}\, A\, e^{-H_{FP}t} | Init\rangle$$

But $\quad \langle - | \hat{\vec{p}} = 0 \quad$ (it's a total derivative in the integral version)

$$\boxed{R(t,t') = 0 \quad \text{if} \quad t<t'}$$

Response of $A(t)$ to a kick at $B(t')$

$$H_J[\vec{s}] \longrightarrow H_J[\vec{s}] + h\cdot B[\vec{s}]$$

$$H_{FP}[\hat{\vec{p}},\hat{\vec{s}}] \longrightarrow H_{FP}^{h=0}[\hat{\vec{p}},\hat{\vec{s}}] + i\mu\, \hat{\vec{p}}\, \frac{\partial B[\vec{s}]}{\partial \vec{s}}\cdot h$$

$$\frac{\delta H_{FP}^{h}}{\delta h} : \longrightarrow + i\mu\, \hat{\vec{p}}\, \frac{\partial B[\vec{s}]}{\partial \vec{s}}$$

Thus,

$$R_{AB}(t,t') = \frac{\delta \langle A(t)\rangle}{\delta h_B(t')}\Big|_{h=0}$$

$$\boxed{R_{AB}(t,t') = \langle - | A\, e^{-H_{FP}(t-t')} \left(+i\mu \hat{\vec{p}}\, \frac{\partial B[\vec{s}]}{\partial \vec{s}}\right) e^{-H_{FP}t'} | Init\rangle} \tag{21}$$

Equilibrium:

If $|Init\rangle = |Eq\rangle \Rightarrow$

$$R_{AB}(t,t') = r(t-t')$$

Response at = times

$$R_{AB}(t,t^-) = -\langle - | A\, i\mu \hat{\vec{p}}\, \frac{\partial B[\vec{s}]}{\partial \vec{s}}\, e^{-H_{FP}t'} | Init\rangle$$

$$= -\mu \langle - | [A, i\hat{\vec{p}}]\, \frac{\partial B[\vec{s}]}{\partial \vec{s}} | \phi(t')\rangle$$

$$= +\mu \langle - | \frac{\partial A}{\partial \vec{s}} \frac{\partial B}{\partial \vec{s}} | \phi(t')\rangle$$

$$[A, i\vec{p}] = -i[\vec{p}, A] = -i(-i)\frac{\partial A}{\partial \vec{s}} = -\frac{\partial A}{\partial \vec{s}}$$

$$\text{if} \begin{cases} A = s_i \\ B = s_i \end{cases} \Rightarrow \quad \frac{\partial A}{\partial s_k} = \delta_{ik}, \quad \frac{\partial B}{\partial s_k} = \delta_{ik}$$

$$\boxed{R_{ii}(t,t^-) = +\mu \langle - | \phi(t')\rangle = +\mu}$$

FDT OF THE 1st kind

$\boxed{R \text{ vs. } C}$ $\boxed{t>t'}$

$$\frac{\partial C_{AB}(t,t')}{\partial t'} = \frac{\partial}{\partial t'} \langle - | A\, e^{-H_{FP}(t-t')}\, B\, e^{-H_{FP}t'} | I_n \rangle$$

$$= \langle - | A\, e^{-H_{FP}(t-t')} [H_{FP}, B]\, e^{-H_{FP}t'} | I_n \rangle$$

WHAT'S $[H_{FP}, B]$?

$$\boxed{[H_{FP}, B]} = -\left[\hat{\vec{p}} \left(-i\mu \vec{F}_J[\vec{s}] - \mu k_B T \hat{\vec{p}}\right), B(\vec{s})\right]$$

$$= -[\hat{\vec{p}}, B] \left(-i\mu \vec{F}_J[\vec{s}] - \mu k_B T \hat{\vec{p}}\right)$$

$$+ (-\hat{\vec{p}}) \left[-i\mu \vec{F}_J[\vec{s}] - \mu k_B T \hat{\vec{p}}, B(\vec{s})\right]$$

(SINCE $B = B[\vec{s}]$).

$$= +i \frac{\partial B}{\partial \vec{s}} \left(-i\mu \vec{F}_J[\vec{s}] - \mu k_B T \hat{\vec{p}}\right)$$

$$+ \hat{\vec{p}} (+\mu k_B T) \left(-i \frac{\partial B}{\partial \vec{s}}\right)$$

$$\boxed{= +i \frac{\partial B}{\partial \vec{s}} \left(-i\mu \vec{F}_J[\vec{s}] - \mu k_B T \hat{\vec{p}}\right) + \mu k_B T \hat{\vec{p}} \left(-i \frac{\partial B}{\partial \vec{s}}\right)} \tag{22}$$

THUS,

$$\frac{\partial C_{AB}(t,t')}{\partial t'} = \langle - | A\, e^{-H_{FP}(t-t')} \left\{ +i \frac{\partial B}{\partial \vec{s}} \left(-i\mu \vec{F}_J[\vec{s}] - \mu k_B T \hat{\vec{p}}\right) + \mu k_B T \hat{\vec{p}} \left(\frac{-i\partial B}{\partial \vec{s}}\right) \right\} e^{-H_{FP}t'} | I_{nit} \rangle \tag{23}$$

if $\boxed{|I_{nit}\rangle = |E_\varphi\rangle}$ $\to$ $e^{-H_{FP}t'} |E_\varphi\rangle = |E_\varphi\rangle$

AND $\left(-i\mu \vec{F}_J[\vec{s}] - \mu k_B T \hat{\vec{p}}\right) |E_\varphi\rangle = 0$. AND THE 1st TERM VANISHES.

$$\frac{\partial C_{AB}(t,t')}{\partial t'} = - \langle - | A\, e^{-H_{FP}(t-t')}\, \mu k_B T\, i\hat{\vec{p}} \left(\frac{\partial B}{\partial \vec{s}}\right) |E_\varphi\rangle$$

$$\left(= f(t-t') \right)$$

THE RESPONSE IS (FROM EQ. (21))

$$R_{AB}(t,t') = \langle - | A\, e^{-H_{FP}(t-t')}\, i\mu \hat{\vec{p}} \frac{\partial B}{\partial \vec{s}} |E_\varphi\rangle$$

COMPARING WE HAVE

$$\boxed{\begin{aligned} R_{AB}(t,t') &= \frac{1}{k_B T} \frac{\partial C_{AB}(t,t')}{\partial t'} \quad \text{OR,} \\ R_{AB}(t-t') &= \frac{1}{k_B T} \frac{\partial C_{AB}(t-t')}{\partial t'} \end{aligned}} \tag{24}$$

$\boxed{\text{FDT OF THE 1st KIND}}$

THUS, IF $|I_{nit}\rangle = |E_\varphi\rangle$, OR EVEN IF $|\phi(t'')\rangle = |E_\varphi\rangle$ FOR $t'' < t'$, THEN FDT FOR THE SYSTEM HOLDS.

WHAT HAPPENS OUT OF EQUILIBRIUM?
A SIMPLER WAY OF GETTING THE LAST RESULT IS THE FOLLOWING:
THAT ALLOWS US TO OBTAIN GENERALIZATIONS.

LET US COMPUTE:

$$\boxed{V_{AB}(t,t')} \equiv k_BT\, R_{AB}(t,t') - \frac{\partial C_{AB}(t,t')}{\partial t'}$$

$$= \langle - | A\, e^{-H_{FP}(t-t')} \left[\cancel{k_BT\, i\mu \vec{p}\, \frac{\partial B}{\partial \vec{s}}} + i \frac{\partial B}{\partial \vec{s}} \left(-i\mu \vec{F}_J[\vec{s}] - \mu k_BT\, \hat{\vec{p}} \right. \right.$$

$$\left. - \cancel{k_BT\, i\mu \vec{p}\, \frac{\partial B}{\partial \vec{s}}} \right] e^{-H_{FP}t'} |\mathrm{Init}\rangle$$

$$= \langle - | A\, e^{-H_{FP}(t-t')} \frac{\partial B}{\partial \vec{s}} \left(\mu \vec{F}_J[\vec{s}] - \mu k_BT\, i\hat{\vec{p}} \right) |\phi(t')\rangle \qquad (25)$$

- IF $|\phi(t')\rangle = |Eq\rangle \Rightarrow V_{AB}(t,t') = 0$ SINCE $(\quad)|Eq\rangle = 0$.
- IF $|\phi(t')\rangle \neq |Eq\rangle$?

WE'LL TRY TO FIND A BOUND TO THE FDT VIOLATION.

CDK COND-MAT/9705002

DISCUSSION ON (N vs. t) LIMITS

— IF N IS FINITE AND $t \to \infty$ $\Rightarrow$ EQUIL WILL BE EVENTUALLY ACHIEVED.

— IF N IS VERY LARGE (eg $O(10^{23})$) AND t' IS LARGE BUT SMALL WRT $t_{eq} = f(N)$

$$\boxed{t' < t_{eq}}$$

THUS, THE RELEVANT ORDER OF LIMITS, TO OBTAIN ASYMPTOTIC RESULTS IS

$$\boxed{\lim_{t' \to \infty} \lim_{N \to \infty}}$$

[TYPICALLY, FOR MF SG ONE EXPECTS $t_{eq} \sim O(e^{\alpha N})$.]

IN THIS SITUATION THE SYSTEM IS OUT OF EQ AT $t' < t < t_{eq}$ AND FDT CAN BE VIOLATED.

NB. THE THERMODYN. LIMIT IS DETERMINANT TO HAVE SUCH A SITUATION!

WE'LL STUDY WHEN WE CAN HAVE FDT-VIOLATIONS

OUR AIM IS TO BOUND $\left| V_{AB}(t,t') \right|$.

LET US FIRST CHOOSE A AND B, FOR THE SAKE OF CLARITY, TO BE

$$\begin{cases} A(t) = s_i(t) \\ B(t') = s_i(t') \end{cases} \Longrightarrow \left| V(t,t') \right| = \left| \frac{1}{N} \sum_i V_{ii}(t,t') \right|$$

IT IS GOING TO BE IMPORTANT, AT LEAST UPTO WHERE WE CAN SHOW THINGS – THAT $A(t)$ DEPENDS ON A <u>FINITE</u> # OF DYN. VARIABLES TO GET A SHARP BOUND.

LET'S NOW GO BACK TO THE FUNCTION NOTATION:

$$\left| V(t,t') \right| \le \frac{1}{N} \sum_i \left| V_{ii}(t,t') \right|$$

$$= \frac{1}{N} \sum_i \left| \int d\vec{s} \int d\vec{s}' \; s_i \, P(\vec{s}, \vec{s}'; t-t') \frac{\left[-\mu \frac{\partial H}{\partial s_i'} - \mu k_B T \frac{\partial}{\partial s_i'} \right]}{P(\vec{s}',t')} \right|$$

$$= \frac{\mu}{N} \sum_i \left| \int d\vec{s} \int d\vec{s}' \; s_i \, P^{1/2}(\vec{s},\vec{s}'; t-t') \, P^{1/2}(\vec{s}',t') \; P^{1/2}(\vec{s},\vec{s}'; t-t') \cdot \frac{\left[- \frac{\partial H}{\partial s_i'} - k_B T \frac{\partial}{\partial s_i'} \right] P(\vec{s}',t')}{P^{1/2}(\vec{s}',t')} \right|$$

$$\le \mu \frac{1}{N} \sum_i \left[\int d\vec{s} \int d\vec{s}' \; s_i^2 \, P(\vec{s},\vec{s}'; t-t') \, P(\vec{s}',t') \right]^{1/2} \left[\cancel{\int d\vec{s}} \int d\vec{s}' \; \cancel{P(\vec{s},\vec{s}'; t-t')} \frac{\left(\left[- \frac{\partial H_J}{\partial s_i'} - \mu k_B T \frac{\partial}{\partial s_i'} \right] P(\vec{s}',t') \right)^2}{P(\vec{s}',t')} \right]$$

$$= \frac{\mu}{N} \sum_i \left\langle s_i^2(t) \right\rangle^{1/2} \cdot \left[\int d\vec{s}' \frac{\left(\left[- \frac{\partial H_J}{\partial s_i'} - k_B T \frac{\partial}{\partial s_i'} \right] P(\vec{s}',t') \right)^2}{P(\vec{s}',t')} \right]$$

$$\le \left(\frac{1}{N} \sum_i \langle s_i^2(t) \rangle \right)^{1/2} \left[\frac{1}{N} \sum_i \int d\vec{s}' \frac{\left[\left(-\mu \frac{\partial H_J}{\partial s_i'} - \mu k_B T \frac{\partial}{\partial s_i'} \right) P(\vec{s}',t') \right]^2}{P(\vec{s}',t')} \right]$$

$$= \sqrt{C(t,t)} \cdot \sqrt{ - \frac{d\mathcal{H}(t')}{dt'} }$$

$$\boxed{ \left| V(t,t') \right| \le \sqrt{C(t,t)} \; \underbrace{\sqrt{ - \frac{d\mathcal{H}(t')}{dt'} }} }$$

ENTROPY PRODUCTION PER SPIN !

$$O(1) \le O(1) \cdot O(1) \quad \underline{\underline{\text{ok}}}$$

$\downarrow$

THE THERMODYN. LIMIT YIELDS A NON-TRIVIAL BOUND.

HOW DO I KNOW THAT

$$\frac{1}{N}\sum_i \int d\vec{s} \; \frac{\left[\left(-\mu\frac{\partial H}{\partial s_i'} - \mu k_B T \frac{\partial}{\partial s_i'}\right) P(\vec{s},t')\right]^2}{P(\vec{s},t')}$$

IS THE ENTROPY PRODUCTION?

LET'S DEFINE:

$$\mathcal{H}(t) = \mu\int d\vec{s} \quad P(\vec{s},t)\left[k_B T \ln P(\vec{s},t) + H_J[\vec{s}]\right]$$

$$\dot{\mathcal{H}}(t) = \mu\int d\vec{s} \left\{ \dot{P}(\vec{s},t)\left[k_B T \ln P(\vec{s},t) + H_J[\vec{s}]\right] + \mu k_B T \cancel{P(\vec{s},t)} \frac{1}{\cancel{P(\vec{s},t)}} \underbrace{\dot{P}(\vec{s},t)}_{0} \right\}$$

$$= \mu\int d\vec{s} \;\; \nabla\left\{\left[P(\vec{s},t)\,\mu\frac{\partial H_J}{\partial \vec{s}}\right] + \mu k_B T\, \vec{\nabla} P(\vec{s},t)\right\} \left(k_B T \ln P(\vec{s},t) + H_J[\vec{s}]\right)$$

$$= -\mu\int d\vec{s} \left\{ P(\vec{s},t)\, \frac{\partial H_J}{\partial \vec{s}} + k_B T\, \vec{\nabla} P(\vec{s},t)\right\} \left(\frac{k_B T}{P(\vec{s},t)} \vec{\nabla} P(\vec{s},t) + \frac{\partial H_J[\vec{s}]}{\partial \vec{s}}\right)$$

$$\dot{\mathcal{H}}(t) = -\int d\vec{s} \;\; \frac{\left(\mu k_B T\, \vec{\nabla} P(\vec{s},t) + \mu P(\vec{s},t)\frac{\partial H_J}{\partial \vec{s}}\right)^2}{P(\vec{s},t)} \leq 0$$

THUS, THE rhs IN THE INEQUALITY IS

$$|V(t,t')| \leq \sqrt{C(t,t)} \;\; \sqrt{-\mu\frac{d\mathcal{H}(t')}{dt'}}$$

(μ-factor)

$$\leq K \cdot \sqrt{-\mu\frac{d\mathcal{H}(t')}{dt'}}$$

- IF THE SYSTEM IS AT EQUILIBRIUM $\Rightarrow$

$$\frac{d\mathcal{H}(t')}{dt'} = 0$$

AND rhs $= 0$ $\Rightarrow$

$$\boxed{|V(t,t')| \leq 0} \Rightarrow \left| \underbrace{k_B T\, R(t,t')}_{\text{Positive}} - \underbrace{\frac{\partial C(t,t')}{\partial t'}}_{\text{Positive.}} \right| = 0 \Rightarrow$$

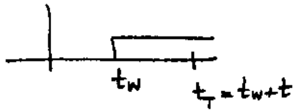
If the diff. doesn't change signs $\rightarrow$

FDT holds

• If $\frac{d\mathcal{H}(t')}{dt'}$ is a decreasing function of t'

What happens?

Rem the experiments



$$h\cdot\int_{t_w}^{t_T} d\tau\; R(t_T,\tau) \equiv m(t,t_w) \quad : \quad \underline{\text{Finite}}$$

And $C(t+t_w,t_w)$ from numerics.

$$\frac{\partial C(t_T,t_w)}{\partial t_w} \geq 0\,, \quad \frac{\partial C(t_T,t_w)}{\partial t_T} \leq 0 \quad \text{(Slower and slower)}$$

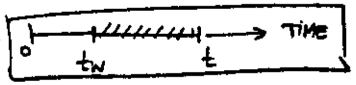
Thus,

$$\left|V(t,t')\right| = \left| k_BT\, R(t,t') - \frac{\partial C(t,t')}{\partial t'} \right| \leq K \sqrt{-\frac{d\mathcal{H}(t')}{dt'}}$$

But $\frac{\partial C(t,t')}{\partial t'} \longrightarrow 0$ and $R(t,t') \longrightarrow 0$ too!

Thus, it's not telling us much ...

Both sides are positive definite ⇒ integrate them over an interval



$[t_w, t]$

$$\boxed{\int_{t_w}^{t} dt'\, \left|V(t,t')\right|} = \int_{t_w}^{t} dt' \left| k_BT\, R(t,t') - \frac{\partial C(t,t')}{\partial t'} \right| \leq k\cdot \int_{t_w}^{t} dt' \sqrt{-\frac{d\mathcal{H}(t')}{dt'}}$$

$$\left| \int_{t_w}^{t} dt' \left(k_BT\, R(t,t') - \frac{\partial C(t,t')}{\partial t'} \right) \right| \leq \int_{t_w}^{t} dt' \left| \qquad \right| \leq$$

$$\leq k\cdot \int_{t_w}^{t} dt' \sqrt{-\frac{d\mathcal{H}(t')}{dt'}}$$

$$\boxed{|I| \equiv \left| k_BT\; m(t,t_w) - C(t,t) + C(t,t_w) \right| \leq k \int_{t_w}^{t} dt' \sqrt{-\frac{d\mathcal{H}(t')}{dt'}}}$$

STRONGER ('INTEGRAL') BOUND !

Let's check some examples:

⊕ If $\mathcal{H}(t') = \mathcal{H}_\infty + c\, t'^{-1}$ (or Slower)

$$\dot{\mathcal{H}}(t') = -c\, t'^{-2}$$

$$K \int_{t_w}^{t} dt' \sqrt{c\, t'^{-2}} = K\sqrt{c}\cdot \ln(t/t_w) \longrightarrow$$

rhs is finite and

$$\lim_{\substack{t\to\infty \\ t_w \text{ fixed}}} \text{rhs}(t,t_w) = \infty$$

$\boxed{\text{FDT viol. allowed}}$

lhs can be finite and <u>there can be</u> FDT Violations

⊕ If $\mathcal{H}(t') = \mathcal{H}_\infty + c\, t'^{-\alpha}$ $\quad \underline{\alpha > 1}$

$$\text{rhs} =: \frac{k\sqrt{\mu c}}{1-\alpha/2} \left(t^{1-\alpha/2} - t_w^{1-\alpha/2} \right) \longrightarrow 0.$$

$\boxed{\text{No FDT violation allowed}}$

→ FDT violations are allowed ∀ slower than t^{-1} decay of $\mathcal{H}(t)$

ENTROPY - ARGUMENTS. WE'VE SHOWN BEFORE (IN COMPLETE GENERALITY)

$$\left| k_B T R(t,t') - \frac{\partial}{\partial t'} C(t,t') \right| \leq K \sqrt{-\dot{H}(t')} \Longrightarrow$$

$$\left| k_B T \int_{t_W}^{t} ds\, R(t,s) - \underbrace{C(t,t)}_{1} + C(t,t_W) \right| \leq K \int_{t_W}^{t} ds \sqrt{-\dot{H}(s)}$$

$$\left| k_B T\, m(t,t_W) - 1 + C(t,t_W) \right| \leq K \int_{t_W}^{t} ds \sqrt{-\dot{H}(s)}$$

Example

if $H(s) = H_\infty + c\, s^{-\alpha}$

$\dot{H}(s) = -c\alpha\, s^{-(\alpha+1)}$

$$\left| k_B T\, m(t,t_W) - (1 - C(t,t_W)) \right| \leq K\sqrt{c\alpha} \int_{t_W}^{t} ds\, s^{-\left(\frac{\alpha+1}{2}\right)}$$

$$\leq \frac{K\sqrt{c\alpha}}{\frac{1-\alpha}{2}} \left(t^{\frac{1-\alpha}{2}} - t_W^{\frac{1-\alpha}{2}} \right)$$

if $\boxed{\alpha < 1}$ FDT VIOLATION IS ALLOWED!

NB: THIS IS THE NOTATION FOR TIMES HERE

$0 \longrightarrow t_W = t' \longrightarrow t = t_T$, $\quad \tau$: from t_W to t

BUT if $\boxed{t = t_W + \tau}$

$$\text{rhs} = K' \left((\tau + t_W)^{\frac{1-\alpha}{2}} - t_W^{\frac{1-\alpha}{2}} \right)$$

$$= k'\, t_W^{\frac{1-\alpha}{2}} \left[\left(1 + \frac{\tau}{t_W}\right)^{\frac{1-\alpha}{2}} - 1 \right]$$

$$\sim k'\, t_W^{\frac{1-\alpha}{2}} \left[\cancel{1} + \left(\frac{1-\alpha}{2}\right)\left(\frac{\tau}{t_W}\right) - \cancel{1} \right] \qquad \text{if } \boxed{\tau \ll t_W}$$

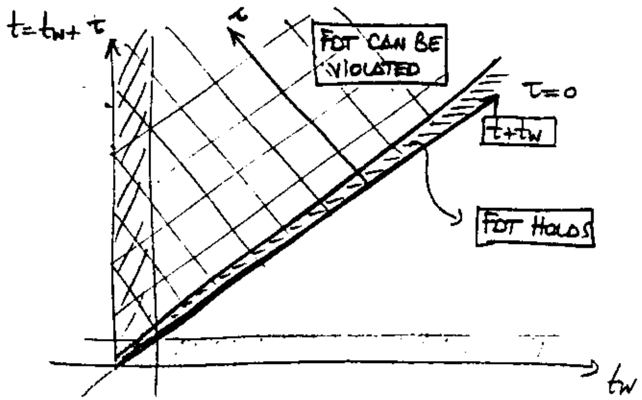
$$= k'' \left(\frac{\tau}{t_W^{\frac{\alpha+1}{2}}} \right)$$

NOW, $\alpha < 1$ IN ORDER TO HAVE THE POSSIBILITY OF FDT VIOLATIONS.

if $\tau \sim t_W^{a} \Rightarrow \text{rhs} \sim t_W^{a - \left(\frac{\alpha+1}{2}\right)}$

$$\begin{cases} a < \frac{\alpha+1}{2} \Rightarrow \text{rhs} \to 0 & \text{FDT HOLDS} \\ a = \frac{\alpha+1}{2} \Rightarrow \text{rhs} = \text{FINITE} & \text{FDT CAN BE VIOL.} \\ a > \frac{\alpha+1}{2} \Rightarrow \text{rhs} \to \infty & \text{"} \end{cases}$$

THIS BOUND DETERMINES WHERE IN THE TWO TIMES SECTOR FDT MUST HOLD.



IF $\tau \sim t_W^{a}$ WITH $a = \frac{\alpha+1}{2}$ $\Rightarrow$ 1st TIME FDT IS VIOL.

ANOTHER WAY TO WRITE IT:

$$CT = t^{\frac{1-\alpha}{2}} - t_w^{\frac{1-\alpha}{2}} = -\ln \exp t_w^{\frac{1-\alpha}{2}} + \ln \exp t^{\frac{1-\alpha}{2}}$$

$$\boxed{C_T = -\ln \left[\frac{\exp t_w^{\frac{1-\alpha}{2}}}{\exp t^{\frac{1-\alpha}{2}}} \right]}$$

IN L.III WE SHOW WHY IT IS RELEVANT TO WRITE IT IN THIS WAY.

THE FDT - VIOLATION

PHENOMENOLOGY.

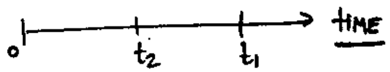
INTERPRETATION

THE CORR FUNCTIONS $\rightarrow$ CORR SCALES.

J. PHYS. A27, 5749 (94) — PRE (MARS 97) 55, 3898 (97)

- SYSTEMS WITH τ_{eq} finite
- SYSTEMS WITH $\tau_{eq} \rightarrow \infty$
  - DOMAIN GROWTH $\tau_{eq}(L) \xrightarrow[L\to\infty]{} \infty$
  - MF SG $\tau_{eq}(N) \xrightarrow[N\to\infty]{} \infty$
  - FINITE D SG $\longrightarrow$?

$\boxed{X(t_1, t_2)}$ $\quad t_2 \le t_1$



IF THERE CAN BE FDT VIOLATIONS, HOW ARE THEY?

$$R(t_1,t_2) = \frac{\tilde{\tilde{X}}(t_1,t_2)}{T} \frac{\partial C(t_1,t_2)}{\partial t_2}$$

GRAL. (DEFINES $\tilde{X}(t_1,t_2)$).

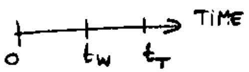
$C(t_1,t_2)$ IS MONOTONOUS wrt t_2 AND t_1 $\Longrightarrow$

FROM $\quad \mathcal{C} = C(t_1,t_2)$ WE CAN OBTAIN:

$$t_2 = f[\mathcal{C}, t_1] \qquad \text{(WITHOUT ANY ASSUMPTION)}$$

$$T R(t_1,t_2) = \tilde{X}\left[C(t_1,t_2), t_1 \right] \cdot \frac{\partial C(t_1,t_2)}{\partial t_2'}$$

$$m^{TRM}(t_T, t_W) = h \cdot \int_{t_W}^{t_T = t_1} dt' \; R(t_T, t')$$



$$= \frac{h}{T} \int_{t_W}^{t_T} dt' \; \tilde{X}\left[C(t_T,t'), t_T \right] \frac{\partial C(t_T, t')}{\partial t'}$$

SUGGESTS TO TAKE $C(t_T, t')$ AS AN INTEGRATION VARIABLE.

$$\lim_{\substack{t_W \to \infty \\ t_T \to \infty / C(t_T,t_W) = \mathcal{C}}} m^{TRM}(t_T, t_W) = \frac{h}{T} \lim_{\substack{t_W \to \infty \\ t_T \to \infty / C(t_T,t_W) = \mathcal{C}}} \int_{C(t_T,t_W) = \mathcal{C}}^{1} dc' \; \tilde{X}[c', t_T] =$$

$$m(\mathcal{C}) = \frac{h}{T} \int_{\mathcal{C}}^{1} dc' \left(\lim_{t_T \to \infty} \tilde{X}[c', t_T] \right) \Longrightarrow \quad (0)$$

IF THIS LIMIT $\exists$ $\Longrightarrow$

$$\boxed{X[c'] \equiv \lim_{t_T \to \infty} \tilde{X}[c', t_T]}$$

EQUIVALENT TO SAYING

$$\boxed{\lim_{\substack{t_2 \to \infty \\ t_1 / C(t_1,t_2) = \mathcal{C}}} \tilde{X}\left[C(t_1,t_2), t_2 \right] = X[\mathcal{C}]} \qquad (2)$$

$\dfrac{X[C]}{T}$ IS GOING TO BE $\neq \dfrac{1}{k_B T}$ IN GLASSY SYSTEMS.

FOR DEFINITE MODELS ONE CAN MAKE DEFINITE PREDICTIONS FOR $\dfrac{X[C]}{T}$

ONE CAN ALSO WRITE (2) ABOVE AS

$$\boxed{\lim_{\substack{t_2 \to \infty \\ t_1 \to \infty / C(t_1,t_2) = \mathcal{C}}} \tilde{X}\left[C(t_1,t_2), t_2 \right] = X[\mathcal{C}] = \lim_{\substack{t_2 \to \infty \\ t_1 \to \infty / C(t_1,t_2) = \mathcal{C}}} \left(\frac{R(t_1,t_2)}{\partial_{t_2} C(t_1,t_2)} \right)} \qquad (3)$$

AND (0) TOGETHER WITH (1) $\Longrightarrow$

$$\boxed{\frac{m(c)}{h} = \frac{1}{T} \int_{c}^{1} dc' \; X[c']} \quad \boxed{\forall c, \; 0 \le c \le 1}$$

BACK TO:

$$R(t_1,t_2') = \frac{\tilde{\tilde{X}}(t_1,t_2)}{T}\frac{\partial C(t_1,t_2)}{\partial t_2}$$ DEFINES $\tilde{\tilde{X}}(t_1,t_2)$

CASES:

* NOTE THAT, FOR ANY SYSTEM, IF t_2 IS SMALL

$$\boxed{\tilde{\tilde{X}}(t_1,t_2) \neq \frac{1}{T}}.$$

EVEN FOR THE HARMONIC OSCILLATOR, IF WE LOOK AT FINITE TIMES (IE t_2 FINITE $< \tau_{eq}$) $\Rightarrow$ THERE WILL BE A FDT VIOLATION.

* FOR A SYSTEM SUCH THAT $\boxed{\tau_{eq} \text{ IS FINITE}}$, IF $t_2' \gg \tau_{eq}$, $t_1 = t_2 + \tau$ WITH τ ANYTHING

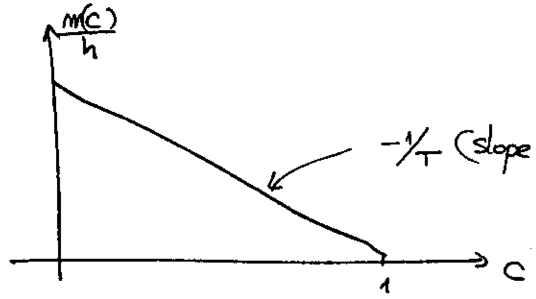
$$\boxed{\frac{\tilde{\tilde{X}}(t_1,t_2)}{T} = \frac{1}{T}.}$$

WE WANT TO STUDY THE BEHAVIOUR OF $\frac{\tilde{\tilde{X}}(t_1,t_2)}{T}$ WHEN $t_2 \to \infty$ AND $t_1 = t_2 + \tau$ MOVES ($t_2 \leq t_1 < \tau_{eq}$). IN THIS LIMIT, IF THE LIMITS ABOVE $\exists$ THIS IS EQUIV. TO STUDYING

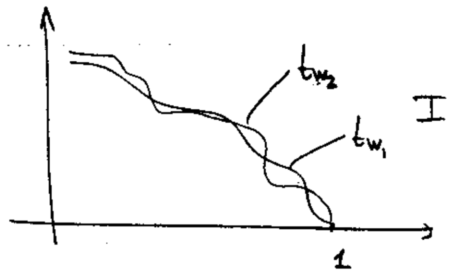
$$\boxed{\frac{m(C)}{h} = \frac{1}{T}\int_C^1 dC'\, x[C']}$$

AS A FUNCTION OF C (FOR LARGE TIMES, ONCE THE SYST. IS IN THE ASYMPTOTIC REGIME).

* IF $\boxed{t_w \gg \tau_{eq}}$



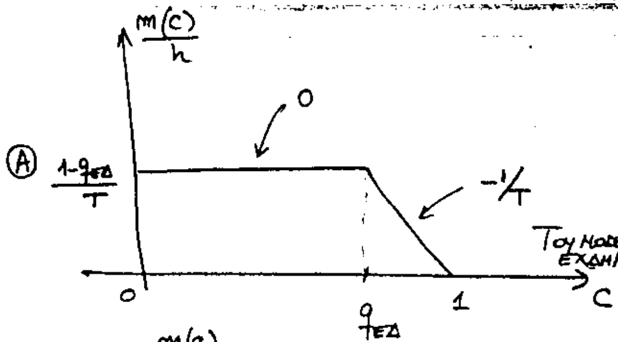
* IF $\boxed{t_w' \text{ finite} \ll \tau_{eq}}$



I DON'T KNOW. (PRE-ASYMPTOTIC REGIME)

* IF $\boxed{t_w \to \infty \ll \tau_{eq}}$

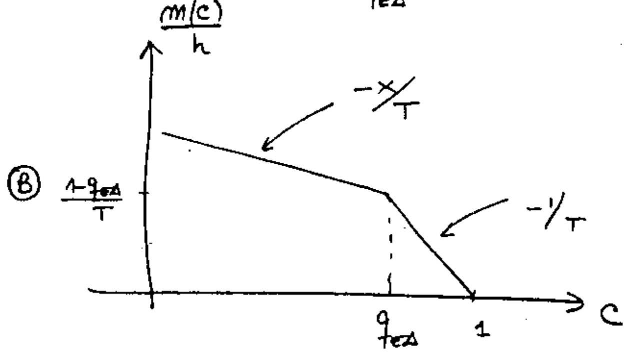
WE HAVE OBSERVED THREE POSIBILITIES. (FOR SYSTEMS WITH $0 \leq C \leq 1$):



('Domain growth')

(Analytic for toy models
A not proven guess, imp.)

Toy model example $\left[\frac{1}{2}\sum_{ij} J_{ij} s_i s_j + \frac{1}{2} z\left(\sum_i s_i^2 - N\right)\right]$



'MF-SG'

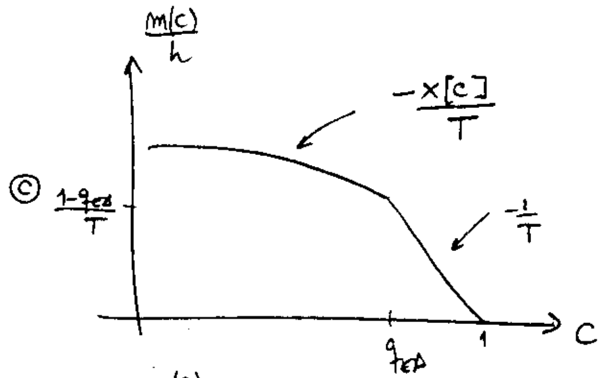
1 step RSB.

Models for glasses

$T_d > T_s$.

(Analytic)

$\sum_{i_1\dots i_p} J_{i_1\dots i_p} s_{i_1}\dots s_{i_p} + z\left(\sum_i s_i^2 - N\right)$ $p\geq 3$



'MF-SG'

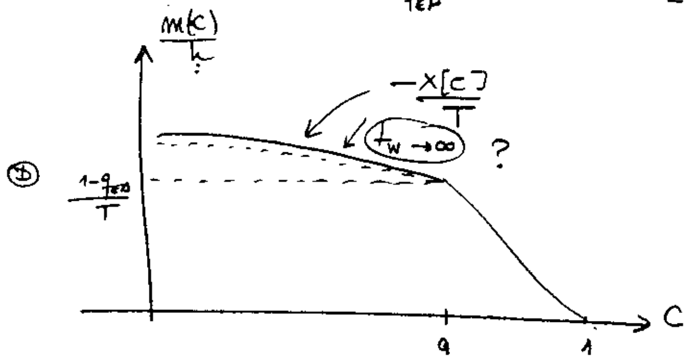
Full RSB

SK-model

$T_d = T_s$

(Analytic)

$\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j$ $s_i=\pm 1$ (SK



3D-EA

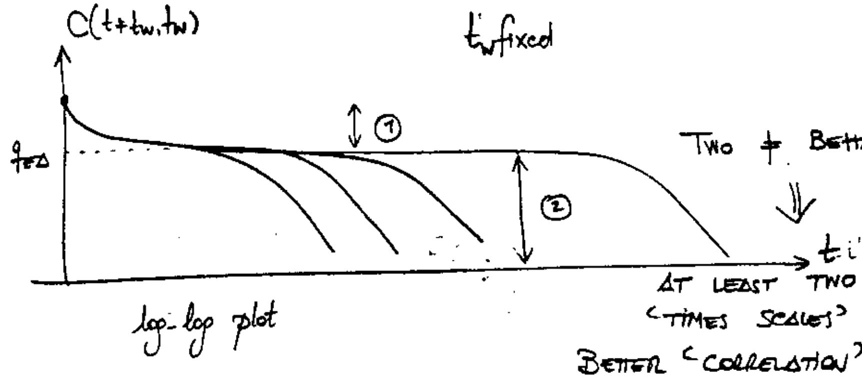
(Numerics)

$t_w \to \infty$?

$\frac{1}{2}\sum_{\langle ij\rangle} J_{ij} s_i s_j$ $s_i=\pm 1$ 3D.

In all these cases there's a range of values of the correlations such that FDT holds $C \in [q_{EA}(T), 1]$ and a range where it doesn't. $C \in [0, q_{EA}]$.

Let's remember the behaviour of $C(t+t_w, t_w)$ with times.

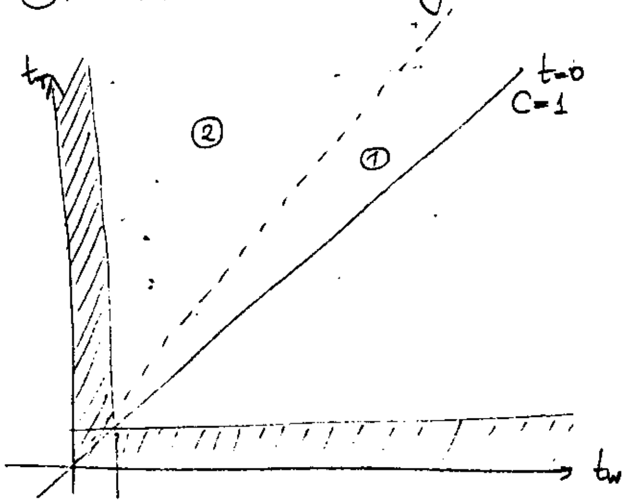


Two ≠ behaviour ⇓ at least two 'times scales' or better 'correlation'-scales.

(1): FDT - stationary times sector → Like Equilibrium!

(2): Aging - non-stationary " " → Non equilibrium.

$\boxed{t\geq 0}$

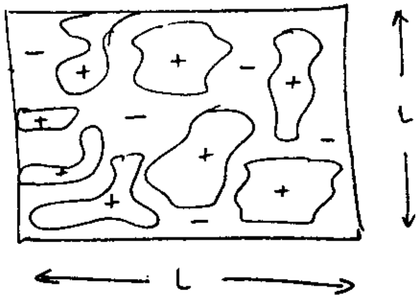


//// : t_w finite or t_T finite ⇒ pre-asymptotic

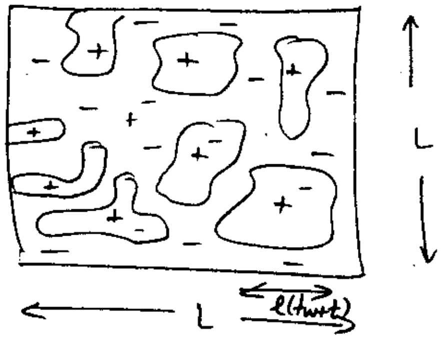
* HOW DO WE UNDERSTAND THE FDT-STATIONARY REGIME?

① DOMAIN GROWTH – 'HAND WAVING ARGUMENT'

$$\begin{cases} T_{eq}(L) \xrightarrow[L\to\infty]{} \infty \\ \text{RANDOM IC} \end{cases}$$



t_w (Large)



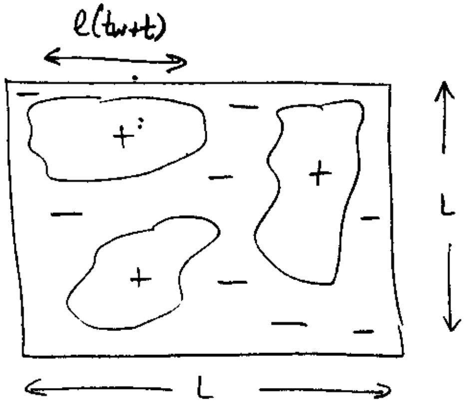
t_w+t $\quad$ t (Small)

$$\boxed{\lim_{t_w\to\infty} C(t+t_w,t_w) = q_{EA}(T) = m^2(T)}$$

THE CONF IS ALMOST THE SAME (DOMAIN WALLS HAVE NOT MOVED). THERE ARE ONLY THERMAL FLUCT. INSIDE THE DOMAINS $\Rightarrow$

$\boxed{\sim \text{EQUIL. DYNAMICS}}$

$\boxed{\text{NB: } T=0 \Rightarrow q_{EA}=1}$



t_w+t $\quad$ t (Large)

$\boxed{\text{NON EQUILIBRIUM}}$

WHAT WE MEAN BY 'Large' DEPENDS ON THE PART. PROBLEM

SCALING WHEN t_w AND $t+t_w$ ARE LARGE AND IN THE TIME REGION SUCH THAT THE WALLS MOVE; THE AUTO-CORR. FUNCTION SCALES AS:

$$C(t+t_w,t_w) \sim \mathcal{C}\left(\frac{\ell(t_w)}{\ell(t+t_w)}\right)$$

WITH $\ell(t)$ THE TYPICAL LENGTH OF A DOMAIN AT TIME t'. (WE'LL SEE LATER HOW THIS SCALING FOLLOWS FROM GENERAL – 'FORMAL' ARGUMENTS.)

WE ALREADY SEE:

- TTI IS VIOLATED → AGING IN C. (SCALING)
- FDT " " "

GUESS

$$\tilde{X}(t_1,t_2) = \tilde{X}[C(t_1,t_2);t_1] \sim$$

$$\circledast \quad \boxed{\tilde{X}(t_1,t_2) \sim f[C]\,\frac{1}{\ell^\theta(t')} \xrightarrow[t'\to\infty]{} 0}$$

GUESS, OBTAINED IN SIMPLE EXAMPLES

$\boxed{\text{SCALING ARGUMENT?}}$

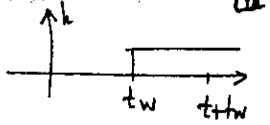
IF THIS IS TRUE $\boxed{\tilde{X}(t_1,t_2) \sim f[C]\,\frac{1}{\ell^\theta(t')}}$:

cfr Alan Bray's argument:

$\circledast$ NB THE FIELD WE APPLY IN THIS CASE DOES NOT FAVOUR ANY OF THE 2 (OR MORE) DOMAINS. IT'S A RANDOM FIELD.

WE DO IT THIS WAY SINCE WE WANT TO COMPARE WITH SG WHEN WE DON'T KNOW HOW ARE (IF THERE ARE) DOMAINS & HENCE WHICH FIELD APP.

$$\boxed{\frac{m_{t_w}(t+t_w)}{h}} = \int_{t_w}^{t+t_w} ds \quad R(t+t_w,s) =$$



$$= \int_{t_w}^{\Delta(t+t_w)} ds \quad f[C(t+t_w,s)] \frac{1}{\ell^\theta(s)} \frac{\partial C(t+t_w,s)}{\partial s} +$$

$$+ \int_{\Delta(t+t_w)}^{t+t_w} ds \quad \frac{1}{T} \frac{\partial C(t+t_w,s)}{\partial s}$$

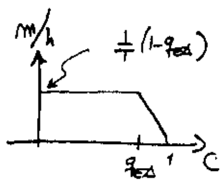
$$= \underbrace{\int_{C(t+t_w,t_w)=C}^{C(t+t_w,\Delta(t+t_w))=q_{EA}} dc' \quad f[c'] \frac{1}{\ell^{-1}(c')\,\ell^\theta(t+t_w)}}_{\text{AGING}} +$$

$$+ \underbrace{\int_{C(t+t_w,\Delta(t+t_w))=q_{EA}}^{1} dc' \quad 1/T}_{\text{FDT}}$$

$$= \frac{1}{T}(1-q_{EA}) + \underbrace{\frac{1}{\ell^\theta(t+t_w)} \cdot \underbrace{f[q_{EA},C]}_{\text{finite}}}_{\longrightarrow 0}$$

$$\longrightarrow \boxed{\frac{1}{T}(1-q_{EA})}$$

THE m VS C CURVE SHOULD LOOK LIKE →
FOR 'ALL' DOMAIN-GROWTH PROBLEMS



$$C' = C(t+t_w,s) = \ell\left(\frac{\ell(s)}{\ell(t+t_w)}\right) \Longrightarrow \ell^{-1}[C']\,\ell(t+t_w) = \ell(s)$$

THE FACT THAT $m(c)$ GOES ABOVE $\frac{1-q_{EA}}{T}$ FOR SG-MODEL REFLECTS A PROP OF THE 'WEAK LONG-TERM MEMORY' SCENARIO FOR GLASSES THAT WE EXPLAIN IN LIV - LV IN THE CONTEXT OF THE p-SPIN MODEL (SEE BELOW)

(B) - (C) - (D)

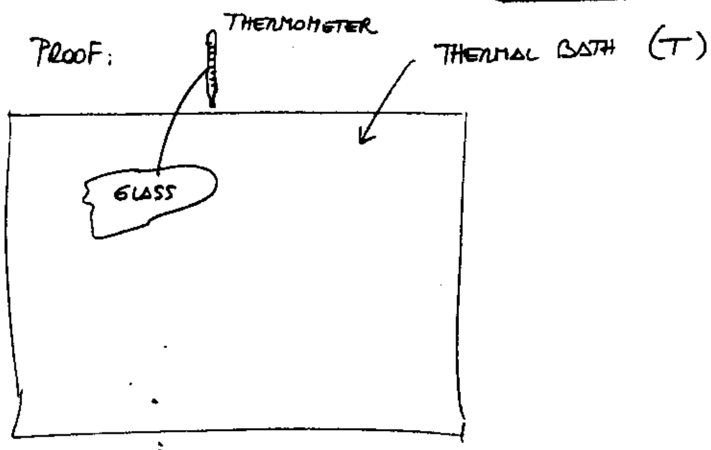
WE DON'T HAVE AN OBVIOUS REAL-SPACE INTERPRETATION OF $X(C)$. BUT WE HAVE AN INTERP IN TERMS OF:

$\boxed{\text{EFFECTIVE TEMPERATURES}}$ (J. KURCHAN'S LECTURE)

WHEN COMPARING $\dfrac{X(C)}{T}$ WITH $\dfrac{1}{T}$

ONE CAN THINK, ABOUT $\boxed{\dfrac{X(C)}{T} = \dfrac{1}{T_{eff}(C)}}$

PROOF:



LET THE GLASS BE OF AGE t_w WHEN THE THERMOMETER IS CONNECTED TO IT.

$$E_{TOTAL} = E_{SYST} + E_{THERMOMETER} + E_{INT}$$

EXAMPLE OF THERMOMETER: HARMONIC OSCILLATOR.

$$E_{THERM} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2$$

$$E_{INT} = -a\, O(\vec{s}) \cdot x$$

OBSERVABLE OF THE SYST.

ACTS AS AN EXTERNAL (DYN) FIELD ON THE SYSTEM.

$$\begin{cases} C_0(t_T, t_w) = \langle O_b(t_T)\, O_b(t_w) \rangle & \text{BARE - CORR.} \\ R_0(t_T, t_w) = \left.\dfrac{\delta \langle O(t_T) \rangle}{\delta a x(t_w')}\right|_{a=0} & \text{" RESPONSE} \end{cases}$$

(OF THE SYSTEM)

$\vec{S}(t) \to$ FOLLOW LANG DYNAMICS THAT IMPLIES $\{C_0, R_0\}$

THE EQ FOR $\vec{x}$ IS

$$\ddot{x}_{t_T} = -\omega_0^2 x_{t_T} - \frac{\delta E_{INT}}{\delta x_{t_T}} = -\omega_0^2 x_{t_T} + a\, O(t_T)$$

$$\boxed{\ddot{x}_{t_T} = -\omega_0^2 x_{t_T} + a\, O(t_T)}$$

$$O(t_T) = O_b(t_T) + a \cdot \int_0^{t_T} ds\, R_0(t_T, s)\, x(s)$$

(ASSUMING LINEAR RESPONSE)

$$\boxed{\ddot{x}_{t_T} = -\omega_0^2 x_{t_T} + a\, O_b(t_T) + a^2 \int_0^{t_T} ds\ R_0(t_T,s)\, x(s)} \qquad (*)$$

EQ FOR THE OSCILLATOR. (DAMPED)

LINEAR IN $x(t_T)$, WITH AN INHOMOG.

GIVEN BY $O_b(t_T)$

THE IDEA IS:

MEASURE $E_{osc}(t_w, \omega_0)$ AND RELATED,

THROUGH EQUIPARTITION (APPLIED UPON THE THERMOM)

TO

$$\boxed{k_B T_{eff}(t_w, \omega_0) = E_{osc}(t_w, \omega_0)}$$

WITHOUT ENTERING INTO ALL THE DETAILS OF THE CALCULATIONS, LET ME DEFINE THE QUANTITIES THAT ENTER THE SOL TO EQ (*) AND GIVE THE SOLUTION

$$\begin{cases} \chi(\omega, t_T)\ e^{i\omega t_T} = \int_0^{t_T} ds\ e^{i\omega s}\ R(t_T,s) \\ \tilde{C}(\omega, t_T)\ e^{i\omega t_T} = \int_0^{t_T} ds\ e^{i\omega s}\ C(t_T,s) \end{cases}$$

NB: THEY ARE 'EXTENDED' FOURIER TRANSFORMS IN THE SENSE THAT THEY ACT UPON THE 2nd (SMALLEST) TIME ONLY.

$$\chi(\omega, t_T) = \underbrace{\chi'(\omega, t_T)}_{\text{IN PHASE}} + i\, \underbrace{\chi''(\omega, t_T)}_{\text{OUT OF PHASE}}$$

THE GREEN FUNCTION IS

$$G(t,t') = \exp\left(-\frac{t-t'}{t_c(t)}\right) \sin\left(\omega_0 (t-t')\right) \theta(t-t')$$

FOR SMALL COUPLING / $a^2 \ll 1$

WITH $t_c(t) = \dfrac{2\omega_0}{a^2 \chi''(\omega_0, t)}$

AND $\langle x^2(t_T)\rangle = a^2 \int_0^{t_T} ds \int_0^{t_T} ds'\ G(t_T,s)\, G(t_T,s')\, C_0(s,s')$

AFTER SEVERAL (BORING) MANIPULATIONS

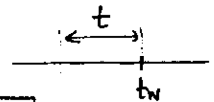
$$\frac{\omega_0^2}{2}\langle x^2(t_T)\rangle = \frac{\omega_0 \tilde{C}(\omega_0, t_T)}{2\, \chi''(\omega_0, t_T)}$$

CHECK: I NEED $\boxed{t_T \gg t_c(t_T)}$ { CHARACT TIME FOR THE EQUIL. OF THE OSCILLATOR

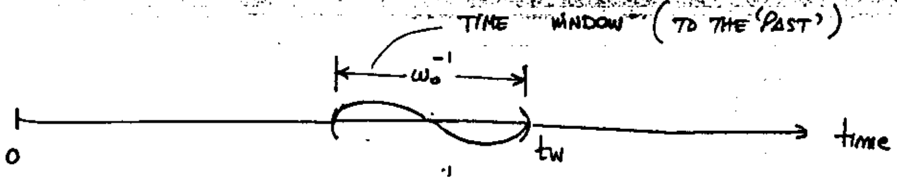
AT THE SAME TIME $a^2 \chi''(\omega_0, t_T)$ SMALL TO DEAL WITH THE INT. ON THE rhs OF $\langle x^2(t_T)\rangle$.

THUS, $$\boxed{k_B T_{eff}(t_w, \omega_0) = \frac{\omega_0 \tilde{C}(\omega_0, t_w)}{\chi''(\omega_0, t_w)}}$$

THIS LOOKS VERY MUCH LIKE IMPLYING



$$\boxed{k_B T_{eff}(t_w, t_w - t) = \frac{\partial_T C(t_w, t_w - t)}{R(t_w, t_w - t)}}$$



'LOCAL' MEASURE IN TIME.

$\boxed{t_c(t_W) \ll t_W}$ CONDITION ON THE COUPLED THERMOM.

NOW, WE COMPARE $\boxed{\omega_0 \text{ WITH } t_W}$

WE WANT TO GO BACK TO TIMES. SUPPOSE THAT FOR A CERTAIN RELATION BETWEEN

$$\boxed{k_B T_{eff}(t_W, \omega_0)\, \chi''(\omega_0, t_W) = \omega_0\, \tilde{C}(\omega_0, t_W)}$$

IF $\boxed{\omega_0 \text{ VS } t_W}$ IS SUCH THAT

$$T_{eff}(t_W, \omega_0) \sim T_{eff}$$

$\Longrightarrow$:

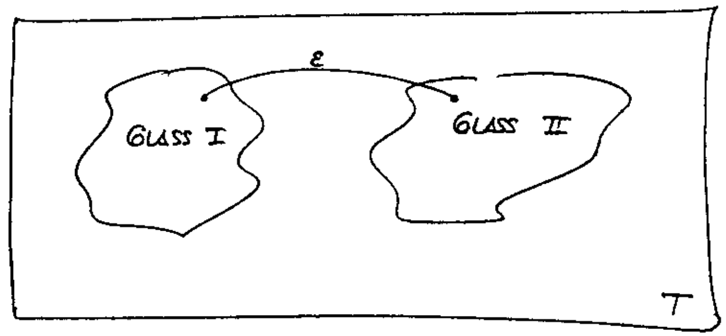
$$\boxed{k_B T_{eff}\, R(t_W, t_W+t) = \frac{\partial C(t_W, t_W+t)}{\partial(t_W - t)}}$$

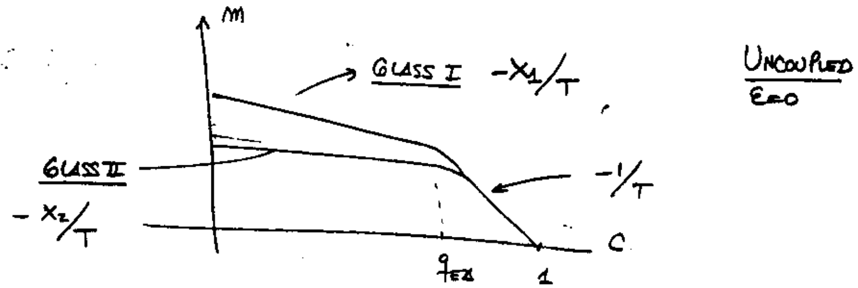
$$\boxed{k_B T_{eff}\, R(t_W+t, t_W) = \left.\frac{\partial C(t_W+t, t')}{\partial t'}\right|_{t'=t_W}}$$

BUT ! $\Big\{$ I CAN CHOOSE ω_0 VS. t_W IN SUCH A WAY TO GET $\neq$ T_{eff}'s ! !

A CHECK: DOES T_{eff} BEHAVE AS A 'TEMPERATURE' ?

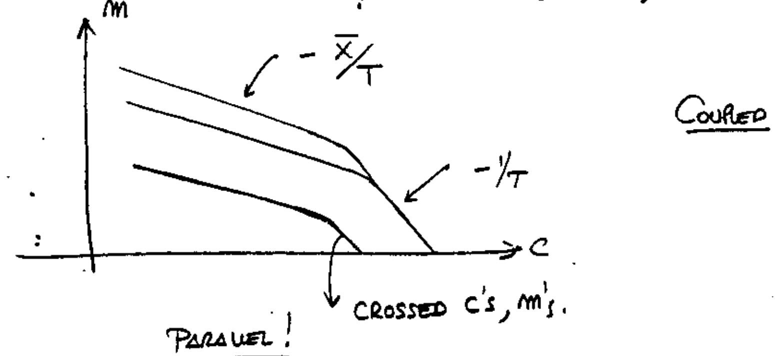
PARTIAL EQUILIBRATION





NOW, CONNECT THEM WITH $\varepsilon \sum_i s_i\, \sigma_i$ (EXAMPLE)

$\Longrightarrow$



WE COMPUTE

$$m^{11}(t,t_w) = h \int_{t_w}^{t+t_w} ds\; R_{11}(t,s) \quad \text{vs} \quad C_{11}(t,t_w)$$

$$m^{22}(t,t_w) = h \int_{t_w}^{t+t_w} ds\; R_{22}(t,s) \quad \text{vs} \quad C_{22}(t,t_w)$$

$$m^{12}(t,t_w) = h \int_{t_w}^{t+t_w} ds\; R_{12}(t,s) \quad \text{vs} \quad C_{12}(t,t_w)$$

$$m^{21}(t,t_w) = h \int_{t_w}^{t+t_w} ds\; R_{21}(t,s) \quad \text{vs} \quad C_{21}(t,t_w)$$

FOR

$$\boxed{H_J[\vec{s},\vec{\sigma}] = -\sum_{i_1\cdots i_p} J_{i_1\cdots i_p}\, s_{i_1}\cdots s_{i_p} - \sum_{i_1\cdots i_p} J_{i_1\cdots i_p}\, \sigma_{i_1}\cdots\sigma_{i_p} + \varepsilon \sum_i s_i \sigma_i}$$

THIS IS IMPORTANT SINCE (SYST 1) & (SYST 2) CAN BE JUST 'PARTS' OR 'OPERATORS' OF A SAME SYSTEM. IF THEY ALWAYS 'EQUILIBRATE' INSIDE A CORR SCALE THEN WE CAN UNDERSTAND WHY $X = CT$ INSIDE A CORR SCALE (MEANING THE TIME SECTOR WHERE C VARIES AS $C(h(t_2)/h(t_1))$, SEE BELOW FOR AN EXPLANATION)

BUT, NOTE THAT NEITHER SYST 1 NOR SYST 2 ARE EQUILIBRATED WITH THE BATH SINCE $X < 1$!

WHAT DOES THIS MEAN FOR $\neq$ OPERATOR OF THE SAME SYSTEM LIKE $E(t)$, $m(t)$?

IT MEANS THAT INITIALLY – SHORT TIMES, PRE-ASYMPTOTIC – THEY MAYBE DIFFERENT BUT FOR LARGE TIMES THEY ('MUST') BECOME THE SAME.

LAST SURPRISE

ONSAGER RELATIONS:

$$\boxed{\langle A(t+t_w)\, B(t_w)\rangle = \langle A(t_w)\, B(t+t_w)\rangle}$$

OK $\begin{cases} \text{NUMERICALLY (3DEA – 3D-ISING)} \\ \text{MF} \end{cases}$

NB ONSAGER RELATIONS LIKE THE ONE ABOVE ARE PROVEN TO HOLD ONLY IN EQUILIBRIUM.

ONE OBSERVES THEY EVEN HOLD $(\forall (t_w, t) / t_w$ IS LARGE$)$ FOR GLASSY SYSTEMS THAT ARE WELL OUT OF EQUILIBRIUM. IS THIS RELATED TO THE 'SMALL ENTROPY PRODUCTION?

OUT OF EQ (THERMODYNAMICS)

$$\dot{H}(t) \text{ SMALL} \Longrightarrow \begin{cases} T_{eff} \\ \text{ONSAGER} \\ \text{SLOW DYN. - AGING.} \end{cases}$$

HOW STRONGLY ARE THESE PROPERTIES RELATED?

STILL AT A GENERAL LEVEL

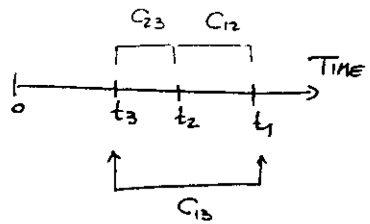
WHAT CAN WE SAY ABOUT THE BEHAVIOUR OF THE CORR FUNCTIONS?

J. PHYS. A27, 5749 (94).

IMMAGINE THAT YOU HAVE 3 TIMES:

$t_1 \geq t_2 \geq t_3 \Longrightarrow$ YOU CAN CONSTRUCT 3 CORR

$$C(t_1,t_2) \Longrightarrow t_1 = g(C(t_1,t_2),t_2)$$
$$C(t_2,t_3) \longrightarrow t_2 = g(C(t_2,t_3),t_3)$$
$$C(t_1,t_3)$$



$$\boxed{C_{13} \leqslant \min(C_{12}, C_{23})}$$

$$C(t_2,t_3) = C\left[g\left(C(t_1,t_2),\ t_1\right);\ t_3\right]$$

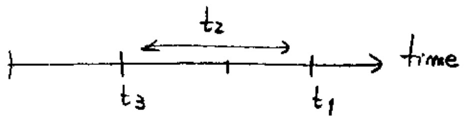
$$= C\left[g\left(C(t_1,t_2),\ g\left(C(t_1,t_3),\ t_3\right)\right);\ t_3\right]$$

$$\boxed{\lim_{t_3\to\infty} C(t_2,t_3) = \bar{f}\left(C(t_1,t_2),\ C(t_1,t_3)\right)}$$

$C(t_1,t_3) = C_{13}$
$C(t_1,t_2) = C_{12}$

Construction

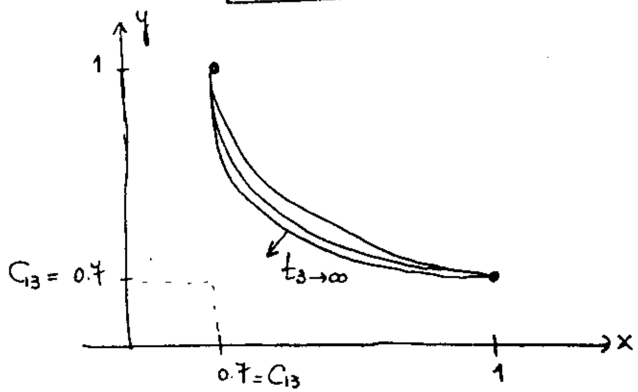
Take $C_{13} = 0.7$



if $\begin{cases} t_2 = t_3 \Rightarrow & C(t_2,t_3) = 1 \\ t_2 = t_1 \Rightarrow & C(t_1,t_2) = 1 \end{cases}$

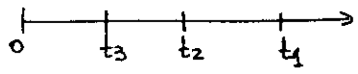
$$\boxed{y = \bar{f}\ (x,\ C_{13})}$$

$y = C(t_2,t_3)$
$x = C(t_1,t_2)$



Choose t_3
↓
Move t until
$C(t,t_3) = 0.7$
that sets $t_1 = t$
Re-do the run computing
$C(t_2,t_3)$ & $C(t_1,t_2)$

If the limit $\exists \Rightarrow$ this construction has an asymptotic form (doesn't oscillate)



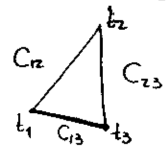
Now, let me introduce the 'inverse' of this function $\boxed{z = \bar{f}\,(x,y)}$

and write

$$\boxed{C_{13} = f(C_{12}, C_{23})}$$

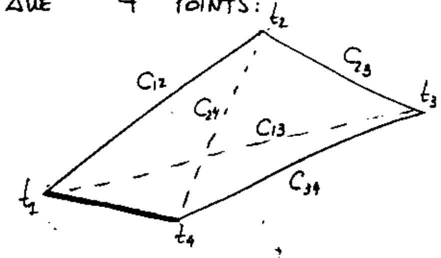
↑
Smallest

Defines a 'triangle'

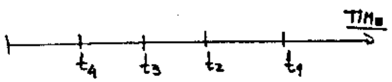


I'll work with f instead of $\bar{f}$ even if it's tricky to take this inverse.

Take 4 points:



$$\boxed{t_1 \geq t_2 \geq t_3 \geq t_4}$$



$C_{14} = f\left(C_{12}, C_{24}\right)$
$C_{14} = f\left(C_{13}, C_{34}\right)$
= They must be equal!

$t_1 < t_2 < t_3 < t_4$ LIII(2)

$$f(C_{12}, C_{24}) = f(C_{13}, C_{34})$$

BUT
$$C_{13} = f(C_{12}, C_{23})$$
$$C_{24} = f(C_{23}, C_{34})$$

$$f\Big(C_{12}, f(C_{23}, C_{34})\Big) = f\Big(f(C_{12}, C_{23}), C_{34}\Big)$$

$$f(x, f(y,z)) = f(f(x,y), z)$$

$$\Downarrow$$

f IS ASSOCIATIVE

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

PROPERTIES:

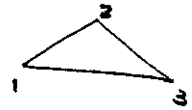
$$C_{13} = f(C_{12}, C_{23})$$

if $t_2 = t_1 \Rightarrow C_{23} = f(1, C_{23}) \Rightarrow x = f(1,x)$

if $t_2 = t_3 \Rightarrow C_{12} = f(C_{12}, 1) \Rightarrow x = f(x,1)$

IT HAS AN IDENTITY

SINCE $C_{13} \leqslant \min(C_{12}, C_{23})$



$$\boxed{f(C_{12}, C_{23}) \leq \min(C_{12}, C_{23})}$$

EXAMPLES:

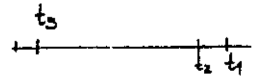
→ $f(x,y) = \min(x,y)$ → Ultrametricity.

→ $f(x,y) = xy$

NO DETAILS SINCE LONG BUT

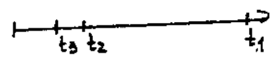
ONE CAN SHOW WITHOUT ANY ASSUMPTION AND QUITE FORMALLY THAT:

IT HAS AT LEAST A ZERO,



IF $t_3 = 0^+ \ll t_1 \Rightarrow C_{13} = 0 = f(C_{12}, \underset{\downarrow 0}{C_{23}}) = f(C_{12}, 0)$

$t_3 = t_2 = 0^+ \ll t_1 \Rightarrow C_{13} = 0 \; f(0, C_{23})$



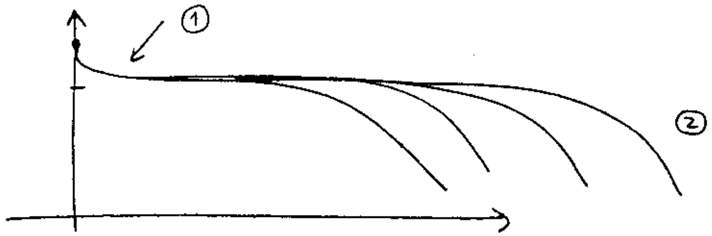
Combinations of these two examples are the <u>unique</u> possibilities for f !

Take t_2 / $C_{12} = C_{23} = a \Rightarrow$

$$f(a,a) \leq \min(a,a) = a$$

We can have 'fixed points' a^* / $f(a^*,a^*) = a^*$
or values $a \neq a^*$ / $f(a,a) < a$.

a^* separate $\neq$ corr scales. Like in the corr. $a^* = q_{EA}$



When $C \geq q_{EA} \Rightarrow C_{①}(t,t') = j^{-1}_{①}\left(\frac{h_{①}(t')}{h_{①}(t)}\right)$

When $C \leq q_{EA} \Rightarrow C_{②}(t,t') = j^{-1}_{②}\left(\frac{h_{②}(t')}{h_{②}(t)}\right)$

This scaling is proven!

Inside a scale, ie between two fixed points:

$$f(a,b) = a \cdot b \iff a = j^{-1}\left(\frac{h(t')}{h(t)}\right)$$

Just a couple of examples:

$\boxed{\text{FDT - STATIONARY SECTOR}}$

⊛ $C_{①}(t,t') = C_F(t-t') = j_F^{-1}\left(\frac{h_F(t')}{h_F(t)}\right)$

Choosing $h_F(t') = \exp(\alpha t')$

$$\frac{h_F(t')}{h_F(t)} = \exp[\alpha(t-t')]$$

$$C_F(t-t') = j_F^{-1}\left[\exp\left(-\alpha(t-t')\right)\right]$$

$$C_F = j_F^{-1} \circ \exp - \alpha$$

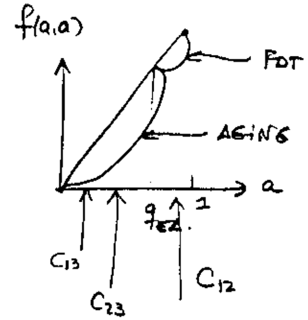
⊛ $C_{②}(t,t') = j^{-1}\left(\frac{h(t')}{h(t)}\right)$

e.g. Domain Growth $\ell\left(\frac{\ell(t')}{\ell(t)}\right)$

⊛ f between $\neq$ corr scales:

I want to compute

$$C_{13} = f(C_{12}, C_{23})$$



$$\begin{cases} C_{12} = j_F^{-1}\left[\exp\left(-\alpha(t_1-t_2)\right)\right] \\ C_{23} = j_A^{-1}\left[\frac{h(t_3)}{h(t_2)}\right] \end{cases}$$

$$C(t_1,t_3) \leq \min\left(C_{12}, C_{23}\right)$$

I NEED TO WRITE $C(t_1,t_3)$

I'LL ASSUME $C_{13} \in$ AGING $\Rightarrow$

$$C_{13} = j_A^{-1}\left(\frac{h(t_3)}{h(t_1)}\right)$$

$$= j_A^{-1}\left(\frac{h(t_3)}{h(t_2)}\,\frac{h(t_2)}{h(t_1)}\right)$$

$$= j_A^{-1}\left[j_A(C_{23})\cdot\frac{h(t_2)}{h(t_1)}\right]$$

$$\frac{h(t_2)}{h(t_1)} = ?$$

$$\ln\left[j_F(C_{12})\right] = -\alpha(t_1-t_2)$$

$$\frac{1}{\alpha}\left[\alpha t_1 + \ln j_F(C_{12})\right] = t_2$$

$$\underbrace{\frac{h\left[t_1 + \frac{1}{\alpha}\ln j_F(C_{12})\right]}{h(t_1)}}_{\downarrow\; 1} = \frac{h(t_2)}{h(t_1)}$$

THUS,

$$\boxed{C_{13} = C_{23} = \min\left(C_{12}, C_{23}\right)}$$

THIS HAS BEEN PROVEN IN FULL GENERALITY.

THIS WILL HELP US SOLVING THE EQUATIONS, IN PARTICULAR

$$C(t,t') = j^{-1}\left(\frac{h(t')}{h(t)}\right).$$

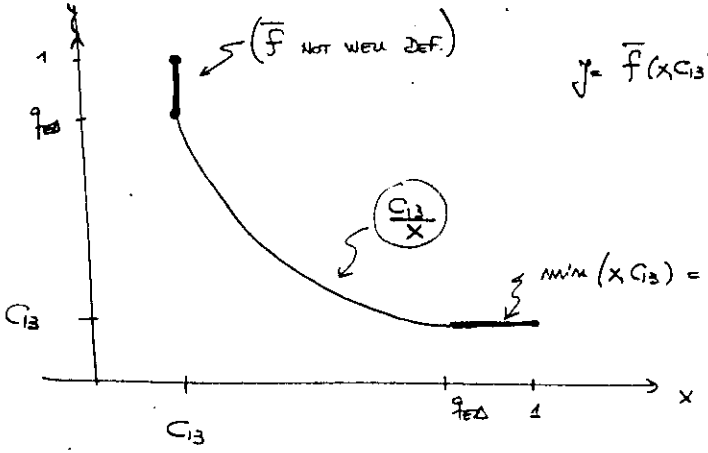
HOW MANY FIXED POINTS ARE, WHAT'S THE FORM OF j^{-1}, h WILL BE DETERMINED BY THE MODEL.

ACTUALLY, FOR THE MOMENT WE HAVE:

$$\begin{cases} 2 \text{ SCALES.} \longrightarrow \text{ALL } p\text{-SPINS, DOMAIN GROWTH} \\ \infty \ne \text{ SCALES} \longrightarrow \text{SK} \end{cases}$$

THIS IS REMINISCENT OF THE STRUCTURE OF REPLICA SYMMETRY BREAKING AT THE STATIC LEVEL.
FORMAL CONNECTION? SUSY.

THUS, STANDARD CASE FDT + SOMETHING, e.g. DOMAIN GROWTH, is



$y = \bar{f}(x C_{13}) = \frac{C_{13}}{x}$

$xy = f(x,y) = C_{13}$

$y = \frac{C_{13}}{x}$

$C_{13} = f(x,y) = \min(x,y) = \min(C_{13}, y) = C_{13}$ (INDEP OF y)

NO INVERSE WELL DEFINED. BUT I CAN PLOT IT AS ABOVE

LECTURES IV & V

- DYN. EQS. FOR DISORDERED & NON-DISORDERED MODELS
- SOLUTION TO A PARTICULAR CASE: THE p-SPIN SPHERICAL MODEL OR A SIMPLE MODE-COUPLING APPROACH TO GLASSES.

AN EXAMPLE p-spin MODEL

$$\boxed{\begin{aligned} H_J[s] &= -\sum_{i_1<\dots<i_p} J_{i_1\dots i_p}\, s_{i_1}\dots s_{i_p} \\ \dot{s}_i(t) &= -\frac{\delta H_J[\vec{s}]}{\delta s_i(t)} - z(t)\, s_i(t) + \xi_i(t) \end{aligned}}$$

HOW DO WE OBTAIN DYN EQS FOR MACROSCOPIC QUANTITIES LIKE $\{C, R\}$? THROUGH THE MARTIN-SIGGIA-ROSE (MSR FORMALISM.

MSR (REM: H. KADANOFF'S LECTURE):

$$Z_{dyn} = 1 = \int \mathcal{D}\xi \, \exp\left[-\frac{1}{4T}\int d\tau\, \vec{\xi}^{\,2}(\tau)\right] \qquad (1)$$

$$\mathcal{D}\xi = \prod_\tau \frac{d\vec{\xi}(\tau)}{\sqrt{4T\pi}}$$

$$\boxed{P[J_{i_1\dots i_p}] = \sqrt{\frac{N^{p-1}}{\pi p!}}\, \exp\left[-\frac{N^{p-1} J^2_{i_1\dots i_p}}{p!}\right]}$$

CIRANO $\boxed{Z_{dyn} \text{ indep of } J!}$

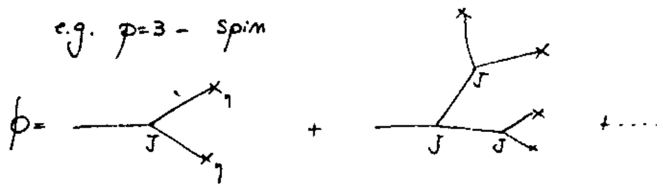
LIST OF STEPS

- Start from (1)
- Fix the gauge à la Fadeev-Popov inserting
  $$\int \mathcal{D}s \;\delta\left(\vec{s} - \vec{E}^J_\phi[\vec{s}]\right) \left|\det\left(\frac{\delta \vec{E}^J_\phi(s)}{\delta \vec{s}}\right)\right| = 1$$
- Write $Z_{dyn} = 1 = \int \mathcal{D}\vec{\xi}\,\mathcal{D}\vec{s}\,\mathcal{D}i\hat{\vec{s}}\;\exp - S^{(1)}_J[\vec{s}, i\hat{\vec{s}}, \vec{\xi}]$
- Int over $\xi \Rightarrow Z_{dyn} = 1 = \int \mathcal{D}\vec{s}\,\mathcal{D}i\hat{\vec{s}}\;\exp - S^{(2)}_J[\vec{s}, i\hat{\vec{s}}]$
- Take $\int dJ\, P[J]\, Z_{dyn} = 1 = \int \mathcal{D}\vec{s}\,\mathcal{D}i\hat{\vec{s}}\;\exp -NS^{(3)}[\vec{s}, i\hat{\vec{s}}]$
- Now, Large N approx $\Rightarrow$ SEVERAL WAYS OF OBTAINING EQS FOR C & R (& m if $h\neq 0$).

ANOTHER WAY

$\boxed{\text{PERTURBATIVELY}}$

e.g. $p=3$ - spin



$$\left.\begin{aligned} C(t,t') &= \overline{\langle \phi\phi \rangle} \\ R(t,t') &= \overline{\langle \phi\eta \rangle} \end{aligned}\right\} \Rightarrow$$ BUT $\overline{}$ $\rightarrow$ ONLY CERTAIN TERMS SURVIVE (SYMM. PROP. OF $P[J]$, ξ

$\Rightarrow$ SCHWINGER-DYSON EQS AND $\left(\begin{matrix}\Sigma \\ D\end{matrix}\right)$.

REMARK: ONE NOTICES THAT

$\boxed{\text{• DISORDERED SYSTEM} \equiv \text{• NON-DISORDER APPROXIMATED. !} \quad \text{(SAME GRAPHS SURVIVE)}}$

$\Downarrow$

$\boxed{\text{SAME PHENOMENOLOGY}}$

(AND A POSSIBLE WAY OF SEEING THE 'UNIVERSALITY' APPEAR)

(KRAICHNAN; KIRKPATRICK & THIRUMALAI, FRANZ & HERTZ, JPB, LFC, JK, MM)

DYNAMICAL EQS. FOR THE p-SPIN SPH MODEL

LFC - JK PRL 71, 173 (93).

NOTATION: $0 \longrightarrow t' \longrightarrow t \longrightarrow$ TIME

$\forall (t,t')$:

$$\partial_t R(t,t') = -z(t)\, R(t,t') + \frac{p(p-1)}{2} \int_{t'}^{t} ds\, R(t,s)\, C^{p-2}(t,s)\, R(s,t') + \delta(t-t') \qquad \text{(EQ-R)}$$

$$\partial_t C(t,t') = -z(t)\, C(t,t') + \frac{p}{2} \int_0^{t'} ds\, C^{p-1}(t,s)\, R(t',s) + \frac{p(p-1)}{2} \int_0^{t} ds\, R(t,s)\, C^{p-2}(t,s)\, C(s,t') + 2T R(t',t) \qquad \text{(EQ-C)}$$

THE EQ FOR THE LAGRANGE MULTIPLIER $z(t)$ FOLLOWS FROM IMPOSING THE SPH CONSTRAINT $C(t,t) = 1$.

IF $C(t,t) = 1 \Rightarrow$

$$\lim_{t' \to t^-} \left(\partial_t C(t,t') + \partial_{t'} C(t,t') \right) = 0$$

$$\lim_{t' \to t^-} \left[\text{EQ-C}(t,t') + \text{EQ-C}(t',t) \right] = 0.$$

$$\Downarrow$$

$$0 = -2z(t) + p^2 \int_0^t ds\, R(t,s)\, C^{p-1}(t,s) + 2T \lim_{t' \to t^-} \left(\underbrace{R(t',t)}_{0} + R(t,t') \right)$$

$$\Downarrow$$

$$\boxed{z(t) = \frac{p^2}{2} \int_0^t ds\, R(t,s)\, C^{p-1}(t,s) + T} \qquad \text{(EQ-z)}$$

SINCE $\lim_{t' \to t^-} R(t',t) = 0, \quad \lim_{t' \to t^-} R(t,t') = 1$ (1)

PROPS OF THE DERIV. OF C AROUND $t \simeq t'$

PROPS OF $\lim_{t' \to t^-} \partial_t C(t,t')$ & $\lim_{t' \to t^+} \partial_t C(t,t')$

LET'S CONSIDER $t > t'$ IN (EQ-C) AND TAKE $\lim_{t' \to t^-}$:

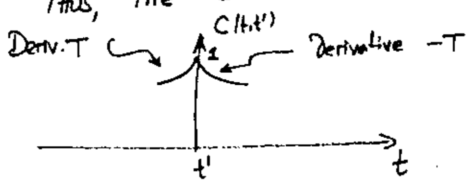
$$\lim_{t' \to t^-} \partial_t C(t,t') = -z(t) + \frac{p^2}{2} \int_0^t ds\, R(t,s)\, C^{p-1}(t,s) = -T$$

(USING EQ-z)

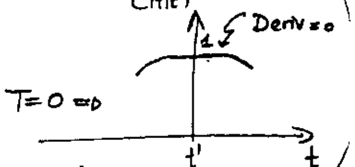
NOW, CONSIDER $t < t'$ IN (EQ-C) AND TAKE $\lim_{t' \to t^+}$:

$$\lim_{t' \to t^+} \partial_t C(t,t') = -z(t) + \frac{p^2}{2} \int_0^t ds\, R(t,s)\, C^{p-1}(t,s) + 2T = T \qquad (3)$$

THUS, THE DERIVATIVE OF C HAS A 'CUSP' IN $t = t'$:



IF $T = 0 \Rightarrow$



(REM: BRAY'S DOMAIN GROWTH AT T=0)

Confront R with $\partial_t C$ at $t' \to t^-$:

$$-\frac{1}{T}\lim_{t'\to t^-} \partial_t C(t,t') = \lim_{t'\to t^-} R(t,t')$$

From (2)-(3), around $t' \simeq t$ one can write: $\boxed{C(t,t') \simeq 1 - T|t-t'|}$

and then

$$\boxed{\lim_{t'\to t^-} R(t,t') = +\frac{1}{T}\lim_{t'\to t^-} \partial_{t'} C(t-t')}$$

ie FDT holds around the diagonal. (1st order in $(t-t')$)

$\forall t' \geq 0$. $\forall T$!

NB This is not true if there's inertia ($m\,\partial_t^2$). In this case one needs t' large to have FDT around $t' \simeq t$.

Rem formal bounds & time-sectors $\Longrightarrow$ ok.

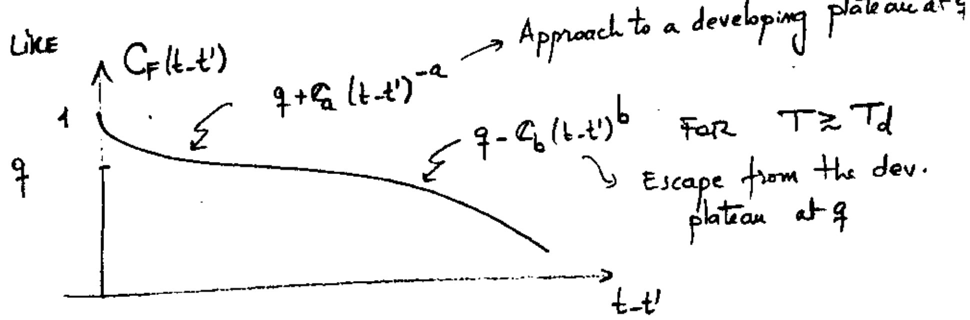
Solution above T_d

One can check (excercise!) that (EQ-C)-(EQ-R)-(EQ-Z) admit, for $T > T_d$ a solution

$$\begin{cases} z(t) \longrightarrow z_\infty \\ R(t,t') \longrightarrow R_F(t-t') \\ C(t,t') \longrightarrow C_F(t-t') \end{cases} \quad \Big\} \; R_F(t-t') = \frac{1}{T}\frac{dC_F(t-t')}{dt'}$$

For large t' and $\forall\,(t-t')$.

In other words, above T_d the system equilibrates — even if $N\to\infty$ first. After a nonequilibrium transient (once the R-eq & the resulting eq for $C_F(t-t')$ (once the R-eq & C-eq have been collapsed to a single one using FDT) is one of the MC-eqs for glasses. The sol looks like



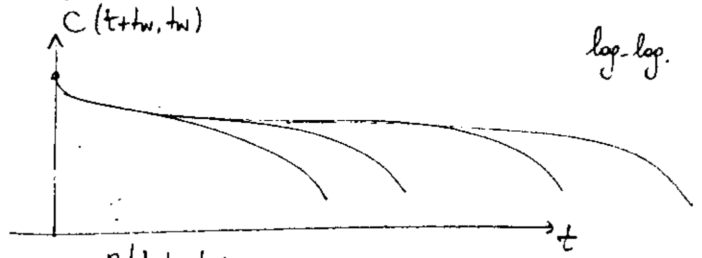
$\longrightarrow$ This sol breaks down below T_d!

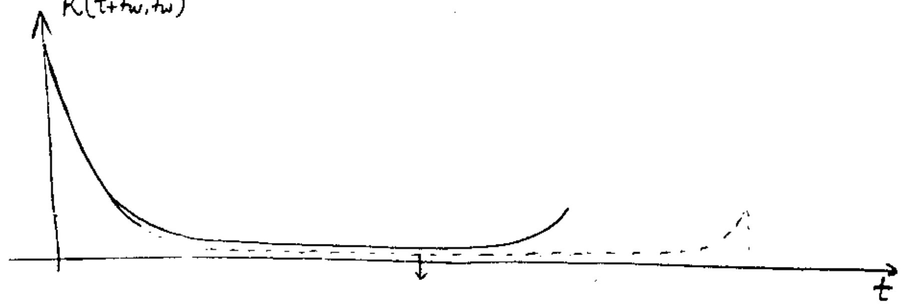
$\boxed{T<T_d}$ → NUMERICAL SOLUTION

NUMERICAL SOLUTION OF THE DYN EQS (CAUSAL) HELP US DEVELOPING AN ANSATZ TO SOLVE THEM. THE MAIN POINTS ARE:

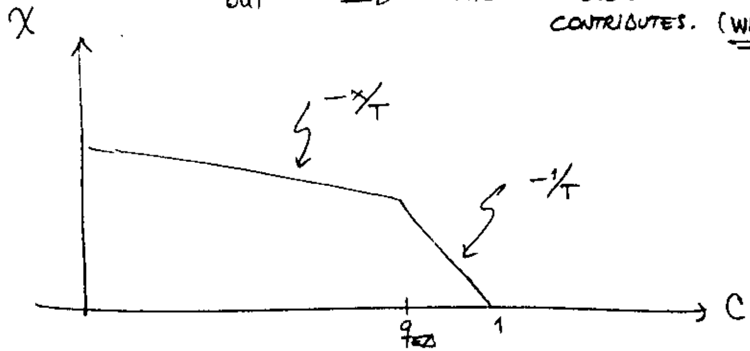
$\boxed{T<T_d}$

NUMERICAL SOLUTION:





BUT ⟹ THE INT OVER A LARGE INTERVALS CONTRIBUTES. (WLTM, SEE BELOW)



$\boxed{T<T_d}$ → THE CORRELATION

LET'S DEF

$$C(t+t_w,t_w) = C_S(t) + C_A(t+t_w,t_w)$$

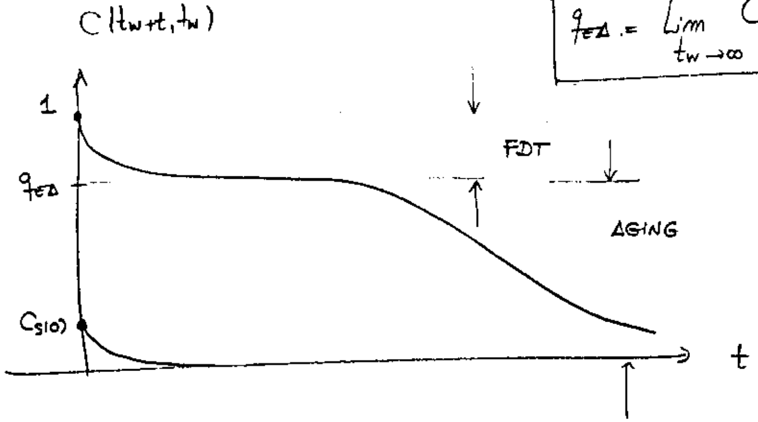
• $\boxed{t=0}$ $\quad C(t_w,t_w) = 1 = C_S(0) + C_A(t_w,t_w)$

let's choose $C_S(0) = 1-q_{EA}$

$C_A(t_w,t_w) = q_{EA} \quad \forall t_w.$

• $\boxed{\lim_{t_w\to\infty}}$ $\quad \boxed{\lim_{t_w\to\infty} C = C_F(t)} = C_S(t) + \lim_{t_w\to\infty} C_A(t+t_w,t_w)$

$$\boxed{q_{EA} = \lim_{t_w\to\infty} C_A(t+t_w,t_w)}$$



$$\lim_{t\to\infty}\lim_{t_w\to\infty} C = q_{EA} = \lim_{\tau\to\infty} C_S(t) + q_{EA}$$

$$\boxed{\lim_{t\to\infty} C_S(t) = 0}$$

$$\boxed{\lim_{t\to\infty} C_F(t) = q_{EA}}$$

• $\boxed{\lim_{t\to\infty}}$ $\qquad \lim_{t\to\infty} C = 0 = \lim_{t\to\infty} \underbrace{\cancel{C_s(t)}}_{0} + \lim_{t\to\infty} C_\Delta$

$$\boxed{\lim_{t\to\infty} C_\Delta(t+t_w,t_w) = 0}$$

C_{su} $\qquad C_F(t) = C_s(t) + q_{EA}$ $\qquad$ (LIVES IN AN INTERVALL

THE RESPONSE

I EXPECT $\qquad R = X(C)\, \partial_{t_2} C \qquad X(C) = \begin{cases} 1/T & C > q_{EA} \\ x/T & C < q_{EA} \end{cases}$

$$R(t+t_w,t_w) = \underbrace{R_s(t)}_{\downarrow} + \underbrace{R_\Delta(t+t_w,t_w)}_{\downarrow}$$

SAME INT AS C_s $\qquad$ SAME INT AS C_Δ.

• $\boxed{t=0^+}$ $\qquad R(t_w,t_w) = 1 = R_s(0) + R_\Delta(t_w,t_w)$

$$\boxed{\begin{array}{l} R_s(0) = 1 \\ R_\Delta(t_w,t_w) = 0. \end{array}}$$

• $\lim_{t_w\to\infty} R(t+t_w,t_w) \equiv R_F(t) = R_s(t) + \underbrace{\lim_{t_w\to\infty} R_\Delta(t+t_w,t_w)}_{0}$ $\quad$ (WLTM)

$$\boxed{R_F(t) = R_s(t) = -\frac{1}{T}\frac{dC_s(t)}{dt} = \frac{-1}{T}\frac{dC_F(t)}{dt}}$$

$$\boxed{\lim_{t\to\infty} R_s(t) = 0.}$$

NO FDT

$$\lim_{\substack{t,t_w\to\infty \\ /C<q_{EA}}} R = \lim_{\substack{t,t_w\to\infty/ \\ C<q_{EA}}} R_\Delta = \frac{x}{T}\left.\frac{\partial C_\Delta(t+t_w,t')}{\partial t'}\right|_{t'=t_w}$$

SLOW DYNAMICS & WLTM

BOTH R_Δ AND $\frac{\partial C_\Delta}{\partial t'} \longrightarrow 0$

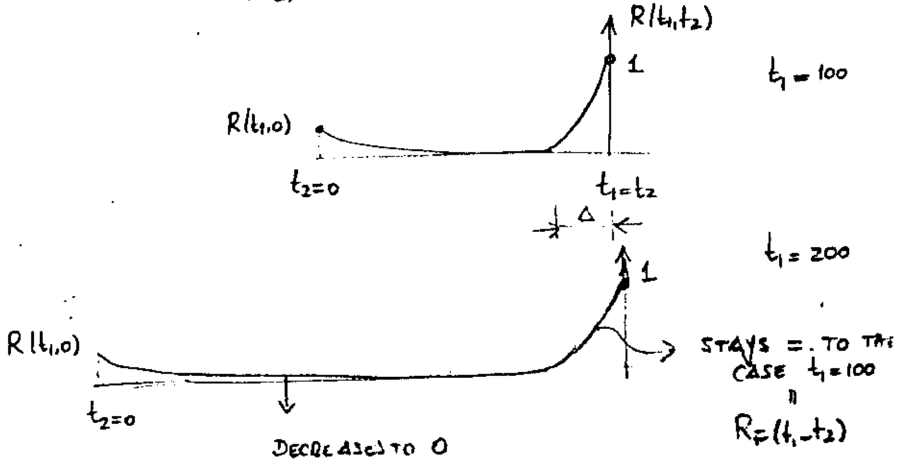
BUT $R_\Delta \Big/ \frac{\partial C_\Delta}{\partial t'} \longrightarrow x/T \neq 0$

$\longrightarrow$ $\boxed{\text{REM}}$ FORMAL DEMONSTRATION & INEQUALITIES IN LII & LIII!

WEAK LONG-TERM MEMORY

EVEN IF $R_A(t_1,t_2) \xrightarrow[t_2\to\infty]{} 0$, ITS INTEGRAL OVER A LONG INTERVAL IS FINITE!

LET'S LOOK AT $R(t_1,t_2)$ FOR t_1 FIXED, AS A FUNCT OF t_1-t_2.



BUT, WHEN WE LOOKED AT $m(C)$ vs C:

- $$\lim_{t_w\to\infty} \int_0^{t^*} ds\, R(t+t_w,s) = 0 \qquad \forall\, t^* \text{ fixed (ie finite wrt } t_w)$$
- $$\lim_{\substack{t,t_w\to\infty \\ C(t,t_w)=C<q_{EA}}} \int_{t_w}^{t+t_w} ds\, R(t+t_w,s) > \frac{1-q_{EA}}{T}$$

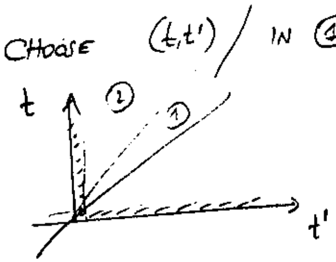
THIS IS IMPORTANT! EVEN IF THE SYSTEM HAS MEMORY OF THE PAST, IT KEEPS IT ONLY IN AN INTEGRATED WAY.
IT MAKES THE PROBLEM DIFFICULT BUT SOLVABLE.

BELOW T_d

TACTICS: ASSUME THERE ARE $\begin{cases}\text{FDT-REGION} \\ \text{AGING} \;\;''\end{cases}$, WRITE THE EQS FOR EACH OF THEM & TRY TO SOLVE THEM (GUIDED BY THE NUMERICAL SOL)

EXPLICITLY. FDT-REGION

CHOOSE (t,t') IN ①, ie $C(t,t') = C_F(t-t') > q_{EA}$.



$\tau = t-t'$

R_F-EQ

$$\frac{d}{d\tau} R_F(\tau) = -z_\infty R_F(\tau) + \frac{p(p-1)}{2} \underbrace{\int_0^\tau d\tau'\, R_F(\tau-\tau')\, C_F^{p-2}(\tau-\tau')\, R_F(\tau')}$$

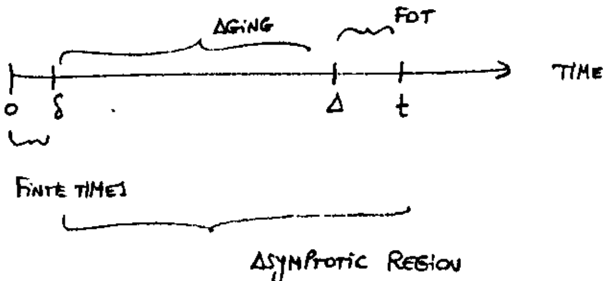
R_F-EQ

Easy: all times fall between (t',t) that are assumed to fall in ①.

NB: z_∞ DISCUSSED JUST BELOW

$\boxed{Z\text{-EQ}}$. IT'S A ONE-TIME QUANTITY : WE ASSUME IT GOES TO A CT.

$$z(t) = T + \frac{p^2}{2}\int_0^t ds\, C^{p-1}(t,s)\, R(t,s)$$



$$\lim_{t\to\infty} z(t) = T + \frac{p^2}{2}\left(\overset{0\ (\text{WLTM})}{\int_0^\delta} + \int_\delta^\Delta + \int_\Delta^t\right)$$

$$\simeq T + \frac{p^2}{2}\lim_{t\to\infty}\left[\int_{0^+}^{\Delta} ds\, \underbrace{R_\Delta(t,s)}_{\Downarrow\ \frac{X}{T}\frac{\partial C_\Delta(t,s)}{\partial s}}\, C_\Delta^{p-1}(t,s) + \int_\Delta^t ds\, \underbrace{R_F(t-s)}_{\Downarrow\ \frac{1}{T}\frac{dC_F(t-s)}{ds}}\, C_F^{p-1}(t-s)\right.$$

$$= T + \frac{p^2}{2}\left[\frac{X}{T}\lim_{t\to\infty}\int_{C_\Delta(t,0^+)\to 0}^{C_\Delta(t,\Delta)\to q_{EA}} dC'_\Delta\, {C'_\Delta}^{p-1} + \right.$$

$$\left. + \frac{1}{T}\lim_{t\to\infty}\int_{C_F(t-\Delta)=q_{EA}}^{C_F(0)=1} dC'_F\, {C'_F}^{p-1}\right]$$

$$= T + \frac{p^{\cancel{2}}}{2}\left[\frac{X}{T\cancel{p}}\, q_{EA}^p + \frac{1}{T\cancel{p}}\left(1-q_{EA}^p\right)\right]$$

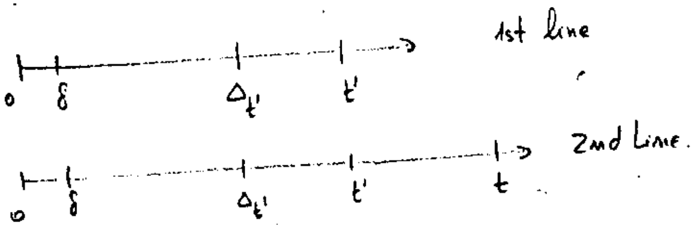
$$\boxed{z_\infty = T + \frac{p}{2}\left[\left(\frac{X-1}{T}\right) q_{EA}^p + \frac{1}{T}\right]} \quad \boxed{z_\infty\text{-EQ}}$$

$\boxed{C_F\text{-EQ}}$ MORE DIFFICULT.

$$\frac{d}{d\tau}C_F(\tau) = -z_\infty C_F(\tau) + \underbrace{\frac{p}{2}\,\text{Int}_{1} + \frac{p(p-1)}{2}\,\text{Int}_{2}}$$

$$\frac{p}{2}\left[\left(\cancel{\int_0^\delta} + \int_\delta^{\Delta_{t'}} + \int_{\Delta_{t'}}^{t'}\right) C^{p-1}(t,s)\, R(t',s)\right] +$$

$$+ \frac{p(p-1)}{2}\left[\left(\cancel{\int_0^\delta} + \int_\delta^{\Delta} + \int_\Delta^{t'} + \int_{t'}^{t}\right) R(t,s)\, C^{p-2}(t,s)\, C(s,t')\right]$$



$$= \frac{p}{2}\left[\int_{0^+}^{\Delta_{t'}} ds\, C_\Delta^{p-1}(t,s)\, R_\Delta(t',s) + \int_{\Delta_{t'}}^{t'} ds\, C_F^{p-1}(t-s)\, R_F(t'-s)\right]$$

$$+ \frac{p(p-1)}{2}\left[\int_{0^+}^{\Delta_{t'}} ds\, R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, C_\Delta(t',s) + \right.$$

$$+ \int_{\Delta_{t'}}^{t'} ds\, R_F(t-s)\, C_F^{p-2}(t-s)\, C_F(t'-s) +$$

$$\left. + \int_{t'}^{t} ds\, R_F(t-s)\, C_F^{p-2}(t-s)\, C_F(s-t')\right]$$

$$= \underbrace{\frac{p\,x}{2T}\int_0^{\Delta} ds\, C_\Delta^{p-1}(t,s)\,\frac{\partial C_\Delta(t',s)}{\partial s}}_{①} + \underbrace{\frac{p}{2T}\int_\Delta^{t'} ds\, C_F^{p-1}(t-s)\,\frac{\partial C_F(t'-s)}{\partial s}}_{②}$$

$$+ \underbrace{\frac{p(p-1)}{2T}\,x\int_{0^+}^{\Delta} ds\,\frac{\partial C_\Delta(t,s)}{\partial s}\,C_\Delta^{p-2}(t,s)\,C_\Delta(t',s)}_{①} +$$

$$+ \underbrace{\frac{p(p-1)}{2T}\int_\Delta^{t'} ds\,\frac{\partial C_F(t-s)}{\partial s}\,C_F^{p-2}(t-s)\,C_F(t'-s)}_{②} +$$

$$+ \frac{p(p-1)}{2T}\int_{t'}^{t} ds\,\frac{\partial C_F(t-s)}{\partial s}\,C_F^{p-2}(t-s)\,C_F(s-t')$$

$$= \underbrace{\frac{p\,x}{2T}\int_{0^+}^{\Delta} ds\,\frac{\partial}{\partial s}\left(C_\Delta^{p-1}(t,s)\,C_\Delta(t',s)\right)}_{①} + \underbrace{\frac{p}{2T}\int_\Delta ds\,\frac{\partial}{\partial s}\left(C_F^{p-1}(t-s)\,C_F(t'\right.}_{②}$$

$$+ \frac{p(p-1)}{2T}\int_{t'}^{t} ds\,\frac{\partial C_F(t-s)}{\partial s}\,C_F^{p-2}(t-s)\,C_F(s-t')$$

$$= \frac{p\,x}{2T}\left[q_{EA}^{p-1}\;q_{EA}\right] + \frac{p}{2T}\;C_F^{p-1}(t-t') - q_{EA}^{p-1}\;q_{EA} +$$

$$+ \frac{p}{2T}\int_{t'}^{t} ds\,\frac{\partial C_F^{p-1}(t-s)}{\partial s}\,C_F(s-t')$$

$$= \frac{p}{2T}\,(x-1)\,q_{EA}^{p} + \frac{p}{2T}\,C_F^{p-1}(t-t') + \frac{p}{2T}\int_{t'}^{t} ds\,\frac{\partial C_F^{p-1}(t-s)}{\partial s}\,C_F(s-t')$$

THUS,

$$\boxed{\frac{dC_F(\tau)}{d\tau} = -z_\infty\,C_F(\tau) - \frac{p}{2T}\,(1-x)\,q_{EA}^{p} + \frac{p}{2T}\,C_F^{p-1}(\tau) + \frac{p(p-1)}{2}\int_0^{\tau} du\;R_F(u)\,C_F^{p-2}(u)\,C_F(\tau-u)}$$

$\boxed{C_F - EQ}$

EXERCISE:

- CHECK $-\frac{1}{T}\frac{d}{d\tau}\left(Eq - C_F\right) \Rightarrow Eq - R_F.$ $\boxed{\text{OK WITH FDT}}$
- IF $T > T_d$ $\quad x = 1$ AND ONE RECOVERS THE HIGH-T EQS.

THESE EQS CONTAIN INFO ABOUT THE AGING REGIME THROUGH z_∞ !

TO SOLVE THIS, LET US FIRST LOOK AT THE AGING REGIME & THEN COME BACK TO $(R_F - EQ) - (C_F - EQ)$.

AGING EQUATIONS

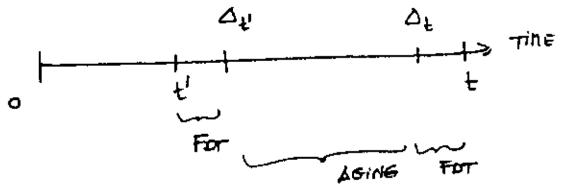
LET'S CHOOSE $(t,t') \,/\, C < q_{EA}$.

$$\frac{\partial C_\Delta(t,t')}{\partial t} = -z_\infty C_\Delta(t,t') + \frac{p}{2}\, \mathrm{INT}_{①} + \frac{p(p-1)}{2}\, \mathrm{INT}_{②}$$

$$\frac{\partial R_\Delta(t,t')}{\partial t} = -z_\infty R_\Delta(t,t') + \frac{p(p-1)}{2}\, \mathrm{INT}_{③}$$

$\boxed{R_\Delta - EQ}$

SINCE (t,t') ARE FAR AWAY, EVEN TO $\mathrm{INT}_{③}$ THE AGING PART CONTRIBUTES

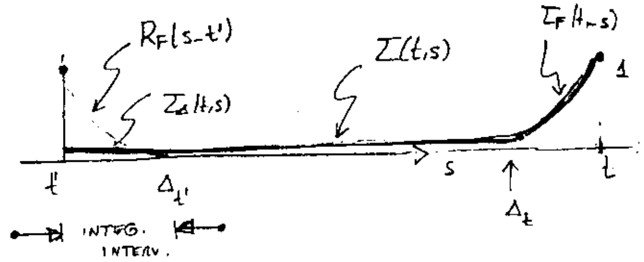


$$\mathrm{INT}_{③} = \frac{p(p-1)}{2} \int_{t'}^{t} ds\; R(t,s)\, C^{p-2}(t,s)\, R(s,t')$$

$$= \frac{p(p-1)}{2} \left\{ \int_{t'}^{\Delta_{t'}} ds\; R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, \underbrace{R_F(s-t')}_{\doteq\, \frac{1}{T}\frac{\partial C_F(s-t')}{\partial t'}} + \int_{\Delta_{t'}}^{\Delta_t} ds\; R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, R_\Delta(s,t') + \int_{\Delta_t}^{t} ds\; \underbrace{R_F(t-s)}_{\doteq\, \frac{1}{T}\frac{\partial C_F(t-s)}{\partial s}}\, C_F^{p-2}(t-s)\, R_\Delta(s,t') \right\}$$

FIRST TERM

$$\boxed{\Sigma(t,s) \equiv R(t,s)\, C^{p-2}(t,s)}$$



SO, WE ARE CONVOLUTING A VERY 'PEAKED' FUNCTION $R_F(s-t')$ WITH A VERY SLOWLY VARYING FUNCTION $\Sigma_\Delta(t,s)$. THUS, WE'LL TAKE $R_F(s-t')$ TO ACT ALMOST AS A 'δ- FUNCTION WITH A WIDTH':

$$\int_{t'}^{\Delta_{t'}} ds\; R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, R_F(s-t') \simeq$$

$$R_\Delta(t,t')\, C_\Delta^{p-2}(t,t') \int_{t'}^{\Delta_{t'}} ds\; \underbrace{R_F(s-t')}_{\doteq\, \frac{1}{T}\frac{\partial C_F(s-t')}{\partial t'}}$$

$$= R_\Delta(t,t')\, C_\Delta^{p-2}(t,t') \left(\frac{-1}{T}\right) \left(C_F(\Delta_{t'} - t') - \cancel{C_F}(0) \right) \quad (1)$$

$$= -\frac{1}{T} R_\Delta(t,t')\, C_\Delta^{p-2}(t,t')\, (q_{EA} - 1)$$

THIRD TERM

SAME PROCEDURE $\Rightarrow$

$$\int_{\Delta_t}^{t} ds\; R_F(t-s)\, C_F^{p-2}(t-s)\, R_\Delta(s,t') \sim R_\Delta(t,t') \int_{\Delta_t}^{t} ds\; R_F(t-s)\, C_F^{p-2}(t-s)$$

$$= R_\Delta(t,t') \left(\frac{+1}{T}\right) \frac{1}{p-1} \int_{\Delta_t}^{t} ds\; \frac{\partial C_F^{p-1}(t-s)}{\partial s}$$

$$= +\frac{1}{T(p-1)} R_\Delta(t,t') \left[1 - q_{EA}^{p-1} \right]$$

2ND TERM

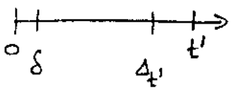
$INT_{(2)}$ = LEAVE IT LIKE IT IS.

FINALLY $\boxed{R_\Delta\text{-EQ}}$.

$$\boxed{\begin{aligned}\frac{\partial R_\Delta(t,t')}{\partial t} = & -Z_\infty R_\Delta(t,t') + \frac{p(p-1)}{2}\left(\frac{1-q_{EA}}{T}\right) R_\Delta(t,t')\, C_\Delta^{p-2}(t,t') \\ & + \frac{p}{2T} R_\Delta(t,t') \left(1-q_{EA}^{p-1}\right) + \\ & + \frac{p(p-1)}{2}\int_{\Delta_{t'}\to t'}^{\Delta_t\to t} ds\; R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, R_\Delta(s,t')\end{aligned}}$$

$\boxed{C_\Delta\text{-EQ}}$.

$$INT_{(1)} = \frac{p}{2}\int_0^{t'} ds\; C^{p-1}(t,s)\, R(t',s)$$



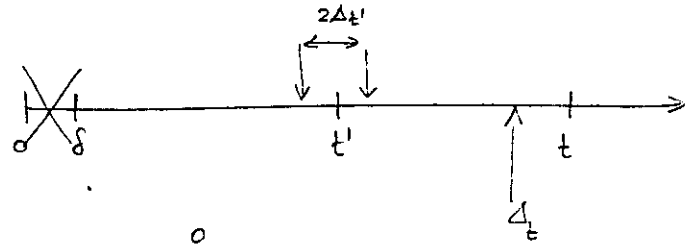
$$= \frac{p}{2}\left\{\left(\cancel{\int_0^\delta} + \int_\delta^{\Delta_{t'}} + \int_{\Delta_{t'}}^{t'}\right) C^{p-1}(t,s)\, R(t',s)\right\}$$

$$= \frac{p}{2}\left\{\int_{0^+}^{\Delta_{t'}} ds\; C_\Delta^{p-1}(t,s)\, R_\Delta(t',s) + \int_{\Delta_{t'}}^{t'} ds\; C_\Delta^{p-1}(t-s)\, R_F(t'-s)\right\}$$

$$= \frac{p}{2}\left\{\int_0^{\Delta_{t'}\to t'} ds\; C_\Delta^{p-1}(t,s)\, R_\Delta(t',s) + C_\Delta^{p-1}(t,t')\int_{\Delta_{t'}}^{t'} ds\; R_F(t'-s)\right\}$$

$$= \frac{p}{2}\int_0^{t'} ds\; C_\Delta^{p-1}(t,s)\, R_\Delta(t',s) + C_\Delta^{p-1}(t,t')\,\frac{p}{2T}\left(1-q_{EA}\right)$$

$$INT_{(2)} = \frac{p(p-1)}{2}\int_0^t ds\; R(t,s)\, C^{p-2}(t,s)\, C(t',s)$$



$$= \frac{p(p-1)}{2}\left\{\cancel{\int_0^\delta} + \int_\delta^{t'-\Delta_{t'}} + \int_{t'-\Delta_{t'}}^{t'+\Delta_{t'}} + \int_{t'+\Delta_{t'}}^{\Delta_t} + \int_{\Delta_t}^{t}\right\}.$$

$$\begin{aligned} = \frac{p(p-1)}{2}\Bigg\{ & \int_{0^+}^{t'-\Delta_{t'}} ds\; R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, C_\Delta(t',s) + \\ & + \int_{t'-\Delta_{t'}}^{t'+\Delta_{t'}} ds\; R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, C_F(t'-s) + \\ & + \int_{t'+\Delta_{t'}}^{\Delta_t} ds\; R_F(t,s)\, C_\Delta^{p-2}(t,s)\, C_\Delta(s,t') + \\ & + \int_{\Delta_t}^{t} ds\; R_F(t-s)\, C_F^{p-2}(t-s)\, C_\Delta(s,t')\Bigg\}.\end{aligned}$$

$\cancel{C_s(t'-s)} + C_\Delta(t'$

$$\begin{aligned}\simeq \frac{p(p-1)}{2}\Bigg\{ & \int_0^{\Delta_t} ds\; R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, C_\Delta(t',s) + \\ & + C_\Delta(t,t')\int_{\Delta_t}^{t} ds\; \frac{1}{T}\frac{\partial C_F(t-s)}{\partial s}\, C_F^{p-2}(t-s)\Bigg\}\end{aligned}$$

$$= \frac{p(p-1)}{2}\int_0^{\Delta_t\to t} ds\; R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, C_\Delta(t',s) + \frac{p}{2T}\, C_\Delta(t,t')\left[1-q_{EA}^{p-1}\right]$$

FINALLY, THE $\boxed{C_\Delta - EQ}$ READS:

$$\boxed{\begin{aligned} \frac{\partial C_\Delta(t,t')}{\partial t} = & - z_\infty C_\Delta(t,t') + \frac{p}{2T}(1-q_{EA})\, C_\Delta^{p-1}(t,t') + \\ & + \frac{p}{2T} C_\Delta(t,t') \left[1 - q_{EA}^{p-1}\right] + \\ & + \frac{p}{2}\int_0^{\Delta t \to t'} ds\, C_\Delta^{p-1}(t,s)\, R_\Delta(t',s) + \\ & + \frac{p(p-1)}{2}\int_0^{\Delta t \to t} ds\, R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, C_\Delta(t',s) \end{aligned}}$$

CHECK: (EXERCISE) IF $\frac{\partial C_\Delta(t,t')}{\partial t} \ll rhs_C$ AND THROW THEM

$\frac{\partial R_\Delta(t,t')}{\partial t} \ll rhs_R$

THEN $R_\Delta(t,t') = \frac{X}{T}\frac{\partial C_\Delta(t,t')}{\partial t'}$ REDUCES THE SET OF

EQS (R_Δ - EQ) – (C_Δ - EQ) TO A SINGLE ONE FOR C_Δ.

WE NOW HAVE 4-EQS (FOR C_F, R_F, C_Δ, R_Δ) WITH

THE MATCHING CONDITIONS $C_F \xrightarrow[t-t'\to\infty]{} q_{EA}$, ETC.

LET'S TRY TO FIND A SOL

- WE START BY ANALYSING THE R_Δ-EQ

LET'S ASSUME $\frac{\partial R_\Delta(t,t')}{\partial t} \ll rhs_R$ (TO BE VERIFIED A POSTERIORI BY THE SOL) & THROW lhs_R VS rhs_R:

ALSO: REM THE NUMERICAL SOL $\Rightarrow$ THE 'VELOCITY' BECOMES SMALLER & SMALLER.

$$\boxed{\begin{aligned} 0 \simeq & - z_\infty R_\Delta(t,t') + \frac{p(p-1)}{2}\left(\frac{1-q_{EA}}{T}\right) R_\Delta(t,t')\, C_\Delta^{p-2}(t,t') + \\ & + \frac{p}{2T} R_\Delta(t,t')\left(1 - q_{EA}^{p-1}\right) + \\ & + \frac{p(p-1)}{2}\int_{t'}^{t} ds.\, R_\Delta(t,s)\, C_\Delta^{p-2}(t,s)\, R_\Delta(s,t') \end{aligned}}$$

CONSIDER NOW $\boxed{\lim_{t'\to t}}$ REM $\lim_{t'\to t} C_\Delta(t,t') = q_{EA}$

$$0 \simeq \Big[- z_\infty + \underbrace{\frac{p(p-1)}{2}\left(\frac{1-q_{EA}}{T}\right) q_{EA}^{p-2} + \frac{p}{2T}\left(1 - q_{EA}^{p-1}\right)}\Big] R_\Delta(t,t')$$

$$\Downarrow$$

$$\overbrace{\frac{p}{2T}\left((p-1)\, q_{EA}^{p-2} - (p-1)\, q_{EA}^{p-1} + 1 - \cancel{q_{EA}^{p-1}}\right)}$$

$$\frac{p}{2T}\left((p-1)\, q_{EA}^{p-2} - p\, q_{EA}^{p-1} + 1\right)$$

TWO SOLUTIONS = $\begin{cases} R_\Delta = 0 & \longrightarrow \quad T > T_d \\ R_\Delta \neq 0 \quad \& \quad [\quad] = 0. & \longrightarrow \quad T < T_d. \end{cases}$

THE BRACKET $= 0 \implies$ AN EQ. FOR $\boxed{(q_{EA}, X, T)}$.

ONE CAN DO THE SAME TRICK IN THE $\boxed{C_A\text{-EQ}}$

$\boxed{\lim_{t' \to t^-} C_A\text{-EQ}} \implies$ ANOTHER EQ FOR $\boxed{q_{EA}, X, T}$

COMPATIBILITY $\implies$

$$\boxed{\begin{array}{l} \dfrac{X}{T} = \dfrac{(p-2)(1-q_{EA})}{q_{EA}\, T} \\[2ex] \dfrac{p(p-1)}{2}\, q_{EA}^{\,p-2}\,(1-q_{EA})^2 = T^2 \end{array}} \begin{array}{l} \to \boxed{\text{EQ}-X} \\[2ex] \to \boxed{\text{EQ}-q_{EA}} \end{array}$$

(NB $\begin{cases} X = 0 & p = 2 \\ q_{EA} = 1 - T & p \end{cases}$)

$\Downarrow$

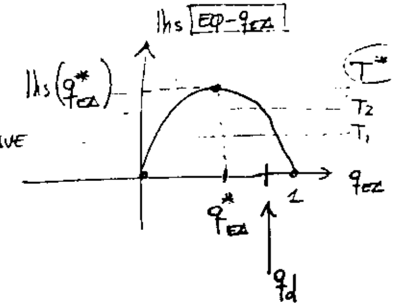
OK, CAN BE OBTAINED SOLVING EXACTLY THE LANG-EQ

AND X IS $\boxed{0 < X \leq 1}$! (AS A FUNCTION OF T)

TRANSITION:

LET'S LOOK AT $\boxed{\text{EQ}-q_{EA}}$ $\boxed{p > 2}$

ONE WAY OF TRYING TO FIND A TRANS. IS TO STUDY WHEN $\boxed{\text{EQ}\ q_{EA}}$ DOES NOT HAVE A SOL.



$$q^*_{EA} = \frac{p-2}{p}$$

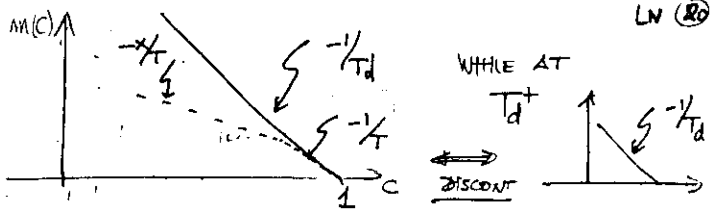
$$\text{lhs}\,(q^*_{EA}) = \frac{2(p-1)}{p}\left(\frac{p-2}{p}\right)^{p-2} = T^2 \longrightarrow T_d.$$

BUT $\dfrac{X(q^*_{EA})}{q_{EA}\, T} = \dfrac{2(p-2)}{\frac{p-2}{\cancel{p}}\,\cancel{p}}\,\dfrac{1}{T} > \dfrac{1}{T}$!

$\longrightarrow$ WE DON'T LIKE $X > 1$. (UNPROVEN, BUT WE THINK X MUST BE ≤ 1) [FORMAL PROOF? .

THUS, THERE'S A TRANSITION BEFORE ARRIVING AT q^*_{EA}, AT A GIVEN q_d, $q^*_{EA} < q_d < 1$, AT WHICH $X = 1$.

---- FOR $T < T_d$
—— FOR T_d



(COULD BE SEEN IN AN EXP...)

FINALLY SOLUTION. WE HAVE NOW (X, q_{EA}). LOOKING AT $\boxed{C_A\text{-EQ}}$ – $\boxed{R_A\text{-EQ}}$. ONE SEES THAT:

$$\boxed{C_A(t,t') = f\left(\frac{h(t')}{h(t)}\right)}$$

IS A SOLUTION. $\forall\ h(t)$. f IS DETERMINED BY THE APPROX. EQ (WHERE WE'VE THROWN $\dfrac{\partial C_A(t,t')}{\partial t}$ wrt rhs)

BUT $h(t)$ IS INDETERMINED.

$\Downarrow$

$\boxed{\text{MATCHING PROBLEM}}$

$\Downarrow$

OPEN.

WHAT DOES

$$\boxed{\begin{aligned} R_A(t,t') &= \frac{X}{T}\,\frac{\partial C_A(t,t')}{\partial t'} \\ \text{WITH }\; C_A(t,t') &= f\left(\frac{h(t')}{h(t)}\right) \end{aligned}}$$

IMPLY FOR THE EXPERIMENTAL MEASUREMENTS?

A SIMPLE CALCULATION OF m^{TRM} OR m^{ZFC}

YIELDS

$\boxed{\text{NON-TRIVIAL AGING}}$.

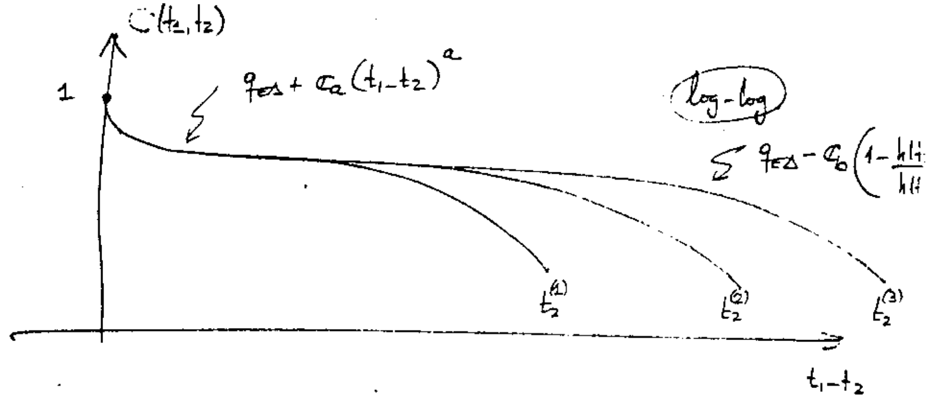
IF $\boxed{h(t) = \exp \ln^k(t/t_0)}$ $\Longrightarrow$

BACK TO LI & LAST FIGURE.

ACTUALLY, FOR THIS MODEL $h(t)$ IS MOST PROBABLY A POWER LAW (FROM NUMERICS). BUT ONE CAN CONSTRUCT OTHERS WHICH COULD LEAD TO

$$h(t) = \exp \ln^k(t/t_0)$$

SUMMARY $\quad T < T_d \quad$ FOR THIS FAMILY OF MODELS



a & b ARE THE LOW-T COUSINS OF (a,b) IN $T > T_d$ (REM MCT). BELOW T_d THEY DEPEND UPON T