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SUMMER COLLEGE IN CONDENSED MATTER ON
" STATISTICAL PHYSICS OF FRUSTRATED SYSTEMS "

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# " Problems on quantum phase transitions with disorder"

presented by:

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# Problems on Quantum Phase Transitions With Disorder 

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1/ Use the Mean field theory described in the lectures to determine $\langle \sigma^x \rangle$ for the Ising model in a transverse field both in the paramagnetic phase and the ferromagnetic phase.

n.b. Your answer should show that $\langle \sigma^x \rangle$ is independent of $T$ in the paramagnetic phase.

2/ Series expansion for the one-dimensional, non-random transverse field Ising model.

$$\mathcal{H} = \underbrace{-J \sum_i \sigma_i^z \sigma_{i+1}^z}_{H_0} \underbrace{- h \sum_i \sigma_i^x}_{V}$$

The ground state of $H_0$ is $\sigma_i^z = 1$ for all $i$. By considering $V$ as a perturbation show that $m \equiv \langle \sigma_i^z \rangle$ at $T=0$ is given by

$$m = 1 - \frac{1}{8}\left(\frac{h}{J}\right)^2 + O\left(\frac{h}{J}\right)^4 .$$

n.b. This is actually the first two terms of the exact result $m = \left[1 - \left(\frac{h}{J}\right)^2\right]^{1/8}$, i.e. the magnetization vanishes at $h_c = J$ with an exponent $\beta = 1/8$ compared with the mean field value, $\beta = 1/2$.

3/ A simple example of the Suzuki-Trotter formula

(a) Show that $\mathrm{Tr}\left\{\exp\left(h_x \sigma_x + h_z \sigma_z\right)\right\}$

$$= 2\cosh\left(\sqrt{h_x^2 + h_z^2}\right)$$

by working in a basis where $\vec{\sigma}$ is quantized along the direction of $\vec{h} = (h_x, 0, h_z)$

(b) Derive the same result using the Suzuki–Trotter formula

$$\mathrm{Tr} \lim_{m\to\infty} \left[ \exp\left(\frac{h_x}{m}\sigma_x\right) \exp\left(\frac{h_z}{m}\sigma_z\right) \right]^m$$

by showing that this can be written

$$\mathrm{Tr} \lim_{m\to\infty} \Big[ \underbrace{1 + \frac{h_x}{m}\sigma_x + \frac{h_z}{m}\sigma_z}_{A} \Big]^m$$

and finding the eigenvalues of A.

4. In a Monte Carlo simulation of a classical Ising model we flip a spin with a probability $P(\Delta E)$, where $\Delta E$ is the energy to flip the spin. Show that the following choices for $P(\Delta E)$ satisfy the detailed balance condition, discussed in the lectures:

(a) Metropolis

$$P(\Delta E) = \begin{cases} e^{-\Delta E} & \text{if } \Delta E \geq 0 \\ 1 & \text{if } \Delta E < 0 \end{cases}$$

(b) Heat Bath.

$$P(\Delta E) = \frac{1}{e^{\Delta E} + 1}$$

n.b. We are taking the Boltzmann factor to be $\exp(-E)$ [No factor of $\frac{1}{k_B T}$ since this does not occur in quantum Monte Carlo]

# 5 Jordan–Wigner Transformation

(a) The Pauli spin matrices are

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You are given that On different sites, Pauli spin matrices commute
Show that, on the same site, Pauli spin matrices anticommute i.e.

$$\{\sigma_i^\alpha, \sigma_i^\beta\} = 0$$

for $\alpha \neq \beta$, where $\alpha = x, y$ or $z$ and $\{A, B\} \equiv AB + BA$

(b) Show that if

$$\left.\begin{aligned} \sigma_i^x &= a_i^\dagger + a_i \\ \sigma_i^y &= i(a_i^\dagger - a_i) \\ \sigma_i^z &= 1 - 2a_i^\dagger a_i \end{aligned}\right\}$$

where $a_i^\dagger$ and $a_i$ are creation and destruction operators of spinless fermions

then the anticommutation relations of the $\sigma_i^\alpha$, discussed in part (a) are reproduced.

(c) Show that if instead of the $a_i^\dagger$ and $a_i$ being simple Fermi operators they are given by

$$a_i^\dagger = c_i^\dagger \exp\left[-i\pi \sum_{j=1}^{i-1} c_j^\dagger c_j\right]$$

$$a_i = \exp\left[-i\pi \sum_{j=1}^{i-1} c_j^\dagger c_j\right] c_i$$

where now it is the $c_i^\dagger$ and $c_i$ which are Fermi operators, then not only are the anticommutation relations of the $\sigma_i^\alpha$ on the same site satisfied but also the commutation relations on different sites

## 6/ Wick's Theorem

Consider non-interacting Fermions in the grand canonical ensemble, and let $A, B, C$ and $D$ be Fermi operators. Then, according to Wick's theorem

$$\langle ABCD \rangle = \langle AB \rangle \langle CD \rangle - \langle AC \rangle \langle BD \rangle + \langle AD \rangle \langle BC \rangle$$

(You should understand the signs)

(a) Without using Wick's theorem explain why

$$\langle c^+ c \, c^+ c \rangle = n = \frac{1}{e^{\beta\epsilon}+1}$$

(n.b. it is $n$, not $n^2$)

(b) Derive the same result using Wick's theorem

(c) (Optional) Show that $\langle (c^+ c)^3 \rangle = n$ using Wick's theorem.

## 7/ Pfaffian

Verify that

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix} = \left(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\right)^2.$$

8/ Consider the simple phenomenological model for slow dynamics, discussed in the appendix to lecture 4, i.e.

$$S(\tau) = A \sum_{n=0}^{\infty} \exp\left(-\frac{n}{\xi_t} + \epsilon_n \tau\right)$$

where $A$ is a constant ensuring $S(0)=1$, and the $\epsilon_n$ are distributed with a power law

$$\pi(\epsilon_n) \propto \epsilon_n^{\lambda}$$

(a) Show that the average correlation function varies as

$$[S(\tau)]_{av} \propto \frac{1}{\tau^{\lambda+1}} \qquad \text{i.e. a power law.}$$

(b) You are given that the distribution of $y = -\ln S(\tau)$ is

$$P(y) \propto \left(\frac{y}{\tau}\right)^{\lambda+1} \exp\left(- \text{const.} \frac{y^{\lambda+2}}{\tau^{\lambda+1}}\right) \qquad (1)$$

Showing that the scaling variable is $x = \frac{y}{\tau^{(\lambda+1)/(\lambda+2)}}$ *

(i.e. $\hat{P}(x)$ only depends on $x$). This is derived in the lecture notes.

(c) Explain why the average is determined only by the extreme limit of the distribution for $y$, near $y=0$, and that the distribution does ~~also~~ given an average varying with a power law, $\tau^{-(\lambda+1)}$, even though the typical value is a stretched exponential.

* i.e. the typical correlation function satisfies

$$S_{typ} \propto \exp\left[- \text{const.}\ \tau^{1/\mu}\right] \qquad \text{(stretched exponential)}$$

where $\mu = \frac{\lambda+2}{\lambda+1} = 1+z$ since $\lambda+1 = \frac{1}{z}$