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INTERNATIONAL ATOMIC ENERGY AGENCY
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.1004/9

SUMMER SCHOOL ON ELLIPTIC CURVES

(11- 29 August 1997)

On the range of $J_0(q)$

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On the rank of $J_0(q)$

I Introduction.

The purpose of this talk is to give account on some very recent progress made quite simultaneously on the question of determining the rank of $J_0^n(q)$, the "new" part of the jacobian of the modular curve $X_0(q)$, by Iwaniec-Sarnak, Kowalski-Michel and J Vanderkam.

Given $A/\mathbb{Q}$ an abelian variety, we know by the Mordell-Weil theorem that $A(\mathbb{Q})$ the group of $\mathbb{Q}$-rational points of A is an finitely generated abelian group, and the natural question which arises is how to compute is rank.
Birch and Swinnerton-Dyer gave this very elegant conjectural solution:
to $A/\mathbb{Q}$ is associated an L-function constructed from Galois representation on the Tate-module of A

$$L(A,s) := \prod_p L_p(A,s)$$ where the local L-functions L_p

are defined (if $p \nmid \ell\,\mathrm{Cond}(A)$) by $L_p(A,s) := \det\left(\mathrm{Id} - p^{-(s-\frac{1}{2})}\mathrm{Frob}_p \mid T_\ell(A)\right)^{-1}$

By the "standard conjectures", $L(A,s)$ should admit an holomorphic continuation to $\mathbb{C}$ with a functional equation relating (with our normalisation $L(A,s)$ to $L(A,1-s)$.
Rmq: this point is known for modular elliptic curves, and more generally for A a quotient of $\mathrm{Jac}(X_0(q))$

Now the Birch-Swinnerton-Dyer conjecture states among other things that

Conj (B-SD) $\quad \operatorname{rank} A(\mathbb{Q}) = \operatorname{ord}_{s=\frac{1}{2}} L(A,s)$

We will call the integer on the right hand side the "analytic rank" of A (It will our main object of study). Note it $\operatorname{rank}_a A$

II The rank of $J_0^n(q)$

Assume (for simplicity) that q is a prime number.
Then $J_0^n(q) = \operatorname{Jac}(X_0(q))$ and the L-function $L(J_0^n(q),s)$ is well known (Eichler-Shimura factorization)

$$L(J_0^n(q),s) = \prod_{f\in S_2^+(q)} L(f,s)$$

where $S_2^+(q)$ is the set of primitive cuspforms of weight 2 for $\Gamma_0(q)$ and if $f(z) = \sum_{n\geq 1} n^{\frac{1}{2}}\lambda_f(n)e(nz)$, $L(f,s) = \sum_{n\geq 1}\lambda_f(n)n^{-s}$
(with this normalization $\frac{1}{2}$ is the critical point of $L(f,s)$)

This factorisation implies

$$\operatorname{rank}_a J_0^n(q) = \sum_{f\in S_2^+(q)} \operatorname{ord}_{s=\frac{1}{2}} L(f,s)$$

So we get here a problem on average over weight two primitive forms of Level q! A much more accessible problem for analytic methods.

II.1 The first result is

Thm 1 ([I+S], [KM2], [KM3], [V]):
$\exists\, c_1, c_2$, positive absolute constants, s.t. $\forall \varepsilon>0$ and q large enough

(1) $\#\{f\in S_2^+(q) \text{ s.t. } \operatorname{ord}_{s=\frac{1}{2}} L(f) = 0\} \geq (c_1-\varepsilon)\dim J_0^n(q)$

(2) $\#\{f\in S_2^+(q) \text{ s.t. } \operatorname{ord}_{s=\frac{1}{2}} L(f) = 1\} \geq (c_2-\varepsilon)\dim J_0^n(q)$

First results of this type are due to Duke [D] which obtained (1) with c_1 replaced by $c_1/\log q$

the best constant $c_1 = \frac{1}{4}$ is due to Iwaniec-Sarnak (in fact their proof work for square-free q) but the method of [K-M] and [I-S] are essentially the same, and based on Peterson's formula.
the best constant $c_2 =$ is due to Kowalski-Michel [K-M3].
Note also that Vanderkam's method is quite different (based on Eichler-Selberg trace formula) and although it yields much weaker constants c_1 and c_2, it ~~is sus~~ should work in quite different situations.

Rmk: the set of f considered in (1) (resp. (2)) is contained in the set of f having root number $+1$ (resp. -1), that is the set of f in the -1 (resp. $+1$) eigenspace of the Atkin-Lehner involution W_q, as such their cardinal is bounded approximatively by $\frac{1}{2} \dim J_0^u(q)$.

Eq. 1 gives a lower bound for the dimension of the winding quotient of Merel which by work of Kolyvagin-Logachev is a rank 0 Modular abelian variety satisfying B.SD.
From work of Gross and Waldspurger, (1) also admit a corollary for $\frac{3}{2}$-weight modular forms: let $M^*_{c,q}$ the Kohnen subspace in the space of modular forms of weight $\frac{3}{2}$ and $\Theta_q < M^*_{c,q}$ the subspace spanned by theta series then

$$\dim \Theta_q \geqslant (2c_1 - \varepsilon) \dim M^*_{c,q}$$

Eq. 2 says that a positive proportion of $L(f,s)$ have a simple zero at $s = \frac{1}{2}$. Then, by work of Gross-Zagier this implies in particular the lower bound for the geometric rank

$$\mathrm{rank}_{\mathbb{Q}} J_0^u(q) \geqslant (c_2 - \varepsilon) \dim J_0^u(q)$$

What is conjectured is

$$\mathrm{rank}_{\mathbb{Q}} J_0^u(q) \sim \frac{1}{2} \dim J_0^u(q)$$

II 2 Upper bounds for the analytic rank

One may also consider the problem for giving ~~lower~~ upper bound. For this the only method I know is that of the Riemann-Weil-Mestre explicit formulas.

Assuming GRH for the automorphic L-functions $L(f,s)$, Brumer proved using the Eichler-Selberg trace formula (in the form given by Skoruppa-Zagier), the upper bound

$$\mathrm{rank}_a J_0^n(q) \leqslant \left(\frac{3}{2}+\varepsilon\right) \dim J_0^n(q)$$

In fact, in [KM1], we reduced the $\frac{3}{2}$ to $\frac{23}{22}$, and assuming also GRH for Dirichlet L-functions, it can be further reduced to 1 (unpublished).

It was a ~~longstan~~ motivating question to know wether one could get rid of GRH, and it is the case:

<u>THM 2</u> ([KM1]) there is a computable absolute constant $C>0$ such that

$$(3) \quad \mathrm{rank}_a J_0^n(q) \leqslant C \dim J_0^n(q)$$

Rmk: the trivial bound would be with C replaced by $C \log q$. At the same time Chamizo and Pomykala obtained weaker (but nontrivial) bounds

In fact proofs of (1) (2) and (3) share some common feature (Estimates for mean squares of L functions) but some additional technology is needed to prove (3), which we shall now describe:

III Proofs

Firstly, the proof of (3) is based on the Riemann-Weil-Mestre explicit formulas: let F a "nice" test function (integrable with compact support) note $\hat{F}(s) = \int_{\mathbb{R}} F(x)e^{sx}dx$ its Laplace transform then one has

$$\sum_{L(f,\rho)=0} \hat{F}\left(\rho-\frac{1}{2}\right) = F(0)\log q - 2\sum_{n\geq 1} \frac{a_n(f)}{\sqrt{n}}\Lambda(n)F(\log n) - \int_0^{\infty}\left(\frac{F(x)e^{-x}}{1-e^{-x}} - \frac{e^{-x}}{x}\right)dx$$

where $$\sum_{n\geq 1} \frac{a_n(f)}{n^s} = \frac{L'}{L}(f,s)$$

Ideally one would like to take $\hat{F}$ supported in a small neighborhood of $\frac{1}{2}$ (of length $\alpha/\log q$) but this is not possible. On GRH, the choice of F and $\hat{F}$ is a real variable one and one F choosen the ~~ev~~ $\frac{3}{2}$ result is ~~mainly~~ obtained after averaging over f. Although Brumer mentionned that density theorems for zeros of automorphic L functions might yield unconditionnal result, no suitable test function and density theorems (which are quite subtle indeed) existed at that time to vindicate him.

III.1 The Perelli-Pomykala test function.

Recently, with the purpose of proving unconditionnal upper bounds for the analytic rank of twisted automorphic L-functions on average Perelly and Pomykala [P.P] constructed such a nice F.
Specifically they shown the existence of a F which is

even, C^∞, non negative, compactly supported with $F(0)=1$

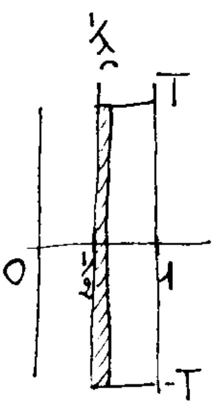
and

(i) $\mathrm{Re}\,\hat{F}(s) \geq 0$ if $|\sigma| \leq 1$ (Positivity)

(ii) $\hat{F}(s) \ll \exp\left(c|\mathrm{Re}\,s| - c'|s|^{3/4}\right)$ (Rapidly decreasing on vertical strips)

Let $\lambda > 1$ a scaling parameter, and choose for test function

$F_\lambda(x) = F\left(\frac{x}{\lambda}\right)$ and set $\varphi_\lambda(s) = \hat{F}_\lambda\left(s-\frac{1}{2}\right) = \lambda \hat{F}\left(\lambda\left(s-\frac{1}{2}\right)\right)$



Using the explicit formula for individuals f one estimates the contribution of the zeros $\rho := \beta + i\gamma$ ~~easily~~ with $|\gamma| \geq T = \log^3 q$ easily then we sum over f, take real part and drop by positivity (i) the sum

$$\sum_f \sum_{\substack{\rho \; |\beta - \frac{1}{2}| < \frac{1}{\lambda} \\ |\gamma| \leq T \\ \rho \neq \frac{1}{2}}} \mathrm{Re}\,\varphi_\lambda(\rho) \geq 0$$

On the right-hand side, use the Eichler-Selberg trace formula (as Brumer originally did), and Ramanujan bound to control this side.

We are left with

$$\lambda \sum_f \mathrm{ord}_{s=\frac{1}{2}} L(f,s) \ll \sum_f \sum_{\substack{|\beta - \frac{1}{2}| \geq \frac{1}{\lambda} \\ |\gamma| \leq T}} \mathrm{Re}\,\varphi_\lambda(\rho) + \text{Admissible Term.}$$

at this point dissect the rectangle $\left[\frac{1}{2}+\frac{1}{\lambda}, 1\right] \times [-T, T]$ into $\lambda \times \lambda T$ squares of side $\frac{1}{\lambda}$ and we achieve the proof thanks to property (ii) of Perelli-Pomykala test function and the following density theorem: $(\lambda = c \log q)$

Thm 3 (KM): $\exists c, B > 0$ absolute st. for $t_2 - t_1 \geq \frac{1}{\log q}$ and $\alpha \geq \frac{1}{2} + \frac{1}{\log q}$ one has

$$\sum_{f \in S_2^+(q)} N(f, \alpha, t_1, t_2) \ll \left(1 + \max(|t_1|, |t_2|)\right)^B q^{1 - c(\alpha - \frac{1}{2})} (\log q) |t_2 - t_1|$$

Rmk: what is important here is that the exponent in $\log q$ is one for then it can be dropped for $|t_2 - t_1| \simeq \log^{-1} q$, moreover the exponent in q is 1 for $\alpha = \frac{1}{2}$ (von Mangolt formula) but decreases enough as $\alpha \to 1/2$. had we a larger exponent in $\log q$, we would get (3) with C replaced by $C \log\log q$.

This **result** is an analog of a 50 year old theorem of Selberg about Dirichlet L-functions. Curiously enough, it seems to have been forgotten by many experts (certainly under the influence of the large sieve which generated zero density theorems with very good exponents in q but weaker in $\log q$).

Selberg's method can be extended and one is reduced to prove estimates of the type:

$$(4) \qquad \sum_f |M(f, \beta + it) L(f, \beta + it)|^2 \ll (1 + |t|)^B q$$

with $\beta = \frac{1}{2} + \frac{1}{\log q}$

and $M(f, s) = \sum_{m \leq M} \frac{x_m}{m^s} \lambda_f(m)$ is a Dirichlet polynomial (so called "Mollifier", which in this case is a truncated version of the Dirichlet series $L^{-1}(f, s)$

Rmk: This kind of estimate for $t=0$ $\beta=\frac{1}{2}$ is exactly the one needed to prove (1) and (2) with the difference that the $(x_m)_{m\leq M}$ are to be choosen with care to optimize c_1 and c_2.

The proof of (4) breaks into 2 steps

Step 1) Note $\displaystyle\sum_{f\in S_2^+(q)}^{h} \alpha_f := \sum_{f\in S_2^+(q)} \frac{\alpha_f}{4\pi(f,f)}$ (h for "harmonic")

(4^h) Prove $\displaystyle S^h = \sum_f^h \left| \quad \right|^2 \ll (1+|t|)^B$

Step 2) Remove the harmonic weight $1/4\pi(f,f) \sim \frac{1}{q}$

Step 1 the functional equation for $L(f,s)$ gives (by contour integration)

$$S^h = \sum_b \frac{\varepsilon(b)}{b} \sum_{m_1, m_2, n} \frac{x_{bm_1}\overline{x_{bm_2}}}{\sqrt{m_1 m_2 n}} \eta_t(n)\, U\left(\frac{2\pi m_1 m_2}{q}\right) \times \left\{ \sum_f^h \lambda_f(m_1 m_2)\lambda_f(n) \right\}$$

U nice

then Peterson's trace formula is applied to the inner sum:

$$\sum_f^h \lambda_f(m)\lambda_f(n) = \delta_{m,n} - \frac{1}{2}\sum_{c>0} \frac{1}{cq} S(m,n;cq)\, J_1\left(\frac{4\pi\sqrt{mn}}{cq}\right)$$

and Weil's bound for Kloosterman sum's gives (with $M=q^{\frac{1}{4}-\varepsilon}$)

$\delta = \frac{1}{\log q}$

$$S^h \simeq \zeta_q(1+2\delta) \sum_k v_\delta(k) \left| \sum_m \frac{\eta_t(m)}{m^{1+\delta}} x_{km} \right|^2 + \zeta_q(1-2\delta) \sum_k v_{-\delta}(k) \left| \sum_m \frac{\eta_t(m)}{m^{1-\delta}} x_{km} \right|^2$$

then we remark with Selberg that $\zeta_q(1-2s)\leqslant 0$ (ζ doesn't vanish on [0,1]) and the second term is dropped by positivity. Finally contour integration techniques (using zero free region of ζ) gives the bound (4^h).

Step II

to remove the $1/4\pi(f,f)$-weigh we appeal to its interpretation in term of the symetric square L-function, we have

$$L(\mathrm{sym}^2 f,1) \simeq \frac{(f,f)}{q}.$$

One has $\displaystyle\sum_f \alpha_f = 4\pi \sum_f{}^h (f,f)\alpha_f \simeq 4\pi q \sum_f{}^h L(\mathrm{sym}^2 f,1)\alpha_f$

We want to replace $L(\mathrm{sym}^2 f,1)$ by a short Dirichlet polynomial at 1

This would follow, for individual f, from Riemann's hypothesis for $L(\mathrm{sym}^2 f,s)$.

However, we can avoid any unproven hypothesis because of the averaging over the f's.

write $L(\mathrm{sym}^2 f,1) = \underbrace{\displaystyle\sum_{n\leqslant y} \rho_f(n)n^{-1}}_{\omega_f(y)} + \text{Error term}$

and decompose $\omega_f(y) = \omega_f(x) + \omega_f(x,y)$ with $x = q^{\varepsilon}$.

As x is so small, we can incorporate $\omega_f(x)$ into the mollifier without changing too many things.

The other term is handled by Hölder's inequality

$$\sum_f^h \omega_f(x,y)\,\alpha_f \leqslant \left(\sum_f \left(\omega_f(x,y)^r\right)^2\right)^{\frac{1}{2r}} \underbrace{\left(\sum_f \left(\frac{\alpha_f}{4\pi(f,f)}\right)^{\frac{2r}{2r-1}}\right)^{1-\frac{1}{2r}}}_{\ll_t q^{-\gamma} \text{ with } \gamma>0, \text{ by the previous result on } S^h \text{ essentially}}$$

Using the multiplicativity of the $\rho_f(n)$'s (which follows from the Euler product for $L(\mathrm{sym}^2 f, s)$) we get

$$\sum_f \left(\left(\omega_f(x,y)\right)^r\right)^2 \simeq \sum_f \left| \sum_{x^r \leqslant n \leqslant y^r} \rho_f(n)\,\frac{c(n)}{n}\right|^2 \quad \left(c(n) \ll \tau(n)^B\right),$$

We finish by applying the following "large sieve inequality" for these GL_3-objects which is due to Duke-Kowalski

<u>Thm</u> (Duke-Kowalski [DK]): for $N > q^9$ and for any complexs $(c_n)_{n\leqslant N}$ one has

$$\sum_{f\in S_2^+(q)} \left|\sum_{n\leqslant N} c_n \rho_f(n)\right|^2 \ll N \log^{16} N \sum_n |c_n|^2.$$

Il ne reste qu'à choisir r assez grand pour que $x^r = q^{r\varepsilon} > q^9$.

□

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