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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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SUMMER COLLEGE IN CONDENSED MATTER ON
" STATISTICAL PHYSICS OF FRUSTRATED SYSTEMS "

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# " Phenomenology of disordered systems "

presented by:

**J.P. BOUCHAUD**
C.E.A.
S.P.E.C. - Service de Physique de l'Etat Condense
Orme des Merisiers
F-91191 Gif sur Yvette
France

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Introduction : simplest example of a disordered sample : point particle in a random pot.

## (1) Random kicks.

$x_{i+1} - x_i = \eta_i$ $\qquad t = i\tau_0 \quad \tau_0$ Correlation time

"Thermal bath" – All that you <u>don't</u> know!

$x_N = \sum_{i=1}^{N} \eta_i$ $\qquad \rho(\eta)$ time <u>independent</u> (cf finance)

$P_N(x_N)$ ? $\quad (N \to \infty)$ : CLT.

$P_2(x_2) = \int d\eta_2\, f(\eta_2)\, f(x_2 - \eta_2)$ : <u>CONVOLUTION</u>

$\hat{P}(k) = \int dx\, e^{ikx} P(x)$

$\hat{P}_N(k) = [\rho(k)]^N$

a) $\langle \eta \rangle = 0$ $\qquad$ Rq Notation $\langle \dots \rangle_\eta$

$\rho(k) \underset{k\to 0}{=} 1 - \frac{k^2\sigma^2}{2} + \dots \quad \sigma^2 = \langle \eta^2 \rangle < +\infty.$

$\simeq e^{-k^2\sigma^2/2 + O(k^3)}$

$P_N(x) = \int \frac{dk}{2\pi} e^{-ikx} \begin{cases} \left[1 - \frac{k^2\sigma^2}{2} + O(k^3)\right]^N \\ e^{-\frac{k^2\sigma^2}{2}N + N O(k^3)} + \dots \end{cases}$

$\left.\begin{array}{l} x = u\sqrt{N} \\ k = q/\sqrt{N} \end{array}\right\} \to \int \frac{dq}{2\pi\sqrt{N}} e^{-iqu - \frac{q^2\sigma^2}{2} + \frac{1}{\sqrt{N}} O(q^3)} + \dots$

$P_N(x) \simeq \frac{1}{\sqrt{2\pi\sigma^2 N}} e^{-u^2/2\sigma^2} = \frac{1}{\sqrt{N}} P(u)$ ← <u>Gaussian</u>

$P(u)\, du = P_N(x)\, dx$

**Corrections** Hyp: All $\langle \eta^k \rangle < +\infty$.

$$P_>(u) = \int_u^\infty du' \, \rho_N(u')$$

$$\Delta P_>(u) = P_>(u) - P_{>G}(u) = \frac{\exp(-u^2/2)}{\sqrt{2\pi}} \Big[ \overset{k=3}{\overset{\downarrow}{\frac{Q_1(u)}{\sqrt{N}}}} + \overset{k=4}{\overset{\downarrow}{\frac{Q_2(u)}{N}}} + \overset{k=5}{\overset{\downarrow}{\cdots}} \Big]$$

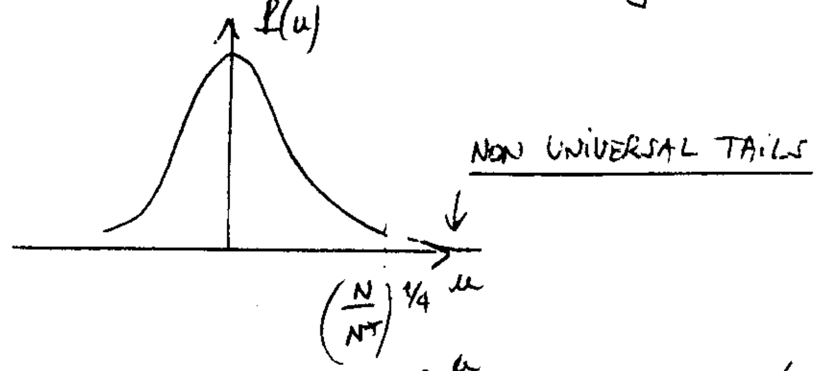
**Ex** $\rho(\eta) = \rho(-\eta)$. $\quad \langle \eta^4 \rangle - 3\langle \eta^2 \rangle^2 = k \, \langle \eta^2 \rangle^2$.

$Q_1(u) = 0 \qquad Q_2(u) = \frac{k}{24}(u^3 - 3u)$

Gaussian is a good approximation when: $(u \sim 1)$: $N \gg N^* = k$

$$\frac{\Delta P_>(u)}{P_{>G}(u)} \ll 1 \quad \text{or } (u \text{ large}): \quad u^4 \ll \frac{N}{N^*}$$

$$\text{or} \quad x \ll \sigma\sqrt{N} \left[\frac{N}{N^*}\right]^{1/4}$$




**Tail Conservation** $\quad \rho(\eta) \simeq \frac{\mu \eta_0^\mu}{\eta^{1+\mu}} \quad \eta \to \infty$ (Note: $\sigma^2 < +\infty \Rightarrow \mu > 2$)

$$\hat{\rho}(k) \underset{k \to 0}{=} 1 + \sum_{n=1}^{E(\mu)} \frac{(ik)^n}{n!} \langle \eta^n \rangle + C_\mu |k|^\mu + \ldots$$

**Example** $\mu < 1$

$$\hat{\rho}(k) = \int dx \, \rho(x) e^{ikx} = \int dx \, \rho(x)(e^{ikx} - 1) + 1$$

③ CONVERGENT

$u = kx$

$$\underset{k\to 0}{\simeq} 1 + \frac{1}{k}\int du\, \rho\left(\frac{u}{k}\right)\left(e^{iu}-1\right) \simeq 1 + \mu k^{\mu} \eta_0^{\mu} \int du \frac{\left(e^{iu}-1\right)}{u^{1+\mu}}$$

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$$\hat{\rho}(k) = e^{\sum_{n=1}^{E(\mu)} \frac{c_n}{n!}(ik)^n + c_\mu |k|^\mu}$$

$\rightarrow [\hat{\rho}(k)]^N$ always has the same non analyticity!

$c_\mu \rightarrow N c_\mu$. (as for all cumulants $c_n$)

$$\Rightarrow P_N(x) \underset{x\to\infty}{\simeq} \frac{\mu N \eta_0^{\mu}}{x^{1+\mu}}.$$

Hence $P_N(x) \xrightarrow{\mu>2}$ Gaussian but $\langle x^k\rangle_{k>\mu} = \infty$.

Width of the Gaussian region : $\sigma\sqrt{N}\,.\,\sqrt{\log N}$

Relative weight in the "tail" : $\dfrac{1}{N^{\frac{\mu}{2}-1}\log^{\frac{\mu}{2}} N}$ NOTE BLOWS UP for $\mu \to 2$!

$\mu < 2$ : LEVY SUMS, see later.

b/m $= \langle\eta\rangle \neq 0$ $\quad x \longrightarrow x - mN$

② The continuous time limit

Gaussian : $N = \dfrac{t}{\tau_0}$ large $\quad \begin{cases} t\to\infty,\ \tau_0 \text{ fixed} & (\text{phys.}) \\ t \text{ fixed},\ \tau_0 \to 0 & (\text{maths}) \end{cases}$

$$\sigma_N^2 = N\sigma^2 = \frac{t}{\tau_0}\sigma^2 \qquad \tau_0 \to 0 \text{ with } \sigma^2 = \sigma_0^2 \tau_0$$

$\longrightarrow \sigma_t^2 = \sigma_0^2 t$ : SCALE INVARIANT PROCESS

$$P(x,t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp - \frac{x^2}{2\sigma_t}$$

Obeys the diffusion equation

$$\frac{\partial P}{\partial t} = D \Delta P \qquad \text{with} \quad 2D = \sigma_0^2$$

large scale equation, valid when the CLT holds

$m = V\tau$ : $\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} V P + D \Delta P$.

Convection term.

$Vt^* \sim \sqrt{Dt^*}$ : diffusion time

$Vt^* = l^+$ : diffusion length $\sim \frac{D}{V}$

Rq Extreme "counter-flow" excursion

$P(\Delta x < 0) \sim e^{-|\Delta x| \frac{V}{D}}$

General result

In the presence of an $x$-dependent mean:

$$\frac{dx}{dt} = \underbrace{\frac{1}{\gamma} F(x)}_{V(x)} + \eta$$

makes sense only if

$F' . \sqrt{D\tau_0} \ll F \Rightarrow \xi \gg \sqrt{D\tau_0}$

Langevin's equation

$\overline{\eta(t)\eta(t')} = 2D\, \delta(t-t')$

## 3 Fokker-Planck equation and thermal equilibrium.

$x$-dependent convection term.

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{1}{\gamma} F(x)\, P\right] + D\Delta P.$$

(i) $F(x) = -\frac{\partial U}{\partial x}$ $\quad \frac{\partial P}{\partial t} = 0 \Rightarrow P = P_{eq}(x) = e^{-\frac{U(x)}{D\gamma}}$

$$\boxed{D\gamma = kT}$$

(ii) $F(x) \neq -\frac{\partial U}{\partial x}$ $\quad P_{st}$ unknown in general.

if $\vec{\nabla} f = 0 \Rightarrow P_{st} = \frac{1}{V}$

## 4 Barrier Crossing and Arrhenius time.

$x_f$

$$\left.\begin{array}{ll} t=0 & x, \hat{x} \sim 0 \\ t=T & x = x_f \end{array}\right\} \qquad \dot{x} = -\frac{\partial U}{\partial x} + \eta$$

$$\gamma = 1 \quad D = T$$

$$P = \int_{x(t=0)=0}^{x(T)=x_f} \mathcal{D}x \, \mathcal{P}\{x(t)\} = \int_{x=0}^{x_f} \mathcal{D}x \int \prod_t d\eta(t) \; e^{-\int_0^T dt \, \eta^2(t)/4D} \, \delta\left(\dot{x} + \frac{\partial U}{\partial x} - \eta\right)$$

$$= \int \mathcal{D}x \; e^{-\int_0^T dt \left[\dot{x} + \frac{\partial U}{\partial x}\right]^2 / 4D}$$

Trajectoire la plus probable $(D \to 0)$ : $-\ddot{x} + U'U'' = 0$

$\dot{x}^2 = U'^2 + C \qquad C \sim 0 \quad (C.I)$

$\dot{x} = \pm U' \qquad - :$ impossible $\qquad + :$

$$P \sim e^{-\int_0^T dt \; 2\dot{x} \, . \, 2U' / 4D} = e^{-[U(x_f)]/D}$$

Generalizations : Higher D objects

Non gaussian noise
e.g. : $P(\eta) \sim e^{-(\eta/\sqrt{T})^\alpha}$ $\qquad P \sim e^{-\frac{x^\alpha}{T^\alpha}}$
$U(x) = \frac{x^2}{2}$

Interpretation : Boltzmann weight incorrect in the tail.

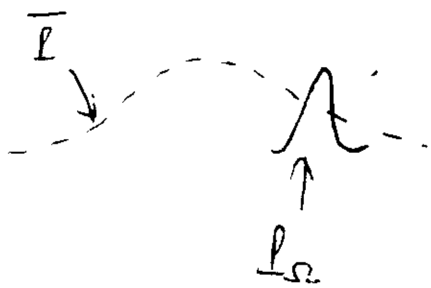
$P(\eta) \neq G(\eta) \to$ Information available
$\Longrightarrow$ Entropy should not be maximized.

(5) Disordered Environments: Two averages.

<u>f(x) : random</u>

Observables : for a given environment $\Omega$ : $\rho_\Omega(x,t)$
Averaged over environments $\overline{\rho}(x,t)$

<u>Structure</u>

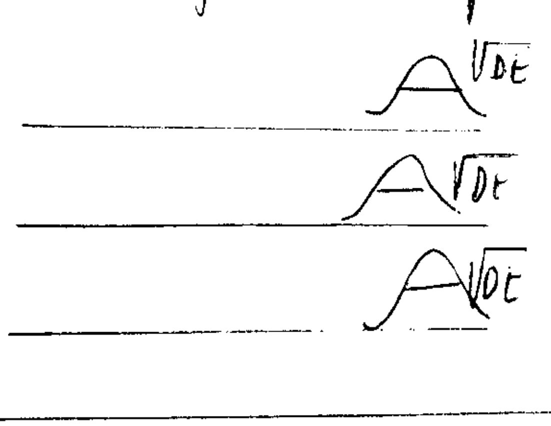



Average diffusion constant (when it exists):

$$2Dt = \overline{\int dx\, x^2 \rho_\Omega(x,t) - \left[\int dx\, x\, \rho_\Omega(x,t)\right]^2}$$

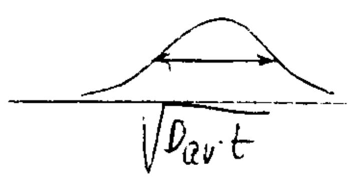
self-averaging when $t \to \infty$

Other definition : spread of the average packet




$$2D_{av}\, t = \int dx\, x^2\, \overline{\rho_\Omega(x,t)} - \left[\int dx\, x\, \overline{\rho_\Omega(x,t)}\right]^2$$

=

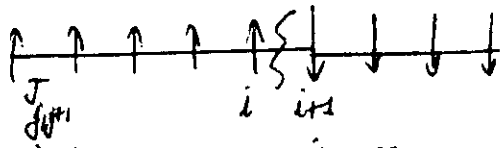



$D_{av} \geq D$.

## 6 Different types of randomness

### Example

Random Bond Ising Model in 1D $\qquad \mathcal{H} = -\sum J_{i,i+1} S_i S_{i+1}$




$$U = -\sum_{j=-\infty}^{i-1} J_{j,j+1} - \sum_{i+1}^{j=+\infty} J_{j,j+1} + J_{i,i+1} = \mathcal{E} + 2J_{i,i+1}$$

J random $\Rightarrow$ U random, short range correlated

$$\overline{U(x)U(y)} = U_0^2 f(x-y) \qquad \overline{\hat{U}(k)\hat{U}(q)} = (2\pi) U_0^2 f(k+q)$$

$$\overline{\hat{f}(k)\hat{f}(q)} = (2\pi) k^2 U_0^2 \delta(k+q)$$

"Blue noise"

Random field Ising Model $\qquad \mathcal{H} = -\sum J S_i S_{i+1} + \sum h_i S_i$

$$U = -\sum_{j=-\infty}^{i} h_j + \sum_{i+1}^{\infty} h_j + J$$

$$\Delta U(i \to i+1) = -F_{i,i+1} = -2h_{i+1}$$

$h_i$ random $\Rightarrow$ f white noise $\Rightarrow$ U random walk.

Other example: kink on a dislocation

$\rightarrow$ Random field.

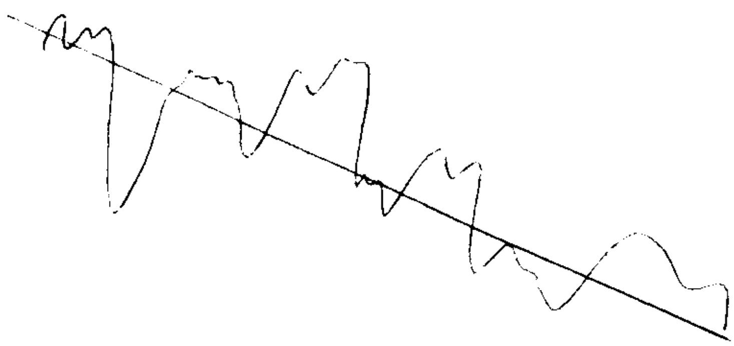
Triple line of a | rough surface / chemically dirty

7 Random forces in 1d : universal features (≠ random V)

$$\langle f \rangle = m$$

$$\langle f(x) f(y) \rangle_c = \sigma^2 \delta(x-y)$$

$$\Rightarrow U(x) = -\int^x f(x')\, dx' \quad : \text{ Random walk}$$



Probability of a large barrier ? "Counterflow" problem above

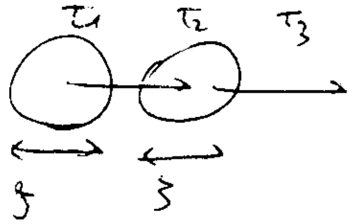
$$P(\Delta U > 0) \hat{=} e^{-\Delta U\, 2m/\sigma^2} \quad (\text{cf: more general})$$

$$\tau = \tau_0 e^{\Delta U/T}$$

$$P(\tau)\, d\tau = P(\Delta U)\, d\Delta U \Rightarrow P(\tau) \sim \frac{\tau_0^{\mu}}{\tau^{1+\mu}}$$

$$\boxed{\mu = \frac{2mT}{\sigma^2}}$$

"Size of a trap". "Diffusion time $\xi = \frac{\sigma^2}{V^2}$"

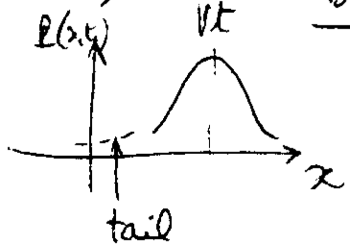


$$t(x) = \sum_{i=1}^{x/\xi} \tau_i$$

$\underline{\mu > 2}$ . CLT $\quad P(t,x) \sim e^{-\left(\frac{t - x/V}{\Delta x}\right)^2}$

$$\Rightarrow P(x,t) \sim e^{-(x-Vt)^2/2Dt}$$

$$V = \frac{\xi}{\langle \tau \rangle}$$




$\underline{\text{But}}$ : $P(x=0,t) = \int_t^\infty P(\tau)\, d\tau \sim \frac{1}{t^\mu}$

exact result for all $\mu$.

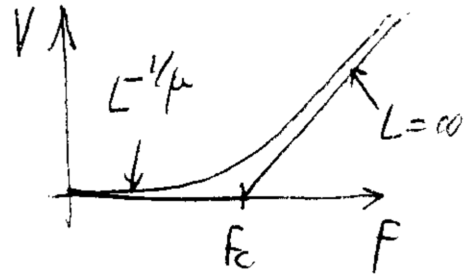
$\underline{\mu < 1}$

$\tau_{max} \sim t \sim \left(\frac{x}{\xi}\right)^{1/\mu} \cdot \tau_0$ [ Single event dominance

$\rightarrow$ levy distribution $l_\mu$

$\Rightarrow x \sim \xi\, t^\mu$ sublinear "creep".

$$V(L) = \frac{L}{t(L)} = \frac{L^{1-\frac{1}{\mu}}}{\tau_0}\, \xi^{1/\mu}$$




More on this anomalous trapping case later.

$\underline{\mu = 0}$ $\quad x \sim \log^2 t$

Golosov phenomenon.

# Lecture 2 — The random energy model

## 1 – Random potential and extreme value statistics

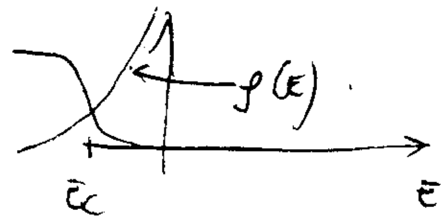
Set of $E_i$ : — — — — — —  $\rho(E) \simeq \frac{A}{|E|^\alpha} \exp - B|E|^\delta$ $(E \to -\infty)$

$E^* = \min\{E_1 \dots E_M\}$   $P_M(E^*)$ ?

$$P_M(E^*) = M \rho(E^*) \left[1 - \int_{-\infty}^{E^*} \rho(E')\, dE'\right]^{M-1}$$
$$= -\frac{d}{dE^*} \left[1 - \int_{-\infty}^{E^*} \rho(E^*)\, dE'\right]^{M}$$

Becomes very small if $M \int_{-\infty}^{E^*} \rho(E')\, dE \gg 1$

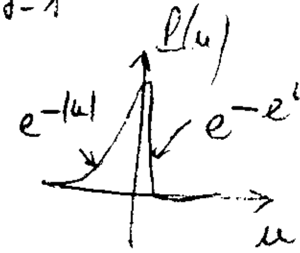
$E_c^*(M)$ : $\int_{-\infty}^{E_c} \rho(E')\, dE' = \frac{1}{M}$




$$E_c(M) \simeq -\left(\frac{\log M}{B}\right)^{1/\delta}$$

$E^* = E_c + \epsilon$   $\epsilon = \frac{u}{B\delta |E_c|^{\delta-1}}$

$$P(u) \underset{M\to\infty}{\sim} e^{u - e^u}$$




$\frac{\epsilon}{E_c} \sim \frac{1}{\log M}$.

universal !
Gumbel distribution.

Note 1 : Slow convergence ($1/\log M$) !
Note 2 : Other universality classes.

a) $\rho(E) = 0$ for $|E| > |E_{max}|$ $\longrightarrow$ Weibull.

b) $\rho(E) \sim \dfrac{\mu E_0^{\mu}}{|E|^{1+\mu}}$, in which case $E_c^* = -\mu E_0 M^{1/\mu}$
Fréchet distribution

⊗ Role of correlations

<u>The random energy model</u>. Thermodynamics of this model

Motivation : "simplest" description of a complicated system (e.g. spin-glasses) — cf "Random matrix theory"

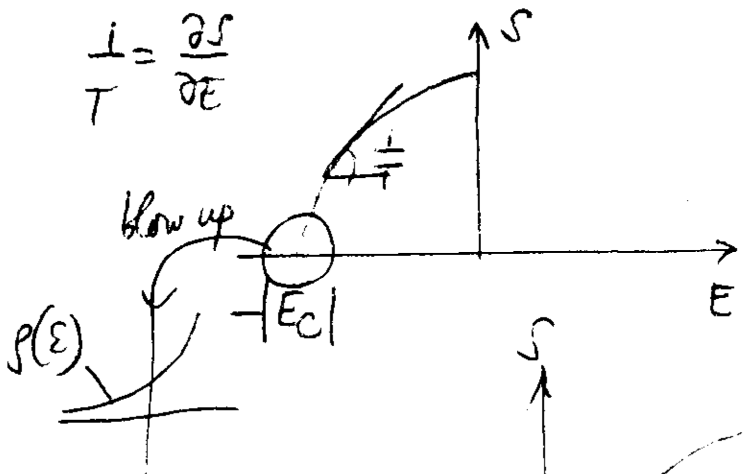
$2^N = M$ states $\qquad E_c \propto N \Longrightarrow B = N^{1-\delta}$

$\Longrightarrow \epsilon = O(1)$ — Non trivial low $T$ properties !

<u>Microcanonical method</u> $\quad S(E) = \log \mathcal{N}(E) = N \log 2 - N\left|\dfrac{E}{N}\right|^{\delta}$

($\delta > 1$)

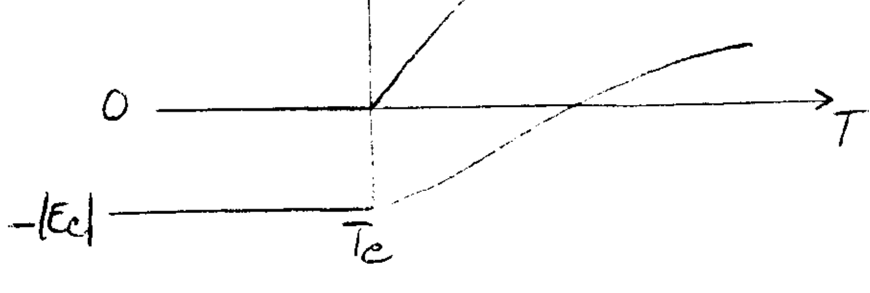
$\dfrac{1}{T} = \dfrac{\partial S}{\partial E}$




Beyond $-|E_c|$ : No more state
$|E_c| = N(\log 2)^{1/\delta}$

$\dfrac{1}{T_c} = \dfrac{\partial S}{\partial E}\Big|_{E_c}$

$= \delta (\log 2)^{\frac{\delta-1}{\delta}}$



$E = E_c + \mathcal{E}$ $\quad O(1)$

$$\rho(\mathcal{E}) = e^{-N\left|\frac{E_c}{N}\right|^{\delta}} e^{\frac{\partial S}{\partial E}\mathcal{E}} \equiv \frac{1}{2^N} e^{\mathcal{E}/T_c}$$

$\to$ Exp. tail of the Gumbel distribution

$$Z = \sum_{i=1}^{2^N} e^{-\mathcal{E}_i/T} = \sum_i z_i$$

$$P(z) \simeq z^{-1-\mu} \qquad \mu = \frac{T}{T_c}$$

$T < T_c \Rightarrow \mu < 1$ : Dominance of a few terms
localisation of the Boltzmann weight

$T > T_c$ : All states contribute to the partition function

$$Y_k = \sum \overline{\frac{z_i^k}{Z^k}} = \frac{\Gamma(k-\mu)}{\Gamma(k)\Gamma(\mu)}$$ (directly from the tail of $P(z)$)

<u>Note</u> $\delta < 1$ : First order transition. $S$ discontinuous.

<u>Alternative method : the Replica Method!</u>

$$\overline{Z^n} = \sum_{i_1 \ldots i_n} \overline{e^{-\sum_{\alpha=1}^{n} E_{i_\alpha}/T}}$$

Take $\delta = 2$ (Gaussian) to simplify (although it does not matter)

weight $e^{-E_i^2/N}$

$$= \sum_{i_1 \ldots i_n} \overline{e^{-\frac{1}{T}\sum_i E_i \sum_\alpha \delta_{i, i_\alpha}}}$$

$$= \sum_{i_1 \ldots i_n} e^{\frac{N}{4T^2}\sum_i \left(\sum_\alpha \delta_{i,i_\alpha}\right)^2}$$

$N \to \infty$ : Saddle point in the "configuration" $i_1 \dots i_n$.
unfortunately, $n \to 0$

Simplest case : all "replicas" play sym. role with respect to all other ones : all $i_\alpha$ are different (largest space a priori). Hence
$$\begin{cases} \sum_\alpha \delta_{i,i_\alpha} = 1 & \text{if } i = i_{\alpha_0} \\ \phantom{\sum_\alpha \delta_{i,i_\alpha}} = 0 & \text{else} \end{cases}$$

$$\overline{Z^n} = [2^N][2^N-1] \dots [2^N-n+1] \; e^{nN/4T^2}$$
$$\simeq e^{nN[\log 2 + \frac{1}{4T^2}]} \underset{n\to 0}{=} e^{-nNf/T}$$

$$f = -\frac{1}{4T} - T\log 2 \qquad S = \frac{\partial f}{\partial T} = \log 2 - \frac{1}{4T^2} \quad \text{idem as above}$$

$S < 0$ for $T < T_c = \dfrac{1}{2\sqrt{\log 2}}$ cf. above expression

<u>RSB</u>

$$\left|\begin{array}{l} i_1 \dots i_m = j_1 \\ i_{m+1} \dots i_{2m} = j_2 \\ \dots \\ i_{n-m+1} \dots i_n = j_{n/m} \end{array}\right. \qquad \frac{n}{m} \text{ "packets" of } m \text{ replicas} \quad \Rightarrow \sum_\alpha \delta_{i,i_\alpha} = \begin{cases} m \\ 0 \end{cases}$$

$$\overline{Z^n} \simeq \text{Combinatorics} \,.\, [2^N]^{\frac{n}{m}} \; e^{\frac{N}{4T^2}\frac{n}{m}m^2}$$
$$\simeq e^{-nNf(\rho)/T} \qquad \rho = \frac{m}{T}$$

$$f(\rho) = -\frac{1}{\rho}\log 2 - \frac{\rho}{4}$$

$$\frac{\partial f}{\partial \rho} = 0 \Rightarrow \rho^2 = 4\log 2 \Rightarrow m = \frac{T}{T_c}$$

Unfortunately a maximum of $f(\rho)$

But $f(T) = f(T_c)$ : same result as above

more precisely: $Y_k = \sum_i \left(\frac{z_i}{Z}\right)^k = \lim_{n\to 0} \sum_i z_i^k Z^{n-k}$

$$= \lim_{n\to 0} \sum'_{i_1\dots i_n} z_{i_1} z_{i_2} \cdots z_{i_n} \frac{1}{n(n-1)\cdots(n-k)}$$

$'$ : $k$ indices must be the same

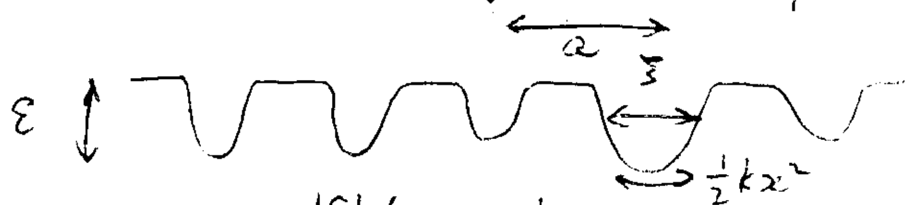
With the above "grouping scheme": $Y_k = \frac{\Gamma(k-m)}{\Gamma(k)\Gamma(1-m)}$

identical as above!

Interpretation * The RSB has managed, in a sense, to compute $\overline{Z^n}$ with $n=\mu$, to get the typical behaviour out of $P(z)$

* "One step" RSB and the Gumbel distribution of extremes are identical, for all $f$

The trap model – Aging, $\alpha$ and $\beta$ relaxation




$\xi : \frac{1}{2}k\xi^2 = T$

$\rho(\varepsilon) = e^{-|\varepsilon|/T_c}$

$\vec{r} = \vec{r}_0 + \delta\vec{r}$

uncorrelated dynamics

$$C_q(t_w+t, t_w) = \langle e^{i\vec{q}(\vec{r}(t_w+t) - \vec{r}(t_w))} \rangle$$

$$= \langle \vec{r} \to \vec{r}_0 \rangle \, e^{-q^2 \delta r^2/2}$$

$$\delta r^2(t) = \xi^2 \left(1 - e^{-t/\tau_0}\right)$$

Assumption $qa >> 1$

$$\Rightarrow C_q(t_w+t, t_w) \approx e^{-q^2\delta r^2/2} \cdot \Pi(t_w+t, t_w)$$

Probability that the particle has NOT jumped between $t_w$ and $t_w+t$

$$\Pi(t_w+t, t_w) = \frac{\int_0^{t_w} dt_\ell \, \ell(t_\ell) \int_{t_w+t-t_\ell}^{\infty} \psi(\tau) d\tau}{\int_0^{t_w} dt_\ell \, \ell(t_\ell) \int_{t_w-t_\ell}^{\infty} \psi(\tau) d\tau}$$

$$\psi(\tau) \simeq \frac{\mu \tau_0^\mu}{\tau^{1+\mu}} \qquad \mu = \frac{T}{T_C}$$

$\ell(t_\ell)$: Probability of the last entry time in the trap

$$\ell(t) = \psi(t) + \int_0^t dt' \, \ell(t') \, \psi(t-t')$$

$\boxed{\mu<1}$ $\ell(E) = \psi(E) + \ell(E)\, \psi(E) \qquad \psi(E) = 1 - c_\mu E^\mu$

$$\ell(E) \sim \frac{1}{c_\mu E^\mu} \quad \Rightarrow \quad \ell(t) \sim t^{\mu-1}$$

$\boxed{\mu>1}$ $\ell(t) \sim$ constant

$\boxed{\mu<1}$ $$\Pi(t_w+t, t_w) \propto \int_0^{t_w} dt_\ell \; t_\ell^{\mu-1} \frac{1}{(t_w+t-t_\ell)^\mu}$$

$$t_\ell = v(t_w+t)$$

Scaling variable $t/t_w$.

$$s = \frac{t_w}{t_w+t} \in [0,1] \qquad \begin{array}{l} s \to 0 \text{ long times} \\ s \to 1 \text{ short times} \end{array}$$

$$\Pi(t_w+t, t_w) = \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^s dv \; v^{\mu-1}(1-v)^{-\mu}$$

$\Pi(s \ll 1) \propto s^\mu \propto \left(\frac{t_w}{t}\right)^\mu$ : slow traps, last to release

$\Pi(s \simeq 1-\epsilon) = 1 - \left(\frac{t}{t_w}\right)^{1-\mu}$ : "fast traps"

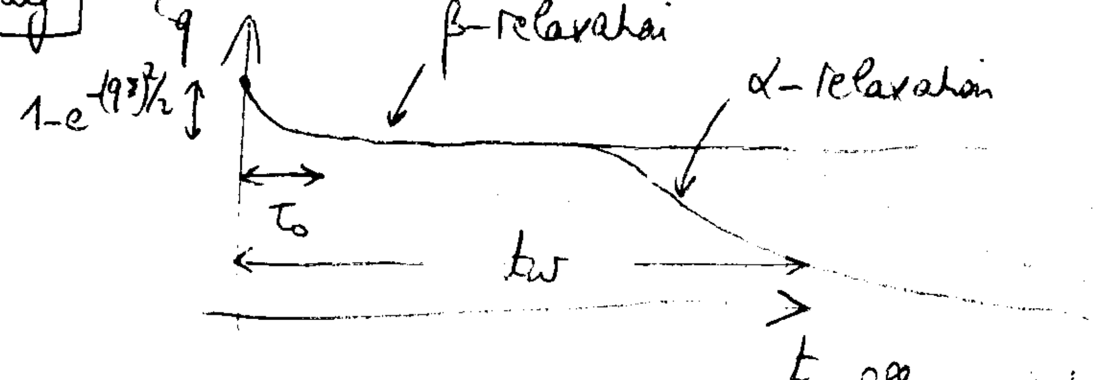
Probability to be in a trap of depth $\tau$ :

non zero only for $\tau \sim t_w$

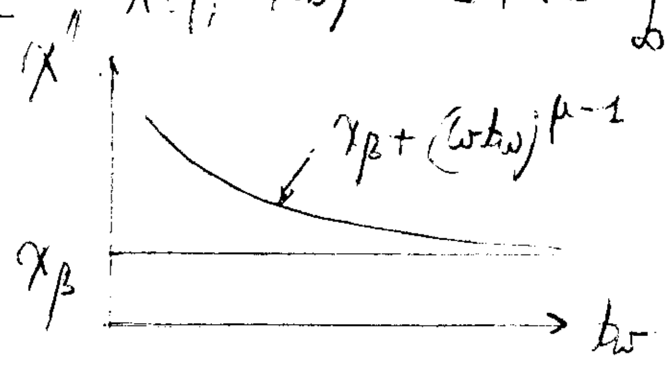
Breaking of Time Translation Invariance (cf. $P(t_e)$)

last trapping date $t_e$ always $\propto t_w$.

$\boxed{\text{Aging}}$



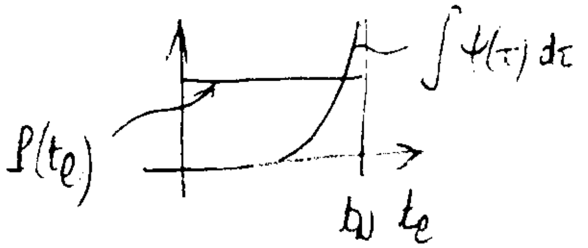

or else. $\chi(q,\omega,t_w) = 1 + i\omega \int_0^\infty dt \; e^{i\omega t} C(q, t_w+t, t_w)$



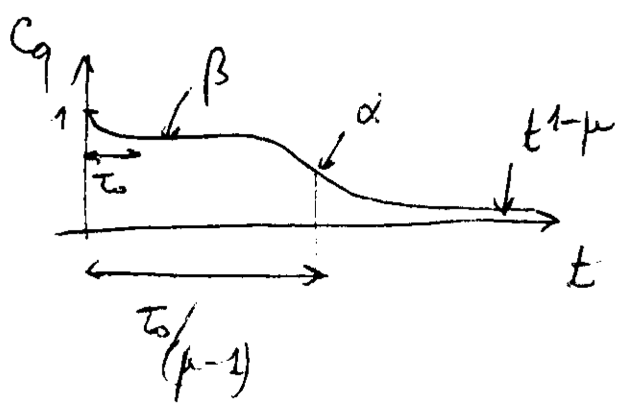

cf. spin-glasses.

## Case $\mu > 1$

Everything changes: last trapping time at date $t_\ell = t_w -$




$\Longrightarrow \quad \Pi(t_w + t, t_w) \underset{t_w \to \infty}{\simeq} \hat{\Pi}(t)$ No longer any $t_w$ dependence




**Note** $\rho(E) \sim e^{-[E/T_c]^\delta}$ $\delta > 1$

"Transient" or "Interrupted" aging

Ex: $\delta = 2$ $\mu = (\log t_w) \left(\frac{T}{T_c}\right)^2$ $\mu = 1$ when $t_w = e^{(T_c/T}$

lecture 3

# Pinning and Metastability

(18)

**Motivation**: Pinning of extended object of various internal dimension $D$

e.g. $D=1$ polymer, dislo chain, triple ligne, vortex fracture front

$D=2$ Bloch wall, magnetic bubble lattice, vortex lat

⋮

**Questions**: – Static : what is the conformation of the object?
what are the energy scale ?

– Dynamics – Response to an external field.

Depinning.

**A simple model**: elasticity is treated in a MF way

$$\frac{d\vec{x}_i}{dt} = \frac{1}{N} k \sum_j (\vec{x}_j - \vec{x}_i) + \vec{F}_i(x_i) + \vec{\eta}_i$$

$$\frac{d\vec{x}_i}{dt} = k(\bar{\vec{x}} - \vec{x}) + \vec{F} + \vec{\eta}$$

In one dimension : $\dfrac{dx}{dt} = k(\bar{x} - x) - \dfrac{\partial U}{\partial x} + \eta$.

$\Rightarrow$ Two particles with a spring:

$$\mathcal{H} = \frac{1}{2} k (x - \bar{x})^2 + U(x)$$

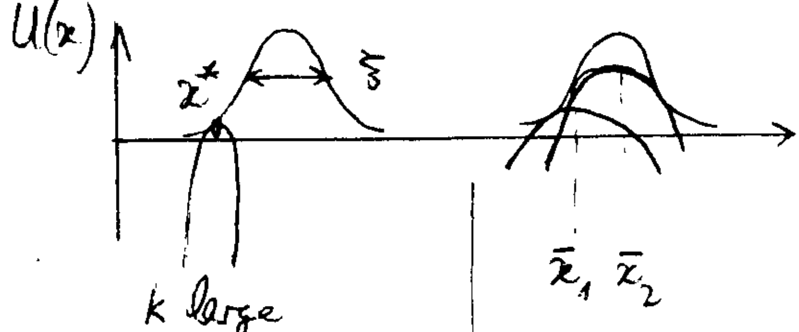
<u>Thermodynamics at fixed $\bar{x}$</u> (cf: control of the center of mass pos.)

$$Z(\bar{x}) = \int dx\, e^{-\beta \mathcal{H}(x, \bar{x})} \equiv e^{-\beta \mathcal{V}(\bar{x})}$$

$\beta \to \infty$ : saddle point method: $x^*$ such that $\mathcal{H}(x^*, \bar{x})$ is minimum.

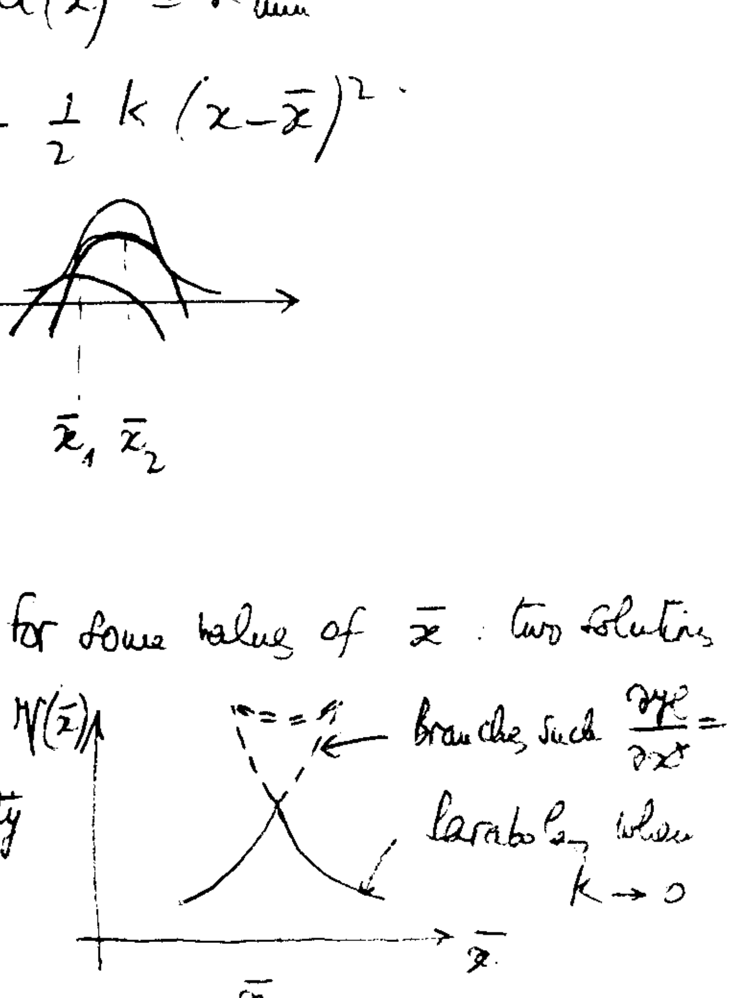
$$\mathcal{H} = \frac{1}{2} k (x - \bar{x})^2 + U(x) = \mathcal{H}_{min}$$

$$\Rightarrow U(x) = \mathcal{H}_{min} - \frac{1}{2} k (x - \bar{x})^2 .$$




$x^* \simeq \bar{x} \Rightarrow \mathcal{V}(\bar{x}) \simeq U(\bar{x})$

for some value of $\bar{x}$: two solutions




Appearance of metastability

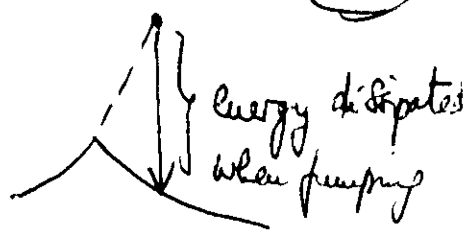
$\Leftrightarrow u \simeq k \xi^2$

"Larkin" criterion (corresponds to a length)

- - - : Metastable branches
= = = : Unstable branch

Note

– A model for static friction ?




Relation with Burgers equation

$$\boxed{\frac{Z(\bar{x})}{\sqrt{4\pi\nu t}} \equiv \int \frac{dx}{\sqrt{4\pi\nu t}} \exp -\frac{(x-\bar{x})^2}{4\nu t} e^{-\frac{1}{2\nu} u(x)}} \quad \left(\beta k = \frac{1}{2\nu t}\right)$$

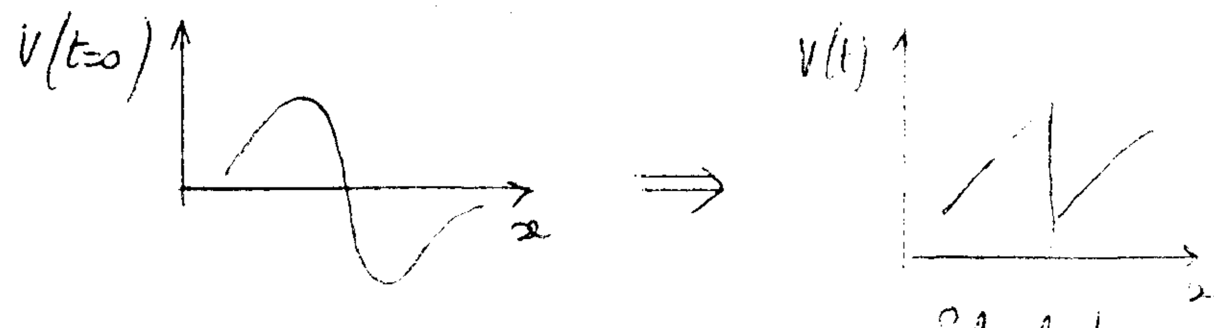
$$T = 2\nu = \frac{1}{\beta}$$

Hence $Z(\bar{x})$ is the solution of

$$\frac{\partial Z}{\partial t} = \nu \frac{\partial^2 Z}{\partial x^2} \qquad \text{with } Z(\bar{x}, t=0) = e^{-\frac{1}{2\nu} u(\bar{x})}$$

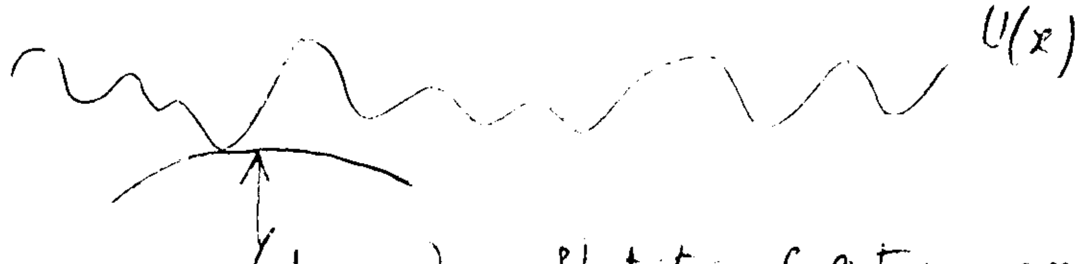
Now : if $\quad v = -2\nu \dfrac{\partial \log Z}{\partial \bar{x}} = + \dfrac{\partial \Psi}{\partial x}$

One has $\quad \dfrac{\partial v}{\partial t} + v \dfrac{\partial v}{\partial x} = \nu \dfrac{\partial^2 v}{\partial x^2}$ : Burgers equation




Schock !

– Evolution of a random initial potential



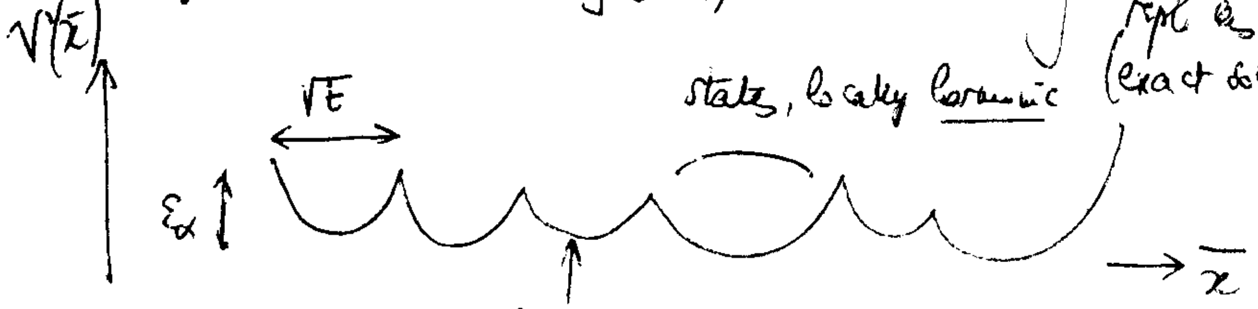

$t \to \infty \quad (k \to 0)$ : Statistics of extremes again

$$\mathcal{V}(\bar{x}) = \frac{1}{\beta} \log \sum_{\alpha} (w_\alpha)\, e^{\frac{\beta k}{2}(\bar{x}-x_\alpha)^2}$$

$x_\alpha$ : poisson set of points

$W_\alpha \doteq e^{+|\varepsilon_\alpha|/T} \qquad \rho(|\varepsilon_\alpha|) = e^{-|\varepsilon_\alpha|/T_c}$

Either through a direct construction or his repl as (exact solution




states, locally harmonic

local dominance of one state $\alpha$

Note $\quad f(\bar{x}) = -\frac{\partial \mathcal{V}}{\partial \bar{x}}$

$$\overline{[f(\bar{x}) f(\bar{y})]^2} \underset{|\bar{x}-\bar{y}| \to 0}{=} \begin{cases} [\bar{x}-\bar{y}]^2 & \text{(no shock between } \bar{x} \text{ and } \bar{y}) \\ \sim 1 & \text{(if there is one)} \end{cases}$$

probability $\rho|\bar{x}-\bar{y}|$

$\simeq \rho|\bar{x}-\bar{y}|$ $\qquad$ But not a random walk.

Remarks

* Dynamics in $V(\bar{x})$: the trap model again.
* ⚠ Different universality classes for extreme value statistics.
* Simplistic RG: elimination of the "fast" mode $x$ to generate an effective potential on the slow mode

For a more sophisticated approach, see D.S. Fisher

BUT the final result is: $\overline{V(\bar{x})\,V(\bar{y})} = A_0 + A_1(\bar{x}-\bar{y}) + A_2\,|\bar{x}-\bar{y}|^3$

Same singularity! $\Rightarrow$ { Same overall picture; replicas and FRG qualitatively agree

~

Conclusion

Replicas: description of rugged energy landscapes

encodes the statistics of extremes