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Hamiltonian Plasma Dynamics and Topological Invariants Self-Similar Solutions Spatial Structures in Collisionless Reconnection

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Hamiltonian plasma dynamics and topological invariants, self-similar solutions, spatial structures in collisionless reconnection

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SCHEMATIC NOTES OF THE LECTURES

1 Hamiltonian structure

In this section we generalize the "reduced" MHD (RMHD) model and include electron and ion diamagnetism, finite ion gyroradii and finite electron mass effects which dominate the plasma dynamics at small scales. In a high-temperature plasma, the electron inertial skin depth is smaller than the gyro-radius of a thermal ion. These equations can be cast in (noncanonical) Hamiltonian form. It is shown that infinite sets of conserved quantities (Casimirs) exist. Sufficient conditions for stability are discussed on the basis of the second variation, at constant Casimirs, of the Hamiltonian functional.

We consider a geometry with magnetic field $\vec{B}$ and electric field $\vec{E}$

$$\vec{B} = B_0(\vec{e}_z + \vec{e}_z \times \nabla\Psi), \qquad \vec{E} = -\nabla\phi + \frac{B_0}{c}\frac{\partial\Psi}{\partial t}\vec{e}_z, \tag{1}$$

where Ψ is the flux function, ϕ is the electrostatic potential. Assuming that the parallel ion velocity is much smaller than the electron velocity v_z, Ampere's law reads (see Appendix 1) $v_z \approx -J_z/en_0 = -(cB_0/4\pi e n_0)\nabla^2\Psi$, where n_0 is a reference value. The parallel momentum balance and the continuity equation of the electrons are

$$\frac{1}{a}\frac{\partial\Psi_e}{\partial t} + [\Phi\,,\,\Psi_e] + [\Psi\,,\,\ln\frac{n}{n_0}] = \frac{\partial\Phi}{\partial z} - \frac{\partial}{\partial z}\ln\frac{n}{n_0}, \tag{2}$$

and

$$\frac{1}{a}\frac{\partial}{\partial t}\ln\frac{n}{n_0} + [\Phi\,,\,\ln\frac{n}{n_0}] - \frac{1}{\beta_e}[\Psi\,,\,J] = \frac{1}{\beta_e}\frac{\partial J}{\partial z}, \tag{3}$$

respectively. Here, $\Psi_e = \Psi - d_e{}^2\nabla_\perp^2\Psi$ is the generalized flux function, $d_e = c/\omega_{pe}$ is the electron inertial skin depth, the brackets are defined by $[f,g] = \vec{e}_z\cdot\nabla f\times\nabla g$, $\Phi = e\phi/T$, $J = \nabla_\perp^2\Psi$, $a = cT/(eB_0)$, and $\beta_e = 4\pi n_0 T/B_0{}^2$. Finite electron mass effects are taken into account. Temperatures are taken to be constant throughout the fluid.
The electron density n is related to the ion density through the quasi-neutrality condition. The ion response is given by:

$$\frac{1}{a}\frac{\partial}{\partial t}(\ln\frac{n}{n_0} - \rho_i{}^2\nabla_\perp^2 h) + [\Phi,\,\ln\frac{n}{n_0}] - \rho_i{}^2\nabla_\perp\cdot[\Phi,\,\nabla_\perp h] = 0, \tag{4}$$

where $\rho_i^2 = T_i m_i c^2/(eB_0)^2$, $\tau_i = T/T_i$, and $h = \tau_i\Phi + \ln n/n_0$. Since Eq.(4) can be viewed as a nonlinear generalization of the Padé approximation to the linear ion response for arbitrary values of the ion gyroradius[1] we assume it to be valid for all values of the gyroradius. Its leading order solution in the large gyroradius limit is the adiabatic response $\ln n/n_0(\vec{x}) + \tau_i\Phi = 0$.

The nonlinear equations [23, 24, 4] can be written in Hamiltonian form[2]

$$\frac{\partial \xi_i}{\partial t} = \Big\{\xi_i \, , \, H\Big\}, \qquad i = 1,2,3, \tag{5}$$

where the noncanonical variables ξ_i are

$$\xi_1 = \Psi_e, \quad \xi_2 = d_e\beta_e^{1/2}\ln\frac{n}{n_0}, \quad \xi_3 = d_e\beta_e^{1/2}\Big(\rho_i{}^2\nabla^2 h - \ln\frac{n}{n_0}\Big). \tag{6}$$

The Hamiltonian is the energy functional

$$H = \frac{1}{2}\int d^3x\Big(-\xi_1\frac{\nabla_\perp^2}{1-d_e^2\nabla_\perp^2}\xi_1 + (1+\frac{1}{\tau_i})\frac{\xi_2^2}{d_e^2} - \frac{1}{d_e^2}(\xi_2+\xi_3)\frac{1}{\tau_i\rho_i^2\nabla_\perp^2}(\xi_2+\xi_3)\Big). \tag{7}$$

The noncanonical Poisson brackets are a generalization of the brackets given in[3,4]

$$\Big\{F \, , \, G\Big\} = \frac{ad_e}{\beta_e^{1/2}}\int d^3x\; W_{ij}\left[\frac{\delta F}{\delta\xi_i}\, , \,\frac{\delta G}{\delta\xi_j}\right] - \frac{ad_e}{\beta_e^{1/2}}\int d^3x\; W_{ij}^{(z)}\;\frac{\delta F}{\delta\xi_i}\frac{\partial}{\partial z}\frac{\delta G}{\delta\xi_j}$$
$$-\frac{ad_e}{\beta_e^{1/2}}\rho_i{}^2\int d^3x\;\xi_2\left[\frac{\partial}{\partial x_k}\frac{\delta F}{\delta\xi_3}\, , \,\frac{\partial}{\partial x_k}\frac{\delta G}{\delta\xi_3}\right], \tag{8}$$

where $x_k = (x,y)$ and the symmetric matrices W_{ij} and $W_{ij}{}^{(z)}$ are defined by

$$W_{ij} = \begin{pmatrix}\xi_2 & \xi_1 & 0\\ \xi_1 & \xi_2 & 0\\ 0 & 0 & \xi_3\end{pmatrix}, \quad W_{ij}^{(z)} = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{9}$$

We consider localized phenomena and disregard boundary effects. The brackets (8) are antisymmetric and satisfy the Jacobi identity $\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0$. Apart from the ξ_2 contribution to the integrand in the second term in Eq.(8), we see from the form of the matrices (9) that the Poisson brackets do not contain any coupling between $\xi_{1,2}$ and ξ_3. Coupling arises when ion currents are no longer negligible and/or when charge neutrality is violated. It can easily be seen that the choice of new variables $\xi_\pm = 1/2(\xi_1 \pm \xi_2)$ diagonalizes the Poisson brackets, but not the Hamiltonian.

The system of Eqs.(5) has two types of invariants: those related with the symmetries of the Hamiltonian and those related with the algebraic structure of the Poisson brackets.
When the Hamiltonian has a continuous translational symmetry in x, y and/or rotational symmetry, the functionals P_x and/or P_y, that generate translations, and the angular momentum functional L_z, that generates rotations in (x, y),

$$P_{(x,y)} = \int d^3x\;(y,-x)(\xi_2+\xi_3), \quad L_z = -\int d^3x\; r^2(\xi_2+\xi_3)/2 \tag{10}$$

are conserved, as is the case for the Hamiltonian (7). The operators $P_{x,y}$ and L_z satisfy the appropriate commutation relations $\{L_z, P_x\} = P_y$, $\{L_z, P_y\} = -P_x$ and $\{P_x, P_y\} = \int d^3x(\xi_2 + \xi_3)$.
The invariants of the second set are called Casimirs and are functionals that commute with all

functionals F, i.e. $\{C, F\} = 0$. We treat 2D cases and take all quantities to depend on t, x, and y. In these cases the brackets (8) admit two infinite sets of Casimirs,

$$C_\pm = \int d^2x\ f_\pm(\xi_1 \pm \xi_2) = \int d^2x\ f_\pm(\Psi_e \pm \beta_e^{1/2} d_e \ln n/n_0), \tag{11}$$

with $f_\pm$ arbitrary functions. These functionals depend only on the dynamics of the electrons and do not depend on the ion response.
In the limits $m_e \to 0$, $\ln n/n_0 \to \rho_s^2 \nabla^2 \Phi$, the Casimir (11) become those of RMHD[4] In the cold ion limit, $T_i \to 0$, the last term in (8) vanishes. Then the system contains an infinite number of Casimirs involving ξ_3

$$C_3 = \int d^3x\ G(\xi_3), \tag{12}$$

with G an arbitrary function. Note that in this limit $\xi_3 = d_e \beta_e^{1/2} (\rho_s^2 \nabla^2 \Phi - \ln(n/n_0))$ with $\rho_s^2 = \tau_i \rho_i^2$. In the model of reduced magnetohydrodynamics (RMHD) ξ_3 vanishes. Then the Casimirs (12) become trivial. Eq.(11) implies that magnetic reconnection in Ψ_e and/or Ψ can occur in the presence of an infinite set of conservation laws. This is related with the fact that inertia is particularly important in regions where the reconnection process can occur. The existence of these infinite sets of Casimirs is equivalent to the special properties of the equations of motion (23, 24) and (4) under time transformations.

In terms of the variables (ξ_+, ξ_-, ξ_3), where $\xi_\pm = \frac{1}{2}(\xi_1 \pm \xi_2)$ so that $\delta H/\delta \xi_\pm = \delta H/\delta \xi_1 \pm \delta H/\delta \xi_2$, the equations of motion (5) read in the limit $T_i \to 0$,

$$\frac{\partial \xi_\pm}{\partial t} = \mp \frac{a d_e}{\beta_e^{1/2}} [\xi_\pm, \frac{\delta H}{\delta \xi_\pm}], \quad \frac{\partial \xi_3}{\partial t} = -\frac{a d_e}{\beta_e^{1/2}} [\xi_3, \frac{\delta H}{\delta \xi_3}], \tag{13}$$

We remark that Eqs.(13) and the corresponding Poisson brackets (8) are invariant under coordinate transformations in the plane that leave the Jacobian equal to unity. The Hamiltonian, however, is only invariant under translations and rotations. Equations (13) also take the same form after the transformations $g_i(\xi_i) d\tau \to d\tau_i$ and $\delta H/\delta \xi_i \to g_i^{-1} \delta H/\delta \xi_i$. This is equivalent to redefining the time in Eq.(5) and changing the diagonal matrix W_{ij} according to $\xi_i \to g_i^{-1} \xi_i$. Since the $g_i(\xi_i)$'s are arbitrary, this invariance is equivalent to the existence of three infinite sets of Casimirs. In terms of the new Poisson brackets the translation and rotation operators (10) become

$$P^g_{(x,y)},\ L^g_z = \int d^2x\ (y, -x, -r^2/2)\ [g_+(\xi_+)\xi_+ - g_-(\xi_-)\xi_- + g_3(\xi_3)\xi_3], \tag{14}$$

The existence of the Casimir invariants restricts the possible plasma motions to hypersurfaces in the infinite dimensional phase space. This restriction can be used to describe the stationary solutions and the stability of the Hamilton equations (5) with the help of a variational principle. Variations of the field variables that conserve automatically all Casimir invariants, have been introduced by Arnold[5] for the Euler equation. A generic variation Δ which conserves all the Casimir invariants is $\Delta \xi_i = \delta \xi_i + \frac{1}{2!} \delta^2 \xi_i + \dots$, with

$$\delta \xi_\pm = [\mu_\pm, \xi_\pm], \qquad \delta \xi_3 = [\sigma\ , \xi_3], \tag{15}$$

$\mu_\pm(x, y)$ and $\sigma(x, y)$ being arbitrary functions. The first variation of H gives the stationary solutions of Eqs. (13),

$$\frac{\delta H}{\delta \xi_\pm} = F_\pm(\xi_\pm), \quad \frac{\delta H}{\delta \xi_3} = U(\xi_3). \tag{16}$$

These equations can be written as $F_\pm(\xi_\pm) = -J \pm \beta_e^{1/2} d_e^{-1}(\ln n/n_0 - \Phi)$, and $U(\xi_3) = -\beta_e^{1/2} d_e^{-1}\Phi$. When the Hamiltonian is minimized under the constraint of a constant operator (14), propagating solutions are obtained where each ξ_i propagates with a velocity that is constant on the corresponding $\xi_i = const$ surfaces. When all $g_i = 1$, all velocities are equal and a stationary equilibrium is obtained.
The second variation of the Hamiltonian functional is

$$\delta^2 H = \int d^2x \, \Big(-\delta\xi_1 \delta J \; + A_1(F'_+ \delta\xi_+ + F'_- \delta\xi_-)^2 + A_2(\delta\xi_+ - \; \delta\xi_-)^2/d_e^2 \\ - \beta_e \delta\Phi \rho_s^2 \nabla^2 \delta\Phi - \; U'(\xi_3)(\delta\xi_3)^2\Big), \qquad (17)$$

where $\delta J = (1 - d_e^2\nabla^2)^{-1}\nabla^2\delta\xi_1$. $\delta\Phi = (d_e \beta_e^{1/2} \rho_s^2 \nabla^2)^{-1}(\delta\xi_2 + \delta\xi_3)$, and $A_1 = -(F'_+ + F'_-)^{-1}$, $A_2 = 1 - d_e^2(F'_+ F'_-)/(F'_+ + F'_-)$. The first and fourth terms in (17) are positive definite. Sufficient conditions for stability are $A_{1,2} \geq 0$

$$F'_+ + F'_- \leq 0, \qquad -(F'_+ + F'_-) \geq -d_e^2 F'_+ F'_-, \qquad U' \leq 0. \qquad (18)$$

In the RMHD limit, ξ_3 is not an independent variable and the last contribution to (17) has to be omitted and the last condition of (18) does not apply. Also in the large-ρ_i limit, ξ_3 is not an independent variable. The equilibrium is given by the first two eqs in (16) and by the relationship $\ln n/n_0(\vec{x}) + \tau_i \Phi = 0$. Instead of (17), we obtain

$$\delta^2 H = \int d^2x \, \left(-\delta\xi_1 \delta J \; + A_1(F'_+ \delta\xi_+ + F'_- \delta\xi_-)^2 + \frac{\hat{A}_2}{d_e^2}(\delta\xi_+ - \; \delta\xi_-)^2\right), \qquad (19)$$

where $\hat{A}_2 = \tau_i^{-1} + A_2$. In this case the sufficient conditions for stability read $A_1, \hat{A}_2 \geq 0$ In the limit of zero electron mass $d_e \to 0$, these two conditions reduce to a single one on the profile of the equilibrium current density[6] $\partial J/\partial \xi_1 = \partial J/\partial \Psi \geq 0$.

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2 Self Similar Solutions

In this section we study a special solution of The hamiltonian equations given above.

When the system is energetically closed, the energy functional is the Hamiltonian and non-canonical Poisson brackets can be defined. The system possesses two infinite sets of invariants

(Casimirs) that arise from the structure of the equations. They reflect the invariance of the topology of the configuration. The plasma dynamics in the neighbourhood of critical points (X- and O-points) of the magnetic configuration is investigated in terms of scale-invariant equations. Their solutions correspond to open systems which, in general, do not have well defined Casimirs. However, the scale-invariant members of the families of Casimirs of the closed system survive. These surviving elements are not related to simple power expansions in the fields. When the fields are analytical, they can be expressed as polynomials and the system has a finite number of degrees of freedom. This truncated system is Hamiltonian and integrable. Most of the initial structures lead to a collapse of the magnetic separatrices with a velocity that grows as $(t-t_0)^{-1}$.

We consider a geometry with magnetic field $\vec{B} = B_0(\vec{e}_z + \vec{e}_z \times \nabla\Psi)$ and electric field $\vec{E} = -\nabla\phi + (B_0/c)(\partial\Psi/\partial t)\,\vec{e}_z$, where $\Psi(x,y,t)$ is the flux function and $\phi(x,y,t)$ the electric potential. Assuming that the parallel ion velocity is much smaller than the electron velocity v_z, Ampere's law reads $v_z \approx -J_z/en_0 = -(cB_0/4\pi e n_0)\nabla^2\Psi$, where n_0 is a reference value. The parallel momentum balance and the continuity equation of the electrons are ([7])

$$\frac{1}{a}\frac{\partial\Psi_e}{\partial t} + [\Phi\ ,\ \Psi_e] + [\Psi\ ,\ \ln\frac{n}{n_0}] = 0\ , \qquad (20)$$

and

$$\frac{1}{a}\frac{\partial}{\partial t}\ln\frac{n}{n_0} + [\Phi\ ,\ \ln\frac{n}{n_0}] - \frac{1}{\beta_e}[\Psi\ ,\ J] = 0\ , \qquad (21)$$

respectively. Here, $\Psi_e = \Psi - {d_e}^2\nabla_\perp^2\Psi$ is the generalized flux function, $d_e = c/\omega_{pe}$ is the electron inertial skin depth, the brackets are defined by $[f,g] = \vec{e}_z\cdot\nabla f\times\nabla g$, $\Phi = e\phi/T$, $J = \nabla_\perp^2\Psi$, $a = cT/(eB_0)$, and $\beta_e = 4\pi\ n_0T/{B_0}^2$. Finite electron mass effects enter through d_e. Temperatures are taken to be constant throughout the plasma. The electron density n is related to the ion density through the quasi-neutrality condition. In the limit of large ion gyroradii, the ion response is adiabatic,

$$\ln n/n_0(\vec{x}) + \tau_i\Phi = 0\ . \qquad (22)$$

Then, with the normalizations $\Phi \to [\beta_e\tau_i(1+\tau_i)]^{1/2}\Phi$ and $t \to [\tau_i\beta_e/a^2(1+\tau_i)]^{1/2}t$, the electron momentum balance (20) and continuity equation (21) become

$$\frac{\partial\Psi_e}{\partial t} + [\Phi,\ \Psi_e] - \frac{\tau_i}{1+\tau_i}d_e^2\ [J,\ \Phi] = 0, \qquad \frac{\partial\Phi}{\partial t} + [\Psi,\ J] = 0\ . \qquad (23)$$

The energy functional is

$$H = \frac{1}{2}\int d^2x\Big(\nabla\Psi^2 + d_e^2(\nabla^2\Psi)^2 + \Phi^2\Big)\ . \qquad (24)$$

In terms of the proper variables, the nonlinear equations (23) can be written in Hamiltonian form with non-canonical Poisson brackets ([6], [7]).

This Hamiltonian system has two types of invariants: those related to the symmetries of the Hamiltonian and those related to the algebraic structure of the equations i.e. of the Poisson brackets. The equations (23) admit the two infinite sets of Lagrangian invariants $F_\pm[\Psi_e \mp (\tau_i/1+\tau_i)^{1/2}d_e\Phi]$, with $F_\pm$ arbitrary functions. These two sets are advected with different velocities. In the case of closed systems, the volume integrals of these invariant functions are the Casimirs of the noncanonical Poisson brackets. In the limit of vanishing electron inertia ($d_e \to 0$), these invariants become

$$\int d^2x\ F(\Psi),\qquad \int d^2x\ \Phi G(\Psi)\ . \qquad (25)$$

In this limit Eqs.(23) do not contain an explicit length scale.

The plasma behaviour in the neighbourhood of critical points, such as X- and O-points, is studied by referring to a class of solutions of Eqs (23) that are scale invariant ([Pegoraro et al. 1994]). As will be shown, this class contains solutions that describe the collapse of the magnetic configuration in a finite time. We look for solutions that leave Eqs(23) unchanged under the transformation $t \rightarrow t,\ (x, y) - (\alpha x,\ \alpha y)$. Adopting polar coordinates, such spatially self-similar solutions are of the form

$$\Psi = r^3\hat{\Psi}(\theta, t), \qquad \Phi = r^2\hat{\Phi}(\theta, t) \ . \tag{26}$$

Equations (23) become

$$\frac{\partial\hat{\Psi}}{\partial t} + 2\hat{\Phi}\hat{\Psi}_\theta - 3\hat{\Phi}_\theta\hat{\Psi} = 0, \qquad \frac{\partial\hat{\Phi}}{\partial t} + 3\hat{\Psi}\ \hat{J}_\theta - \hat{\Psi}_\theta\hat{J} = 0 \ , \tag{27}$$

where $\hat{J} = 9\hat{\Psi} + \hat{\Psi}_{\theta\theta}$ is the current density, and the subscript denotes differentiation. It is clear that this self-similar system is not energetically closed, since fluxes will cross the boundaries of the system. In general, such open systems will not possess invariant integrals over a fixed domain. However, one expects that in a scale-invariant system the integrals of Lagrangian invariants that scale inversely with the volume will still be conserved. This is because the fluxes through nested, closed surfaces are equal. This leaves the following two invariants

$$\oint \hat{\Psi}^{-2/3}d\theta, \qquad \oint \hat{\Phi}\hat{\Psi}^{-4/3}d\theta \ . \tag{28}$$

These integrals reflect the geometrical structure of the configuration. Their values are controlled by the magnetic separatrices $\Psi = 0$. It can easily be verified that, independently of the relationship between Ψ and J, these integrals are indeed constants of the motion described by Eq.(27). The invariants (28) are not related to power expansions of the functions in Eq.(25). The additional invariant $\oint \hat{\Phi}d\theta$ is the remnant of the rotational invariance of the Hamiltonian (24).

By expanding $\hat{\Psi}$ and $\hat{\Phi}$ in a Fourier series in θ, we obtain from Eqs.(27) a set of coupled ordinary differential equations for the Fourier coefficients $\Psi_m = \Psi^*_{-m}$ and $\Phi_m = \Phi^*_{-m}$

$$\dot{\Psi}_m = i\sum_l (5l - 2m)\Psi_{m-l}\Phi_l \ , \tag{29}$$

$$\dot{\Phi}_l = i\sum_m (l - 4m)(9 - m^2)\Psi_{l-m}\Psi_m \ . \tag{30}$$

where a dot denotes time differentiation. This infinite system becomes finite if it consists only of the harmonics $\Psi_{\pm 1}$, $\Psi_{\pm 3}$, Φ_0 and $\Phi_{\pm 2}$. Note that Φ_0 is a constant. The resulting set of equations describes a finite-dimensional system with six degrees of freedom. In Cartesian coordinates the fluxes are given by

$$\Psi(x, y, t) = \Psi_{30}(t)x^3 + \sqrt{3}\Psi_{21}(t)x^2y + \sqrt{3}\Psi_{12}(t)xy^2 + \Psi_{03}(t)y^3 \ , \tag{31}$$

$$\Phi(x, y, t) = \Phi_{20}(t)x^2 + \Phi_{11}(t)xy + \Phi_{02}(t)y^2 \ . \tag{32}$$

These fluxes correspond to linear velocity fields and to magnetic fields with either one or three real separatrices. This finite Fourier system is Hamiltonian with conjugate variables

$$\{q_i\} = (\Psi_{-1},\ 1/2\ \Phi_2,\ 3^{1/2}\Psi_3), \qquad \{p_i\} = (\Psi_1,\ 1/2\ \Phi_{-2},\ 3^{1/2}\Psi_{-3}) \ , \tag{33}$$

that are also each others' complex conjugates. The Hamiltonian is

$$I_3 == 8\sqrt{3}i(p_1q_2p_3 + q_1p_2q_3) - 8i({p_1}^2p_2 + {q_1}^2q_2) + 2i\Phi_0(q_1p_1 - 3q_3p_3) \ . \tag{34}$$

This Hamiltonian does not depend explicitly on the angle θ and, thus, is invariant under rotations $\theta \to \theta + \varphi$. As a consequence, the equations for the Fourier amplitudes are invariant under the transformation $\xi_m \to \epsilon^m \xi_m$ with $\xi_m = (\Psi_m, \Phi_m)$ and group parameter $\epsilon = \exp i\varphi$. This symmetry implies the conservation of the momentum

$$I_2 = 2(-q_1p_1 + 2q_2p_2 + 3q_3p_3) = -2\Psi_1\Psi_{-1} + 18\Psi_3\Psi_{-3} + \Phi_2\Phi_{-2} \ , \tag{35}$$

which is an energy-like variable. It can easily be verified that the standard canonical Poisson brackets $\{I_2, I_3\}_{p,q}$ vanish.

Introducing the new variables

$$Q_1 = 2(\Phi_2 + \Phi_{-2}) = \Phi_{02} - \Phi_{20} \ , \qquad Q_2 = 2i(\Phi_2 - \Phi_{-2}) = \Phi_{11} \ , \tag{36}$$

Eqs(29)and (30) give

$$\ddot{Q}_1 = -32I_2Q_1 + 2Q_1(Q_1^2 + Q_2^2) - 4\Phi_0\dot{Q}_2 \tag{37}$$

and

$$\ddot{Q}_2 = -32I_2Q_2 + 2Q_2(Q_1^2 + Q_2^2) + 4\Phi_0\dot{Q}_1 \ . \tag{38}$$

Equations (37) and (38) are Hamiltonian in terms of the variables Q_1, Q_2 and the conjugate momenta $P_1 = \dot{Q}_1 + 2\Phi_0Q_2$, $P_2 = \dot{Q}_2 - 2\Phi_0Q_1$, with Hamiltonian

$$I_4 = 1/2(P_1 - 2\Phi_0Q_2)^2 + 1/2(P_2 + 2\Phi_0Q_1)^2 - 1/2[16I_2 - (Q_1^2 + Q_2^2)]^2 \ . \tag{39}$$

In terms of this reduced set of canonical variables, I_3 is proportional to the angular momentum M,

$$I_3 = -(i/32)M, \quad \text{with} \quad M = 2I_2\Phi_0 - (Q_1P_2 - Q_2P_1) \ . \tag{40}$$

Expressing I_4 in terms of the canonical variables (33), it turns out that it depends only on the Fourier components of Ψ. In addition it can be shown that the invariants I_2, I_3 and I_4 are independent and in convolution. Thus the scale-invariant system is integrable.

An exact solution of Eqs.(37, 38) can be obtained by writing them in the form of the reduced nonlinear Schrödinger equation

$$4i\Phi_0\dot{\Phi}_2 + \ddot{\Phi}_2 + 32I_2\Phi_2 - 32\Phi_2 \mid \Phi_2 \mid^2 = 0 \ , \tag{41}$$

where $\mid \Phi_2 \mid^2 \equiv \Phi_2\Phi_{-2}$. Equation (41) has the following integrals of motion

$$J = \mid \Phi_2 \mid^2 + \frac{i}{4\Phi_0}\left(\Phi_2\dot{\Phi}_2^* - \Phi_2^*\dot{\Phi}_2\right) \equiv I_2 - \frac{i}{\Phi_0}I_3 \ , \qquad I_4 = \mid \dot{\Phi}_2 \mid^2 - 16\left(\mid \Phi_2 \mid^2 - I_2\right)^2 \ . \tag{42}$$

Substituting $\Phi_2 = \phi^{1/2}\exp(i\omega)$ into (41), with ϕ and ω real functions, we obtain

$$\dot{\omega} = 2\Phi_0(J/\phi - 1) \ , \quad \dot{\phi}^2/4 + \phi^2\dot{\omega}^2 - 16\phi(\phi - I_2)^2 = I_4\phi \ , \tag{43}$$

which can be solved for ϕ explicitly in terms of the Weierstrass elliptic functions.

However, it may be more transparent to look for approximate solutions of Eqs. (37)-(38). For simplicity, we take $\Phi_0 = 0$. Expressing the Hamiltonian (39) in terms of the radial variable $R \equiv (Q_1^2 + Q_2^2)^{1/2}$ and of the invariants I_2 and $M = R^2\dot{\xi}$, with ξ the polar angle, we obtain

$$I_4 = \dot{R}^2/2 + 16I_2R^2 - R^4/2 + (M^2/2R^2) \ . \tag{44}$$

The maximum and minimum values of the effective potential in (44) are determined by the roots of a 3^{rd} order polynomial in R^2. The sign of the discriminant is given by $M^2 - (32I_2/3)^3$. If the discriminant is positive, there are no positive real roots and the potential is everywhere repulsive. This means that each initial configuration collapses in a finite time, i.e., that the magnitude R of the velocity potential Φ becomes infinite with a $(t_0 - t)^{-1}$ behaviour for large R. If the discriminant is negative, there exist, besides the collapsing solutions, also oscillatory solutions.

In these collapsing solutions ξ vanishes as $(t_0 - t)^2$ and thus the system tends to become one-dimensional and the coordinate system can be chosen in such a way that asymptotically $R \rightarrow Q_2$. The structure of the collapsing magnetic configurations can be studied more conveniently in terms of the Cartesian forms (31) and (32).

The magnetic separatrices of the configuration correspond to $\Psi = 0$. Because of flux conservation, they remain distinct at all times. Their number is determined by the discriminant of the resulting equation. In the present case of a cubic equation, it can be shown that this discriminant is proportional to the invariant I_4. The configuration has a single separatrix when $I_4 > 0$ and three separatrices when $I_4 < 0$.

The behavior of a magnetic collapse can be described in the asymptotic state where Φ_2 and $-\Phi_{-2}$ remain bounded as $t \rightarrow t_0$, and $\xi \approx 0$ i.e. where Φ_{02}, Φ_{20} and $3\Psi_{30}\Psi_{03} - \Psi_{21}\Psi_{12}$ remain bounded. Under these conditions the leading order terms in the equations of motion expressed in Cartesian variables take the simple form $\dot{\Psi}_{ij} \approx b_{ij}\Phi_{11}\Psi_{ij}$, with $b_{30} = 3$ and $b_{21} = 1$. Recalling that $R \approx \Phi_{11} \approx (t_0 - t)^{-1}$, the solution of these equations for t close to the collapse time t_0 is

$$\Psi_{30}(t) \propto (t_0 - t)^{-3} \;, \quad \Psi_{21}(t), \Phi_{11}(t) \propto (t_0 - t)^{-1}. \tag{45}$$

The other cartesian coefficients remain bounded. It can be concluded that all separatrices collapse towards the y-axis.

The treatment given in this paper can be extended to fluxes Ψ and Φ that also contain lower powers in x and y. The resulting system is now only scale-invariant order by order. In this case electron inertia can be included. A hierarchal set of equations for the time dependence of the coefficients of the polynomial representation is obtained, where the equations for the higher powers do not depend on the coefficients of the lower powers. An analysis shows that the lower powers do not collapse faster than the leading order terms. The lower powers cause a splitting of the cubic X-point into two quadratic X-points at a distance determined by electron inertia. This analysis is analogous to the self-similar dynamics studied by [Bulanov et al. 1984], [Bulanov et al. 1985], [Bulanov et al. 1992] for the 3D MHD- and EMHD-equations by assuming a linear, time-dependent relationship between Eulerian and Lagrangian fluid variables and a polynomial representation for the dependence of the magnetic field on spatial coordinates.

We have shown, on a specific plasma model that applies to a strongly magnetized plasma and to scale-lengths smaller than the ion thermal gyro-radius, that the plasma dynamics in the neighbourhood of a critical point is integrable, i.e. non chaotic, at least for the case of spatially self-similar motions and that critical points tend to collapse to one-dimensional configurations in a finite time.

A connection has been identified, based on scale invariance, between the integrability of the self-similar solutions, that have a finite number of degrees of freedom, and the infinite number of integral invariants (Casimirs) of the starting equations, that have an infinite number of degrees of freedom.

The present approach can be extended in principle to reduced MHD (2D incompressible MHD): in this framework the solutions given here are to be seen as local solutions valid close to the critical points to be connected, through a non-scale-invariant transition region described e.g. by a Padé-type representation, to global scale-invariant solutions valid in the limit of small ion thermal gyro-radii. Parity arguments require that X-points of the type described above appear in these global solutions in pairs.

An open problem under investigation is whether the integrability of the self-similar solutions is a special property of our dynamical plasma model or whether it occurs also in more general cases, such as 3D-MHD and EMHD, being simply a remnant of the infinite number of invariants (Casimirs) of these systems with infinite degrees of freedom.

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3 Nonlinear Collisionles reconnection

Recently it was shown [1],[9] theoretically that electron inertia can account for the fast reconnection time scales observed in laboratory experiments, e.g. the fast sawtooth crash in Tokamak plasmas. The first nonlinear simulation [3] of electron inertia reconnection neglected the electron pressure gradient in the generalized Ohm law. This term was considered in subsequent contributions [4],[5], which revealed a further enhancement of the reconnection rate, although the physical interpretation of the simulation results remained unclear.

A related problem is the role played by the invariants (Casimirs) of the collisionless plasma evolution. It can be shown that collisionless models of the type adopted in [2],[5]-[7] admit an Hamiltonian structure. Clearly, while the magnetic flux is reconnected in the course of plasma evolution, the invariant fields preserve their initial topology. The nature of collisionless reconnection under these circumstances is entirely different form that of resistive (dissipative) reconnection.

The aim of this last section is twofold. First, we present new numerical results in regimes where both electron inertia and the electron pressure tensor are important. We confirm that the nonlinear growth of magnetic islands is even faster than in the case where the electron pressure is neglected, and that this fast growth is accompanied by the formation of microscales below the electron inertia skin depth, as in the purely inertial case [2]. A significant finding [8] is the splitting of the current and vorticity sheets in two branches crossing at the stagnation point of the plasma flow. A similar behavior was also observed in Ref. [4] in the context of resistive reconnection with electron pressure effects. Secondly, we interpret these results on the basis of the Hamiltonian structure of the adopted plasma model. In particular, we show that the spatial structures are the consequence of the presence of Casimirs advected by effective velocity fields.

Our investigation considers an extension of reduced MHD on a two dimensional slab, where electron inertia, proportional to the square of the inertial skin depth, $d_e = c/\omega_{pe}$ and the electron stress tensor are retained in the generalized Ohm law. Diamagnetic effects are neglected here. Thus, the pressure effect we consider has to do with electron space charge perturbations along the field lines, balanced by ions streaming across the field lines in order to preserve quasineutrality. This process is associated with the characteristic scale length. $\varrho_s = \sqrt{T_e/m_i}/\omega_{ci}$. The equations we solve are

$$\partial F/\partial t + [\varphi, F] = \varrho_s^2[U, \Psi] \tag{46}$$

$$\partial U/\partial t + [\varphi, U] = [J, \Psi] \tag{47}$$

The quantities appearing in these equations are dimensionless. The normalization is based on the Alfvén time, τ_A, and the slab width L_x. The magnetic field is $\mathbf{B} = B_0\mathbf{e}_z + \nabla\Psi \times \mathbf{e}_z$, with B_0 a constant component along the ignorable z-direction, φ is the e.s. potential, $U = \nabla^2\varphi$ is the vorticity, $J = -\nabla^2\Psi$ is the current density and $F = \Psi + d_e^2 J$ is the generalized canonical momentum [2] along z. The Poisson brackets are defined as $[A, B] = \mathbf{e}_z \cdot \nabla A \times \nabla B$. Note that the l.h.s. of Eqs. (46,47) represent the time derivatives of U and F, advected by the $\mathbf{E} \times \mathbf{B}$ velocity field, $\mathbf{v}_\perp = \mathbf{e}_z \times \nabla\varphi$. This system of equations can be written in a conservative form for the quantities

$$G_\pm \equiv F \pm d_e\varrho_s U. \tag{48}$$

Defining the generalized stream function

$$\varphi_\pm = \varphi - \varrho_s^2 U^2 \mp d_e\varrho_s J \tag{49}$$

multiplying Eq. (47) by $d_e\varrho_s$ and adding and subtracting the resulting equation to Eq. (46) we obtain

$$\partial G_\pm/\partial t + [\varphi_\pm, G_\pm] = 0 \tag{50}$$

Thus, the quantities $G_\pm$, are conserved along fluid elements in motion with the effective velocity fields $\mathbf{v}_\pm = \mathbf{e}_z \times \nabla\varphi_\pm$, so the topology of $G_\pm$ remains "frozen" during time evolution.

It can be shown [5]-[7] that Eqs. (46,47) can be cast in non canonical Hamiltonian from. The Hamiltonian is

$$H = \frac{1}{2}\int d^2x \left(|\nabla\Psi|^2 + d_e^2 J^2 + |\nabla\varphi|^2 + \varrho_s^2 U^2\right) \tag{51}$$

The associated generalized Poisson brackets admit two infinite sets of Casimirs

$$C_\pm = \int d^2x h_\pm(\xi_1 \pm \xi_2) = \int d^2x h_\pm(G\pm) \tag{52}$$

with $h_\pm$ arbitrary functions. In the limit of vanishing ϱ_s, upon expanding $h_\pm$ to first order, we find the Casimirs of the purely inertial case, $C_1 = \int d^2x h_1(F)$ and $C_2 = \int d^2x U h_2(F)$. Thus, for $\varrho_s = 0$, the generalized momentum F is conserved and its topological structure is preserved in time. When $\varrho_s \neq 0$, the fields $G_\pm$ are topologically invariant, while F can undergo reconnection.

The linearized system of equations (46,47) was solved analytically in Ref. [9]. Let us assume double periodic boundary conditions at the frontier of a rectangular slab with aspect ratio $\epsilon = L_x/L_y$ and equilibrium magnetic flux $\Psi_{eq} = \cos x$, $\varphi_{eq} = U_{eq} = 0$. Considering perturbations of the type $\tilde\Psi(x, y) = \hat\Psi(x)e^{\gamma t}\cos ky$, one finds for the "outer solution" (i.e. neglecting ϱ_s and

d_e), $\Delta' \equiv \lim_{\delta\to 0} |d \ln \hat{\Psi}/dx|_{-\delta}^{\delta} = 2\sigma \tan(\sigma \frac{\pi}{2})$ where $\sigma = \sqrt{1-k^2}$. In the relevant limit $\varrho_s > d_e$, considering the large Δ' regime defined by $\Delta' d_e > (d_e/\varrho_s)^{1/3}$, the linear growth rate, normalized to the Alfvén time, is $\gamma_L \sim (2 d_e \varrho_s^2/\pi)^{1/3}$. The mode structure exhibits macroscopic convection cells, with $L_{cell} \sim L_x$, the vorticity profile has a width $\sim \varrho_s$, while the current density has a width of order d_e, with a tail extending over a distance $\sim \varrho_s$. Note that, already in linear theory, one finds an enhancement of the reconnection rate, by a factor $(\varrho_s/d_e)^{2/3}$ compared with the growth rate obtained for $\varrho_s < d_e$.

In this paper, we present the numerical solution of the nonlinear model (46,47) obtained on the basis of a finite difference scheme with variable grid. In Figs. 1-7 we present the solution for a case with $\epsilon = 1/2$, $d_e = 0.08$ and $\varrho_s = 3d_e$. In particular, Figs. 1-4 show the contour plots at $t = 50\tau_A$ of the fields φ, Ψ, J and U. Figs. 5-7 show the contour plots of G_+, φ_+ and F.

Two features of this solution are common to the behavior obtained for $\varrho_s = 0$ [2]. Firstly, the mode grows very rapidly in the early non linear stage. In fact, the reconnection rate is even faster when ϱ_s is larger at fixed d_e. This faster growth, already noted in the linear solution, can be attributed to a broadening of the flow layer, from a width of order d_e when $\varrho_s = 0$ to a width of order ϱ_s. Secondly, the mode structure develops a microscale rapidly shrinking in time. Similarly to the case with $\varrho_s = 0$, we attribute this behavior to the presence of the conserved quantities, $G_\pm$.

When $\varrho_s \neq 0$, the generalized momentum F is no longer conserved, indeed F changes topology , with an O-point forming at $x = y = 0$ and four Y-points forming symmetrically around the origin, as shown in Fig. 7. On the other hand, the initial topology of the $G_\pm$ fields is preserved, as expected.

The most striking difference between $\varrho_s = 0$ and $\varrho_s \neq 0$ is the formation, in the latter case, of cross-shaped current density and vorticity layers. This structure is already visible when $0 < \varrho_s < d_e$. The two branches of the current and vorticity layers lie on the separatrix of F and not on that of Ψ. Also, the separation between the two branches scales with ϱ_s.

We can establish a link between the Casimirs and the spatial structures that form nonlinearly. This link can clarify an important difference between Hamiltonian and dissipative reconnection. Both these processes require the localized violation of the topological constraints that involve the magnetic flux Ψ. However, electron inertia (and the electron stress tensor) make field line reconnection possible, but do not eliminate these topological constraints. Simply they now involve different fields (F, or $G_\pm$ when $\varrho_s \neq 0$, instead of Ψ).

The difference between the conserved fields and Ψ consists of a current density and of a plasma vorticity term. Thus, reconnection of Ψ can only proceed unimpeded by the conservation of F (or of $G_\pm$) if current and vorticity layers are formed. In the presence of dissipation there are no fields conserved locally and thus these layers have a minimum diffusive width. On the contrary, in Hamiltonian reconnection the presence of the locally conserved fields makes these layers increasingly sharper and leads to a cascade towards smaller and smaller microscales. Eventually, this cascade must be limited by kinetic and dissipative effects. In this sense, the collisionless model is incomplete from a physics point of view. One can draw an analogy with Landau damping and phase mixing, where smaller and smaller scale lengths are produced that are eventually wiped out by collisions.

The cross-shaped structure of the current and vorticity layers can be interpreted on the basis of the advection of the invariants $G_\pm$. At equilibrium, $G_\pm = G_\pm(x)$. As the instability evolves, G_+ and G_- rotate in opposite directions, advected by the effective velocities $\mathbf{v}_+$ and $\mathbf{v}_-$. Note that these velocities introduce a rotation, as the corresponding stream functions, $\varphi_\pm$, add terms with mixed parity with respect to x and y (by contrast, the potential $\varphi(x,y)$ is odd in x and even in y: indeed, the convection cells in Fig. 1 do not exhibits any rotation). Since the instability evolution is slow, the potentials $\varphi_\pm$ remain largely aligned with $G_\pm$. The structure of J, U and

F follows that of $G_\pm$ and $\varphi_\pm$, as $2\varrho_s d_e J = \varphi_+ - \varphi_-$, $2\varrho_s d_e U = G_+ - G_-$ and $2F = G_+ + G_-$.

In conclusion, magnetic reconnection in 2D collisionless regimes remains a fast process in the early nonlinear stage, in marked contrast with the standard Sweet-Parker model developed within the context of resistive MHD. We have established a link between the topological constraints of the collisionless model and the spatial structure that are formed non linearly. This link can clarify important differences between Hamiltonian and dissipative reconnection.

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4 Appendix

$$\mathbf{B} = B_0\mathbf{e}_z + \nabla\Psi \times \mathbf{e}_z \tag{53}$$

$$[\mathbf{B}\cdot\nabla\Psi = 0]$$

$\Psi(x,y,t) = const$ magnetic surfaces

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\Psi}{\partial t}\mathbf{e}_z \tag{54}$$

Φ electrostatic potential

quasi neutrality $n_i = n_e = n$

Electron fluid equations

$$\frac{\partial n}{\partial t} + \nabla\cdot n\mathbf{v} = 0 \tag{55}$$

$$m_e n(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}) = -en(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}) - T\nabla n - \nabla\cdot\mathbf{\Pi} \tag{56}$$

Stress tensor $\mathbf{\Pi}$

$$\mathbf{v}_\perp = \frac{c}{B_0}\mathbf{e}_z \times \nabla\Phi + c\mathbf{e}_z \times \nabla\Psi - \frac{cT}{eB_0}\mathbf{e}_z \times \frac{\nabla n}{n} \tag{57}$$

$$\omega_* = \frac{kcT}{eB_0L_n} = \frac{k\varrho_i v_{thi}}{L_n}\frac{T}{T_i} \tag{58}$$

k perpendicular wave number
L_n density gradient scalelength

$\varrho_i = \frac{\sqrt{T_i m_i}c}{eB_0}$ ion Larmor radius
$\varrho_s = \frac{c_s}{\omega_{ci}}$
$c_s = \left(\frac{T_e}{m_i}\right)^{1/2}$

$$n(\mathbf{x},t) = n_0(x)[(1+\tilde{n}(\mathbf{x},t)]$$

$\nabla\tilde{n} \sim \nabla \ln\frac{n_0(x)}{n_0}$

Ampère's law $J_z = -en_0v_z = -\frac{c}{4\pi}\nabla^2\Psi$

$$m_en(\frac{\partial v_z}{\partial t} + \mathbf{v}\cdot\nabla v_z) = -en(E_z + \frac{1}{c}\mathbf{v}\times\mathbf{B}\cdot\mathbf{e}_z) - \nabla nT\cdot\mathbf{e}_z - (\nabla\cdot\mathbf{\Pi})\cdot\mathbf{e}_z \tag{59}$$

$$(\nabla\cdot\mathbf{\Pi})_z = \frac{cm_e}{eB_0}(\mathbf{e}_z\times\nabla nT)\cdot\nabla v_z \tag{60}$$

$$\frac{\partial(\Psi+d_e^2J)}{\partial t} + \frac{c}{B_0}\nabla\Phi\times\nabla(\Psi+d_e^2J)\cdot\mathbf{e}_z = \frac{Tc}{eB_0}\nabla\left(\ln\frac{n}{n_0}\right)\times\nabla\Psi\cdot\mathbf{e}_z \tag{61}$$

$$d_e = \frac{c}{\omega_{pe}} = \sqrt{\frac{m_ec^2}{e^24\pi n_0}}$$

J normalized to $4\pi/c$
Define

$$F = \Psi + d_e^2J \tag{62}$$

and

$$[f,g] = \nabla f\times\nabla g\cdot\mathbf{e}_z = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \tag{63}$$

$$\frac{\partial F}{\partial t} + \frac{c}{B_0}[\Phi,F] = \frac{Tc}{eB_0}[\ln\frac{n}{n_0},\Psi] \tag{64}$$

$$\frac{\partial}{\partial t}\ln\frac{n}{n_0} + \frac{c}{B_0}[\Phi,\ln\frac{n}{n_0}] = \frac{c}{4\pi en_0}[\Psi,J] \tag{65}$$

Ion Equations

$$\frac{\partial}{\partial t}\left(\ln\frac{n}{n_0} + \tau_i(1-\Gamma_0)\frac{e}{T}\Phi\right) + \frac{c}{B_0}\left[\frac{e}{T}\Phi,\ln\frac{n}{n_0} + \tau_i(1-\Gamma_0)\frac{e}{T}\Phi\right] +$$

$$\frac{cT}{eB_0}L_j\left[L_j\frac{e}{T}\Phi,\ln\frac{n}{n_0}\right] = 0 \tag{66}$$

$$\tau_i = T/T_i$$

$$\Gamma_0 = \frac{1}{1 - \varrho_i^2 \nabla^2} \tag{67}$$

$$L_j = \varrho_i \partial_j (\Gamma_0)^{1/2} \tag{68}$$

For $\varrho_i \ll 1$: $\quad \Gamma_0 \approx 1 + \varrho_i^2 \nabla^2$

$$\frac{\partial}{\partial t}\left(\ln \frac{n}{n_0} - \tau_i \varrho_i^2 \nabla^2 \frac{e}{T}\Phi\right) + \frac{c}{B_0}\left(1 + \varrho_i^2 \nabla^2\right)\left[\frac{e}{T}\Phi, \ln \frac{n}{n_0}\right] -$$

$$\frac{cT}{eB_0} \nabla \cdot \left[\frac{e}{T}\Phi, \nabla\left(\ln \frac{n}{n_0} + \tau_i \frac{e}{T}\Phi\right)\right] = 0 \tag{69}$$

Take

$$\ln \frac{n}{n_0} = \tau_i \varrho_i^2 \nabla^2 \Phi = \varrho_s^2 \nabla^2 \Phi \tag{70}$$

$$\frac{\partial F}{\partial t} + \frac{c}{B_0}[\Phi, F] = \frac{c}{B_0}\varrho_s^2 [\nabla^2 \Phi, \Psi] \tag{71}$$

$$F = \Psi - \nabla^2 \Psi \tag{72}$$

$$\frac{\partial}{\partial t}\nabla^2 \Phi + \frac{c}{B_0}[\Phi, \nabla^2 \Phi] = \frac{cT}{4\pi e^2 n_0 \varrho_s^2}[J, \Psi] \tag{73}$$