UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL ATOMIC ENERGY AGENCY
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE



H4.SMR/1011 - 17



Fourth Workshop on Non-Linear Dynamics
and Earthquake Prediction

6 - 24 October 1997

# *Abelian Sandpile Models: From Earthquakes and Avalanches to Graphs, Games, and Groebner Bases*



*A. GABRIELOV*

Purdue University
Dept. of Mathematics
Earth and Atmospheric Sciences
West Lafayette, Indiana
U.S.A.

MAIN BUILDING STRADA COSTIERA, 11 TEL. 2240111 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 2249911 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 2240311 TELEFAX 2240310 TELEX 460392
ENRICO FERMI BUILDING VIA BEIRUT, 6 (TELEPHONE, FAX AND TELEX THROUGH MAIN BUILDING)

Physica A 195 (1993) 253–274
North-Holland

PHYSICA A

# Abelian avalanches and Tutte polynomials

Andrei Gabrielov

*Department of Geology, Cornell University, Ithaca, NY 14853, USA*
*and Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA*

Received 15 November 1992

We introduce a class of deterministic lattice models of failure, Abelian avalanche (AA) models, with continuous phase variables, similar to discrete Abelian sandpile (ASP) models. We investigate analytically the structure of the phase space and statistical properties of avalanches in these models. We show that the distributions of avalanches in AA and ASP models with the same redistribution matrix and loading rate are identical. For an AA model on a graph, statistics of avalanches is linked to Tutte polynomials associated with this graph and its subgraphs. In the general case, statistics of avalanches is linked to an analog of a Tutte polynomial defined for any symmetric matrix.

## 1. Introduction

Different cellular automaton models of failure (sand piles, avalanches, forest fires, etc.), starting with Bak, Tang and Wiesenfeld (BTW) [1], were introduced in connection with the concept of self-organized criticality [2]. Traditionally, all of these models are considered on uniform cubic lattices of different dimensions. Recently Dhar [3] suggested a generalization of the BTW model with an arbitrary (modulo some natural sign restrictions) matrix $\Delta$ of redistribution of accumulated particles during an avalanche. An important property of this Abelian sand pile (ASP) model is the presence of an Abelian (commutative) group governing its dynamics. Abelian sandpiles were studied in ref. [4], and one special case is treated in ref. [5]. In a non-dissipative case ($\Sigma_j \Delta_{ij} = 0$, for all $i$) an avalanche in the ASP model coincides with a chip-firing game on a graph [6] where $\Delta$ is a Laplace matrix of the underlying graph.

Another class of lattice models of failure, slider block models introduced in ref. [7] and studied in ref. [8], as well as models [9–13] which are equivalent to quasistatic block models, have continuous time and some quantity which accumulates and is redistributed at lattice sites. This quantity is called the slope, height, stress or energy by different authors. In slider block models it corresponds to force [11]. We use the term *height* as in ref. [3].

0378-4371/93/$06.00 

We introduce here a class of deterministic lattice models with continuous time and height values at the sites of the lattice, and with an arbitrary redistribution matrix. For a symmetric matrix, these models are equivalent to arbitrarily interconnected slider block systems. One of these models, which in the case of a uniform lattice coincides with models studied in ref. [10] and in ref. [13] (as series case *a*), is characterized by the same Abelian property as ASP models. We call this the *Abelian avalanche* (AA) model.

The stationary behavior of the AA model is periodic or quasiperiodic, depending on the loading rate vector. We show, however, that the distribution of avalanches for a discrete, stochastic ASP model is *identical* to the distribution of avalanches for an arbitrary quasiperiodic trajectory (or to its average over all periodic trajectories) of a continuous, deterministic AA model with the same redistribution matrix and loading rate.

For the AA model on a graph, the combinatorial structure of the phase space and the corresponding statistics of avalanches is described in terms of the invariants of the graph and its subgraphs called "Tutte polynomials" [14]. In the general case, the same is true for an analog of a Tutte polynomial defined for any symmetric matrix.

In the second section, we introduce different types of avalanche models. In the third section, we investigate the properties of AA models. In the fourth section, we study the structure of the set of recurrent configurations and derive analytic formulas for the mean number of avalanches in the AA model. Some of our results are new also for ASP models. In the fifth section, we establish the equivalence of distributions of avalanches for AA and ASP models. In the sixth section, we describe the structure of the phase space for AA models on a graph in terms of Tutte polynomials. In the seventh section, we describe the distribution of avalanches in the AA model in terms of an analog of a Tutte polynomial for an arbitrary symmetric matrix. The proofs of the different statements are given in the appendix.

## 2. Avalanche models

Let $V$ be a finite set of $N$ elements (sites), and let $\Delta$ be a $N \times N$ real matrix with indices in $V$, with the following properties:

$$\Delta_{ii} > 0 \,, \quad \text{for all } i; \qquad \Delta_{ij} \leq 0 \,, \quad \text{for all } i \neq j \,; \tag{1}$$

$$s_i = \sum_j \Delta_{ij} \geq 0 \,, \qquad \text{for all } i \,. \tag{2}$$

The value $s_i$ is called the *dissipation* at a site $i$.

At every site $i$, we define a positive real value $h_i$ (height). The set $\boldsymbol{h} = \{h_i\}$ is called the *configuration* of the system. For every site $i$, a threshold $H_i$ is defined, and configurations with $h_i < H_i$ are called *stable*. For every stable configuration, the height $h_i$ increases in time with a constant rate $v_i \geqslant 0$ until it exceeds a threshold $H_i$ at a site $i$. Then the site $i$ breaks, and the heights are redistributed as follows:

$$h_j \to h_j - \Delta_{ij}\,, \qquad \text{for all } j\,. \tag{3}$$

If after this redistribution, any heights exceed thresholds at some other sites, these sites also break according to (3), and so on, until we arrive at a stable configuration and the loading resumes. The sequence of breaks is called an *avalanche*.

The model (3) has the important *Abelian* property (see below): the stable configuration of the system after an avalanche, and the number of breaks at any site during an avalanche, *do not depend* on the order of breaks during the avalanche. We call this model an *Abelian avalanche* (AA) model.

It may happen that an avalanche continues without end. We can avoid this possibility by suggesting that the system is *weakly dissipative* in the following sense. We require that from every non-dissipative site $i$, $s_i = 0$, there exists a path to a dissipative site $j$, $s_j > 0$, i.e. a sequence $i_0, \ldots, i_m$ with $i_0 = i$, $i_m = j$ and $\Delta_{i_{k-1}i_k} < 0$, for $k = 1, \ldots, m$. It is easy to show that in a weakly dissipative system every avalanche is finite.

We suppose also that the system is *properly loaded*, i.e. for every site $j$, there exists a path from a loaded site $i$, $v_i > 0$, to the site $j$. If this is not the case, some parts of the system do not evolve in time. For a properly loaded system, the rate of breaks at every site is positive.

In the case of a symmetric matrix $\Delta$ and $v_i = s_i$, for all $i$, this model is equivalent to a system of blocks where the $i$th block is connected to the $j$th block by a coil spring of rigidity $\Delta_{ij}$ and to a slab moving with a unit rate by a leaf spring of rigidity $s_i$. For every block, a static friction force $H_i$ is defined, and a block is allowed to move by one unit of space when the total force $h_i$ applied to this block from other blocks and the moving slab exceeds $H_i$. The dissipation property means that the loading rate is positive at least for one block in every connected component of the system.

*Remark 1.* The previous definition can be also reformulated for the model where

$$h_j \to h_j - \Delta_{ij}h_i\,, \qquad \text{for } j \neq i \quad \text{and} \quad h_i \to 0 \tag{4}$$

at a break of the $i$th site, studied in refs. [11,12] and in ref. [13] as series case $b$. This corresponds to a system of blocks in which every block stops when the total force acting on it vanishes. In this case, in addition to the redistribution rule, the choice of one or several (e.g. all) possible breaks in fast time should be specified.

Finally, we can introduce a system with parallel redistribution by [13] considering continuous fast time $\theta$ and redistribution rules

$$\frac{\partial h_j}{\partial \theta} \to \frac{\partial h_j}{\partial \theta} - \Delta_{ij}\,, \qquad \text{for all } j\,, \tag{5}$$

when the $i$th element breaks at $h_i = H_i$, and

$$\frac{\partial h_j}{\partial \theta} \to \frac{\partial h_j}{\partial \theta} + \Delta_{ij}\,, \qquad \text{for all } j\,, \tag{6}$$

when the $i$th element heals at $h_i = 0$. This corresponds to a system of blocks where several blocks are allowed to slip simultaneously during an avalanche.

These two models, i.e. specified by eq. (4) and by eqs. (5) and (6), are not Abelian.

*Remark 2.* All the models introduced here are deterministic. If we replace uniform loading in time by random loading then a class of stochastic models can be defined. Many of the properties of the deterministic AA model are valid also for the stochastic case.

## 3. Abelian avalanches

We want to establish the Abelian properties of the model (3). Many of our arguments are similar to those in ref. [3].

The dynamics of the model does not change if we replace the values $H_i$ by some other values, and add the difference to all configuration vectors. For convenience we take $H_i = \Delta_{ii}$. In this case, $h_i \geq 0$ for any trajectory of the system when the $i$th element has been broken at least once. Hence only configurations with non-negative heights at all sites are relevant for the long-term dynamics. Let $\mathcal{S} = \{0 \leq h_i < \Delta_{ii}\}$ be the set of all stable configurations in $\mathbf{R}_+^V = \{h_i \geq 0, \text{ for all } i\}$.

Let $h(t)$ be a trajectory of the model (3), and let $n = \{n_i(t),\ i \in V\}$ be the number of breaks of a site $i$ during a time interval $t$. It is easy to show (see appendix A) that the average rate of breaks per unit time $r = n(t)/t$ satisfies

$$\Delta' r \to v\,, \qquad \text{for } t \to \infty\,. \tag{7}$$

Here $\Delta'$ is transpose of $\Delta$, and $v$ is the loading rate in a deterministic model, or the mean loading rate in a stochastic model.

As the rate of breaks at every site is positive for a properly loaded system, this implies that $\Delta'(\mathbf{R}_+^V) \supset \mathbf{R}_+^V$. In particular, $\Delta$ is nonsingular. We have also $\det(\Delta) > 0$ because the set of all weakly dissipative matrices satisfying (1) and (2) is a convex domain containing a unit matrix.

Let $h$ be any configuration in $\mathbf{R}_+^V$. Let $i_1, \ldots, i_m$ be an avalanche started at $h$, i.e. a sequence of consecutive breaks (3) such that configurations after all breaks but the last are unstable and the configuration $h' \in \mathcal{S}$ after the $m$th break is stable.

It can be shown (see appendix B) that $h' = \mathcal{A}h$ does not depend on the possible choice of breaks, and is completely determined by the initial configuration $h$. More precisely, let $n_i$ be the number of times the site $i$ breaks during the avalanche. Then

$$n_i \text{ depends only on } h\,, \quad \text{for all } i\,. \tag{8}$$

Hence an *avalanche* operator

$$\mathcal{A} : \mathbf{R}_+^V \to \mathcal{S} \tag{9}$$

is defined.

For any vector $u \in \mathbf{R}_+^V$ we define a *loading* operator $\mathcal{B}_u h = h + u$. We call $\mathcal{C}_u = \mathcal{A} \circ \mathcal{B}_u$ a *load-avalanche* operator.

We claim that every pair of load-avalanche operators commutes. More precisely, for any $u \in \mathbf{R}_+^V$ and $v \in \mathbf{R}_+^V$, we have

$$\mathcal{C}_u \circ \mathcal{C}_v = \mathcal{C}_{u+v}\,. \tag{10}$$

The proof (see appendix B) follows the arguments of ref. [6] for chip-firing games.

Following ref. [3], we define *recurrent* configurations of the AA model as those stable configurations that can be reached after arbitrarily long time intervals.

We claim that for a weakly dissipative, properly loaded system, the set $\mathcal{R}$ of all of these configurations does not depend on $v$ and has volume $\det(\Delta)$.

Let $\delta_i = (\Delta_{i1}, \ldots, \Delta_{iN})$ be the $i$th row vector of the matrix $\Delta$. Integer combinations of vectors $\delta_i$ generate a lattice $\mathcal{L}$ in $\mathbf{R}^V$. Two configurations $h$ and $h'$, are called *equivalent* if $h' - h$ belongs to $\mathcal{L}$. A subset in $\mathbf{R}^V$ is called a *fundamental domain* for $\mathscr{L}$ if, for every configuration $h \in \mathbf{R}^V$, it contains exactly one configuration equivalent to $h$. The volume of every fundamental domain is equal to $\det(\Delta)$.

The rule (3) for breaks can be rewritten as $h \to h - \delta_i$. For every $h \in \mathbf{R}_+^V$, configuration $\mathscr{A}h$ is equivalent to $h$ and belongs to $\mathscr{S}$. Hence $\mathscr{S}$ contains a fundamental domain for $\mathscr{L}$.

Let $u \in \mathbf{R}_+^V$ and $\mathscr{S}_u = \mathscr{S} + u$. Then $\mathscr{S}_u$ contains a fundamental domain for $\mathscr{L}$ because this property is translation-invariant. Hence $\mathscr{A}(\mathscr{S}_u) = \mathscr{C}_u(\mathscr{S})$ contains a fundamental domain for $\mathscr{L}$.

It can be shown (see appendix C) that

$$C_u(\mathscr{S}) \text{ is a fundamental domain for } \mathscr{L} \text{ if } u_i \geqslant \Delta_{ii}\,, \quad \text{for all } i\,. \tag{11}$$

The Abelian property (10) implies that the intersection of images of any two load-avalanche operators $\mathscr{C}_u$ and $\mathscr{C}_v$ contains a fundamental domain for $\mathscr{L}$, because it contains $\mathscr{C}_{u+v}(\mathscr{S})$, hence the two images coincide when both vectors $u$ and $v$ have large enough components.

This proves that $\mathscr{R}$ is a fundamental domain for $\mathscr{L}$, hence its volume is $\det(\Delta)$, when all components of the loading rate vector $v$ are positive, because all components of $vt$ are large enough for large values of $t$. If some of $v_i$ are 0, the proper loading condition guarantees that, for large $t$, there exists an avalanche starting at $vt$ and passing through a vector with large enough components, hence $\mathscr{R}$ is a fundamental domain for $\mathscr{L}$ also in this case.

The dynamics of the system on $\mathscr{R}$ in a deterministic model is defined by the break rate vector $r = \Delta'^{-1}v$. If this vector is collinear to an integer vector, $Tr = n$, then $vT = \Delta' n$, hence every trajectory is periodic, with a period $T$, and a site $i$ breaks $n_i$ times during a period $T$, for every periodic trajectory. Otherwise, every trajectory is quasiperiodic.

In any case, the measure $\mathrm{d}h = \Pi_{i \in V}\, \mathrm{d}h_i$ on $\mathscr{R}$ is invariant under the dynamics of the system. This is also true for the random loading.

As a result, (7) has the following implication:

$$\frac{1}{\det(\Delta)} \int_{\mathscr{R}} n(h + v)\, \mathrm{d}h = \Delta'^{-1} v\,, \tag{12}$$

where $n(h) = \{n_j(h)\}$, and $n_j(h)$ is the number of breaks at a site $j$ during an avalanche started at $h$.

Let

$$\mathscr{R}_i = \bar{\mathscr{R}} \cap \{h_i = \Delta_{ii}\} \tag{13}$$

be the set of (unstable) configurations where the recurrent avalanches with a first break at $i$ start. Here $\bar{\mathcal{R}}$ is the closure of $\mathcal{R}$. For any quasiperiodic trajectory of the system (in the periodic case, for a randomly chosen periodic trajectory), the mean (per unit time) number of times it crosses a domain $D \subset \mathcal{R}_i$ is equal to

$$p_i(D) = v_i \operatorname{vol}(D)/\operatorname{vol}(\mathcal{R}) = v_i \operatorname{vol}(D)/\det(\Delta) . \tag{14}$$

Here vol($D$) is the volume in $(|V|-1)$-dimensional space $\{h_i = \Delta_{ii}\}$.

In particular, the mean number of avalanches started at $i$ is equal to

$$p_i(\mathcal{R}_i) = v_i \operatorname{vol}(\mathcal{R}_i)/\det(\Delta) . \tag{15}$$

The mean number of breaks at a site $j$ per unit time can be computed from (14) as

$$r_j = \sum_i \frac{v_i}{\det(\Delta)} \int_{\mathcal{R}_i} n_j(\boldsymbol{h}) \, \mathrm{d}\boldsymbol{h}'_i , \tag{16}$$

where $\mathrm{d}\boldsymbol{h}'_i = \mathrm{d}\boldsymbol{h}/\mathrm{d}h_i$ is the measure on $\mathcal{R}_i$ and $n_j(\boldsymbol{h})$ is the number of breaks at a site $j$ during an avalanche started at $\boldsymbol{h}$.

Due to (7), $\boldsymbol{r} = \Delta'^{-1}\boldsymbol{v}$. Hence

$$\int_{\mathcal{R}_i} n_j(\boldsymbol{h}) \, \mathrm{d}\boldsymbol{h}'_i = \det(\Delta)\,(\Delta^{-1})_{ij} , \tag{17}$$

and the mean (per avalanche) number of breaks at a site $j$ during avalanches started at a site $i$ is

$$m_{ij} = \frac{v_i}{\det(\Delta)} \int_{\mathcal{R}_i} n_j(\boldsymbol{h}) \, \mathrm{d}\boldsymbol{h}'_i = \frac{\det(\Delta)\,(\Delta^{-1})_{ij}}{\operatorname{vol}(\mathcal{R}_i)} . \tag{18}$$

The value of vol($\mathcal{R}_i$) is found in the next section, for the case when every site breaks at most once during an avalanche.

*Remark.* In the periodic case, a single trajectory can contain avalanches of different sizes, and for a large system, in a time interval shorter than its period, it can be indistinguishable from a chaotic trajectory. This effect (called "periodic chaos") was found in ref. [13] for a uniform lattice.

## 4. Recurrent configurations

To investigate the structure of the set $\mathcal{R}$ of recurrent configurations, we note first that, for any $h \in \mathcal{R}$ and any vector $u \in \mathbf{R}_{+}^{V}$, configuration $h + u$ belongs to $\mathcal{R}$ if it belongs to $\mathcal{S}$.

Let $Q = (\Delta_{11}, \ldots, \Delta_{NN})$. For an integer vector $n$, let $P = Q_n = Q - \Delta' n$ be a configuration equivalent to $Q$, and let $\mathcal{V}_n = \{h_i < P_i\}$ be an open negative octant with a vertex at $P$. It can be shown (see appendix D) that

$$\mathcal{R} = \mathcal{S} \setminus \cup' \mathcal{V}_n , \tag{19}$$

where the union $\cup'$ is taken over all $n$ with at least one positive component. If $0 \leqslant n_i \leqslant 1$, for all $i$, the sets $\mathcal{S} \cap \mathcal{V}_n$ coincide with *stable forbidden subconfigurations* [3]

$$\mathcal{F}_X = \left\{ h \in \mathcal{S}, h_j < - \sum_{i \in X,\, i \neq j} \Delta_{ij} , \quad \text{for } j \in X \right\} . \tag{20}$$

Here $X$ is the set of sites $i$ with $n_i = 1$. Dhar [3] argues that the union of sets (20) over all nonempty subsets of $V$ coincides with $\mathcal{S} \setminus \mathcal{R}$.

In general, this is not true. For a $2 \times 2$ matrix $\Delta$ with $\delta_1 = (2, -1)$ and $\delta_2 = (-3, 4)$ we have $Q_{(2,1)} = (1, 2)$. Hence configurations with $h_1 < 1$ and $h_2 < 2$ are not recurrent, and only configurations with $h_2 < 1$ are forbidden.

It can be shown, however (see appendix E) that

$$\mathcal{R} = \mathcal{S} \setminus \bigcup_{X \subset V} \mathcal{F}_X , \tag{21}$$

i.e. all *allowed* stable configurations (i.e. those that do not contain any forbidden subconfigurations) are recurrent, when

$$\sum_{i \in V} \Delta_{ij} \geqslant 0 , \qquad \text{for all } j . \tag{22}$$

In particular, this is true when $\Delta$ is symmetric.

Suppose now that

$$\sum_{i \in V} \Delta_{ij} > 0 , \qquad \text{for all } j . \tag{23}$$

In this case, the configuration $Q - \Sigma_{i \in V} \delta_i$ is stable. Hence every site can break at most once in an avalanche started at any configuration $h$ with $h_i \leqslant \Delta_{ii}$, for all $i$.

Let $\mathcal{R}_i$ be the set (13) of recurrent configurations initiating avalanches with a first break at $i$. If (23) holds, the values of $\boldsymbol{h}' = \{h_j,\ j \in V,\ j \neq i\}$ in $\mathcal{R}_i$ are defined, due to (21), by the same inequalities as the set of all allowed configurations for a model (3) on $V \backslash \{i\}$ with a matrix $\Delta(i)$, where $\Delta(i)$ is $\Delta$ with the $i$th row and the $i$th column deleted. Hence $\mathcal{R}_i$ coincides with the set of all recurrent configurations for $\Delta(i)$, and

$$\mathrm{vol}(\mathcal{R}_i) = \det[\Delta(i)] = \det(\Delta)\,(\Delta^{-1})_{ii}\,. \tag{24}$$

Due to (15), the mean number of avalanches started at a site $i$ per unit time is

$$p_i(\mathcal{R}_i) = v_i \det[\Delta(i)]/\det(\Delta) = v_i(\Delta^{-1})_{ii}\,. \tag{25}$$

Hence the mean number of avalanches in the system per unit time is equal to

$$\sum_i v_i(\Delta^{-1})_{ii}\,. \tag{26}$$

Due to (18), we have

$$m_{ij} = (\Delta^{-1})_{ij}/(\Delta^{-1})_{ii}\,, \tag{27}$$

where $m_{ij}$ is the mean (per avalanche) number of breaks at a site $j$ during avalanches started at a site $i$.

*Remark.* In case (22) holds but (23) is not valid, the volume of $\mathcal{R}_j$ is less than $\det[\Delta(j)]$. Due to (21), for every subset $F \subset V \backslash \{j\}$ such that $\Delta_{jj} + \Sigma_{i \in F} \Delta_{ij} = 0$, configurations with $h_j = \Delta_{jj}$ and $h_\nu < -\Delta_{j\nu} - \Sigma_{i \in F} \Delta_{i\nu}$, for $\nu \in F$, do not belong to $R_j$. It can be shown, however, that the formulas (25)–(27) are still valid for the following modification of the model.

We allow every site to break at most once in an avalanche. At the end of an avalanche started at a site $i$, the value $h_i$ can be still at the threshold level $\Delta_{ii}$. In this case we immediately start a new avalanche at a site $i$, and so on until finally we arrive at a stable configuration.

For the original model, eqs. (25)–(27) are true if we count every avalanche with the multiplicity of the number of breaks at its starting site.

## 5. Distributions of avalanches in AA and ASP models

There is obvious similarity between the properties of the deterministic, continuous AA model and the stochastic, discrete ASP model. We want to show that, for a matrix $\Delta$ with integer elements, the distribution of avalanches in the AA model is *identical* to the distribution of avalanches in the ASP model with the same matrix $\Delta$ and the same loading rate vector $\boldsymbol{v}$.

For a matrix $\Delta$ with integer elements satisfying (1) and (2), and a loading vector $\boldsymbol{v}$ with $\Sigma\, v_i = 1$, the ASP model is defined as follows. The height $h_i$ at every site $i \in V$ is an integer, $0 \leqslant h_i < \Delta_{ii}$. At every (discrete) time step, we choose a site $i$ with a probability $v_i$ and add a particle at the site $i$, i.e. add 1 to the height $h_i$. If $h_i = \Delta_{ii}$ after this operation, we start an avalanche according to the rule (3). After termination of an avalanche, we proceed with adding the next particle. Only uniform loading (all $v_i$ equal) was considered in ref. [3]. However, the generalization to any proper loading rate vector is straightforward.

As is shown in ref. [3], the recurrent configurations for the ASP model are precisely the integer points in the set $\mathcal{R}$ of recurrent configurations of the AA model with the matrix $\Delta$, every point is attended with equal probability, and the total number of these points $\#(\mathcal{R})$ is equal to $\det(\Delta) = \text{vol}(\mathcal{R})$.

For the ASP model, a recurrent configuration with $h_i = \Delta_{ii}$ starting an avalanche at a site $i$ belongs to the set of integer points in the set $\mathcal{R}_i$ defined in (13). For a randomly chosen configuration in $\mathcal{R}$, the probability of initiating an avalanche at a site $i$ at any time step is equal to $p_i = v_i \#(\mathcal{R}_i)/\det(\Delta)$. Due to (19), the number of integer points $\#(\mathcal{R}_i)$ in $\mathcal{R}_i$ is equal to $\text{vol}(\mathcal{R}_i)$. Hence $p_i = p_i(\mathcal{R}_i)$ coincides with the mean number of avalanches initiated at $i$ defined in (15) for the AA model. For any integer vector $k = \{k_j\}$, the set of points $\mathcal{R}_{i,k} \subset \mathcal{R}_i$ where an avalanche with $k_j$ breaks at a site $j$, for all $j \in V$, starts coincides with

$$\Big(\mathcal{R} + \sum_j k_j \delta_j\Big) \cap \mathcal{R}_i\,, \tag{28}$$

where $\delta_j$ is the $j$th row vector of $\Delta$. Due to (19), the number of integer points in (28) coincides with its volume. Hence the mean number per time step of avalanches started at a site $i$, with $k_j$ breaks at a site $j$, which is equal to $v_i \#(\mathcal{R}_{i,k})/\det(\Delta)$ for the ASP model, coincides with the mean number per unit time of avalanches of the same type for a quasiperiodic trajectory (in the periodic case, for a randomly chosen periodic trajectory) in the AA model, which is equal to

$$p_i(\mathcal{R}_{i,k}) = v_i\, \text{vol}(\mathcal{R}_{i,k})/\det(\Delta)\,, \tag{29}$$

according to (14).

This equivalence implies, in particular, that the size distributions of avalanches in AA and ASP models, with the same matrix $\Delta$ and loading rate $\boldsymbol{v}$, coincide.

## 6. Avalanches on graphs

A graph $G$ is a set $V(G)$ of *vertices* and a set $E(G)$ of *edges* with a relation or rule of incidence which associates with every edge in $E(G)$ two vertices in $V(G)$ called its *ends*. An edge with different ends is a *link*. Its ends are called *adjacent* vertices, or *neighbors*. An edge with identical ends is a *loop*. For every vertex $i$ its *degree* $d_i$ is equal to the number of the edges incident to it, loops counting twice.

If $V' \subseteq V(G)$ and $E' \subseteq E(G)$ with the ends of any edge in $E'$ belonging to $V'$ then a graph $H$ with $V(H) = V'$, $E(H) = E'$ and the incidence relation induced from $G$ is called a subgraph of $G$. If $V' = V$ then H is a *spanning* subgraph. If $U \subseteq V(G)$ then the subgraph $H = G[U]$ with $V(H) = U$ and $E(H)$ consisting of all edges of $G$ with both ends in $U$ is called an *induced* subgraph of $G$.

An $n$-arc is a graph with vertices $i_1, \ldots, i_n$ and edges $e_1, \ldots, e_{n-1}$ where $i_\nu$ and $i_{\nu+1}$ are the ends of the edge $e_\nu$. An *n-circuit* is an $n$-arc with additional edge $e_n$ with ends $i_n$ and $i_1$. A graph $G$ is *connected* if any two of its vertices belong to an arc in G. A *tree* is a connected graph without circuits.

The *tree number* $T(G)$ of a graph $G$ is defined as the total number of different spanning trees of this graph. $T(G) > 0$ only for connected graphs, $T(G) = 1$ for any tree, and $T(G) = n$ for a circuit of order $n$.

For every edge $e$ in a graph $G$, the operation of *deletion* $G-e$ is defined by removing $e$ from $E(G)$, and the operation of *contraction* $G/e$ is defined by removing $e$ and identifying the ends of $e$ in $V(G)$. It is easy to show (ref. [14], p. 40) that

$$T(G) = T(G - e) + T(G/e) \tag{30}$$

for every link $e$ of $G$. The functions with this property are often called Tutte polynomials.

We consider only loopless graphs, with multiple edges, and define $G/e$ as a graph with loops removed after contraction of $e$. The property (30) of $T(G)$ remains valid for this operation. We define the order $|G| = |V(G)|$ of a graph $G$ as the number of its vertices.

The Laplace matrix $\Delta(G)$ of a graph $G$ is defined as $\Delta(G)_{ii} = d_i$ and $-\Delta(G)_{ij}$ equal to the number of links between vertices $i$ and $j$, for $i \neq j$. We have $\Sigma_j \Delta(G)_{ij} = 0$, for all $i$.

With any diagonal matrix $S$, with non-negative elements $S_{ii} = s_i$, we can associate an avalanche model with a symmetric matrix $\Delta_S = \Delta(G) + S$. (The model with non-symmetric matrices can be associated with directed graphs, but we do not consider this here.)

Let $v = \{v_i,\ i \in V(G)\}$ be a loading vector for this model. Suppose that $v_i = s_i$, for all $i$, as in a slider block model. Then every recurrent trajectory of the AA model with matrix $\Delta_S$ and loading rate $v$ is periodic, with a period $T = 1$, and every vertex of $G$ breaks once during this period.

Let $G_1, \ldots, G_m$ be an arbitrary partition of $G$ into induced subgraphs, and $i_\nu$ a selected vertex in $G_\nu$. We claim (see appendix F) that the total volume occupied by recurrent configurations generating periodic trajectories with an ordered sequence of avalanches covering sets $V(G_1), \ldots, V(G_m)$ initiated at sites $i_1, \ldots, i_m$ is equal to

$$s_{i_1} T(G_1) \cdots s_{i_m} T(G_m)/m! \tag{31}$$

In particular, this volume does not depend on the order of $G_\nu$.

Hence the total volume occupied by periodic trajectories with avalanches constituting a partition of $G$ into induced subgraphs $G_1, \ldots, G_m$ is equal to

$$S(G_1)\, T(G_1) \cdots S(G_m)\, T(G_m)\,, \tag{32}$$

where $S(G_\nu) = \Sigma_{i \in V(G_\nu)} s_i$.

As the total volume of all recurrent configurations is $\det(\Delta_S)$, we have

$$\sum_{m \geq 1} \sum_{G_1, \ldots, G_m} \prod_{\nu=1}^{m} S(G_\nu)\, T(G_\nu) = \det(\Delta_S)\,, \tag{33}$$

where the sum is taken over all partitions of $G$ into induced subgraphs.

The linear term in $S$ in this expression appears for $m = 1$ and $G_1 = G$. It is equal to $S(G)\, T(G)$. Hence

$$T(G) = \left.\frac{\partial \det(\Delta_S)}{\partial S_i}\right|_{S=0} = \det[\Delta_i(G)]\,, \qquad \text{for every } i \in V\,. \tag{34}$$

Here $\Delta_i(G)$ is the matrix $\Delta(G)$ with the $i$th row and column removed. This is the matrix-tree theorem for graphs (see ref. [14], p. 141).

The mean number of avalanches in a randomly chosen periodic trajectory is equal to

$$\det(\Delta_S)^{-1} \sum_{m \geq 1} m \sum_{G_1, \ldots, G_m} \prod_{\nu=1}^{m} S(G_\nu)\, T(G_\nu)\,. \tag{35}$$

Comparing expressions (35) and (26) for the mean number of avalanches, we have an identity

$$\sum_{m \geq 1} m \sum_{G_1,\ldots,G_m} \prod_{\nu=1}^{m} S(G_\nu)\, T(G_\nu) = \det(\Delta_S) \sum_i s_i (\Delta_S^{-1})_{ii}\,, \tag{36}$$

where the sum in the left part is taken over all partitions of $G$ into induced subgraphs.

Let $H$ be an induced subgraph of $G$, and let $G\backslash H$ be an induced subgraph of $G$ with $V(G\backslash H) = V(G)\backslash V(H)$.

For $i \in V(H)$, the total volume of all periodic trajectories with an avalanche started at $i$ covering $H$ is equal, due to (31), to

$$X(i, H) = s_i T(H)\Big(\sum_{m \geq 1} \sum_{G_1,\ldots,G_m} \prod_{\nu=1}^{m} S(G_\nu)\, T(G_\nu)\Big)\,, \tag{37}$$

where the sum is taken over all partitions of $G\backslash H$ into induced subgraphs.

Applying (33) to $G\backslash H$, we have

$$X(i, H) = s_i T(H) \det[\Delta_S(G\backslash H)]\,. \tag{38}$$

Here $\Delta_S(G\backslash H)$ is the Laplace matrix of $G\backslash H$ with $s_\nu$ added to diagonal elements, for $\nu \in V(G)\backslash V(H)$.

We have also $X(i, H) = s_i \operatorname{vol}(\mathcal{R}_{i,H})$, where $\mathcal{R}_{i,H}$ is the subset of $\mathcal{R}_i$ where the avalanches covering $H$ start. Hence, if $s_i > 0$ then

$$\operatorname{vol}(\mathcal{R}_{i,H}) = T(H) \det[\Delta_S(G\backslash H)]\,. \tag{39}$$

Due to (14), the mean (per unit time) number of avalanches started at $i$ covering $H$ is $p_i(\mathcal{R}_{i,H}) = v_i \operatorname{vol}(\mathcal{R}_{i,H}) / \det[\Delta_S(G)]$, for arbitrary loading rate $\boldsymbol{v}$. Hence

$$p_i(\mathcal{R}_{i,H}) = v_i T(H) \det[\Delta_S(G\backslash H)] / \det[\Delta_S(G)]\,, \tag{40}$$

for an arbitrary loading rate vector. Due to section 5, this is also equal to the mean (per time step) number of avalanches started at $i$ and covering $H$ in the ASP model if all $s_\nu$ are integer.

Finally, we have the following expression for the mean number of avalanches of size $k$, both in the AA and ASP models:

$$\frac{1}{\det[\Delta_S(G)]} \sum_{U \subset V(G),\, |U| = k} T(G[U]) \det[\Delta_S(G\backslash G[U])] \sum_{i \in U} v_i\,. \tag{41}$$

This gives a purely combinatorial expression for the distribution of the avalanches of different sizes. Explicit formulas for this distribution are found in ref. [13] for a circuit of arbitrary order.

*Remark.* The formulas (40) and (41) are valid when $s_i > 0$, for all $i$ with $v_i > 0$ (because (39) is not true when $s_i = 0$). When $s_i = 0$ and $v_i > 0$ for some $i$, these formulas are still valid for the modification of the model suggested in remark at the end of section 4, when every site is allowed to break at most once during an avalanche.

## 7. Tutte polynomials for matrices

For any symmetric matrix $\Delta$ with indices in a set $V$, we define a symmetric matrix $\Delta' = D_{ij}(\Delta)$ (deletion of $(i, j)$) as

$$\Delta'_{ii} = \Delta_{ii} + \Delta_{ij}\,, \qquad \Delta'_{jj} = \Delta_{jj} + \Delta_{ij}\,, \qquad \Delta'_{ij} = 0\,, \tag{42}$$

with other elements of $\Delta$ unchanged, and a symmetric matrix $\Delta'' = C_{ij}(\Delta)$ (contraction of $(i, j)$) as

$$\Delta''_{ii} = \Delta_{ii} + \Delta_{jj} + 2\Delta_{ij}\,, \quad \Delta''_{ik} = \Delta_{ik} + \Delta_{jk}\,, \qquad \text{for } k \neq i, j\,, \tag{43}$$

with the $j$th row and column of $\Delta$ removed and other elements of $\Delta$ unchanged.

For $\Delta = \Delta(G)$, the Laplace matrix of a graph $G$, the matrix $D_{ij}(\Delta)$ is the Laplace matrix of $G$ after deletion of all edges connecting $i$ and $j$, and $C_{ij}(\Delta)$ is the Laplace matrix of $G$ after contraction of all edges with ends at $i$ and $j$.

For every symmetric matrix $\Delta$, let $s_i = \Sigma_j \Delta_{ij}$, and let $\Delta_0$ be the matrix $\Delta$ with the diagonal terms $\Delta_{ii}$ replaced by $\Delta_{ii} - s_i$, for all $i$. Then

$$C_{ij}(\Delta_0) = C_{ij}(\Delta)_0\,, \qquad D_{ij}(\Delta_0) = D_{ij}(\Delta)_0\,, \tag{44}$$

the operation $C_{ij}$ does not change the values of $s_\nu$, and the operation $D_{ij}$ replaces $s_i$ by $s_i + s_j$ leaving the other values $s_\nu$ unchanged.

We call a function $F(\Delta)$ on the set of symmetric matrices a Tutte polynomial if the following properties hold:

(A) For every pair of distinct indices $i$ and $j$,

$$F(\Delta) = F(D_{ij}(\Delta)) - \Delta_{ij} F(C_{ij}(\Delta))\,. \tag{45}$$

(B) Let $\Delta'$ and $\Delta''$ be two matrices with indices in $V'$ and $V''$, and let

$\Delta = \Delta' \times \Delta''$ be a matrix with indices in disjoint union of $V'$ and $V''$, $\Delta_{ij} = \Delta'_{ij}$, for $i, j \in V'$, $\Delta_{ij} = \Delta''_{ij}$ for $i, j \in V''$, $\Delta_{ij} = 0$ otherwise. Then

$$F(\Delta' \times \Delta'') = F(\Delta')\, F(\Delta'') . \tag{46}$$

Let $T(\Delta)$ be a function of a symmetric matrix which does not depend on the values of $s_i$, satisfies (A), and is equal to 1 for a $1 \times 1$ matrix $\Delta$ and to zero for any diagonal matrix of size greater than 1. Then $T(\Delta)$ coincides with the tree number $T(G)$ of a graph $G$ in the case $\Delta = \Delta(G)$.

Let $\Delta$ satisfy (1) and (23). Consider the set of periodic trajectories of the AA model with the matrix $\Delta$ and the loading rate $v_\nu = s_\nu$, for all $\nu$. The same arguments as in appendix F show that the volume $X(\Delta;\, V_1, \ldots, V_m;\, i)$ occupied by all recurrent configurations generating periodic trajectories with an ordered set of avalanches covering subsets $V_1, \ldots, V_m$ of $V$ starting at sites $i_\nu \in V_\nu$ satisfies

$$X(\Delta; V_1, \ldots, V_m; i) = X(C_{ij}(\Delta); V_1, \ldots, V_m; i) - \frac{\Delta_{ij} s_i}{s_i + s_j} X(D_{ij}(\Delta); V_1, \ldots, \hat{V}_\nu, \ldots, V_m; i) \tag{47}$$

when $i = i_\nu$, $j \in V_\nu$, $j \neq i$, $\hat{V}_\nu = V_\nu \setminus \{j\}$, and

$$X(\Delta; V_1, \ldots, V_m; i) = s_1 \cdots s_m \tag{48}$$

when $V_\nu = \{i_\nu\}$, for all $\nu$. Hence

$$X(\Delta; V_1, \ldots, V_m; i) = s_{i_1} T(\Delta_1) \ldots s_{i_m} T(\Delta_m) , \tag{49}$$

where $\Delta_\nu$ is the minor of $\Delta$ with indices in $V_\nu$.

Let $f_m(\Delta)$ be the volume of all periodic trajectories with $m$ avalanches. Due to (49),

$$f_m(\Delta) = \sum_{V_1, \ldots, V_m} S(V_1)\, T(\Delta_1) \cdots S(V_m)\, T(\Delta_m) . \tag{50}$$

Here the sum is taken over all partitions of $V$ into $m$ subsets, $S(V_\nu) = \Sigma_{i \in V_\nu} s_i$.

Let $F(z)(\Delta) = \Sigma_m f_m(\Delta)\, z^m$. Due to (50), $F(z)(\Delta)$ satisfies (45) and (46), i.e. $F(z)$ is a Tutte polynomial. In particular, the total volume of all periodic trajectories is equal to $F(1) = \det(\Delta)$. Hence $\det(\Delta)$ satisfies (45). This implies, in particular, the following identity:

$$F(z)(\Delta) = \det(\Delta_0 + zs) , \tag{51}$$

where $s$ is a diagonal matrix with $s_{ii} = s_i$.

Computing linear terms in $S$ in the expression for $F(1)$, we have the matrix-tree theorem

$$T(\Delta) = \det[\Delta_0(i)] , \qquad \text{for every } i \in V(G) . \tag{52}$$

Here $\Delta_0(i)$ is the matrix $\Delta_0$ with the $i$th row and column removed.

Finally, the mean number of avalanches per unit time in a randomly chosen periodic trajectory is equal to

$$F'(1)/\det(\Delta) = \sum_i (\Delta^{-1})_{ii} \sum_j \Delta_{ij} . \tag{53}$$

## 8. Conclusions

We introduce a class of deterministic lattice models of failure with continuous phase variables, Abelian avalanche (AA) models, with Abelian properties similar to those of the discrete, stochastic Abelian sandpile (ASP) models. We investigate analytically the dynamics, distributions of avalanches and the structure of the phase space of AA models. Depending on the loading rate vector, the steady state dynamics of the AA model can be periodic or quasiperiodic. However, periodic trajectories can contain sequences of avalanches with non-trivial time-space-size distributions. We call this phenomenon "periodic chaos". We show, in particular, that the distribution of avalanches for an ASP model is identical to the distribution of avalanches for an AA model with the same redistribution matrix and loading rate vector, after averaging over all periodic trajectories. We present a proof of Dhar's conjecture on the description of the set of recurrent configurations of an Abelian model in terms of forbidden subconfigurations. Recurrent combinatorial formulas for the distributions of avalanches are given, in terms of operations on matrices corresponding to deletion and contraction operations in graph theory. Corresponding combinatorial expressions are known in graph theory as Tutte polynomials. Several identities for these combinatorial expressions, in terms of determinants of various matrices, are derived.

## Acknowledgements

This work was performed when the author was visiting the Department of Geology, Cornell University, under NSF grant #EAR-91-04624, and the

Institute for Theoretical Physics, UCSB, under NSF grant #PHY89-04035. The author thanks D.L. Turcotte, W.I. Newman, L. Billera, A. Zelevinsky, and G. Narkounskaya for discussions and correspondence, and especially W.I. Newman for many suggestions during the preparation of the manuscript.

## Appendix

*A. Proof of (7).* Let $\boldsymbol{n} = \{n_i, i \in V\}$ where $n_i$ is the number of breaks of a site $i$ during a time interval $t$, starting from a stable configuration $\boldsymbol{h}$, and let $\boldsymbol{h}'$ be the stable configuration after these breaks. Then $\boldsymbol{h}' = \boldsymbol{h} + \boldsymbol{v}t - \Delta'\boldsymbol{n}$ where $\boldsymbol{v}$ is the loading rate vector in a deterministic model or its mean value during a time interval $t$ in a stochastic model.

As both configurations $\boldsymbol{h}$ and $\boldsymbol{h}'$ belong to $\mathcal{S}$, the distance between $\boldsymbol{h}''$ and $\boldsymbol{h}$ remains bounded, hence

$$\Delta'\boldsymbol{n}/t = \boldsymbol{v} + (\boldsymbol{h} - \boldsymbol{h}')/t \to \boldsymbol{v} \qquad \text{as } t \to \infty .$$

*B. Proof of (8).* We want to prove that any two avalanches starting at a point in $\mathbf{R}_+^V$ terminate at the same stable point. In this case the two avalanches automatically contain equal number of breaks for every site. Let $\boldsymbol{i} = (i_1, \ldots, i_l)$ be an avalanche of minimum size $l$ such that there exists another avalanche $\boldsymbol{j} = (j_1, \ldots, j_m)$ starting at the same point $q$ with a different end. Let $q' \neq q''$ be the ends of $\boldsymbol{i}$ and $\boldsymbol{j}$. Then $i_1 \neq j_1$, otherwise $l$ is not a minimum. We want to show that $p = q - \delta_{i_1} - \delta_{j_1}$ belongs to $\mathbf{R}_+^V$. As $q - \delta_{i_1} \in \mathbf{R}_+^V$, only the coordinate $j_1$ of $p$ may be negative. The same argument with $i_1$ and $j_1$ interchanged shows that only the coordinate $i_1$ of $p$ may be negative. As $i_1 \neq j_1$, $p \in \mathbf{R}_+^V$. In particular, $l > 1$ and $m > 1$. Let $\boldsymbol{k}$ be an avalanche initiated at $p$. An avalanche $\boldsymbol{i}' = (i_2, \ldots, i_l)$ initiated at $q - \delta_{i_1}$ has size $l - 1$, hence an avalanche $(j_1, \boldsymbol{k})$ initiated at $q - \delta_{i_1}$ has the same end $q'$ and size $l - 1$ as $\boldsymbol{i}'$. Next, its end coincides with the end of the avalanche $(i_1, \boldsymbol{k})$ initiated at $q - \delta_{j_1}$. As the size of this last avalanche is $l - 1$, the avalanche $(j_2, \ldots, j_m)$ initiated at $q - \delta_{j1}$ has the same end $q'$, i.e. $q' = q''$ in contradiction with our hypothesis.

If an avalanche contains a loading vector $\boldsymbol{v}$ with $v_i \geq 0$, for all $i$ at some site, the same argument shows that an avalanche with the vector $v$ displaced one site towards the starting point always belongs to $\mathbf{R}_+^V$. This proves commutativity of load-avalanche operators.

*C. Proof of (11).* Let $\boldsymbol{h}$ be a configuration in $\mathcal{S}_u$, and let $\boldsymbol{h}'$ be any configuration with large enough components equivalent to $\boldsymbol{h}$. We want to show that

$$\mathcal{A}h = \mathcal{A}h' . \tag{54}$$

This shows that avalanches started at any two equivalent configurations in $\mathcal{S}_u$ terminate at the same stable configuration, hence $\mathcal{A}(\mathcal{S}_u)$ is a fundamental domain for $\mathcal{L}$.

To prove (54), we note that condition (11) implies that $h - \mathcal{A}h' = \Sigma\, n_i \delta_i$, where all $n_i$ are non-negative. Because the components of $h$ and $\mathcal{A}h'$ and the values of $n_i$ are bounded, we can suppose that any sequence of breaks, with at most $n_i$ breaks at a site $i$, applied to $h'' = h' + h - \mathcal{A}h'$ is contained in $\mathbf{R}_+^V$ (this is the exact meaning of "large enough" components of $h'$). Hence there exists an avalanche started at $h''$ passing through $h'$. Due to the Abelian property (8), this yields $\mathcal{A}h'' = \mathcal{A}h'$. At the same time, an avalanche from $h'$ to $\mathcal{A}h'$ shifted by $h - \mathcal{A}h'$ (due to the condition of (11), all components of this vector are non-negative) connects $h''$ with $h$. Due to the Abelian property (8), this yields $\mathcal{A}h'' = \mathcal{A}h$. Hence, $\mathcal{A}h' = \mathcal{A}h$, q.e.d.

*D. Proof of (19).* We call a (stable or unstable) configuration $h$ *reachable* if there exists an avalanche passing through $h$ and starting at a configuration with arbitrarily large components.

Let us show first that configurations in $\mathcal{V}_n$ are not reachable. If some of the configurations in $\mathcal{V}_n$ are reachable, then all configurations in $\mathcal{V}_n$ close to $Q_n$ are reachable. There exists a configuration $h$ in $\mathcal{R}$ arbitrarily close to $Q$. The configuration $h_n = h - \Delta' n$ is equivalent to $h$, belongs to $\mathcal{V}_n$ and is close to $Q_n$. Any avalanche starting at a configuration with large enough components that passes through $h_n$ should terminate at $h \in \mathcal{R}$. But this is possible only if all components of $h$ are non-positive. Hence $\mathcal{R} \subset \mathcal{S} \setminus \bigcup{}' \mathcal{V}_n$. It is easy to show that for any two equivalent configurations in $\mathcal{S}$ at least one belongs to $\bigcup{}' \mathcal{V}_n$. As $\mathcal{R}$ is a fundamental domain for the lattice $\mathcal{L}$, this yields $\mathcal{R} = \mathcal{S} \setminus \bigcup{}' \mathcal{V}_n$.

*E. Proof of (21).* Due to (19),

$$\mathcal{R} \subset \mathcal{S} \setminus \bigcup_{X \subset V} \mathcal{F}_X . \tag{55}$$

Hence it is enough to show that the volume of the right side in (55) is $\det(\Delta)$ if (22) holds. We have

$$\operatorname{vol}\Big(\mathcal{S} \setminus \bigcup_{X \subset V} \mathcal{F}_X\Big) = \prod_{i \in V} \Delta_{ii} + \sum_{l \geq 1} (-1)^l \sum_{X_1, \ldots, X_l} \operatorname{vol}(\mathcal{F}_{X_1} \cap \ldots \cap \mathcal{F}_{X_l}) . \tag{56}$$

Here the sum is taken over all unordered collections $X_1, \ldots, X_l$ of distinct nonempty subsets of $V$.

If (22) holds then, for any two subsets $X'$ and $X''$ of $V$,

$$\mathcal{F}_{X'} \cap \mathcal{F}_{X''} \subset \mathcal{F}_{X' \cup X''} . \tag{57}$$

This implies that only terms with

$$X_1 \subset \ldots \subset X_l \tag{58}$$

can be left in (56).

To show this, let $<$ be any ordering of subsets of $V$ such that $X' < X''$ when $|X'| < |X''|$ and the sets of equal size are arbitrarily ordered. Then $X_j$ in (56) can be arranged in increasing order

$$X_1 < \cdots < X_l . \tag{59}$$

Let $X_j$ be the first term in (59) such that $X_j \not\subset X_{j+1}$. Then $X_{j+1} < X_j \cup X_{j+1}$. If the sequence (59) contains $X_j \cup X_{j+1}$, we remove it from the sequence, otherwise we add it to the sequence. Due to (57) this operation does not change the value of the corresponding term in (56) but does change its sign. Hence all terms but (58) annihilate in (56).

If (22) holds then

$$\mathrm{vol}(\mathcal{F}_{X_1} \cap \ldots \cap \mathcal{F}_{X_l}) = \prod_{j \not\in X_l} \Delta_{jj} \prod_{i=1}^{l} \prod_{j \in X_i \setminus X_{i-1}} \Big(-\sum_{\nu \in X_i, \nu \neq j} \Delta_{\nu j}\Big) , \tag{60}$$

for $X_1 \subset \ldots \subset X_l$, appears in (56) with the sign $(-1)^l$. If we add an empty set $\emptyset$ as $X_0$ to every sequence (58) and define $\mathcal{F}_\emptyset = \mathcal{S}$, then (56) can be rewritten as

$$\sum_{l \geq 0} (-1)^{|V|-|X_l|} \sum_{X_0 \subset X_1 \subset \ldots \subset X_l} \prod_{j \not\in X_l} \Delta_{jj} \prod_{i=1}^{l} \prod_{j \in X_i \setminus X_{i-1}} \sum_{\nu \in X_i, \nu \neq j} \Delta_{\nu j} . \tag{61}$$

We claim that this is equal to $\det(\Delta)$, for any matrix $\Delta$.

Expanding all the sums and products in (61) we can rewrite it in the following way:

$$\sum_{\varphi} \varepsilon(\varphi) \prod_{j \in V} \Delta_{\varphi(j), j} , \tag{62}$$

where $\varphi$ runs over all maps from $V$ to itself, and the coefficient $\varepsilon(\varphi)$ is defined as follows. Let $V^\varphi = \{ j \in V : \varphi(j) = j \}$ be the set of fixed points of $\varphi$. Then

$$\varepsilon(\varphi) = (-1)^{|V \setminus V^\varphi|} \sum_{X} (-1)^{l(X)} , \tag{63}$$

where the sum is taken over all $\varphi$-invariant flags

$$X=\{\emptyset=X_0\subset X_1\subset\ldots\subset X_l=V\backslash V^\varphi\}\,, \tag{64}$$

and $l(X)=l$ is the length of $X$. Here the $\varphi$-invariance of $X$ means that $\varphi(X_i)\subseteq X_i$, for all $i$.

If $\varphi$ is a permutation of $V$ then every $\varphi$-invariant set in $V\backslash V^\varphi$ is identified by a subset of the set $W$ of cycles of $\varphi$ with length greater than 1.

All subsets of $W$ can be identified with vertices of an $|W|$-dimensional cube if we set an $i$th component of a vertex equal to 1 when $i$ belongs to a subset and 0 otherwise. All flags of subsets of $W$ constitute a simplicial complex (with the length of a flag as the dimension of a simplex) which is a simplicial subdivision of this cube. The flags starting with $\emptyset$ and ending with $W$ constitute an open $|W|$-dimensional cube. Its Euler characteristics $\chi=(-1)^{|W|}$.

Hence $\varepsilon(\varphi)$ is equal to $(-1)^{|V|-k}$ where $k$ is the number of all cycles of $\varphi$ which is the usual sign of a permutation.

In case $\varphi$ is not a permutation, there exist two $\varphi$-invariant subsets $A\supset B$ in $V\backslash V^\varphi$ such that $|A|=|B|+1$, $\varphi(A)=B$ and $B$ does not contain non-trivial $\varphi$-invariant subsets. For every flag $X$, we find $i$ such that $X_{i+1}\supseteq A$, $X_i\not\supseteq A$. Then $X_i\subseteq Y=(X_{i+1}\backslash A)\cup B$. If $X_i=Y$ we remove it from the flag, otherwise we add $Y$ to the flag between $X_i$ and $X_{i+1}$. This operation defines a sign-changing isomorphism of the set of terms in the sum (63). Hence $\varepsilon(\varphi)=0$.

*F. Proof of (31).* First, it is obvious that (31) is positive only when all subgraphs $G_\nu$ are connected. Also, avalanches covering a non-connected graph have zero measure, because to start such an avalanche at least one site at every connected component has to be at the threshold level.

Configurations generating trajectories with a sequence of single breaks $i_1,\ldots,i_N$ occupy a simplex

$$0<t_1=\frac{\Delta_{i_1i_1}-h_{i_1}}{s_{i_1}}<\cdots<t_N=\frac{\Delta_{i_1i_N}+\cdots+\Delta_{i_Ni_N}-h_{i_N}}{s_{i_N}}\leqslant 1\,, \tag{65}$$

where $t_\nu$ are the time moments of breaks at $i_\nu$. Its volume is equal to $s_{i_1}\cdots s_{i_N}/N!$. Hence (31) is true in this case.

For partitions where at least one subgraph has order greater than 1, we proceed by induction on the number of edges of the graph $G$ and suppose that the statement is true for both $G-e$ and $G/e$ where $e$ is any edge of $G$.

Let $G_1,\ldots,G_m$ be a partition of $G$ into induced connected subgraphs, $|G_\nu|>1$, and $i_\nu\in G_\nu$ a site starting an avalanche. Then there exists an edge $e$ of $G$ with one end $i_\nu$ and another end $j\in G_\nu$.

Let $h(t) = \{h_i(t),\ i \in V(G)\}$, be a periodic trajectory with avalanches $G_1, \ldots, G_m$ and starting sites $i_1, \ldots, i_m$, and let $\boldsymbol{h} = \{h_i\} = \boldsymbol{h}(t_\nu)$ where $t_\nu$ is the time moment of the avalanche $G_\nu$. Then $h_{i_\nu} = \Delta_{i_\nu i_\nu}$.

There are two possibilities.

(a) $h_j < \Delta_{jj} - 1$.

Replacement of $h_{i_\nu}(t)$ by $h_{i_\nu}(t) - 1$ defines a one-to-one correspondence between trajectories for the system on $G$ satisfying (a) and all trajectories for a system on $G - e$, with the same values of $s_i$, generating the partition induced from $G$, with the same starting points and time moments of breaks. Due to inductional conjecture, the volume of the configurations generating trajectories satisfying (a) is equal to

$$s_{i_1} T(G_1) \cdots s_{i_\nu} T(G_\nu - e) \cdots s_{i_m} T(G_m)/m!\,. \tag{66}$$

(b) $\Delta_{jj} > h_j \geqslant \Delta_{jj} - 1$.

We can identify the trajectory $\boldsymbol{h}(t)$ with a periodic trajectory for a system on $G/e$, with the site $j$ removed and all the edges adjacent to it connected to the site $i_\nu$, passing at the time moment $t_\nu$ through the configuration obtained from $\boldsymbol{h}$ replacing $h_{i_\nu} = \Delta_{i_\nu i_\nu}$ by the threshold value $\Delta_{i_\nu i_\nu} + \Delta_{jj} + 2\Delta_{i_\nu j} - s_j$ for $G/e$. This trajectory generates the partition of $G/e$ induced from $G$, with the same starting points and time moments of breaks. The correspondence represents the set of trajectories for $G$ satisfying (b) as a prism of height 1 over the set of trajectories for $G/e$ generating partition induced from $G$, with the same starting points and time moments of breaks. From the induction conjecture, the volume of the configurations generating trajectories satisfying (b) is equal to

$$s_{i_1} T(G_1) \cdots s_{i_\nu} T(G_\nu/e) \cdots s_{i_m} T(G_m)/m!\,. \tag{67}$$

Then, due to (30), the sum of (66) and (67) is equal to (31). This proves our claim.

## References

[1] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. 59 (1987) 381; Phys. Rev. A 38 (1988) 364.

[2] C. Tang and P. Bak, Phys Rev. Lett. 60 (1988) 2347; J. Stat. Phys. 51 (1988) 797.
L.P. Kadanoff, S.R. Nagel, L. Wu and S. Zhou, Phys. Rev. A 39 (1989) 6524.
B. Chhabra, M.J. Feigenbaum, L.P. Kadanoff, A.J. Kolan and I. Procaccia, Sandpiles, Avalanches and the Statistical Mechanics of Non-Equilibrium Stationary States, preprint (1992).

T. Hwa and M. Kardar, Phys. Rev. Lett. 62 (1989) 1813; Physica D 38 (1989) 198; Phys. Rev. A 45 (1992) 7002.
S.S. Manna, J. Stat. Phys. 59 (1990) 509; Physica A 179 (1991) 249.
C.-H. Liu, H.M. Jaeger and S.R. Nagel, Phys. Rev. A 43 (1991) 7091.
[3] D. Dhar, Phys. Rev. Lett. 64 (1990) 1613.
[4] D. Dhar and S.N. Majumdar, J. Phys. A 23 (1990) 4333.
S.N. Majumdar and D. Dhar, J. Phys A 24 (1991) L357; Physica A 185 (1992) 129.
D. Dhar, Physica A 186 (1992) 82.
P. Grassberger and S.S. Manna, J. Phys. (Paris) 51 (1990) 1077.
M. Creutz, Nucl. Phys. B, Proc. Suppl. 20 (1991) 758; Comput. Phys. 3 (1991) 198.
E. Goles, Ann. Inst. Henri Poincaré Phys. Theor. 56 (1992) 75.
H.F. Chau and K.S. Cheng, Phys. Rev. A 44 (1991) 6233; Phys. Rev. A 46 (1992) R2981.
[5] B. Barriere and D. L. Turcotte, Geophys. Res. Lett. 18 (1991) 2011.
[6] A. Bjorner, L. Lovasz and P.W. Shor, Europ. J. Combinatorics 12 (1991) 283.
A. Bjorner and L. Lovasz, Chip firing games on directed graphs, preprint (1991).
[7] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. 57 (1967) 341.
[8] J. Carlson and J.S. Langer, Phys. Rev. Lett. 62 (1989) 2632; Phys. Rev. A 40 (1989) 6470.
J.M. Carlson, Phys. Rev. A 44 (1991) 6226.
[9] H. Nakanishi, Phys. Rev. A 41 (1990) 7086; Phys. Rev. A 43 (1991) 6613.
M. Matsuzaki and H. Takayasu, J. Geophys. Res. 96 (1991) 19925.
J. Lomnitz-Adler, L. Knopoff and G. Martinez-Mekler, Phys. Rev. A 45 (1992) 2211.
[10] H.J.S. Feder and J. Feder, Phys. Rev. Lett. 66 (1991) 2669.
A. Diaz-Guilera, Phys. Rev. A 45 (1992) 8551.
[11] Z. Olami, H.J.S. Feder and K. Christensen, Phys. Rev. Lett. 68 (1992) 1244.
K. Christensen and Z. Olami, Phys. Rev. A 46 (1992) 1829.
[12] Y.-C. Zhang, Phys. Rev. Lett. 63 (1989) 470.
L. Pietronero, P. Tartaglia and Y.-C. Zhang, Physica A 173 (1991) 22.
[13] A. Gabrielov, W.I. Newman and L. Knopoff, Lattice models of failure: sensitivity to the local dynamics, preprint (1992).
[14] W.T. Tutte, Graph Theory (Addison-Wesley, Reading, MA, 1984).

PHYSICAL REVIEW E VOLUME 50, NUMBER 1 JULY 1994

# Lattice models of failure: Sensitivity to the local dynamics

Andrei Gabrielov*
*International Institute of Earthquake Prediction Theory and Mathematical Geophysics, Russian Academy of Sciences, Moscow, Russia;*
*Department of Geology, Cornell University, Ithaca, New York 14853;*
*Institute of Geophysics and Planetary Physics, University of California, Los Angeles, California 90024*

William I. Newman
*Departments of Earth and Space Sciences, Astronomy, and Mathematics, University of California, Los Angeles, California 90024*

Leon Knopoff
*Institute of Geophysics and Planetary Physics and Department of Physics, University of California, Los Angeles, California 90024*
(Received 8 March 1994)

Quasistatic lattice models to explore scaling and other properties of failure events, including earthquakes, are characterized by the presence of two or more time scales. We show that there is a remarkable degree of variability in the qualitative behavior of these models: In one model with periodic boundary conditions, which simulates a moving dislocation or avalanche model of fracture, the trajectories are always periodic in both one and two dimensions. In another quasistatic model which simulates a growing coherent crack, the trajectories are either periodic or chaotic, depending on the initial conditions. Characteristic behaviors derived from analytic results are illustrated by numerical simulations.

PACS number(s): 05.45.+b, 46.30.Nz, 64.60.Ak, 91.60.Ba

## I. INTRODUCTION

In the model of self-organized criticality due to Bak *et al.* [1], a homogeneous lattice system organizes itself into a critical state via fluctuations regulated by the local rules for the dynamics. If the local rules obey a conservation law, the system will develop a scale-independent distribution of sizes of events that is valid on all scales. Bak and Tang [2] proposed that this model had application to the simulation of earthquake faulting. Kadanoff *et al.* [3] showed that the size distributions are sensitive to the local dynamics and may belong to different universality classes. In the model [1], dissipation is only possible at the edge of the lattice, implying the existence of a massive inhomogeneity at the edge of an otherwise homogeneous system. In models with nonconservative local dynamics, dissipation can take place in the interior of the lattice, and a number of models with dissipative dynamics have been explored [4–7]. These dissipative models are similar to a dynamical lattice model of earthquake occurrence introduced by Burridge and Knopoff [8] and explored in depth by Carlson and Langer [9,10] and Carlson [11]. Matsuzaki and Takayasu [12] and Lomnitz-Adler *et al.* [13] have explored the ability of a dissipative lattice system without inertia but with nonlocal dynamics, to organize itself into a critical state. The latter authors also found that the presence or absence of a critical state depends delicately on the choice of the rules, and is very sensitive to small changes. Other models explore systems that involve nonlocal dynamics, other fracture geometries, etc. [14–16].

In this paper, we consider lattice models similar to the above with dissipation but without inertia, i.e., the system is massless. We are concerned with failure or avalanche models, in general, and earthquake models, in particular. Readers are advised to consider terms such as earthquake events and fracture as metaphors for a broad range of phenomena including but not confined to seismicity and material failure. Let "slow" or tectonic time be the interval between fracture events, during which the loading increases the level of stress in the solid. This stress is then released through a sequence of one or more breaks in "fast" time, which is the time needed for fractures to develop. In the case of real earthquakes, these two time scales differ by seven orders of magnitude or more—the slow time scale can be the order of a century, while the fast time scale may be as long as one or two minutes for the largest earthquakes.

We study a class of deterministic models where both time and stress are *continuous* variables and where the loading proceeds at a uniform rate at all lattice sites. This was first done by Burridge and Knopoff [8] and later by others [5–7,17]. This class of models, as with those considered in [9–11,15] is (stress) dissipative, in contrast with the stress conservative model of [1]. Olami *et al.* [7] have shown that this class of lattice models is formally

*Now with the Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5R 2P7.

 

equivalent to the zero-mass limit of the Burridge-Knopoff spring-block model with different friction laws. For a scalar particle model with uniform spacing, the value of the total force and the stress on a lattice site are equivalent. For convenience we use the term stress for this quantity.

We employ a lattice model whose elements slowly accumulate stress up to a certain threshold of strength; the accumulated stress is then rapidly released in fast time. Part of the released stress is redistributed between other elements and part is dissipated locally. We construct a sequence of the slow times and intensities of the fracture events. In all of our experiments we consider uniform lattices with periodic boundary conditions.

We consider two variants of the model, which we call *series* and *parallel* models, both having identical structure and laws of failure; the only difference between the models is the order of stress release and redistribution in fast time.

In the series model, the stress is fully released immediately when the stress on an element achieves the threshold of strength. Then part of the released stress is added to the neighboring elements. The same procedure is now applied to all neighbors that have stress greater than or equal to the threshold. The process is repeated until no elements have stresses exceeding the threshold. In this case, the stress release rate is presumed to be faster than the stress redistribution rate.

In the parallel model, an element breaks when the stress arrives at the threshold of strength, and starts to release its stress in fast time. In this case, part of the released stress is *simultaneously* added to the neighboring elements as the stress on the first element drops. This process continues until all the stress on the broken element is released or until the stress on some of its neighbors arrives at the threshold. In the latter case, the neighboring element breaks and begins to release and redistribute its stress in the same manner, until finally no more elements break. This corresponds to the case when the stress release rate is slower than the stress redistribution rate.

We show that these two models lead to markedly different dynamical histories. For a uniform lattice with periodic boundary conditions, we find and can prove analytically that for most initial conditions the parallel model produces essentially chaotic behavior while the series model always converges to a complex but periodic trajectory.

## II. MODELS

In these quasistatic models, the process of readjustment of the accumulated stress at a site involves two physically different time scales, which are the time needed for the bond at that site to fail, or the stress release time, and the time needed for the released stress to be transferred to the neighboring sites, or the stress transfer time. For simplicity and without loss of generality, we set the stress accumulation rate and the stress threshold, i.e., the strength of the elements, numerically equal to unity.

Both the stress release and stress transmission time scales are much smaller than the loading or tectonic time scale. The ratio of these two fast times is a parameter describing a broad class of models of failure with two limiting cases of particular interest.

### A. Series model

In the first case, which is similar to most other quasistatic lattice models [1,4–7], the stress transmission time is much longer than the stress release time. In an extended lattice, this may correspond to the quasistatic model for a moving dislocation or to the avalanche analogue of a fracture. In this case, we observe a "series" of stress release events, followed by a series of transmission events followed by a series of release events, and so on. Dissipation is introduced through a dissipation parameter $S$ which is the fraction of the accumulated stress that is lost to the system; the remaining fraction $1 - S$ is redistributed equally among the nearest neighbors.

Since a site that receives transferred stress from a neighboring ruptured site can achieve a stress level greater than its threshold, there are at least two variants of the series model. In case $a$, the stress level at a fracturing site is reduced by a fixed amount, while in case $b$, the stress level at the new site is reduced to zero. Let $\sigma_{ij}$ be the stress level at a given site $(i,j)$. In case $a$,

$$\sigma_{ij} \to \sigma_{ij} - 1 \text{ and } \sigma_{nn} \to \sigma_{nn} + p \quad , \tag{2.1}$$

if $\sigma_{ij} \geq 1$, where the interaction parameter $p \equiv (1-S)/2d$ and $d$ is the dimensionality of the lattice. This model is similar to the models explored by Refs. [1,5] and [18]; we present some new results for this model. In case $b$,

$$\sigma_{ij} \to 0 \text{ and } \sigma_{nn} \to \sigma_{nn} + p\sigma_{ij} \tag{2.2}$$

if $\sigma_{ij} \geq 1$. This model can also be related to [6,7,19], and again we present new results for this model.

In case $a$, it can be shown that the sequence of events in slow time is *independent* of the order of breaks in fast time. In this respect, the model is similar to the abelian sandpiles introduced in [20]. As a result of this commutative property, it is not necessary for the stress level of newly released sites to be adjusted synchronously, as long as the stress drop experienced by each of these sites is the same. In case $b$, it is essential that stresses be reduced at failed sites in proper sequence; otherwise, the behavior obtained is quantitatively, but evidently not qualitatively, dependent upon the order of the stress drop. In the numerical and analytic results below we discuss these variants of the series model.

### B. Parallel model

In the second case, the stress release time is much longer than the transmission time. As a given element releases its accumulated stress, its nearest neighbors immediately receive a proportionate increase in their stress.

The stress release and redistribution take place in "parallel" among the appropriate elements. In this model, a neighboring site can reach the threshold of rupture while the site which initiated the release is still in the process of releasing stress. In an extended lattice, this may correspond to a quasistatic model for a growing crack, with motions correlated along its extent, and not dissimilar from the models of [12,13]. In this redistribution model, we cannot distinguish between cases $a$ and $b$ as in the series model, because failure always occurs at precisely the stress threshold, which is reached gradually within fast time. Again we define the failure threshold by $\sigma_{ij} = 1$. When an element fails, the elements respond in fast time $\tau$ according to the relations

$$\frac{d\sigma_{ij}}{d\tau} \to \frac{d\sigma_{ij}}{d\tau} - 1 \text{ and } \frac{d\sigma_{nn}}{d\tau} \to \frac{d\sigma_{nn}}{d\tau} + p \quad . \qquad (2.3)$$

These relations hold until $\sigma_{ij} = 0$ whereupon

$$\frac{d\sigma_{ij}}{d\tau} \to \frac{d\sigma_{ij}}{d\tau} + 1 \text{ and } \frac{d\sigma_{nn}}{d\tau} \to \frac{d\sigma_{nn}}{d\tau} - p \quad . \qquad (2.4)$$

When $d\sigma_{ij}/d\tau = 0$ for all sites, we return to the slow time scale.

## III. EXAMPLE: THREE-ELEMENT CONFIGURATION

We begin by looking at a configuration of three elements connected in a closed chain, which has both a series and a parallel realization, before proceeding to the case of large lattices. When any element fails, a fraction $p$ of its released stress is transferred to each of the other two elements, and a fraction $S = 1 - 2p$ is dissipated. The simple three dimensional phase space of this reduced system permits detailed analytic investigation.

This model can be described by a Poincaré surface of section, defined without loss of generality, by plotting the stress level $\sigma_2$ and $\sigma_3$ of each of the surviving elements 2 and 3 when element 1 fails. In Fig. 1, we show the geometry of the Poincaré surface of section for series case $a$. The regions with indicial notations correspond to various periodic trajectories. The indicial notation "1, 2, 3" describes periodic trajectories in which each element fails separately in the order given; we refer to this sequence of *isolated* failures as "single breaks." On the other hand, "1-2, 3" refers to a situation where the failure of element 1 causes element 2 to fail instantly, and then after some time element 3 fails. Finally, "1-2-3" refers to a situation in which all three elements fail simultaneously. One can readily show that all of these trajectories are periodic. The hatched region is not periodic, but subsequent breaks will bring any point from the hatched region into one of the periodic trajectory domains.

In case $b$, only the shaded regions "1, 2, 3" and "1, 3, 2" (i.e., single breaks) are periodic and we can show that *all* other regions of the $\sigma_2$-$\sigma_3$ plane arrive, after a finite number of breaks, in one of these two periodic domains.

In the parallel case, the single break situations "1, 2, 3" or the inverse, remain periodic while the behavior in the other cases is more complicated. It can be shown that

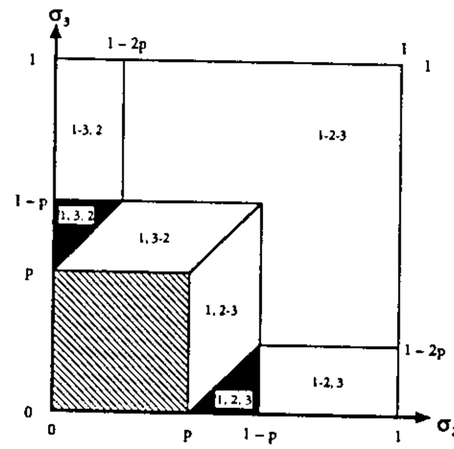


FIG. 1. The Poincaré section $\sigma_1 = 1$ for the three-element configuration, series redistribution model for both cases $a$ and $b$. Areas designated by indices correspond to different types of periodic trajectories. Every trajectory starting in the hatched area is unstable and moves into one of the periodic areas. In case $b$, only areas with single breaks, i.e., those designated by 1, 2, 3 and 1, 3, 2, (shaded in the figure) are stable.

the mapping is area preserving and hence the complexity which it manifests has the signature of Hamiltonian chaos. The map for the case $p = 2/5$ ($S = 1/5$) is shown in Fig. 2. We exploit the fact that Fig. 2 is symmetric about a diagonal. Below the diagonal, the black area identifies a chaotic region. We represent here one chaotic trajectory; it is everywhere dense, but does not have pos-

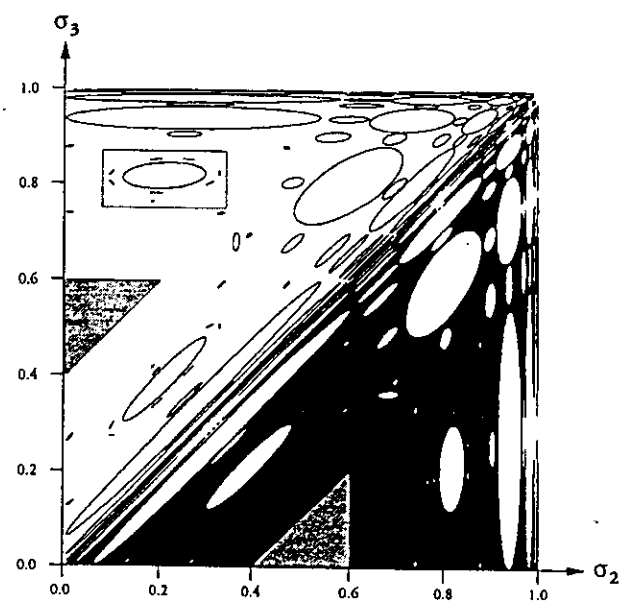


FIG. 2. Poincaré section $\sigma_1 = 1$ for the three-element configuration, parallel redistribution model, $p = 2/5$ or $S = 1/5$. The white triangles correspond to periodic trajectories with single breaks (as in Fig. 1), while the white ellipses correspond to families of invariant tori with quasiperiodic behavior. The figure is symmetric about the diagonal, but the above- and below-diagonal portions highlight different features of the problem. Below the diagonal, the black area defines the chaotic region, while above the diagonal, the boundary between the chaotic and regular behavior is displayed.

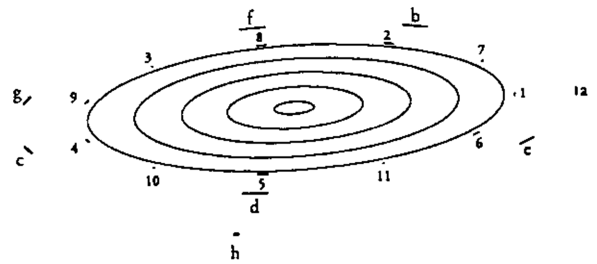

FIG. 3. Expansion of the boxed region of Fig. 2. One of the families of invariant tori, with two satellite families consisting of 11 and eight elements, are shown. The numbers 1 − 11 and letters $a - h$ specify the ordering of quasiperiodic trajectories in the two satellite families.

itive measure because it is a countable set. Indeed, a continuum of chaotic trajectories coexist in the same region. Above the diagonal, we describe the boundaries between the regions of chaotic and regular behavior. The triangular areas of periodic behavior (shaded) shown in Fig. 1 also appear in Fig. 2. There is as well an infinite number of elliptical areas, each of which is filled by a family of invariant tori with quasiperiodic behavior. In Fig. 3, we show in detail one of these regular regions, which is identified by a box in the upper left of Fig. 2. This is an area having a continuum of tori together with "satellites" which are themselves families of invariant tori. In the figure, we identify one group of 11 satellites designated by the numbers 1–11, and another of eight satellites designated by the letters $a$–$h$. Any orbit which originates in one of these satellites progresses to another satellite in the same group in the order shown in the figure.

Unlike many dynamical systems encountered in physics for which empirical evidence of chaotic behavior has been advanced, we are able to show *rigorously* the presence of chaos in the parallel case. We exploit the inherent symmetry of this system as follows. A trajectory may take us from one face of the unit cube, say with $\sigma_1 = 1$ to another with $\sigma_2 = 1$ or $\sigma_3 = 1$. Suppose the trajectory goes from $\sigma_1 = 1$ to $\sigma_2 = 1$; then, we map $\sigma_3 \to \sigma_2$, $\sigma_1 \to \sigma_3$ cyclically, and so on. The symmetry of this system guarantees that this map, which we denote by $P$, is well defined, i.e., is one to one, and is area preserving although it is not continuous.

To prove that the behavior of the three-element parallel configuration is chaotic, consider the polygon (see Fig. 4) with vertices

$$A = \left(\frac{7}{20}, \frac{3}{4}\right),\ B = \left(\frac{1}{5}, \frac{3}{5}\right),\ C = \left(0, \frac{3}{5}\right),$$
$$D = \left(0, \frac{66}{95}\right),\ E = \left(\frac{16}{35}, \frac{6}{7}\right), \tag{3.1}$$

lying in the area between the shaded triangle and the boxed domain in Fig. 2. The line $AC$ separates this polygon into a triangle $ABC$ and a quadrangle $ACDE$. It can be shown that $Q \equiv P \circ P$ (i.e., the map $P$ taken twice) is linear on $ABC$ and on $ACDE$, that is,

$$Q(\sigma_2, \sigma_3) = \left(\sigma_2 - \frac{7}{3}\sigma_3 + \frac{7}{5}, \sigma_3\right) \quad \text{in } ABC \quad , \tag{3.2}$$

and

$$Q(\sigma_2, \sigma_3) = \left(\frac{4}{3}\sigma_2 - \frac{19}{9}\sigma_3 + \frac{22}{15}, -\sigma_2 + \frac{7}{3}\sigma_3 - \frac{4}{5}\right) \tag{3.3}$$

in ACDE.

Images of $ABC$ and $ACDE$ intersect both $ABC$ and $ACDE$ producing a "horseshoe [21]." As a result of Smale's Theorem [21], a subset in the unit square, invariant under $Q$ and $Q^{-1}$, has the characteristic structure of a Cantor set.

The point $X = (\frac{4}{25}, \frac{18}{25})$ in $ACDE$ is a fixed point of $Q$ with eigenvectors $(3 \mp \sqrt{85}, 6)$ and eigenvalues $(11 \pm \sqrt{85})/6$, which define the stable and unstable separatrices of $Q$. A segment $XY \subset ACDE$ of the unstable separatrix of $Q$ extends to $XY_2$ under $Q$. Let $YZ$ be the intersection of $XY_2$ with $ABC$. Then, the image $Y_1Z_1$ of $YZ$ under $Q$ intersects transversely the segment $VX$ of a stable separatrix of $Q$, producing a "homoclinic" structure [21], which satisfies the formal criterion for (homoclinic) chaos. Images of the unstable separatrix under eight iterations of $Q$ are shown in Fig. 4.

From this formal demonstration of the chaotic character of the three-element configuration, we turn our attention to the numerical results for the series and parallel lattice models.

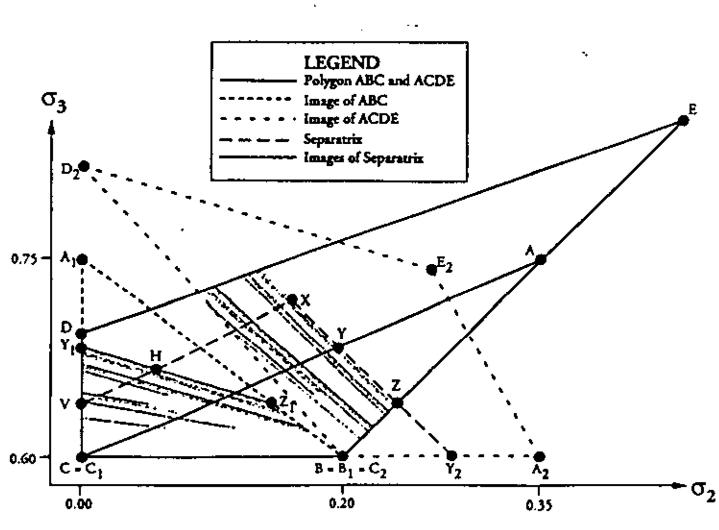


FIG. 4. Homoclinic chaos in the three-element configuration, parallel redistribution model for $p = 2/5$ ($S = 1/5$). $X$ is a fixed point of the Poincaré map, while $H$ is a homoclinic point, i.e., it is the intersection of the stable and unstable separatrices of the Poincaré map at $X$.

## IV. RESULTS: PERIODIC LATTICE

Consider a periodic lattice of arbitrary dimensionality $d$ and interaction parameter $p$, under conditions of case $a$ of the series redistribution model with fixed stress drop. Let the system, starting in slow time from a state $\sigma_i(t)$, undergo an arbitrary sequence of consecutive breaks that includes all the elements in the system, each of them only once, during a time period $S$. Then one can show that $\sigma_i(t+S) = \sigma_i(t)$ for all $i$ where $t$ denotes the time and the state $\sigma_i(t)$ belongs to a periodic trajectory with period $S$.

Such a trajectory, where each element fails only once in a period, will be referred to as a *normal trajectory*. For any initial state (see Appendix A), the system arrives at a normal trajectory after a finite number of breaks.

The period of the trajectory is $S$ because the units of stress and time are chosen so that the stress drop is equal to $S$ and the rate of stress loading is unity. In an equivalent block model formulation [7], the value $S$ corresponds to the rigidity of the loading spring. If we choose the units to make the stress drop and the velocity of loading constant, then the period scales as $1/S$.

What is notable is that the periodic part of the trajectory appears to be "chaotic" within a single period. We refer to this phenomenon as "periodic chaos." In Fig. 5, we show the distribution of break sizes as a function of time during one period for a $300\times300$ lattice with $S = 0.01$ for two randomly chosen initial stress distributions. Both time histories appear to be "chaotic" but are manifestly different from each other. There is no hint in this figure that the trajectory could be periodic. Indeed, this result poses a warning that abbreviated investigations of these phenomena can lead to grossly incorrect conclusions about the nature of the phenomena.

Size-frequency diagrams generated from one periodic trajectory appear to be crudely power law in character and display a characteristic logarithmic slope near $-1$ in

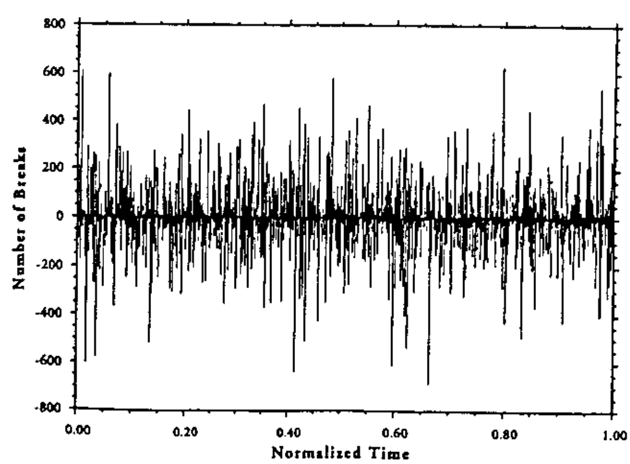


FIG. 5. Two typical realizations of break size as a function of time for one period of a periodic trajectory in the series redistribution model case $a$, $300 \times 300$ lattice and $S = 0.01$ displaying periodic chaos. The results shown are for two randomly selected periodic trajectories of the same model: the two sets of results are displayed above and below the time axis, respectively.

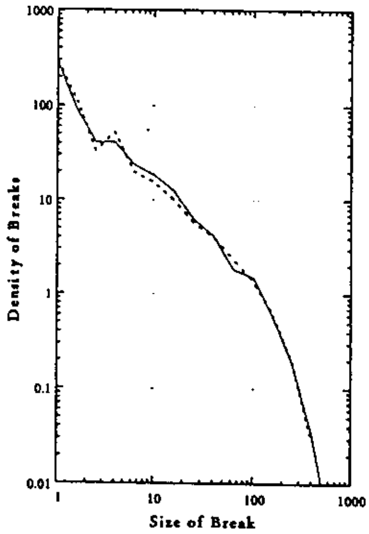


FIG. 6. The size-frequency distribution for two periodic trajectories periodic chaos in the series redistribution model case $a$; $300 \times 300$ lattice with $S = 0.01$. Size is defined to be the number of elements that break in a single cascade event.

the central part of the distribution when $S$ is sufficiently small. This is illustrated in Fig. 6 for the two periodic trajectories shown in Fig. 5. The two distributions are remarkably similar, despite having been produced by two randomly chosen periodic trajectories.

For a one-dimensional lattice, the structure of the phase space in case $a$, i.e., with fixed stress drop, of the series redistribution model can be investigated analytically (see Appendix B). Consider a one-dimensional lattice of size $N$ subdivided into $n$ (connected) clusters of different sizes. The volume of all normal trajectories with $n$ clusters is equal to

$$V(n,N) = \frac{N}{n}\binom{N+n-1}{2n-1} S^n p^{N-n} \quad . \tag{4.1}$$

Let $V(N) = \sum_{n=1}^{N} V(n,N)$ be the total volume of all periodic trajectories. Then

$$V(N) \propto \left(\frac{1+\sqrt{1-4p^2}}{2}\right)^N \quad \text{as } N \to \infty \quad . \tag{4.2}$$

An estimate of the mean value $A_j(N)$ of the number of clusters of size $j$ for a randomly chosen periodic trajectory is the analytic size-frequency distribution:

$$A_j(N) = \frac{jN}{V(N)} \sum_{n=1}^{N} \binom{N+n-j-2}{2n-3} S^n p^{N-n} \quad , \tag{4.3}$$

so that

$$A_j(N) \propto jNS\sqrt{\frac{1-2p}{1+2p}}\left(\frac{2p}{1+\sqrt{1-4p^2}}\right)^{j-1} \tag{4.4}$$

for $N \gg j$. This analytic result is substantially different from the size-frequency distribution obtained numerically in the two-dimensional case, a feature also noted by Bak *et al.* [1]. As $S \to 0$, the large events dominate the distribution, while small events scale in proportion to size (a feature also noted by Bak *et al.* [1]). These results demonstrate a fundamental difference between one- and two-dimensional models.

In case *b* of the series model, as well as in the case of parallel redistribution, periodic behavior appears for a much smaller set of states, in which only single breaks are allowed. (It is easy to show that both of the series cases and the parallel case are equivalent on the set of single breaks [22].) If $N$ is the number of elements in the lattice, then the part of the phase space occupied by normal trajectories with single breaks has volume $S^N$ and consists of $N!$ identical simplexes, a result that has been confirmed analytically (see Appendix C).

Computer simulations show that, in the case of series redistribution with a fixed healing threshold, the set of these simplexes is a global attractor, and a trajectory with *any* initial state becomes periodic after a finite, but generally very large number of breaks. In the short term, the trajectory becomes "almost" periodic, with multiple-breaks occurring with a "period" shorter than $S$. With the passage of time, each multiple break decomposes into a sequence of single breaks; however, the sequence of single breaks preserves the memory of its multiple-break origin and remains localized in space and time, as shown in Fig. 7 for a one-dimensional system of 200 elements with $S = 0.1$. Figure 8 shows two time intervals extracted from the sequence shown in Fig. 7 to illustrate the transition of multiple-break behavior into single-break behavior.

In the parallel redistribution model, the behavior outside of these simplexes is, in general, seemingly chaotic. We considered situations where the initial values of the stress are taken to be uniformly distributed between 0 and 1. In Fig. 9, we show a time stress plot for a $20 \times 20$ lattice with $S = 0.1$. There is a natural time scale which coincides with $S$ in the "stress-accumulation-release cycle." On the other hand, intermittency on a time scale that is long compared with $S$ is evident. This may be observed in two noncontiguous segments of this time series,

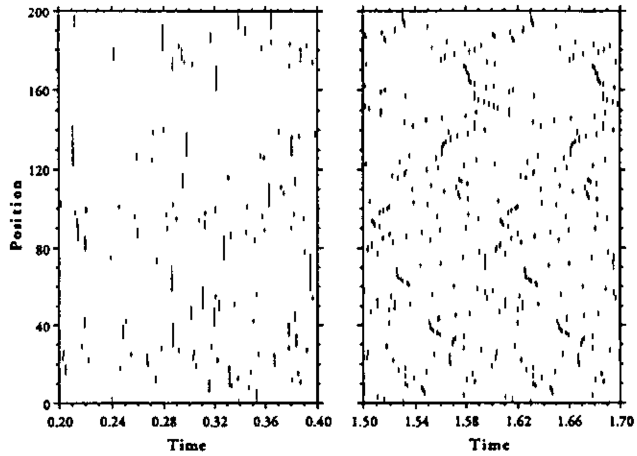


FIG. 8. Two time intervals selected from Fig. 7 to illustrate the transition from multiple- to single-break behavior.

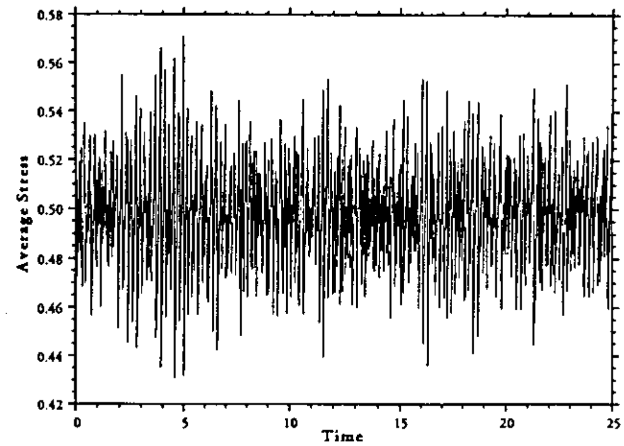


FIG. 9. Intermittency in the parallel model, on a $20 \times 20$ lattice for $S = 0.1$.

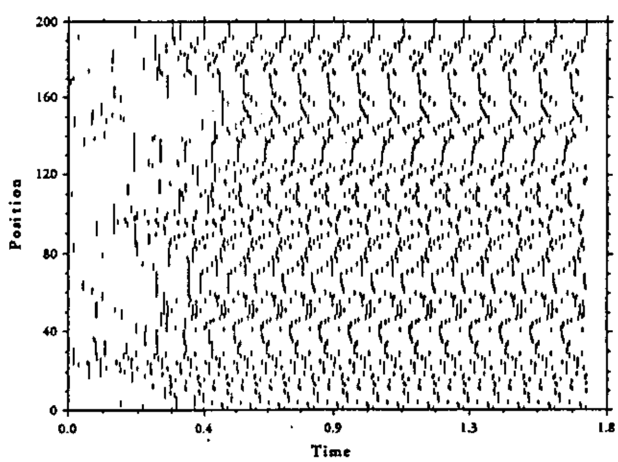


FIG. 7. The space-time distribution for a periodic trajectory in the series model, case *b*, displaying significant localization. For sufficiently long times, all events are single breaks.

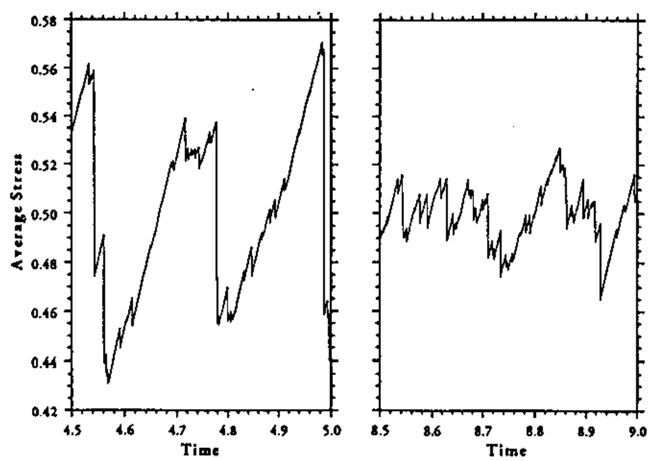


FIG. 10. Two noncontiguous time segments from the previous plot, revealing the nonstationary character of the intermittency.

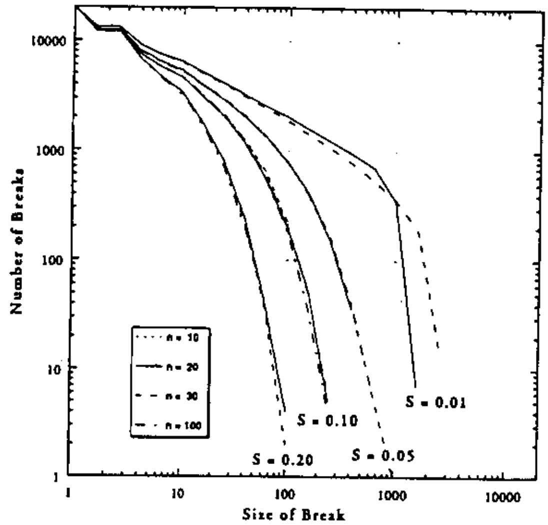


FIG. 11. The cumulative size-frequency distribution for the parallel case for different sizes of the two-dimensional lattice, and for different values of $S$. As the lattice size $n$ increases, the distributions seem to converge to the $\Gamma$ distribution.

which we display in Fig. 10. Finally, we have computed cumulative size-frequency distributions for this case for different values of the dissipation parameter $S$ and lattice size $n$ (Fig. 11); the distributions are independent of $n$ for a given value of $S$. The distribution appears to be consistent with a power law with an exponential cutoff which depends upon the value of $S$; this distribution is called the gamma distribution in the statistical literature. However, for small values of $S$, the exponential cutoff migrates to infinity and, in the apparent power-law regime of the distribution, the power-law index lies between $-1$ and $-2$, a result that is quite different from the situation encountered in the series model.

## V. CONCLUSIONS

Quasistatic lattice models of failure with dissipation have been employed recently by a number of authors to explore the scaling and other properties of catastrophic events including earthquakes. These models are representative samples of a broad spectrum of physical situations described by the presence of two or more characteristic time scales: a "slow" time with the interval between failure events associated with tectonic loading and a "fast" time associated with the fracture event itself. There are two broad classes of models characterized by their fast time dynamics: (i) the series model characterized by an instantaneous release of stress at a failed lattice site followed by the transfer of some fraction of that stress to neighboring lattice sites; and (ii) the parallel model characterized by a *relatively* slow release of stress at a failed site and an instantaneous transfer of some fraction of that stress to neighboring lattice sites. We have considered two versions of the series model; in case $a$ the stress level at the failed lattice site is reduced to zero, and in case $b$ the stress level is reduced by a fixed amount. These models yielded remarkably different qualitative behavior.

In the series model of a uniform periodic lattice containing $N$ elements, part of the phase space consists of *periodic* trajectories. This part of the phase space is of positive measure, i.e., it occupies a finite part of the entire phase space. This set of periodic trajectories is a *global* attractor of the system: *any* initial state arrives at a periodic trajectory in finite time. In case $a$, these periodic trajectories may contain multiple breaks, in which many elements fail at the same time. For $N$ large, the behavior during one period may appear to be chaotic, and displays an approximate "size-frequency" law $1/f$ as the dissipation parameter $S$ goes to zero. We refer to this remarkable situation as "periodic chaos." In case $b$, the set of periodic trajectories is much smaller than in $a$, occupying a fraction $S^N$ of the phase space. The rate of convergence to the periodic trajectory is much smaller In case $b$ than in case $a$. The trajectory quickly enters an intermediate phase of behavior which includes multiple breaks that gradually decompose into a series of single breaks which are localized in space and in time.

In the parallel model, the set of periodic trajectories is also composed solely of single breaks. (We note that once a periodic trajectory composed of single breaks is achieved, the parallel model and both cases of the series model are equivalent.) In the phase space outside the periodic regime, most trajectories are chaotic producing a size-frequency relation which converges to a power law as the dissipation parameter $S$ approaches zero, but the power-law differs from the former $1/f$ situation. Moreover, at time scales that are large compared with S, there is substantial intermittent behavior.

Finally, the three-element model is particularly remarkable in that it provides such complexity. Analytically, we have shown it to exhibit Hamiltonian chaos including homoclinic points and the presence of Smale horseshoes. In addition, we have derived an analytic size-frequency distribution from combinatoric considerations, which is markedly different from the two-dimensional case (cf. [1]).

## ACKNOWLEDGMENTS

This work was initiated when A.G. visited UCLA and W.I.N. visited the Institute of Physics of the Earth, Moscow, under Environmental Protection Agreement 02.09-13 between the United States and the Union of Soviet Socialist Republics (host institutions: USGS and UCLA, and the USSR Academy of Sciences). Part of this work was performed on a visit of A.G. to UCLA under support of the Institute of Geophysics and Planetary Physics, UCLA, and on a visit of A.G. to Cornell University, under NSF Grant No. EAR-91-04624. We thank A. Zelevinsky for suggestions relevant to the combinatoric calculations. This research has been supported in part by the Southern California Earthquake Center, Pub. No. 4170, Institute of Geophysics and Planetary Physics, University of California, Los Angeles.

## APPENDIX A: NORMAL TRAJECTORIES

Let us consider a periodic lattice of arbitrary dimensionality $d$, with an interaction parameter $p$ and a dissipation parameter $S = 1 - 2dp > 0$. For any two sites $i$ and $j$ of the lattice, we define $p_{i,j} = p$ if $i$ and $j$ are nearest neighbors, $p_{i,j} = 0$ otherwise.

*Proposition 1.* In case $a$ of the series redistribution model, let the system, starting in slow time from a state $\sigma = \{\sigma_i(t)\}$, undergo an arbitrary sequence of consecutive breaks that includes all of the elements in the system, each of them only once, during a time period $S$. Then $\sigma_i(t+S) = \sigma_i(t)$, for all $i$, and the state $\sigma$ belongs to a periodic trajectory with period $S$. For arbitrary initial state, the system arrives at one of these periodic trajectories after a finite number of breaks.

*Proof.* After all of the elements break, at the moment in time $t+S$, the stress of the $i$th element becomes $\sigma_i(t+S) = \sigma_i(t)$ due to the loading in slow time $S$, wherein the stress increases by $2dp$ from the $2d$ broken neighbors, and decreases by 1 due to the stress drop.

Suppose now that the initial state $\sigma$ does not satisfy the condition of Proposition 1. The same arguments as before show that no element can break twice during the time period $S$. Thus, if $N-K$ elements break during this period, $K > 0$, then the total stress of the system during the time period $S$ increases by $KS$. Then, during each time interval $S$, the system would experience a net stress increase. As the total stress of the system cannot exceed $N$, we obtain a contradiction. Therefore, we must come to a periodic trajectory after a finite number of breaks.

## APPENDIX B: PHASE SPACE STRUCTURE

We describe here the structure of the set of all periodic trajectories in case $a$ of the series redistribution model for a uniform 1D periodic lattice of size $N$.

Consider an arbitrary subdivision of $(1,\ldots,N)$ into $n$ clusters of different sizes, and let $n_i \geq 0$ be the number of clusters of size $i$, so $n_1 + \cdots + n_N = n$, $\quad n_1 + 2n_2 + \cdots + Nn_N = N$.

*Proposition 2.* The volume of all states generating periodic trajectories with sequence of $n$ breaks containing $n_i$ events of size $i$ is equal to

$$V_{n_1,\ldots,n_N} = N(n-1)!\prod_{i=1}^{N}\frac{i^{n_i}}{n_i!} \times S^n p^{N-n} \quad . \tag{B1}$$

*Proof.* Let us identify an initial state defined in Proposition 1 with an ordered sequence of $n$ nonoverlaping connected clusters (i.e., fast time episodes) covering the lattice, containing $n_i$ clusters of size $i$, and one distinguished (i.e., special) element in each cluster, namely, the first broken element. Simple combinatorial considerations show that the number of these objects is equal to

$$Nn!(n-1)!\prod_{i=1}^{N}\frac{i^{n_i}}{n_i!} \quad . \tag{B2}$$

Here, $n!/\prod n_i!$ is the number of possible unordered distributions of clusters, $\prod i^{n_i}$ corresponds to the choice of an element in each cluster, $(n-1)!$ defines order in the set of clusters (due to periodicity of the lattice, the choice of the first cluster should be made in advance), and $N$ identifies the position of the head of the first cluster on the lattice.

An exception is the case with only one cluster of size $N$ where there are only $N$ different objects, not $N^2$ as suggested by the formula (B2), because this cluster has no head. However, the formula (B1) is valid also in this case.

For any ordered sequence of $n$ nonintersecting connected clusters $K_1,\ldots,K_n$ of sizes $i_1,\ldots,i_n$ covering the lattice, with a distinguished element $m_j \in K_j$, the set of all states with an ordered sequence of $n$ breaks of sizes $i_1,\ldots,i_n$ starting with elements $m_1,\cdots,m_n$ is defined by

$$1 > \sigma_{m_1} + p_1(m_1) > \sigma_{m_2} + p_2(m_2) > \cdots > \sigma_{m_n} + p_n(m_n) \geq 1 - S \quad , \tag{B3}$$

$$\sigma_{m_j} + p_j(m_j) + p > \sigma_k + p_j(k) + p \geq \sigma_{m_j} + p_j(m_j) \quad \text{for } k \in K_j,\ k \neq m_j \quad . \tag{B4}$$

Here, $p_j(k) = \sum_{i\in K_1\cup\cdots\cup K_{j-1}} p_{i,k}$ is the stress increment at a site $k \in K_j$ due to the breaks in the first $j-1$ clusters. The volume of this set is $S^n p^{N-n}/n!$.

Once again, an exception appears in the case of one cluster of size $N$ where the last broken element receives stress $2p$ in fast time. In this case, in addition to the set

$$1 > \sigma_{m_1} \geq 1 - S;\ \sigma_{m_1} + p > \sigma_k + p \geq \sigma_{m_1} \quad \text{for } k \neq m_1 \quad , \tag{B5}$$

the following $N-1$ sets appear for $j \neq m_1$:

$$1 > \sigma_{m_1} \geq 1 - S,\ \sigma_{m_1} + p > \sigma_j + 2p \geq \sigma_{m_1},\quad \sigma_{m_1} + p \geq \sigma_k + p \geq \sigma_{m_1} \tag{B6}$$

for $k \neq m_1, j$. The total volume of sets (B5) and (B6) is equal to $NSp^{N-1}$. This, in combination with (B2), proves (B1).

*Theorem 1.* (a) The volume of all periodic trajectories with $n$ clusters is equal to

$$V(n,N) = \frac{N}{n}\binom{N+n-1}{2n-1} S^n p^{N-n} \quad . \tag{B7}$$

(b) Let $V(N) = \sum_n V(n,N)$ be the total volume of all periodic trajectories. Then

$$\sum_{N=1}^{\infty} V(N) z^N = \frac{Sz(1+pz)}{(1-pz)(1-z+p^2z^2)} \tag{B8}$$

and

$$V(N) \propto \left(\frac{1+\sqrt{1-4p^2}}{2}\right)^N \quad \text{as } N \to \infty \quad . \tag{B9}$$

*Theorem 2.* Let

$$A_j(N) = \frac{1}{V(N)} \sum_{n_1+2n_2+\cdots+Nn_N=N} n_j V_{n_1,\ldots,n_N} \tag{B10}$$

be the mean value of the number of clusters of size $j$ for a randomly chosen periodic trajectory. Then

$$A_j(N) = \frac{jN}{V(N)} \sum_{n=1}^{N} \binom{N+n-j-2}{2n-3} S^n p^{N-n} \quad , \quad \text{(B11)}$$

$$\sum_{N=j}^{\infty} A_j(N) \frac{V(N)}{N} z^N = \frac{jSp^{j-1}(1-pz)^2 z^j}{1-z+p^2z^2} \quad , \quad \text{(B12)}$$

and

$$A_j(N) \propto jNS\sqrt{\frac{1-2p}{1+2p}} \left(\frac{2p}{1+\sqrt{1-4p^2}}\right)^{j-1} \quad \text{(B13)}$$

for $N \gg j$.

*Lemma.* (See Zelevinsky [23]).

$$\sum_{n_1,\ldots,n_N} \prod_{i=1}^{N} \frac{i^{n_i}}{n_i!} = \frac{1}{n!} \binom{N+n-1}{2n-1} \quad , \quad \text{(B14)}$$

where the sum is taken over all integer sequences $(n_1, \ldots, n_N)$ with $n_i \geq 0$ for all $i$, $n_1 + \cdots + n_N = n$, $n_1 + 2n_2 + \cdots + Nn_N = N$.

*Proof.* Consider the generating function $g(x,y) \equiv \sum_{n,N=0}^{\infty} \nu(n,N) x^n y^N$, where $\nu(n,N)$ denotes the left side of (B14). We have

$$g(x,y) = \sum_{n_1,\ldots,n_N} \prod_{i=1}^{N} \frac{i^{n_i}}{n_i!} x^{n_i} y^{in_i} = \prod_{i=1}^{\infty} \exp(ixy^i)$$

$$= \exp\left(\sum_{i=1}^{\infty} ixy^i\right) = \exp\frac{xy}{(1-y)^2}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{i=0}^{\infty} \binom{2n+i-1}{2n-1} y^{n+j}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{N=n}^{\infty} \binom{N+n-1}{2n-1} y^N. \quad \text{(B15)}$$

Extracting the coefficient for $x^n y^N$, we have (B14).

*Proof of Theorem 1.* According to (B1),

$$V(n,N) = \sum_{n_1,\ldots,n_N} V_{n_1,\ldots,n_N}$$

$$= N \sum_{n_1,\ldots,n_N} n_j(n-1)!$$

$$\times \prod_{i=1}^{N} \frac{i^{n_i}}{n_i!} \times S^n p^{N-n} \quad , \quad \text{(B16)}$$

where the sum is taken over $n_1 + \cdots + n_N = n$, $n_1 + 2n_2 + \cdots + Nn_N = N$. Thus (B7) follows from (B14).

In order to prove (B8) we observe

$$-\ln\left(1 - \frac{xy}{(1-y)^2}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{N=n}^{\infty} \binom{N+n-1}{2n-1} y^N$$

$$= \sum_{N=1}^{\infty} \sum_{n=1}^{N} \frac{1}{n} \binom{N+n-1}{2n-1} x^n y^N. \quad \text{(B17)}$$

Taking derivative over $y$ of both sides of (B17) and multiplying by $y$, we have

$$\frac{xy(1+y)}{(1-y)[(1-y)^2 - xy]} = \sum_{N=1}^{\infty} \sum_{n=1}^{N} \frac{N}{n} \binom{N+n-1}{2n-1} x^n y^N.$$

(B18)

Substituting $x = S/p$, $y = pz$ in (B18) we have (B8).

To prove (B9) we represent the right side of (B8) as

$$\frac{Sz(1+pz)}{(1-pz)(1-z+p^2z^2)} = \frac{z_1}{z_1 - z} + \frac{z_2}{z_2 - z} - \frac{2}{1-pz} \quad ,$$

(B19)

where

$$z_{1,2} = \frac{1 \pm \sqrt{1-4p^2}}{2p^2} \quad \text{(B20)}$$

are the roots of $p^2z^2 - z + 1 = 0$. As $z_2 < p^{-1} < z_1$, the second term in this sum defines the asymptotics of coefficients in the power series $\sum V(N)z^N$ for large $N$, so $V(N) \propto z_2^{-N} = (p^2 z_1)^N$.

*Proof of Theorem 2.* According to (B1),

$$A_j(N)V(N) = N \sum_{n=1}^{N} \sum_{n_1+\cdots+n_N=n} n_j(n-1)!$$

$$\times \prod_{i=1}^{N} \frac{i^{n_i}}{n_i!} \times S^n p^{N-n} \quad , \quad \text{(B21)}$$

where $n_1 + 2n_2 + \ldots + Nn_N = N$. In order to prove (B11), we have to show that

$$\sum_{n_1+\cdots+n_N=n} n_j \prod_{i=1}^{N} \frac{i^{n_i}}{n_i!} = \frac{j}{(n-1)!} \binom{N+n-j-2}{2n-3} \quad .$$

(B22)

Consider the generating function $g_j(x,y) = \sum_{n,N=0}^{\infty} \nu_j(n,N) x^n y^N$, where $\nu_j(n,N)$ denotes the left side of (B22). We have

$$g_j(x,y) = \sum_{n_1,\ldots,n_N} n_j \prod_{i=1}^{N} \frac{i^{n_i}}{n_i!} x^{n_i} y^{in_i} = jxy^j \prod_{i=1}^{\infty} \exp(ixy^i) = jxy^j \exp\left(\sum_{i=1}^{\infty} ixy^i\right)$$
$$= jxj^j \exp\frac{xy}{(1-y)^2} = \sum_{n=0}^{\infty} \frac{jx^{n+1}}{n!} \sum_{i=0}^{\infty} \binom{2n+i-1}{2n-1} y^{n+i+j}$$
$$= \sum_{n=1}^{\infty} \frac{jx^n}{(n-1)!} \sum_{N=n+j-1}^{\infty} \binom{N+n-j-2}{2n-3} y^N \quad . \tag{B23}$$

Extracting the coefficient for $x^n y^N$, we have (B22)

To prove (B12) we first observe that

$$\frac{1}{1-\frac{xy}{(1-y)^2}} = \sum_{n=0}^{\infty} x^n \sum_{i=0}^{\infty} \binom{2n+i-1}{2n-1} y^{n+i} \quad . \tag{B24}$$

Hence,

$$\frac{jxy^j}{1-\frac{xy}{(1-y)^2}} = j\sum_{n=0}^{\infty} x^{n+1} \sum_{i=0}^{\infty} \binom{2n+i-1}{2n-1} y^{n+i+j}$$
$$= j\sum_{n=1}^{\infty} x^n \sum_{N=n+j-1}^{\infty} \binom{N+n-j-2}{2n-3} y^N. \tag{B25}$$

Substituting $x = S/p$, $y = pz$ in (B25) we have (B12).

To prove (B13) we represent the right side of (B12) as

$$\frac{jSp^{j-1}(1-pz)^2 z^j}{1-z+p^2z^2} = jSp^{j-1}z^j \times\left[1+\sqrt{\frac{1-2p}{1+2p}}\left(\frac{1}{z_2-z} - \frac{1}{z_1-z}\right)\right], \tag{B26}$$

where $z_1$ and $z_2$ are the same as in the proof of Theorem 1. As $z_2 < z_1$, the term with $1/(z_2 - z)$ defines asymptotics of coefficients in the power series in (B26).

## APPENDIX C: STRUCTURE OF SIMPLEXES

In case $b$ of the series redistribution model, as well as in the case of parallel redistribution, the analog of proposition 1 is valid for essentially smaller set of states, when only events of size 1 (or degenerate cases when several non-neighboring elements break independently) are allowed. Let $\tau = (i_1, \cdots, i_N)$ be an arbitrary permutation of $(1,\ldots,N)$, $i_j \equiv \tau(j)$. For every $i$, let $p_i = \sum_{j:\tau(j)<\tau(i)} p_{\tau(i),\tau(j)}$. Then the set of states $\sigma$ generating periodic trajectories with an ordered sequence $(i_1,\ldots,i_N)$ of single breaks is defined by

$$1 > \sigma_{\tau(1)} + p_1 > \ldots > \sigma_{\tau(N)} + p_N \geq 1 - S \quad . \tag{C1}$$

The volume of this $N$-dimensional simplex with side $S$ is equal to $S^N/N!$. The union of the sets (C1) over all permutations $\tau$ coincides with the set of all periodic trajectories with single breaks. Its volume is $S^N$.


[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).

[2] P. Bak and C. Tang, J. Geophys. Res. **94**, 15 635 (1989).

[3] L.P. Kadanoff, S.R. Nagel, L. Wu, and S. Zhou, Phys. Rev. A. **39**, 6524 (1989).

[4] K. Ito and M. Matsusaki, J. Geophys. Res. **95**, 6853 (1990); H. Nakanishi, Phys. Rev. A **41**, 7086 (1990); H. Nakanishi, *ibid.* **43**, 6613 (1991); H. Takayasu and M. Matsuzaki, Phys. Lett. A **131**, 244 (1988); S.R. Brown, C.H. Scholz, and J.B. Rundle, Geophys. Res. Lett. **18**, 215 (1991); H. Takayasu, I. Nishikawa, and H. Tasaki, Phys. Rev. A **37**, 3110 (1988).

[5] H.J.S. Feder and J. Feder, Phys. Rev. Lett. **66**, 2669 (1991).

[6] K. Christensen and Z. Olami, Phys. Rev. A **46**, 1829 (1992).

[7] Z. Olami, H.J.S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992).

[8] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967).

[9] J. Carlson and J.S. Langer, Phys. Rev. Lett. **62**, 2632 (1989).

[10] J.M. Carlson and J.S. Langer, Phys. Rev. A **40**, 6470 (1989).

[11] J.M. Carlson, Phys. Rev. A **44**, 6226 (1991).

[12] M. Matsuzaki and H. Takayasu, J. Geophys. Res. **96**, 19 925 (1991).

[13] J. Lomnitz-Adler, L. Knopoff, and G. Martinez-Mekler, Phys. Rev. A **45**, 2211 (1992).

[14] A.M. Gabrielov, V.I. Keilis-Borok, T.A. Levshina, and V.A. Shaposhnikov, Comput. Seismology **19**, 168 (1986).

[15] A.M. Gabrielov, T.A. Levshina, and I.M. Rotvain, Phys. Earth Planet. Int. **61**, 18 (1990).

[16] K. Chen, P. Bak, and S.P. Obukhov, Phys. Rev. A **43**, 625 (1991); H.-J. Xu, B. Bergersen, and K. Chen, J. Phys. B **25**, L1251 (1992); B. Barriere and D.L. Turcotte, Geophys. Res. Lett. **18**, 2011 (1991).

[17] A.M. Gabrielov, International Center for Theoretical Physics Report No. H4.SMR/303-36, Trieste, 1988.

[18] A. Diaz-Guilera, Phys. Rev. A **45**, 8551 (1992).

[19] Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989); L. Pietronero, P. Tartaglia, and Y.-C. Zhang, Physica A **173**, 22 (1991).

[20] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).

[21] P. Holmes, Phys. Rep. **193**, 137 (1990).

[22] In this case, our models are also equivalent to that introduced by G. Narkounskaia, J. Huang, and D.L. Turcotte, J. Stat. Phys. **67**, 1151 (1992).

[23] A. Zelevinsky (private communication).