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Extreme Deviations and Applications: Fragmentation and Extreme Events

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Extreme Deviations and Applications

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Abstract. — Stretched exponential probability density functions (pdf), having the form of the exponential of minus a fractional power of the argument, are commonly found in turbulence and other areas. They can arise because of an underlying random multiplicative process. For this, a theory of *extreme* deviations is developed, devoted to the far tail of the pdf of the sum X of a finite number n of independent random variables with a common pdf $e^{-f(x)}$. The function $f(x)$ is chosen (i) such that the pdf is normalized and (ii) with a strong convexity condition that $f''(x) > 0$ and that $x^2 f''(x) \to +\infty$ for $|x| \to \infty$. Additional technical conditions ensure the control of the variations of $f''(x)$. The tail behavior of the sum comes then mostly from individual variables in the sum all close to X/n and the tail of the pdf is $\sim e^{-nf(X/n)}$. This theory is then applied to products of independent random variables, such that their logarithms are in the above class, yielding usually stretched exponential tails. An application to fragmentation is developed and compared to data from fault gouges. The pdf by mass is obtained as a weighted superposition of stretched exponentials, reflecting the coexistence of different fragmentation generations. For sizes near and above the peak size, the pdf is approximately log-normal, while it is a power law for the smaller fragments, with an exponent which is a decreasing function of the peak fragment size. The anomalous relaxation of glasses can also be rationalized using our result together with a simple multiplicative model of local atom configurations. Finally, we indicate the possible relevance to the distribution of small-scale velocity increments in turbulent flow.

1. Introduction

Consider the sum

$$S_n \equiv \sum_{i=1}^{n} x_i, \tag{1}$$

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where the x_i are independent identically distributed (iid) random variables with probability density function (pdf) $p(x)$ and mean value $\langle x \rangle$. The central limit theorem ensures, with suitable conditions such as the existence of finite second-order moments or refinements thereof [1, 2], that as $n \to \infty$ the pdf of $\frac{S_n - n\langle x \rangle}{\sqrt{n}}$ becomes Gaussian. In other words, the "typical" fluctuations of S_n/n around its mean value are Gaussian and $O(1/\sqrt{n})$. The large deviations theory is concerned with events of much lower probability when S_n/n deviates from its mean value by a quantity $O(1)$ [3–6]. In non-technical terms, for large n,

$$\mathrm{Prob}\left[\frac{S_n}{n} \simeq y\right] \sim \mathrm{e}^{n s(y)}, \tag{2}$$

where $s(y) \leq 0$ is the Cramér function (also called "rate function").

In this paper we are concerned with "extreme deviations", that is the régime of *finite* n and *large* $S_n = X$. This régime exists only when the pdf extends to arbitrary large values of x (*i.e.* has noncompact support). In other words, we are interested in tail behavior. While it is common in statistics to consider test probabilities of the order of 1%, much smaller probabilities are of interest in many areas in which crisis may ensue. If, for instance, one wishes to investigate whether a chemical substance causes cancer, one will be interested in very small test probabilities to make a convincing case. In the field of reliability, failure and rupture, for instance of industrial plants, very small probabilities are the rule. Examples are the calculation of the probability of a defect item passing an inspection system and the calculation of the reliability of a system. 10^{-6} is the probability threshold beyond which the U.S. Food & Drug administration considers that any risk from a food additive is considered too small to be of concern. In the same spirit, the legal U.S. maximum man-made risk to public is 5×10^{-5}.

The main result about extreme deviations for sums of random variables is presented in Section 2. The relation to large deviations theory and to other work on extreme deviations is briefly discussed in Section 3. Multiplication of random variables is considered in Section 4. Applications are presented in Section 5. The Appendix is devoted to a rigorous derivation of the main result for sums of independent variables.

2. Extreme Deviations for Sums of Random Variables

We are interested in the tail behavior for large arguments x of sums of iid random variables x_i, $i = 1, 2, \ldots$. We shall only consider large positive values of x. All the results can be adapted *mutatis mutandis* to large negative values. We assume that the common probability distribution of the x_i's has a pdf, denoted $p(x)$, which is normalized:

$$\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1, \tag{3}$$

and which can be represented as an exponential:

$$p(x) = \mathrm{e}^{-f(x)}, \tag{4}$$

where $f(x)$ is indefinitely differentiable ([1]). We rule out the case where $f(x)$ becomes infinite at finite x; this would correspond to a distribution with compact support which has no extreme deviations.

The key assumptions are now listed. All statements involving a limit are understood to be for $x \to +\infty$.

([1]) As we shall see in the Appendix, this assumption can be relaxed.

(i) $f(x) \to +\infty$ sufficiently fast to ensure the normalization (3);

(ii) $f''(x) > 0$ (convexity), where f'' is the second derivative of f ([2]);

(iii) $\lim \frac{f^{(k)}(x)}{(f''(x))^{k/2}} = 0$, for $k \geq 3$, where $f^{(k)}$ is the kth derivative of f. The $k = 3$ instance of (iii) will be denoted by $\widetilde{\text{(iii)}}$.

An important consequence of $\widetilde{\text{(iii)}}$ is

$$\lim x^2 f''(x) = +\infty, \tag{5}$$

the proof of which is given in the Appendix (Lemma 1).

We introduce now the pdf $P_n(x)$ of $S_n = \sum_1^n x_i$ which may be written as a multiple convolution:

$$P_n(x) = \underbrace{\int \cdots \int}_{n} \mathrm{e}^{-\sum_{i=1}^n f(x_i)}\, \delta\left(x - \sum_{i=1}^n x_i\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{6}$$

All integrals are from $-\infty$ to $+\infty$. The delta function expresses the constraint on the sum. We shall show that, under assumptions (i)–(iii), the leading-order expansion of $P_n(x)$ for large x and finite $n \geq 1$ is given by

$$P_n(x) \simeq \mathrm{e}^{-nf(x/n)} \frac{1}{\sqrt{n}} \left(\frac{2\pi}{f''(x/n)}\right)^{\frac{n-1}{2}}, \quad \text{for} \quad x \to \infty \quad \text{and} \quad n \quad \text{finite.} \tag{7}$$

Furthermore, we shall show that the leading contribution comes from individual terms in the sum which are *democratically localized.* By this we understand that the conditional probability of the x_i's, given that the sum is x, is localized, for large x, near

$$x_1 \simeq x_2 \cdots \simeq x_n \simeq \frac{x}{n}. \tag{8}$$

In this section we give a derivation of this result using a formal asymptotic expansion closely related to Laplace's method for the asymptotic evaluation of certain integrals [8] ([3]). In the Appendix we shall give a rigorous proof.

To evaluate (6) for $n \geq 2$, we define new variables

$$h_i \equiv x_i - \frac{x}{n}, \quad \text{for } i = 1, \ldots, n-1 \tag{9}$$

$$h_n \equiv -(h_1 + \ldots + h_{n-1}), \tag{10}$$

and the function

$$g_n(x; h_1, \ldots, h_{n-1}) \equiv \sum_{i=1}^n f\left(\frac{x}{n} + h_i\right). \tag{11}$$

We can then rewrite (6) as

$$P_n(x) = \underbrace{\int \cdots \int}_{n-1} \mathrm{e}^{-g_n(x;h_1,\ldots,h_{n-1})}\, \mathrm{d}h_1 \cdots \mathrm{d}h_{n-1}. \tag{12}$$

([2]) This is called log-concavity of the density by Jensen [7] whose work we shall comment on in Section 3.

([3]) This method is sometimes referred to as "steepest descent", an inadequate terminology when $f(x)$ is not analytic.

The function g_n has the following Taylor expansion in powers of the h_i's:

$$g_n = nf\left(\frac{x}{n}\right) + \frac{1}{2!}f''\left(\frac{x}{n}\right)\sum_{i=1}^{n} h_i^2 + \frac{1}{3!}f'''\left(\frac{x}{n}\right)\sum_{i=1}^{n} h_i^3 + \cdots . \tag{13}$$

Note the absence of the term linear in the h_i's since, by (10), $\sum_{i=1}^{n} h_i = 0$.

If we momentarily ignore the terms of order higher than two in (13), we obtain for $P_n(x)$ a Gaussian integral the convergence of which is ensured by the convexity condition (ii). This integral is evaluated by setting $y = 0$ and $\lambda = (1/2)f''(x/n)$ in the identity (4)

$$\underbrace{\int \cdots \int}_{n-1} e^{-\lambda\left[h_1^2+\cdots+h_{n-1}^2+(y-h_1-\cdots-h_{n-1})^2\right]} \mathrm{d}h_1 \cdots \mathrm{d}h_{n-1} = \frac{1}{\sqrt{n}}\left(\frac{\pi}{\lambda}\right)^{\frac{n-1}{2}} e^{-\frac{\lambda}{n}y^2}. \tag{14}$$

We thereby obtain the desired asymptotic expression (7) for $P_n(x)$.

We now show that higher than second order terms in the taylor expansion (13) do not contribute to the leading-order result. The quadratic form $(1/2)f''(x/n)\sum_{i=1}^{n} h_i^2$ in the $n-1$ variables $h_1, \ldots, h_{n-1}$ can be diagonalized (it is just proportional to the square of the Euclidean norm in the subspace $\sum_{i=1}^{n} h_i = 0$). One can show by recurrence that it has $n-2$ eigenvalues equal to $(1/2)f''(x/n)$ and one eigenvalue n times larger. Hence, the Gaussian multiple integral comes from h_i's which are all $O(1/\sqrt{f''(x/n)})$ or smaller. For such h_i's, it follows from the assumption (iii) that all higher order terms are negligible for large x. Furthermore, the scatter of the x_i's around the value x/n, measured by the the rms value of the h_i's is $O(1/\sqrt{f''(x/n)})$. By (5), this is small compared to x, which proves the *democratic localization* property (8).

We shall also make use of a weaker result obtained by taking the logarithm of (7), namely

$$\ln P_n(x) \simeq -nf(x/n), \quad \text{for} \quad x \to \infty \quad \text{and} \quad n \quad \text{finite}. \tag{15}$$

This weaker form holds only if

$$\frac{\ln f''(x/n)}{f(x/n)} \to 0. \tag{16}$$

We make a few remarks. Our derivation is reminiscent of the derivation of Laplace's asymptotic formula for integrals of the form $\int e^{-\lambda f(x)}\, \mathrm{d}x$ when $\lambda \to \infty$, as given, *e.g.*, in reference [8]. The main difference is that in Laplace's method, when f is Taylor expanded around its minimum, terms of order higher than two give contribution smaller by higher and higher inverse powers of λ, so that a single small parameter $1/\lambda$ is enough to justify the expansion, whereas here we made an infinite number of assumptions (iii) for all $n \geq 3$. Actually, it will be shown in the Appendix that the sole assumption $\widetilde{\text{(iii)}}$ with a slight strengthening of (5) is enough to derive the leading-order term (7).

It is easily checked that our result is not equivalent to the well-known fact that the most probable increment $\Delta x/\Delta t$ of a random walk conditioned to go from (x,t) to (x',t') is constant and equal to the average slope $(x'-x)/(t'-t)$; in other words, the most probable path is then a straight line, corresponding to a constant reduced running sum.

The convexity of $f(x)$ at large x is essential for our result to hold. For instance, pdf's with powerlaw tails $p(x) \propto x^{-(1+\mu)}$ give $f(x) = (1+\mu)\ln x$ which is concave. The extreme deviations of the sum S_n are then controlled by realizations where just one term in the sum dominates.

(4) This identity if obtained, after proper normalization, by evaluating the n-fold convolution of a Gaussian distribution of variance $1/(2\lambda)$ with itself, which is a Gaussian of variance $n/(2\lambda)$.

This extends to *arbitrary* exponents μ, in this extreme deviations régime, the well-known result that the breakdown of the central limit theorem for $\mu < 2$ stems from the dominance of a few large terms in the sum. The breakdown of democratic localization far in the tail also happens for pdf's with finite moments of all orders, for example, when $p(x) \propto x^{-\ln x}$ at large x. Here, again the function $f(x) = \ln^2 x$ is not convex.

The result (15) can be formally ([5]) generalized to the case of dependent variables with nonseparable pdf's $p(x_1, x_2, \ldots, x_i, ..., x_n) = \exp[-f(x_1, x_2, ..., x_i, ..., x_n)]$ where $f(x_1, x_2, ..., x_i, ..., x_n)$ is symmetric and convex. Indeed, $\frac{\partial f}{\partial x_i}|_{x_1=x_2=...=x_n=S_n/n}$ is then independent of i and the matrix of second derivatives $\partial^2 f/\partial x_i^2$ evaluated at $x_1 = x_2 = \ldots = x_n = S_n/n$ is positive, ensuring that f is minimum at $x_1 = x_2 = \ldots = x_n = S_n/n$, thereby providing the major contribution to the convolution integral.

3. Relation with the Theory of Large Deviations

We now assume, in addition to conditions (i)–(iii) of Section 2, that the characteristic function

$$Z(\beta) \equiv \langle e^{-\beta x} \rangle = \int_{-\infty}^{\infty} e^{-\beta x} p(x)\,dx \tag{17}$$

exists for all real β's (Cramér condition). Recall that the Cramér function $s(y)$ is determined by the following set of equations (see, *e.g.*, Refs. [4,6,9]):

$$s(y) = \ln Z(\beta) + \beta y, \tag{18}$$

$$\frac{ds(y)}{dy} = \beta. \tag{19}$$

Hence, $s(y)$ is the Legendre transform of $\ln Z(\beta)$.

Comparison of (2) with (15) shows that the Cramér function $s(y)$ becomes equal to $-f(y)$ for large y. We can verify this statement by inserting the form $p(x) = e^{-f(x)}$ into (17) to get $Z(\beta) \sim \int_{-\infty}^{\infty} dx\, e^{-\beta x - f(x)}$. For large $|\beta|$, we can then approximate this integral by Laplace's method, yielding $Z(\beta) \sim e^{-\min_x(\beta x + f(x))}$. Taking the logarithm and a Legendre transform, we recover the identification that $s(y) \to -f(y)$ for large y. Laplace's method is justified by the fact that $|y| \to \infty$ corresponds, in the Legendre transformation, to $|\beta| \to \infty$. A number of more precise results are known, which relate the tail probabilities of random variables to the large-y behavior of the Cramér function. For example, Broniatowski and Fuchs [10] give conditions for the asymptotic equivalence of $s(y)$ (called by them the "Chernov function") and of $-\ln \bar{F}(y)$ where $\bar{F}(y) \equiv \int_y^{\infty} p(x)\,dx$.

A consequence is that the large and extreme deviations régime overlap when taking the two limits $n \to \infty$ and $S_n/n \to \infty$. Indeed, large deviations theory usually takes $n \to \infty$ while keeping S_n/n finite, whereas our extreme deviations theory takes n finite with $S_n \to \infty$. Our analysis shows that, in the latter régime, Cramér's result already holds for finite n. The true small parameter of the large deviations theory is thus not $1/n$ but $\min(1/n, n/S_n)$.

A paper by Borokov and Mogulskii [11] contains a result resembling somewhat ours. Their equation (12) of Section 1 states, in our notation, that

$$s_n(y) = ns(y/n), \tag{20}$$

([5]) Additional assumptions are then needed to make sure that higher than second-order terms in the Taylor expansion are not contributing.

where $s_n(y)$ is the Cramér function for the sum of n independent and identically distributed copies of a random variable with Cramér function $s(y)$. If we identify the tail of the Cramér function with minus the logarithm of the (tail of the) pdf, their result becomes identical with (15). However, their result makes no use of the convexity assumption without which our result will generally not hold.

Broniatowski and Fuchs [10] derive a more general but weaker theorem on the cumulative distribution of S_n for finite n, which ressembles somewhat our result on the *democratic localization* property (8). It is more general because it is valid for pdf's not obeying the convexity condition (5), for example, the Cauchy distribution. It is weaker because it states only that there is a number $\alpha_n > 0$ such that

$$\ln \mathrm{Prob}\,(S_n \geq nx) = \alpha_n[1 + o(1)] \ln \mathrm{Prob}\,(\min(x_1, \ldots, x_n)) \geq x) \ , \tag{21}$$

for $x \to \infty$. Roughly speaking, (21) means that the main contributions to the event $S_n \geq nx$ come from the realizations where *all* variables constituting the sum are larger than x, a much weaker statement than the property of *democratic localization* (8).

Jensen [7] also considers the case where n is finite and the tail probability tends to zero, for particular choices of the pdf. Jensen is able to show in a few examples that, even though there is no asymptotics, *i.e.* there is no n tending to infinity, the saddlepoint expansion allows one to get the correct order of the probabilities in the tail, using the so-called tilted density introduced by Esscher [12]. Coupled with the Edgeworth expansion, this leads to results similar to ours. Our work generalizes and systematizes these partial results by providing general conditions of applications, in particular not requiring that f be Taylor expandable to all orders (see the Appendix).

4. Multiplications of Random Variables

Consider the product

$$X_n = m_1 m_2 \cdots m_n \tag{22}$$

of n independent identically distributed positive ([6]) random variables with pdf $p(x)$. Taking the logarithm of X_n, it is clear that we recover the previous problem (1) with the correspondence $x_i \equiv \ln m_i$, $S_n \equiv \ln X_n$ and $-f(x) = \ln p(\mathrm{e}^x) + x$. Assuming again the set of conditions (i), (ii) and (iii) on f, we can apply the extreme deviations result (15) which translates into the following form for the pdf $P_n(X)$ of X_n at large X:

$$P_n(X) \sim [p(X^{1/n})]^n, \qquad \text{for} \quad X \to \infty \quad \text{and} \quad n \quad \text{finite}. \tag{23}$$

(In this section we omit prefactors; this amounts to using (15) instead of (7).) Equation (23) has a very intuitive interpretation: the tail of $P_n(X)$ is controlled by the realizations where all terms in the product are of the same order; therefore $P_n(X)$ is, to leading order, just the product of the n pdf's, each of their arguments being equal to the common value $X^{1/n}$.

When $p(x)$ is an exponential, a Gaussian or, more generally, of the form $\propto \exp(-Cx^\gamma)$ with $\gamma > 0$, then (23) leads to stretched exponentials for large n. For example, when $p(x) \propto \exp(-Cx^2)$, then $P_n(X)$ has a tail $\propto \exp(-CnX^{2/n})$.

([6]) What follows is immediately extended to the case of signed m_i's with a symmetric distribution.

Note that (23) can be obtained directly by recurrence. Starting from $X_{n+1} = X_n x_{n+1}$, we write the equation for the pdf of X_{n+1} in terms of the pdf's of x_{n+1} and X_n:

$$\begin{aligned} P_{n+1}(X_{n+1}) &= \int_0^\infty \mathrm{d}X_n P_n(X_n) \int_0^\infty \mathrm{d}x_{n+1} p(x_{n+1}) \delta(X_{n+1} - X_n x_{n+1}) \\ &= \int_0^\infty \frac{\mathrm{d}X_n}{X_n} P_n(X_n) p\left(\frac{X_{n+1}}{X_n}\right). \end{aligned} \tag{24}$$

The maximum of the integrand occurs for $X_n = (X_{n+1})^{(n+1)/n}$ at which $X_n^{1/n} = X_{n+1}/X_n$. Assuming that $P_n(X_n)$ is of the form (23), the formal application of Laplace's method to (24) then directly gives that $P_{n+1}(X_{n+1})$ is of the same form ([7]). Thus, the property (23) holds for all n to leading order in X.

Some generalizations are easily obtained. For instance, for exponential distributions, we can allow for different characteristic scales α_j defined by $p_j(x) = \alpha_j \mathrm{e}^{-\alpha_j x_j}$. Equation (23) then becomes

$$P_n(X) \sim \exp\left(-n\left[X \prod_{j=1}^n \alpha_j\right]^{1/n}\right) \quad \text{for} \quad X_n > \prod_{j=1}^n \frac{1}{\alpha_j}. \tag{25}$$

Similarly, if $p_j(x) = \frac{2}{\sqrt{2\pi}\sigma_j} \mathrm{e}^{-x_j^2/2\sigma_j^2}$, with $x_j \geq 0$, we obtain

$$P_n(X) \sim \exp\left(-\frac{n}{2}\left[\frac{X^2}{\prod_{j=1}^n \sigma_j^2}\right]^{\frac{1}{n}}\right) \quad \text{for} \quad X_n > \prod_{j=1}^n \sigma_j. \tag{26}$$

5. Applications

Considering the simplicity and robustness of the results derived above, we expect the extreme deviation mechanism to be at work in a number of physical or other systems. We are thinking in particular of the application of our result to simple multiplicative processes, that might constitute zeroth-order descriptions of a large variety of physical systems, exhibing anomalous pdf and relaxation behaviors. There is no generally accepted mechanism for their existence and their origin is still the subject of intense investigation. The extreme deviations régime may provide a very general and essentially *model-independent* mechanism, based on the extreme deviations of product of random variables.

Fragments are often found to be distributed according to power law distributions [13]: In Section 5.1, we propose a multiplicative fragmentation model in which the exponent is controlled by the depth of the cascading process. Anomalous relaxations in glasses have been largely documented to occur according to stretched exponentials [14,15]. In Section 5.2, we construct a relaxation model based on the idea that a complex disordered system can be divided into an ensemble of local configurations, each of them hierarchically ordered. Stretched exponential pdf are observed in turbulent flow (see, *e.g.*, Ref. [9]) and our extreme deviation theory provides a simple scenario (Sect. 5.3). Let us finally mention the question of stock market prices and their distribution. Here, the very nature of the pdf's is still debated [16,17]. While price variations at short time scales (minutes to hours) are well-fitted by truncated Lévy laws [18], other alternative have been proposed [16]. We have found that a stretched exponential pdf provides an economical and accurate fit to the full range of currency price variations at the daily intermediate time scale. We will come back in future work to document this claim and to describe the relevance of the multiplicative processes studied here.

([7]) Control over higher-order terms in the asymptotic expansion requires, of course, the same conditions (i)–(iii) as in Section 2.

5.1. FRAGMENTATION. — Fragmentation occurs in a wide variety of physical phenomena from geophysics, material sciences to astrophysics and in a wide range of scales. The simplest (naive) model is to ignore conservation of mass and to view fragmentation as a multiplicative process in which the sizes of children are fractions of the size of their parents. If we assume that the succession of breaking events are independent and concentrate on a *given generation rank* n, our above result (23) applies to the distribution of fragment size X, provided we take X to zero rather than to infinity. Indeed, the factors $m_1, m_2, \ldots, m_n$ are all less or equal to unity ([8]). If we take, for example, $p(m) \propto \exp(-cm^a)$ for small m, we obtain $P_n(X) \propto \exp\left(-cnX^{a/n}\right)$. For values of X which are order unity, large deviations theory applies when $n \to \infty$. This does not, in general, lead to a log-normal distribution, because central limit arguments are inapplicable, except in the very neighborhood of the peak of the pdf of X (see, *e.g.*, Ref. [9], Sect. 8.6.5).

Next, we observe that most of the measured size distribution of fragments, *not conditioned by generation rank*, display actually power-law behavior $\propto X^{-\tau}$ with exponents τ between 1.9 and 2.6 clustering around 2.4 [19]. Several models have been proposed to rationalize these observations [13, 20] but there is no accepted theoretical description.

Here, we would like to point out a very simple and robust scenario to rationalize these observations. We again neglect the constraint that the total mass of the children is equal to that of the parent and use the simple multiplicative model. Indeed, the constraint of conservation becomes less and less important for the determination of the pdf as the generation rank increases. To illustrate what we have in mind, consider a comminution process in which, with a certain probablity less than unity, a "hammer" repetitively strikes all fragments simultaneously. Then the generation rank corresponds to the number of hammer hits. In real experiments, however, each fragment has suffered a specific number of effective hits which may vary greatly from one fragment to the other. The measurements of the size distribution should thus correspond to a superposition of pdf's of the form (23) in the tail $X \to 0$. Recent numerical simulations of lattice models with disorder [21] show indeed that, for sufficient disorder, the fragmentation can be seen as a cascade branching process.

Let us now assume that the tail of the size distribution for a fixed generation rank n is given by (23) and that the mean number $N(n)$ (per unit volume) of fragments of generation rank n grows exponentially: $N(n) \propto e^{\lambda n}$ with $\lambda > 0$. It then follows that the tail of the unconditioned size distribution is given by

$$P_{\rm size}(X) \sim \sum_{n=0}^{\infty} [p(X^{1/n})]^n e^{\lambda n} \sim \int_0^{\infty} dn\, e^{n \ln p(X^{1/n}) + n\lambda}. \tag{27}$$

Application of Laplace's method in the variable n, treated as continuous, gives a critical (saddle) point

$$n_* = -\frac{1}{\alpha} \ln X, \tag{28}$$

where α is the solution of the transcendental equation

$$\lambda + \ln p\left(e^{-\alpha}\right) + \alpha e^{-\alpha} \frac{p'\left(e^{-\alpha}\right)}{p\left(e^{-\alpha}\right)} = 0. \tag{29}$$

The leading-order tail behavior of the size distribution is thus given by

$$P_{\rm size}(X) \sim X^{-\tau}, \tag{30}$$

([8]) When taking the logarithm, the tail for $X \to 0$ corresponds to the régime where the sum of logarithms goes to $-\infty$. Although $X \to 0$, is not strictly speaking a "tail", we shall still keep this terminology.

with an exponent

$$\tau = \frac{1}{\alpha}\left[\ln p\left(\mathrm{e}^{-\alpha}\right) + \lambda\right]. \tag{31}$$

This solution (30) holds for λ smaller than a threshold λ_c dependent on the specific structure of the pdf $p(x)$. For instance, consider $p(x) \propto \exp\left(-Cx^{\delta}\right)$ for $x \to 0$, with $\delta > 0$. This corresponds to a pdf going to a constant as $x \to 0$, with a vanishing slope ($\delta > 1$), infinite slope ($\delta < 1$) or finite slope ($\delta = 1$). The equation (29) for α becomes $\lambda/C = (1+\alpha\delta)\mathrm{e}^{-\alpha\delta}$. This has a solution only for $\lambda \leq C$, as the function $(1+x)\mathrm{e}^{-x}$ has its maximum equal to 1 at $x = 0$. For λ approaching C from below, the exponent of the power law distribution is given by $\tau = C\delta + O(\sqrt{C-\lambda})$. At the other end $\lambda \to 0^+$, we get $\tau \to C\delta e$. In between, for $0 \leq \lambda \leq C$, the quantity $\tau/(C\delta)$ goes continuously from $e \approx 2.718$ to 1. It is interesting that τ depends on the parameters of the pdf $p(x)$ only through the product $C\delta$.

What happens for $\lambda > C$? To find out, we return to the expression (27) giving the tail of the unconditioned size distribution and find that the exponential in the integral reads $\mathrm{e}^{n(\lambda - CX^{\delta/n})}$. In the limit of small fragments $X \to 0$, the term $X^{\delta/n}$ is dominated by the large n limit for which it is bounded by 1. Thus, $\lambda - CX^{\delta/n} \leq \lambda - C$. For $\lambda > C$, the larger n is, the larger the exponential is, while for $\lambda < C$ there is an optimal generation number n_*, for a given size X, given by (28). For $\lambda \geq C$, the critical value n_* moves to infinity. Physically, this is the signature of a shattering transition occurring at $\lambda = C$: for $\lambda > C$, the number of fragments increases so fast with the generation number n (as $\mathrm{e}^{\lambda n} > \mathrm{e}^{Cn}$) that the distribution of fragment sizes develops a finite measure at $X = 0$. This result is in accordance with intuition: it is when the number of new fragments generated at each hammer hit is sufficiently large that a dust phase can appear. This *shattering transition* has been obtained first in the context of mean field linear rate equations [22].

Consider another class of pdf $p(x) \propto \exp\left(-Cx^{-\delta}\right)$ for $x \to 0$, with $\delta > 0$. The pdf $p(x)$ goes to zero faster than any power law as $x \to 0$ (*i.e.* has an essential singularity). The difference with the previous case is that, as the multiplicative factor $x \to 0$ occurs with very low probability in the present case, we do not expect a large number of small fragments to be generated. This should be reflected in a negative value of the exponent τ. This intuition is confirmed by an explicit calculation showing that τ becomes the opposite of the value previously calculated, *i.e.* $\tau/(C\delta)$ goes continuously from $-e \approx -2.718$ to -1 as λ goes from 0 to C.

In sum, we propose that the observed power-law distributions of fragment sizes could be the result of the natural mixing occurring in the number of generations of simple multiplicative processes exhibiting extreme deviations. This power-law structure is very robust with respect to the choice of the distribution $p(x)$ of fragmentation ratios, but the exponent τ is not universal. The proposed theory leads us to urge the making of experiments in which one can control the generation rank *of each* fragment. We then predict that the fragment distribution will not be (quasi-) universal anymore buton the contrary characterize better the specific mechanism underlying the fragmentation process.

The result (30) only holds in the "tail" of the distribution for very small fragments. In the center, the distribution is still approximately log-normal. We can thus expect a relationship between the characteristic size or peak fragment size and the tail structure of the distribution. It is in fact possible to show that the exponent τ given by (31) is a *decreasing* function of the peak fragment size: the smaller is the peak fragment size, the larger will be the exponent (the detailed quantitative dependence is a specific function of the initial pdf). This prediction turns out to be verified by the measurements of particle size distributions in cataclastic (*i.e.* crushed and sheared rock resulting in the formation of powder) fault gouge [23]: the exponent τ of the finer fragments from three different faults (San Andreas, San Gabriel and Lopez Canyon) in Southern California was observed to be correlated with the peak fragment size, with finer gouges tending to have a larger exponent. Furthermore, the distributions were found to be a power law for the smaller fragments and log-normal by mass for sizes near and above the peak size.

5.2. STRETCHED EXPONENTIAL RELAXATION. — We would like to suggest a possible application of the stretched exponential distribution to rationalize stretched exponential relaxations. *A priori*, we are speaking of a different kind of phenomenon: so far we were discussing distributions, while we now consider the time dependence of a macroscopic variable relaxing to equilibrium. In contrast to simple liquids where the usual Maxwell exponential relaxation occurs, "complex" fluids [24], glasses [14, 15, 25], porous media, semiconductors, *etc.*, have been found to relax with time t as $e^{-at^{\beta}}$, with $0 < \beta < 1$, a law known under the name Kohlrausch–Williams–Watts law [14, 15]. Even, the Omori $1/t$ law for aftershock relaxation after a great earthquake has recently been challenged and it has been proposed that it be replaced by a stretched exponential relaxation [26]. This ubiquitous phenomenon is still poorly understood, different competing mechanisms being proposed. An often visited model is that of relaxation by progressive trapping of excitations by random sinks [15]. Models of hierarchically constrained dynamics for glassy relaxations [25] suggest the relevance of multiplicative processes to account for the relaxation in these complex, slowly relaxing, strongly interacting materials. Our model offers a simple explanation for the difference in β measured by the same method on different materials in terms of the dependence of β on the typical number of levels of the hierarchy as we now show.

We assume that a given system can be viewed as an ensemble of states, each state relaxing exponentially with a characteristic time scale. Each state can be viewed locally as corresponding to a given configuration of atoms or molecules leading to a local energy landscape. As a consequence, the local relaxation dynamics involves a hierarchy of degrees of freedom up to a limit determined by the size of the local configuration. In phase space, the representative point has to overcome a succession of energy barriers of statistically increasing heights as time goes on; this is at the origin of the slowing down of the relaxation dynamics. The characteristic time t_i to overcome a barrier ΔE_i is given by the Arhenius factor $t_i \sim \tau_0 e^{\Delta E_i/kT}$, where k is the Boltzmann constant, T the temperature and τ_0 a molecular time scale. For a succession of barriers increments, we get that the characteristic time is given by a multiplicative process, where each step corresponds to climbing the next level of the hierarchy. In other words, the characteristic relaxation time of a given cluster configuration is obtained by a multiplicative process truncated at some upper level. It is important to notice that our model is fundamentally different from the idea of diffusion of a representative particle in a random potential with potential barriers increasing statistically at long times, as in Sinai's anomalous diffusion [27]. We consider rather that the system can be divided into an ensemble of local configurations, each of them hierarchically ordered.

In this simple model, the times $T_n = \tau_0(t_1/\tau_0)\dots(t_n/\tau_0)$ are thus log-normally distributed in their center with stretched exponential tails according to our extreme deviation theory. Now, in a macroscopic measurement, one gets access to the average over the many different local modes of relaxation, each with a simple exponential relaxation: an observable $\mathcal{O}$ is thus relaxing macroscopically as $\mathcal{O} \sim \langle e^{t/T_n} \rangle$, where the average of the observable $\mathcal{O}$ is carried out over the distribution of T_n. For large t (compared to the molecular time scale), Laplace's method gives the leading-order behavior

$$\mathcal{O} \sim e^{-at^{\beta}},$$

with $\beta = \frac{\alpha}{n+\alpha}$ for a distribution of T_n given by $e^{-an(T_n/T_0)^{\alpha/n}}$. In this calculation, we have assumed that all local configuration clusters are organized hierarchically according to a fixed number of n levels. We envision that this organization reflects the local atomic or molecular arrangement such that the system can be subdivised into a set of essentially mutually independent local configurations. These configurations can tentatively be identified with the locally ordered structures observed in randomly packed particles [28], macromolecules [29], glasses and spinglasses [30]. The ultrametric structure found to describe the energy landscape of the spinglass phase of mean field models also leads to a multiplicative cascade [31, 32]. Notice that if a system possesses *multiple* configuration levels n, then by the same mechanism which in fragmentation led to (27), the relaxation becomes a power law instead of a stretched exponential.

The often encountered value $\beta \approx 1/2$ corresponds, in our model, to the existence of $n \approx \alpha$ levels of the hierarchy. It is noteworthy that the factor α can be determined quantitatively from the pdf $p(t_i/\tau_0) \sim \exp[-a(t_i/\tau_0)^\alpha]$ of the multiplicative factors, thus giving the potential to measure the number of levels of the hierarchy that are visited by the dynamical relaxation process. This could be checked for instance in multifragmentation in nuclear collisions, utilizing techniques sensitive to the emission order of fragments [33].

Hierarchical structures are also encountered in evolutionary processes [34], computing architectures [35] and economic structures [36] and, as a consequence, it is an interesting question whether to expect dynamical slowing down of the type described above.

5.3. TURBULENCE. — In fully developed turbulence, random multiplicative models were introduced by the Russian school [37–39] and have been studied extensively since. Indeed, their fractal and multifractal properties provide a possible interpretation for the phenomenon of intermittency [40, 41] (see also Ref. [9]). The pdf's of longitudinal and tranverse velocity increments clearly reveal a Gaussian-like shape at large separations and increasingly stretched exponential tail shapes at small separations, as shown in Figure 1 [42–46].

Within the framework of random multiplicative models, our theory suggests a natural mechanism for the observed stretched exponential tails at small separations as resulting from extreme deviations in a multiplicative cascade. However, this mechanism cannot account for *all* properties of velocity increments. For example, random multiplicative models are not consistent with the additivity of increments over adjacent intervals. Indeed, the pdf of velocity increments δv cannot become much larger than the single-point pdf, as it would if the former were $\propto \exp\left(-C|\delta v|^\beta\right)$ with $0 < \beta < 2$ while the latter would be close to Gaussian (see the Appendix of Ref. [46]). Nevertheless, stretched exponentials could be working in an intermediate asymptotic range of not too large increments, the controlling parameter of this intermediate asymptotics being the separation over which the increment is measured.

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Appendix

Proof of the Main Results for Extreme Deviations

Our aim is to prove (7) without necessarily assuming that the function $f(x)$, which defines the pdf of the individual variables though (4), is Taylor expandable to all orders. Specifically,

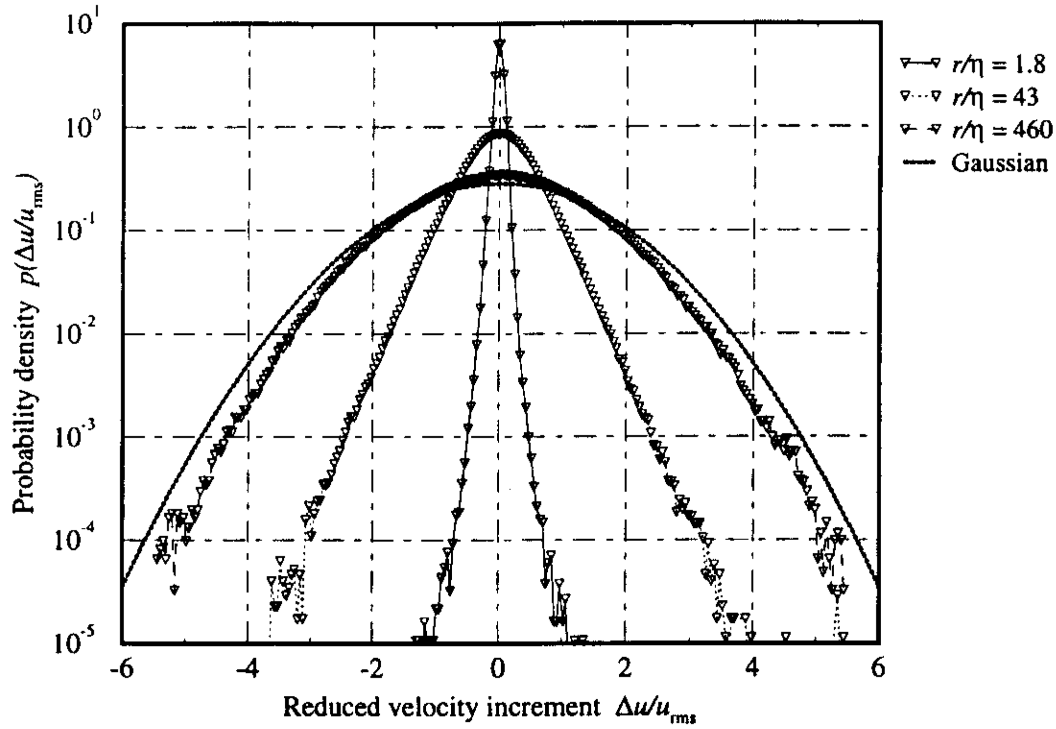

Fig. 1. — pdf of transverse velocity increments reduced by the rms velocity at various separations in units of the Kolmogorov dissipation scale η. (From Ref. [46].)

we assume that f is three times continuously differentiable and satisfies the following conditions when $x \to +\infty$:

(i) $f(x) \to +\infty$ sufficiently fast to ensure the normalization (3);

(ii) $f''(x) > 0$ (convexity), where f'' is the second derivative of f;

$\widetilde{\text{(iii)}}$ $\lim \frac{f'''(x)}{(f''(x))^{3/2}} = 0$;

(iv) there exists $C_1 > 0$ such that, for $x < y$ large enough, $x^2 f''(x) / \left(y^2 f''(y)\right) < C_1$;

(v) there exist $\beta > 0$ and $C_2 > 0$ such that $x^{2-\beta} f(x) > C_2$ for large enough x.

Assumptions (i) and (ii) are just the same as made in Section 2. Assumption $\widetilde{\text{(iii)}}$ is an instance of (iii) corresponding to the third derivative. Note that nothing is assumed about higher order derivatives. Assumptions (iv) and (v) are new and will be seen to be slight strengthenings of a corollary of $\widetilde{\text{(iii)}}$ ([9]).

We begin by proving various lemmas.

Lemma 1 *Assumption* $\widetilde{\text{(iii)}}$ *implies*

$$\lim x^2 f''(x) = +\infty. \tag{A.1}$$

([9]) There are weaker formulations of (iv) and (v) for which our results hold, which we do not make explicit here, as they are quite involved and do not bring any additional insight.

To prove this result, we start from $\widetilde{\text{(iii)}}$ which may be rewritten as

$$\lim_{x\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\sqrt{f''(x)}} = 0. \tag{A.2}$$

It follows that for any $\epsilon > 0$ there exists $X(\epsilon)$ such that, for $x > X(\epsilon)$, the argument of the limit in (A.2) is less than ϵ in absolute value. We take $X(\epsilon) < y < x$ and apply the mean value theorem to get

$$\left|(f''(x))^{-1/2} - (f''(y))^{-1/2}\right| = (x-y)\left|\frac{\mathrm{d}}{\mathrm{d}\xi}(f''(\xi))^{-1/2}\right|, \tag{A.3}$$

with $y < \xi < x$. The rhs of (A.3) is less than $(x-y)\epsilon$. Dividing by x and letting $x \to \infty$, we obtain

$$\limsup_{x\to\infty} \frac{1}{x\sqrt{f''(x)}} \le \epsilon. \tag{A.4}$$

Letting $\epsilon \to 0$, we obtain (A.1). QED

Lemma 2 *Under assumptions* (ii), $\widetilde{\text{(iii)}}$ *and* (iv),

$$|x-y| = C(f''(x))^{-1/2} \tag{A.5}$$

implies, that

$$\frac{f''(y)}{f''(x)} \to 1, \quad \textit{for } x \to +\infty \textit{ and fixed } C. \tag{A.6}$$

For the proof, let us first assume that $y < x$. By the mean value theorem, we have

$$f''(x) - f''(y) = (x-y)f'''(\xi), \quad \text{with } y < \xi < x. \tag{A.7}$$

It follows from Lemma 1 and (A.5), that $x/y \to 1$ and thus $\xi/x \to 1$ as $x \to +\infty$. Dividing (A.7) by $f''(x)$ and using (A.5), we obtain

$$\frac{f''(x) - f''(y)}{f''(x)} = C\frac{f'''(\xi)}{(f''(\xi))^{3/2}}\left(\frac{f''(\xi)}{f''(x)}\right)^{3/2}. \tag{A.8}$$

By (iv), the rightmost factor on the rhs is less than $C_1^{3/2}(x/\xi)^3$, which remains bounded as $x \to +\infty$, while, by $\widetilde{\text{(iii)}}$, the leftmost factor on the rhs tends to zero. Hence, the rhs tends to zero. This implies (A.6).

For the case $x < y$, (A.7) holds similarly with $x < \xi < y$. We then multiply (A.7) by $(f''(x))^{1/2}/(f''(y))^{3/2}$. The rhs tends again to zero. If follows that

$$\frac{f''(x) - f''(y)}{f''(y)}\left(\frac{f''(x)}{f''(y)}\right)^{1/2} \to 0, \tag{A.9}$$

which implies again (A.6). QED.

Lemma 3 *Let h_i, $i = 1, \ldots, n$ be real variables, not all vanishing, such that $\sum_{i=1}^{n} h_i = 0$. The subset of $p \le n-1$ indices i_j such that $h_{i_j} \ge 0$ satisfies*

$$\sum_{j=1}^{p} h_{i_j}^2 \ge \frac{1}{n}\sum_{i=1}^{n} h_i^2. \tag{A.10}$$

Let $i_{p+1}, \ldots, i_n$ denote the subset of indices such that $h_{i_j} < 0$. We set $h'_{i_j} = -h_{i_j} > 0$, so that

$$\sum_{j=1}^{p} h_{i_j} = \sum_{j=p+1}^{n} h'_{i_j}. \tag{A.11}$$

We have

$$\begin{aligned}\sum_{i=1}^{n} h_i^2 &= \sum_{j=1}^{p} h_{i_j}^2 + \sum_{j=p+1}^{n} h_{i_j}^{\prime 2} \\ &\leq \sum_{j=1}^{p} h_{i_j}^2 + \left(\sum_{j=p+1}^{n} h'_{i_j}\right)^2 \\ &= \sum_{j=1}^{p} h_{i_j}^2 + \left(\sum_{j=1}^{p} h_{i_j}\right)^2 \\ &\leq \sum_{j=1}^{p} h_{i_j}^2 + p\sum_{j=1}^{p} h_{i_j}^2 \\ &\leq n\sum_{j=1}^{p} h_{i_j}^2. \end{aligned} \tag{A.12}$$

In deriving (A.12), we have used $p \leq n-1$ and the following inequality for a set of p nonnegative variables $y_1, \ldots, y_p$

$$(y_1 + \cdots + y_p)^2 \leq p(y_1^2 + \cdots + y_p^2). \tag{A.13}$$

Lemma 3 follows from (A.12). QED.

We now turn to the derivation of the main result (7), rewritten here in a slightly different form as:

$$\lim_{x\to+\infty} \frac{P_n(x)}{P_n^{\text{as.}}(x)} = 1, \tag{A.14}$$

where

$$P_n^{\text{as.}}(x) = \mathrm{e}^{-nf(x/n)} \frac{1}{\sqrt{n}} \left(\frac{2\pi}{f''(x/n)}\right)^{\frac{n-1}{2}}. \tag{A.15}$$

We start from the representation (12) of the pdf of the sum of n iid variables as an $(n-1)$-fold integral. The function g_n, given by (11) can be rewritten as

$$g_n = nf(x/n) + \sum_{i=1}^{n} \int_{\frac{x}{n}}^{\frac{x}{n}+h_i} \mathrm{d}z \int_{\frac{x}{n}}^{z} f''(y)\,\mathrm{d}y. \tag{A.16}$$

Observe that, by (10), we have $\sum_{i=1}^{n} h_i = 0$, so that, ignoring contributions of zero measure, at least one of the h_i's must be positive. Furthermore, all the terms involving double integrals are positive.

The proof goes now as follows. By Lemma 2, the second derivative $f''(y)$ can be replaced by the second derivative at the minimizing point x/n as long as all the h_i's are not too large, that is are in the set $\mathcal{A}_H$ defined by

$$|h_i| \leq H = C(f''(x/n))^{-1/2}, \quad \text{for all } i. \tag{A.17}$$

By (A.1), (A.17) expresses that all the individual random terms in the sum stay within a distance of x/n which is small compared to x, that is, what we have called the *democratic localization* property. The substitution of $f''(x/n)$ for $f''(y)$ amounts to using the second-order truncation of the Taylor series (13) for g_n, which leads to $P_n^{\text{as.}}(x)$. It follows from Lemma 2 that the error committed in this substitution is small for large x.
Since $f''(x) > 0$, the contribution of the complementary set $\overline{\mathcal{A}_H}$ to the pdf $P_n(x)$, denoted $P_n^{(>H)}(x)$, is estimated from above by estimating g_n from below, keeping only the contributions from the subset i_j $(j = 1, \ldots, p \leq n-1)$ of indices such that $h_{i_j} \geq 0$. We thus obtain

$$g_n \geq nf(x/n) + \sum_{j=1}^{p} \int_{\frac{x}{n}}^{\frac{x}{n}+h_{i_j}} \mathrm{d}z \int_{\frac{x}{n}}^{z} f''(y)\,\mathrm{d}y. \tag{A.18}$$

By (iv), for $x/n \leq y \leq x/n + h_{i_j}$, we have

$$y^2 f''(y) \geq C_1^{-1}(x/n)^2 f''(x/n). \tag{A.19}$$

Using (A.19) in (A.18), we obtain

$$g_n \geq nf(x/n) + C_1^{-1}\left(\frac{x}{n}\right)^2 f''(x/n) \sum_{j=1}^{p} q(nh_{i_j}/x), \tag{A.20}$$

where

$$q(\alpha) \equiv \alpha - \ln(1+\alpha). \tag{A.21}$$

Note that $q(\alpha) = \alpha^2/2 + O(\alpha^3)$ for small α and $q(\alpha) < \alpha$ for large α. Assumption (v) is used to show that, for large x, the overwhelming contribution to $P_n^{(>H)}(x)$ comes from h_{i_j}'s such that nh_{i_j}/x is small compared to unity. Using (A.20) and Lemma 3, we obtain the following estimate

$$P_n^{(>H)}(x) \leq \mathrm{e}^{-nf(x/n)} \underbrace{\int \cdots \int}_{n-1} \mathrm{e}^{-\frac{C_1^{-1}}{2n} f''(x/n) \sum_{i=1}^{n} h_i^2} \mathrm{d}h_1 \cdots \mathrm{d}h_{n-1}, \tag{A.22}$$

where the domain of integration is over $\overline{\mathcal{A}_H}$, so that at least one of the $|h_i| \geq H = C(f''(x/n))^{-\frac{1}{2}}$. As a consequence, it is easily checked that the bounding integral is less than $P_n^{\text{as.}}(x)$ multiplied by a factor $O(\mathrm{e}^{-C^2/n})$, which tends to zero very quickly for large C. This proves (A.14) and the democratic localization property.

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