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SUMMER COLLEGE IN CONDENSED MATTER ON
" STATISTICAL PHYSICS OF FRUSTRATED SYSTEMS "

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" Solutions to the problems"

presented by:

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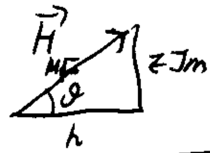
Solutions to the Problems, Peter Young. ①

1) $\vec{H}_{MF} = h\,\hat{x} + zJm\,\hat{z}$



$|\langle\vec{\sigma}\rangle| = \tanh \beta H_{MF}$ $\qquad H_{MF} = \sqrt{h^2 + (zJm)^2}$

$m = |\langle\vec{\sigma}\rangle| \cos\theta = \tanh\beta H_{MF} \times \dfrac{Jzm}{H_{MF}}$

$\Rightarrow \begin{cases} \text{either } \dfrac{\sqrt{h^2+(zJm)^2}}{Jz} = \tanh\left\{\dfrac{\sqrt{h^2+(zJm)^2}}{k_BT}\right\} \quad ① \\ \text{or } \quad m = 0 \end{cases}$ ← see the lecture notes

For $\langle\sigma_x\rangle$,

$\langle\sigma_x\rangle = |\langle\vec{\sigma}\rangle| \sin\theta = \tanh \beta H_{MF} \times \dfrac{h}{H_{MF}}$

$\Rightarrow \begin{cases} \text{if } m=0 \quad \underline{\langle\sigma_x\rangle = \tanh(h/k_BT)} \quad \text{(Paramagnetic)} \\ \text{or } \quad \langle\sigma_x\rangle = \dfrac{h}{\sqrt{h^2+(zJm)^2}} \tanh\left(\dfrac{\sqrt{h^2+(zJm)^2}}{k_BT}\right) \end{cases}$

From ①, this last expression is $\underline{\langle\sigma_x\rangle = \dfrac{h}{zJ}}$ (ferromagnet)

which is <u>independent of T</u>

2) Classical Ground state. $|\psi_0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$

$$\mathcal{H} = \underbrace{-J\sum \sigma_i^z \sigma_{i+1}^z}_{H_0} \underbrace{- h\sum_i \sigma_i^x}_{V}$$

The transverse field term flips the spins.

consider $|\psi_i\rangle = |\uparrow\uparrow\cdots\underset{i}{\downarrow}\cdots\uparrow\rangle$

$E_i - E_0$ (evaluated with H_0) is $4J$ (2 bonds broken each costs an energy $2J$)

Matrix element $\langle \downarrow_i | \sigma^x | \uparrow_i \rangle = 1$.

Hence, 1st order perturbation theory in the wavefunction

$$|\psi\rangle \simeq |\psi_0\rangle + h \sum_i \frac{\langle \psi_i | \sigma^x_i | \psi_0 \rangle}{E_i - E_0} + \cdots$$

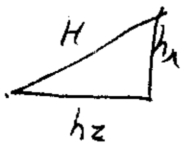
$$= |\uparrow\uparrow\cdots\uparrow\rangle + \frac{h}{4J} \sum_i |\uparrow\uparrow\cdots\downarrow_i\cdots\uparrow\rangle + \cdots$$

(not normalized)

Hence $\langle \sigma_i^z \rangle = \dfrac{\langle \psi | \sigma_i^z | \psi \rangle}{\langle \psi | \psi \rangle} = \dfrac{\cancel{1+}}{1 + O\left(\left(\frac{h}{J}\right)^2\right)}$

$$= \frac{1 + \left(\frac{h}{4J}\right)^2 \{(N-1) + (-1)\} + \cdots}{1 + \left(\frac{h}{4J}\right)^2 N + \cdots} = \underline{\underline{1 - \frac{1}{8}\left(\frac{h}{J}\right)^2 + \cdots}}$$

(N dependence drops out, as it must)

3 (a) $H \equiv |\vec{H}| = \sqrt{h_x^2 + h_z^2}$ 

, $\vec{\mu} = -\vec{H} \cdot \vec{\sigma}$.

Work in a basis where the spin is quantized along $\vec{H}$.
The energy levels are clearly $\pm H = \pm\sqrt{h_x^2 + h_z^2}$

Hence $\mathrm{Tr}\left\{ \exp(h_x \sigma_x + h_z \sigma_z) \right\} = e^{H} + e^{-H}$

$$= 2\cosh H = \underline{2\cosh\left(\sqrt{h_x^2 + h_z^2}\right)}$$

(b) ~~It~~ Now let's evaluate the Suzuki-Trotter formula:

$$\mathrm{Tr} \lim_{m \to \infty} \left[\exp\left(\frac{h_x}{m} \sigma_x\right) \exp\left(\frac{h_z}{m} \sigma_z\right)^m \right]$$

$$= \mathrm{Tr} \lim_{m\to\infty} \left[1 + \frac{h_x}{m} \sigma^x + \frac{h_z}{m} \sigma^z + O\left(\frac{1}{m^2}\right) \right]^m$$

The $O(1/m^2)$ give zero for $m \to \infty$ (even when the expression is taken to the m-th power.

$$\Rightarrow \mathrm{Tr} \lim_{m\to\infty} [\underbrace{1 + \frac{1}{m}(h_x\sigma^x + h_z\sigma^z)}_{A}]^m = \lim_{m\to\infty} \lambda_+^m + \lambda_-^m \quad (1)$$

where $\lambda_\pm$ are the evalues of A

$$A = \begin{pmatrix} 1 + \frac{h_z}{m} & h_x/m \\ \frac{h_x}{m} & 1 - \frac{h_z}{m} \end{pmatrix}$$

e-values of A $\Rightarrow$ $\begin{vmatrix} 1 + \frac{h_z}{m} - \lambda & h_x/m \\ h_x/m & 1 - \frac{h_z}{m} - \lambda \end{vmatrix} = 0$

$(\lambda - 1)^2 - \left(\frac{h_z}{m}\right)^2 - \left(\frac{h_x}{m}\right)^2 = 0$ so $\lambda_\pm = 1 \pm \frac{1}{m}\sqrt{h_x^2 + h_z^2}$.

From (1) $\Rightarrow \lim_{m\to\infty} \lambda_+^m + \lambda_-^m = \lim_{m\to\infty} \left[\left(1 + \frac{1}{m}\sqrt{h_x^2+h_z^2}\right)^m + \left(1 - \frac{1}{m}\sqrt{h_x^2+h_z^2}\right)^m \right]$

$$= e^{\sqrt{h_x^2+h_z^2}} + e^{-\sqrt{h_x^2+h_z^2}} \quad \underline{= 2\cosh\sqrt{h_x^2+h_z^2}}$$

in agreement with part (a).

4/ (a) Metropolis. Consider 2 states, a and b, with energies E_a and E_b and assume for now that $E_a > E_b$

Then $P_{a\to b} = 1$ since $\Delta E = E_b - E_a < 0$

$P_{b\to a} = e^{-(E_a - E_b)}$ since $\Delta E = E_a - E_b > 0$

Hence $\underline{\dfrac{P_{a\to b}}{P_{b\to a}} = e^{(E_a - E_b)}}$, the detailed balance condition

Repeating the argument for $E_a < E_b$ again leads to the detailed balance condition.

(b) Heat bath

$$P_{a\to b} = \frac{1}{e^{(E_b - E_a)}+1} = \frac{e^{(E_a-E_b)}}{1+e^{E_a-E_b}}$$

$$P_{b\to a} = \frac{1}{e^{(E_a-E_b)}+1}$$

Hence $\dfrac{P_{a\to b}}{P_{b\to a}} = e^{(E_a-E_b)}$, the detailed balance condition.

5. $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(a)

$$\{\sigma^x, \sigma^z\} \equiv \sigma^x\sigma^z + \sigma^z\sigma^x$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \underline{\underline{0}}$$

Similarly for the other components.
Notice also that $(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^z)^2 = 1$

(b)

$\sigma^z = a^+ + a$

$\sigma^y = i(a^+ - a)$ where the a's are

$\sigma^x = 1 - 2a^+a$ fermi operators

Note first that $(\sigma^z)^2 = (a^+ + a)^2 = a^+a^+ + aa^+ + a^+a + aa$.
But for fermions, $a^+a^+ = aa = 0$ and $a^+a + aa^+ = 1$
Hence $(\sigma^z)^2 = 1$ as required.
Similarly $(\sigma^y)^2 = 1$
Also $(\sigma^x)^2 = (1-2a^+a)^2 = 1 - 4a^+a + 4a^+a\,a^+a$.
But, for fermions, $(a^+a)^2 = a^+a$ since there are only 2 states, $|1\rangle$, where ~~such~~ that $(a^+a)^2|1\rangle = (a^+a)|1\rangle = |1\rangle$ and $|0\rangle$, where $(a^+a)^2|0\rangle = (a^+a)|0\rangle = \underline{\underline{0}}$

Hence $(\sigma^x)^2 = 1$ as desired.

~~Now~~ Furthermore,

$$\{\sigma^x, \sigma^y\} = i\{a^+ + a,\ a^+ - a\}$$

$$= i\Big[\underbrace{\{a^+, a^+\}}_{0} - \underbrace{\{a, a\}}_{0} - \underbrace{\{a^+, a\}}_{1} + \underbrace{\{a, a^+\}}_{1}\Big]$$

$$\underline{= 0}$$

Also

$$\{\sigma^z, \sigma^x\} = \{a^+ + a,\ 1 - 2a^+a\}$$

$$= (a^+ + a)(1 - 2a^+a) + (1 - 2a^+a)(a + a^+)$$

$$= 2(a^+ + a) - 2\underbrace{a^+a^+}_{=0}a - 2aa^+a - 2a^+\underbrace{aa}_{=0} - 2a^+aa^+$$

$$= 2(a^+ + a) - 2a\underbrace{a^+a}_{1 - aa^+} - 2a^+\underbrace{aa^+}_{1 - a^+a}$$

$$= 2(a^+ + a) - 2a + 2\underbrace{aa}_{=0}a^+ - 2a^+ + 2\underbrace{a^+a^+}_{=0}a$$

$$\underline{\underline{= 0}}$$

Similarly $\underline{\{\sigma^z, \sigma^y\} = 0}$

(c) Now $a_i^+ = c_i^+ \overbrace{\exp\Big[-i\pi \sum_{j<i} c_j^+ c_j\Big]}^{\text{"string" operator}} = c_i^+ \prod_{j<i} (-1)^{\hat{n}_j}$

$$a_i = \exp\Big[-i\pi \sum_{j<i} c_j^+ c_j\Big] = \prod_{j<i} (-1)^{\hat{n}_j}\, c_i$$

where $\hat{n}_j = c_j^+ c_j$ is the number operator.

Note that $[\hat{n}_j, \hat{n}_k] = 0$ for $j \neq k$ and hence

$$\big[(-1)^{\hat{n}_j}, (-1)^{\hat{n}_k}\big] = 0$$

Now

$$(-1)^{\hat{n}_j} = \begin{array}{cc} |0\rangle & |1\rangle \end{array}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1 - 2c_j^+ c_j$$

Following Lieb Schultz + Mattis we define

$$A_j = c_j^\dagger + c_j$$

$$B_j = c_j^\dagger - c_j$$

Using basic properties of fermi operators, $c_j^\dagger c_j^\dagger = c_j c_j = 0$, $c_j^\dagger c_j + c_j c_j^\dagger = 1$ one has

$$(-1)^{\hat{n}_j} = 1 - 2c_j^\dagger c_j = A_j B_j = -B_j A_j$$

Now $c_j^\dagger$ anticommutes with A_i and with B_i (for $i \neq j$) and hence $c_j^\dagger$ <u>commutes</u> with $A_i B_i$ (again for $i \neq j$)

Hence

$$a_i^\dagger = c_i^\dagger \left(\prod_{j<i} A_j B_j\right) = \left(\prod_{j<i} A_j B_j\right) c_i^\dagger$$

$$a_i = \left(\prod_{j<i} A_j B_j\right) c_i = c_i \left(\prod_{j<i} A_j B_j\right)$$

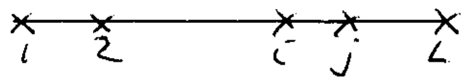
and hence

$$\sigma_i^z = \prod_{j<i} (A_j B_j)\, A_i$$

$$\sigma_i^y = i \prod_{j<i} (A_j B_j)\, B_j$$

$$\sigma_i^x = 1 - 2c_i^\dagger c_i$$

Now lets verify that spin operators on different sites <u>commute</u>, as required.



Consider $\left[\sigma_i^z, \sigma_j^z\right]$, for $j > i$

which equals

$$= \left[\left(\prod_{k<i} A_k B_k\right) A_i,\ \left(\prod_{l<j} A_l B_l\right) A_j\right]$$

Commuting through the factors of $A_k B_k$ etc we get

$$\left(\prod_{k=i+1}^{j-1} A_k B_k\right) \left[A_i,\ \underbrace{A_i B_i}\, A_j\right] \qquad (1)$$

we cannot commute this factor through because

Because it does not commute with A_i.
Note that without the string operators ① would be

$$[A_i, A_j]$$

which is not zero because A_i anticommutes with A_j and the commutator is not simple.
However, with the string operators ① is proportional to

$$[A_i, A_i B_i A_j] = \underbrace{A_i A_i}_{1} B_i A_j - A_i B_i A_j A_i$$

$$= B_i A_j + A_i B_i A_i A_j \quad \text{(since } A_i \text{ anticommutes with } A_j\text{)}$$

$$= B_i A_j - \underbrace{A_i A_i}_{1} B_i A_j \quad \text{(since } A_i \text{ anticommutes with } B_i\text{)}$$

$$= B_i A_j - B_i A_j = \underline{\underline{0}} \ (\checkmark)$$

A similar calculation shows that other spin components on different sites commute.

6. $\langle ABCD \rangle = \langle AB \rangle \langle CD \rangle - \langle AC \rangle \langle BD \rangle + \langle AD \rangle \langle BC \rangle$ (Wick's theorem)

(a) $\langle 1 | c^+ c\, c^+ c | 1 \rangle = 1 = \langle 1 | c^+ c | 1 \rangle$

$\langle 0 | c^+ c\, c^+ c | 0 \rangle = 0 = \langle 0 | c^+ c | 0 \rangle$

Hence $\hat{n}^2 = \hat{n}$ where $\hat{n} = c^+ c$ is the number operator

Hence $\langle \hat{n}^2 \rangle = \langle \hat{n} \rangle = n = \dfrac{1}{e^{\beta \epsilon} + 1}$

(b) Now use Wick's theorem $\quad A = c^+ \quad B = c \quad C = c^+ \quad D = c$

$$\langle c^+ c\, c^+ c \rangle = \langle c^+ c \rangle \langle c^+ c \rangle - \underbrace{\langle c^+ c^+ \rangle \langle c c \rangle}_{0} + \langle c^+ c \rangle \underbrace{\langle c c^+ \rangle}_{1-n}$$

$$= n^2 + n(1-n) = \underline{\underline{n}}$$

6(c) $\langle (c^\dagger c)^3 \rangle = \langle \underset{A}{c^\dagger}\, \underset{B}{c}\, \underset{C}{c^\dagger}\, \underset{D}{c}\, \underset{E}{c^\dagger}\, \underset{F}{c} \rangle$

The non-zero pairings in Wick's theorem are

$$= \langle \underset{A}{c^\dagger} \underset{B}{c} \rangle \langle \underset{C}{c^\dagger} \underset{D}{c} \rangle \langle \underset{E}{c^\dagger} \underset{F}{c} \rangle + \langle \underset{A}{c^\dagger} \underset{B}{c} \rangle \langle \underset{C}{c^\dagger} \underset{F}{c} \rangle \langle \underset{D}{c} \underset{E}{c^\dagger} \rangle$$

$$+ \langle \underset{A}{c^\dagger} \underset{D}{c} \rangle \langle \underset{B}{c} \underset{C}{c^\dagger} \rangle \langle \underset{E}{c^\dagger} \underset{F}{c} \rangle - \langle \underset{A}{c^\dagger} \underset{D}{c} \rangle \langle \underset{B}{c} \underset{E}{c^\dagger} \rangle \langle \underset{C}{c^\dagger} \underset{F}{c} \rangle$$

$$+ \langle \underset{A}{c^\dagger} \underset{F}{c} \rangle \langle \underset{B}{c} \underset{C}{c^\dagger} \rangle \langle \underset{D}{c} \underset{E}{c^\dagger} \rangle + \langle \underset{A}{c^\dagger} \underset{F}{c} \rangle \langle \underset{B}{c} \underset{E}{c^\dagger} \rangle \langle \underset{C}{c^\dagger} \underset{D}{c} \rangle$$

$$= n^3 + n^2(1-n) + n^2(1-n) - n^2(1-n) + n(1-n)^2 + n^2(1-n)$$
$$= n^3 + 2n^2(1-n) + n(1-n)^2$$
$$= n\left[n^2 + 2n(1-n) + (1-n)^2\right] = n\left[\cancel{n^2} + \cancel{2n} - \cancel{2n^2} + 1 - \cancel{2n} + \cancel{n^2}\right]$$
$$\underline{= n}$$

(which is also obtained by direct calculation along the lines of part (a))

7. $$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix} = -a_{12} \begin{vmatrix} -a_{12} & a_{23} & a_{24} \\ -a_{13} & 0 & a_{34} \\ -a_{14} & -a_{34} & 0 \end{vmatrix}$$
$$+ a_{13} \begin{vmatrix} -a_{12} & 0 & a_{24} \\ -a_{13} & -a_{23} & a_{34} \\ -a_{14} & -a_{24} & 0 \end{vmatrix} - a_{14} \begin{vmatrix} -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \\ -a_{14} & -a_{24} & -a_{34} \end{vmatrix}$$

$$= -a_{12}\left(-a_{14} a_{23} a_{34} + a_{34}\left(-a_{34} a_{12} + a_{13} a_{24}\right)\right)$$
$$+ a_{13}\left(-a_{12} a_{24} a_{34} + a_{24}\left(a_{13} a_{24} - a_{14} a_{23}\right)\right)$$
$$- a_{14}\left(-a_{12} a_{23} a_{34} + a_{23}\left(a_{13} a_{24} - a_{14} a_{23}\right)\right)$$

$$= q_{12}^2 q_{34}^2 + q_{13}^2 q_{24}^2 + q_{14}^2 q_{23}^2$$
$$+2\left(q_{12} q_{34} q_{14} q_{23} - q_{13} q_{24} q_{14} q_{13} - q_{12} q_{13} q_{34} q_{24}\right)$$
$$= \underline{\underline{\left(q_{12} q_{34} - q_{13} q_{24} + q_{14} q_{23}\right)^2}}$$

8. $S(\tau) \propto \sum_{n=0}^{\infty} \exp\left(-\frac{n}{\xi} + \varepsilon_n \tau\right)$

where $\pi(\varepsilon_n) \propto \varepsilon_n^{\lambda}$ (probability distribution for ε_n)

(a) Average each term separately

$$[S(\tau)]_a \propto \underbrace{\sum_{n=0}^{\infty} e^{-n/\xi}}_{\text{const}} \underbrace{\int_0^1 \varepsilon^{\lambda} e^{-\varepsilon\tau} d\varepsilon}_{\text{let } \varepsilon\tau = x}.$$

The integral is $\frac{1}{\tau^{\lambda+1}} \underbrace{\int_0^{\tau} x^{\lambda} e^{-x} dx}_{\text{a number}}$ (replace by ∞)

Hence $\underline{\underline{[S(\tau)] \propto \frac{1}{\tau^{\lambda+1}}}}$, a power law.

(b) given that $P(y) \propto \left(\frac{y}{\tau}\right)^{\lambda+1} \exp\left(-\text{const.} \frac{y^{\lambda+2}}{\tau^{\lambda+1}}\right)$

where $y = -\ln S(\tau)$

Change variables to $x = \frac{y}{\tau^{1/\mu}}$ where $\mu = \frac{\lambda+2}{\lambda+1}$

$P(y)\, dy = \bar{P}(x)\, dx$ so

$\underline{\bar{P}(x) \propto x^{\lambda+1} \exp\left(-\text{const.}\, x^{\lambda+2}\right)}$ i.e. scaling variable is x.

Hence typical value (where $p(x)$ has significant probability) is $x \sim$ constant, i.e.

$$-\ln S(\tau) = \text{constant } \tau^{1/\mu}$$

$$\text{so } \underline{S(\tau) \sim e^{-\text{const.}\,\tau^{1/\mu}}}, \text{ a stretched exponential.}$$

This is much faster than the average, so the latter must be dominated by rare regions with unusually large values of $S(\tau)$, i.e. $\frac{-\ln S(\tau)}{\tau^{1/\mu}}$ close to 0, i.e. the small x region above.

$\bar{P}(x) \propto x^{\lambda+1}$ or $P(y) \propto \left(\frac{y}{\tau}\right)^{\lambda+1}$

Now $S(\tau) = e^{-y}$ so

$$[S(\tau)] = \int_0^\infty P(y) \cdot \cancel{\left(\frac{y}{\tau}\right)} e^{-y}\, dy$$

$$\propto \frac{1}{\tau^{\lambda+1}} \underbrace{\int_0^\infty y^{\lambda+1} e^{-y}\, dy}_{\text{const.}}$$

$$\propto \underline{\underline{\frac{1}{\tau^{\lambda+1}}}} \quad (\checkmark)$$

9. The ~~1-d~~ transverse Ising model with 2 spins

$$\overset{h}{\times}\overset{J}{—}\overset{h}{\times}$$
$1 \quad 2$

$$\mathcal{H} = -J\sigma_1^z\sigma_2^z - h(\sigma_1^x + \sigma_2^x)$$

(a) Find the eigenvalues by diagonalization

The 4 basis states are $|\uparrow\uparrow\rangle \quad |\uparrow\downarrow\rangle \quad |\downarrow\uparrow\rangle \quad |\downarrow\downarrow\rangle$

The Hamiltonian matrix is

$$\mathcal{H} = \begin{array}{c} \begin{array}{cccc} |\uparrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \end{array} \\ \begin{vmatrix} -J & h & h & 0 \\ h & J & 0 & h \\ h & 0 & J & h \\ 0 & h & h & -J \end{vmatrix} \end{array}$$

The eigenvalues are given by

$$\begin{vmatrix} -J-\lambda & h & h & 0 \\ h & J-\lambda & 0 & h \\ h & 0 & J-\lambda & h \\ 0 & h & h & -J-\lambda \end{vmatrix} = 0$$

Subtract the 3rd column from the 2nd

$$\begin{vmatrix} -J-\lambda & 0 & h & 0 \\ h & J-\lambda & 0 & h \\ h & -(J-\lambda) & J-\lambda & h \\ 0 & 0 & h & -J-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - J)\begin{vmatrix} -J-\lambda & 0 & h & 0 \\ h & 1 & 0 & h \\ h & -1 & J-\lambda & h \\ 0 & 0 & h & -J-\lambda \end{vmatrix} = 0$$

add 3rd row to 2nd row

$$\Rightarrow (\lambda - J)\begin{vmatrix} -J-\lambda & h & 0 \\ 2h & J-\lambda & 2h \\ 0 & h & -J-\lambda \end{vmatrix} = 0$$

Subtract the 3rd row from the first

$$(\lambda - J)\begin{vmatrix} -J-\lambda & 0 & J+\lambda \\ 2h & J-\lambda & 2h \\ 0 & h & -J-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - J)(\lambda + J)\begin{vmatrix} -1 & 0 & 1 \\ 2h & J-\lambda & 2h \\ 0 & h & -J-\lambda \end{vmatrix} = 0$$

Add the 3rd column to the first

$$\Rightarrow (\lambda - J)(\lambda + J)\begin{vmatrix} 4h & J-\lambda \\ -J-\lambda & h \end{vmatrix} = 0$$

$$\Rightarrow \underline{\underline{\lambda = \pm J, \quad \pm\sqrt{J^2 + (2h)^2}}}$$

(b) Now use the fermion method, as in the notes

$$\Rightarrow \mathcal{H} = -J\left(c_1^\dagger - c_1\right)\left(c_2^\dagger + c_2\right) + h\left(c_1^\dagger c_1 - c_1 c_1^\dagger + c_2^\dagger c_2 - c_2 c_2^\dagger\right) \qquad (1)$$

We wish to find new fermion creation and destruction operators $\gamma_a^\dagger, \gamma_a, \gamma_b^\dagger, \gamma_b$ such that

$$\mathcal{H} = \epsilon_a\left(\gamma_a^\dagger \gamma_a - \tfrac{1}{2}\right) + \epsilon_b\left(\gamma_b^\dagger \gamma_b - \tfrac{1}{2}\right) \qquad (2)$$

We have only 2 sites and therefore expect that the $\gamma_a^\dagger$ etc. will involve the combinations $c_1^\dagger \pm c_2^\dagger$ etc. so we write

$$\gamma_a^\dagger = \cos\theta \frac{1}{\sqrt{2}}\left(c_1^\dagger + c_2^\dagger\right) + \sin\theta \frac{1}{\sqrt{2}}\left(c_1 - c_2\right)$$

$$\gamma_a = \cos\theta \frac{1}{\sqrt{2}}\left(c_1 + c_2\right) + \sin\theta \frac{1}{\sqrt{2}}\left(c_1^\dagger - c_2^\dagger\right)$$

$$\gamma_b^\dagger = \cos\theta \frac{1}{\sqrt{2}}\left(c_1^\dagger - c_2^\dagger\right) + \sin\theta \frac{1}{\sqrt{2}}\left(c_1 + c_2\right)$$

$$\gamma_b = \cos\theta \frac{1}{\sqrt{2}}\left(c_1 - c_2\right) + \sin\theta \frac{1}{\sqrt{2}}\left(c_1^\dagger + c_2^\dagger\right)$$

One can check that $\{\gamma_a^\dagger, \gamma_a\} = \{\gamma_b^\dagger, \gamma_b\} = 1$

$\{\gamma_a^\dagger, \gamma_b^\dagger\} = \{\gamma_a^\dagger, \gamma_b\}$ etc $= 0$

The transformation from the c's to the γ's is orthogonal so the inverse transformation, from the the γ's to c's is given by the transpose of the matrix of coefficients, i.e.

$$\left.\begin{aligned} c_1^\dagger &= \cos\theta \frac{1}{\sqrt{2}}\left(\gamma_a^\dagger + \gamma_b^\dagger\right) + \sin\theta \frac{1}{\sqrt{2}}\left(\gamma_a + \gamma_b\right) \\ c_2^\dagger &= \cos\theta \frac{1}{\sqrt{2}}\left(\gamma_a^\dagger - \gamma_b^\dagger\right) + \sin\theta \frac{1}{\sqrt{2}}\left(\gamma_a - \gamma_b\right) \\ & c_1 \text{ and } c_2 \text{ are the Hermitian conjugates} \end{aligned}\right\} \quad (3)$$

Substitute (3) into (2) and require that the terms $\gamma_a^\dagger \gamma_b^\dagger$ and $\gamma_a \gamma_b$ have zero coefficient.

This gives $\tan 2\theta = \dfrac{J}{2h}$

Comparison with Eq. (2) then gives

$$\left.\begin{aligned} \epsilon_a &= \sqrt{(2h)^2+J^2} - J \\ \epsilon_b &= \sqrt{(2h)^2+J^2} + J \end{aligned}\right\}$$ ~~Check! if J=0, $\epsilon_a = \epsilon_b = 2h$ (✓)~~

From (2) the 4 energy levels of the transverse Ising model are

$$\pm \frac{\epsilon_a}{2} \pm \frac{\epsilon_b}{2}$$

$$\Rightarrow \underline{\underline{\pm\sqrt{(2h)^2+J^2}, \ \pm J}}$$, in agreement with part (a)

<u>Check!</u> if $J=0$, levels are $\pm 2h$, 0, 0 (✓)
if $h=0$, levels are $\pm J$, twice

<u>n.b.</u> The calculation of the fermion energy levels was a bit messy because we used free, rather than periodic, boundary conditions. However, in general, there is an additional <u>complication</u> ~~from~~ in applying the fermion method to periodic boundary conditions due to the string operator which wraps all the way round the lattice, see eg Ref. 26 of the notes.