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H4-SMR 1012 - 11

AUTUMN COLLEGE ON PLASMA PHYSICS

13 October - 7 November 1997

STATISTICHAL MECHANICS AND SELF-ORGANIZED PLASMA

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These are lecture notes, intended for distribution to participants.

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PHYSICAL REVIEW E VOLUME 53, NUMBER 5 MAY 1996

Statistical mechanics of magnetohydrodynamics

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(Received 18 April 1995; revised manuscript received 11 September 1995)

A statistical mechanical formulation for the steady state of self-organized magnetohydrodynamic plasma is studied based on the empirical variational principle, $\delta(E-\lambda H)=0$, for the steady state, where E and H denote the energy and the helicity of a magnetic field. The eigenfunctions of the curl operator are shown to span the phase space of a magnetic field in a bounded system, and the invariant measure is found. The classical ensemble theory is formulated assuming the Shannon or Rényi entropy. To avoid the divergence of the expectation values at nonzero temperature, Bose-Einstein statistics is also phenomenologically treated. It is implied that the spectra of the energy, helicity, and the helicity fluctuation obey the power law for a multiply connected domain with a nonzero cohomological field. For the toroidal system, these powers are implied to be three, three, and two, respectively. The invariant measure for the incompressible flow in a bounded domain is also given.

PACS number(s): 95.30.Qd, 02.30.Sa, 02.50.Cw, 05.20.−y

I. INTRODUCTION

The magnetohydrodynamical systems generate macroscopically ordered states from random disordered states. These phenomena are primarily due to the dynamical laws of such systems, that is, the magnetohydrodynamics equations (MHD equations). These equations are, however, not simple and they determine the behavior of the system more precisely than we expect. What we want to know is not a microscopic structure which fluctuates much under the change of minute conditions but the macroscopic coarse-grained structure which is stable under the microscopic changes.

Such separation of the scale is usually impossible. Strong experimental or mathematical conditions are indispensable. In some MHD systems, its self-organization phenomena seem to allow us to postulate the possibility of self-contained and self-consistent descriptions in the macroscopic level without referring to the microscopic details. More explicitly, we have a quantitative phenomenological variational principle which determines the macroscopic structure of a magnetic field [1,2]:

$$\delta(E-\lambda H)=0, \tag{1.1}$$

where E and H denote the energy and the helicity of a magnetic field, respectively. This variational principle first appeared when Chandrasekhar and Woltjer [1] discussed the minimum energy state of magnetic flux tubes tangled in a stellar plasma with introducing the magnetic helicity to characterize the twist of magnetic fields. With a fixed gauge, we write the magnetic field $\boldsymbol{B}=\nabla\times\boldsymbol{A}$. The helicity density is $h=\boldsymbol{A}\cdot\boldsymbol{B}$, and the helicity in a fixed domain Ω is $H=\int_\Omega h dx$. He assumed that the plasma relaxes into the minimum energy state with a given (prescribed) helicity. In a low-pressure charge-neutral plasma, the energy is dominated by $E=(2\mu_0)^{-1}\int_\Omega B^2 dx$ (μ_0, vacuum permeability). By formal calculations of the variation with appropriate boundary conditions, the minimum energy state is shown to satisfy the Beltrami condition

$$\nabla\times\boldsymbol{B}=\lambda\boldsymbol{B}, \tag{1.2}$$

where λ is a real constant (corresponding to the Lagrange multiplier). Since the current density $\boldsymbol{j}=(\mu_0)^{-1}\nabla\times\boldsymbol{B}$ in steady state, (1.2) implies the force-free condition ($\boldsymbol{j}\times\boldsymbol{B}=\boldsymbol{0}$), which had been considered to be obeyed by the relaxed magnetic field in a plasma [3]. If λ is an eigenvalue of the curl operator, Eq. (1.2) implies that $\boldsymbol{B}$ is the corresponding eigenfunction. And it was shown that the state becomes unstable when $|\lambda|>\lambda_{\min}$ [4], where $\lambda_{\min}$ is the nonzero and minimum absolute value of the curl eigenvalues, without charging some conditions which fix the modes with absolutely smaller eigenvalues than λ. This $\lambda_{\min}$ is proved to be positive (nonzero) [5]. So theoretically and experimentally interesting problems are the state for $0<|\lambda|<\lambda_{\min}$.

Exactly the same equation as (1.2) was found to describe the relaxed state of turbulent plasmas in laboratory experiments. Taylor [2] conjectured that a selective dissipation of the magnetic energy with respect to the helicity yields such a relaxed state. By Maxwell's equations, we obtain "Poynting's law" for the helicity,

$$\partial_t h=-\nabla\cdot(\phi\boldsymbol{B}+\boldsymbol{E}\times\boldsymbol{A})+2\boldsymbol{E}\cdot\boldsymbol{B}, \tag{1.3}$$

where $\boldsymbol{E}$ ($=-\partial_t\boldsymbol{A}-\nabla\phi$) is the electric field and ϕ is the scalar potential. Assuming a perfectly conductive wall at the boundary $\partial\Omega$, we obtain, using (1.3),

$$\frac{d}{dt}H=\int_\Omega 2\boldsymbol{E}\cdot\boldsymbol{B}dx. \tag{1.4}$$

In a highly conductive hydrodynamic plasma, $E_{\parallel} = \boldsymbol{E} \cdot \boldsymbol{B}/|\boldsymbol{B}| \approx 0$, and hence H is conserved. Furthermore, under the perfectly conductive boundary condition, $\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0}$, the helicity can be formulated to be a gauge invariant quantity. The conservation of the helicity imposes an essential restriction on the dynamics of the plasma. If H remains constant while the magnetic energy E achieves its minimum, the relaxed state is characterized by the minimizer of $F = E - \lambda' H$, and the formal Euler-Lagrange equation becomes (1.2).

The dynamical process of the relaxation was studied by computer simulations based on three-dimensional magnetohydrodynamic model equations [6].

Let us now revisit the thermodynamics and its statistical mechanics. Many experiments and speculations supported that the thermal equilibrium state is determined by the variational principle

$$\delta(F) = 0, \tag{1.5}$$

where F denotes the free energy. The thermodynamics itself is a self-consistent and self-contained theory within the macroscopic quantities like volume, pressure, and entropy. The statistical mechanics gives the relations between the microscopic dynamics and the macroscopic thermodynamics by assuming appropriate ensembles. The Boltzmann distribution is a kind of working hypothesis. It reproduces the correct results and its mathematical structure is now accepted to be natural. The additivity of the energy and the importance of the energy as the principal integral of the equation of motion imply the Boltzmann distribution with appropriate invariant measure. So most physicists already accepted that the ensemble and the Boltzmann distribution have sufficient reason to be regarded as the *reality*.

The purpose of this paper is to elucidate the ensemble description for a MHD system starting from the formal similarity between Eq. (1.1) and Eq. (1.5). It is to propose a statistical mechanics for a MHD system. There are pioneering works [7–9] towards such statistical mechanics already, which will be discussed at the end of this paper.

In this paper, we will make a statistical treatment only for the magnetic field. The velocity field is not treated explicitly in our formalism, because it does not appear in Eq. (1.1) explicitly. The variational principle (1.1) is interpreted as the zero-(helicity)-temperature form of the thermodynamic variational principle of the helicity ensemble. Appropriate space for this purpose is analyzed in the next section and a good phase space with invariant measure is given. A related topic of this phase space is given in the Appendix. In this space, the solution of Eq. (1.1) is studied in the third section. This solution is considered as the zero-temperature ground state. The fourth section proposes a simple quantal statistics after we see that the simplest classical statistics shows difficulty. Some connections to the experimental verification of this statistical mechanics are given in the fifth section.

II. PHASE SPACE

When an equilibrium or steady state exists, there are two key steps towards the statistical mechanical transcription of a variational principle. One is to find the relevant additively conserving quantity to characterize the state. In our case, the energy and the helicity of magnetic field play this role. The other is to find the invariant measure of the temporal evolution equation. It corresponds to Liouville's theorem in the classical Hamilton mechanics. In this section, it is proved that the expansion coefficient of the magnetic field, $\boldsymbol{B}$, with the complete orthogonal functions described below is a natural phase space and its volume element is temporally invariant.

Let Ω ($\subset \mathbf{R}^3$) be a bounded domain with a smooth boundary $\partial\Omega$. We denote by $\boldsymbol{n}$ the outward unit normal vector onto $\partial\Omega$. We consider a function space of real solenoidal vector fields in Ω,

$$L^2_\sigma(\Omega) = \{\boldsymbol{u} \in L^2(\Omega); \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ on } \partial\Omega\}, \tag{2.1}$$

which is a Hilbert space endowed with the standard L^2 innerproduct ( ). If Ω is multiply connected, we obtain the subspace of harmonic vector fields,

$$L^2_H(\Omega) = \{\boldsymbol{u} \in L^2(\Omega); \nabla \cdot \boldsymbol{u} = 0, \nabla \times \boldsymbol{u} = \boldsymbol{0} \text{ in } \Omega, \boldsymbol{n} \cdot \boldsymbol{u} = 0 \text{ on } \partial\Omega\}, \tag{2.2}$$

which represents the cohomology class, and the dimension of this $L^2_H(\Omega)$ is equal to the first Betti number ν of Ω. We write $L^2_\sigma(\Omega) = L^2_H(\Omega) \oplus L^2_\Sigma(\Omega)$, where $L^2_\Sigma(\Omega)$ is defined as the orthogonal complement of $L^2_H(\Omega)$. For these function spaces, the following lemma is proved [5].

Lemma 1. (1) When we consider eigenvalue problem

$$\nabla \times \boldsymbol{u} = \lambda \boldsymbol{u}, \quad \boldsymbol{u} \in L^2_\Sigma(\Omega), \tag{2.3}$$

we obtain a complete orthogonal set of eigenfunctions to span $L^2_\Sigma(\Omega)$. All eigenvalues are real, nonzero, and discrete.

(2) For every $\boldsymbol{u} \in L^2_\sigma(\Omega)$, we have an orthogonal expansion

$$\boldsymbol{u} = \sum_j (\boldsymbol{u}, \boldsymbol{\varphi}_j)\boldsymbol{\varphi}_j + \sum_{\ell=1}^{\nu} (\boldsymbol{u}, \boldsymbol{h}_\ell)\boldsymbol{h}_\ell, \tag{2.4}$$

where $\boldsymbol{\varphi}_j \in L^2_\Sigma(\Omega)$ is the eigenfunction of the curl operator and $\boldsymbol{h}_\ell$ is the orthogonal basis of $L^2_H(\Omega)$.

In the following, the subscript j for the nonzero eigenvalue and its eigenfunction of curl operator is assumed to run over all integers except zero, and this numbering is assumed to follow the order of the eigenvalue. Negative and positive subscripts are assumed to correspond to negative and positive eigenvalues, respectively. That is, the eigenvalue numbering looks like

$$\cdots \le \lambda_{-3} \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots. \tag{2.5}$$

These are unbounded and go to $\pm\infty$ when $j \to \pm\infty$.

Now we can find the phase space of the magnetic field.

Lemma 2. Let $\boldsymbol{v}(x,t)$ be a smooth vector field in Ω. Suppose that a solenoidal vector field $\boldsymbol{f}(x,t)$ obeys

$$\partial_t \boldsymbol{f} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{f}) \quad \text{in } \Omega, \tag{2.6}$$

$$\boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{f}) = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{2.7}$$

Using the eigenfunctions of the curl operator, we expand

$$\boldsymbol{f}(\boldsymbol{x},t) = \sum_j c_j(t)\boldsymbol{\varphi}_j(\boldsymbol{x}) + \sum_{\ell=1}^{\nu} \check{c}_\ell(t)\boldsymbol{h}_\ell(\boldsymbol{x}) \tag{2.8}$$

(see Lemma 1). Then, $dC = d\check{c}_1 \cdots d\check{c}_\nu \prod_j dc_j$ is an invariant measure.

In fact, by the boundary condition (2.7), we observe

$$\frac{d}{dt}\check{c}_\ell = 0 \quad (\forall \ell). \tag{2.9}$$

Using (2.6) and (2.7), we obtain

$$\begin{aligned}\frac{d}{dt}c_j &= (\boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{f}), \boldsymbol{\varphi}_j) = (\boldsymbol{v} \times \boldsymbol{f}, \boldsymbol{\nabla} \times \boldsymbol{\varphi}_j) \\ &= \lambda_j(\boldsymbol{v} \times \boldsymbol{f}, \boldsymbol{\varphi}_j) \\ &= \lambda_j \left[\sum_k c_k(\boldsymbol{v} \times \boldsymbol{\varphi}_k, \boldsymbol{\varphi}_j) + \sum_{\ell=1}^{\nu} \check{c}_\ell(\boldsymbol{v} \times \boldsymbol{h}_\ell, \boldsymbol{\varphi}_j)\right].\end{aligned} \tag{2.10}$$

Since $(\boldsymbol{v} \times \boldsymbol{\varphi}_j) \cdot \boldsymbol{\varphi}_j \equiv 0$, we find

$$\partial(dc_j/dt)/\partial c_j = 0 \quad (\forall j). \tag{2.11}$$

Hence the measure dC is invariant. This implies that these $\check{c}$ and c are a good set of coordinates in the phase space in the statistical mechanical sense.

In a MHD system, $\boldsymbol{v}$ and $\boldsymbol{f}$ are the velocity field of the fluid motion and the magnetic field, $\boldsymbol{B}$, respectively. The velocity field is now treated to be a separated freedom from the magnetic field. This treatment will be good when the magnetic field has the most energy in the system, and then the velocity field acts as a perturbation or as a fluctuation generator to the magnetic field. When we expand a magnetic field as

$$\boldsymbol{B}(\boldsymbol{x}) = \sum_j c_j\boldsymbol{\varphi}_j(\boldsymbol{x}) + \sum_{\ell=1}^{\nu} \check{c}_\ell\boldsymbol{h}_\ell(\boldsymbol{x}), \tag{2.12}$$

the second summation term over the harmonic field in the RHS is the same for all possible $\boldsymbol{B}$ because of the boundary condition $\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0}$. It is called the cohomology field. So we do not treat $\check{c}_\ell$ as a dynamical variable, instead, as a constant. Only c_j's are treated as dynamical variables, and the first summation in the RHS of Eq. (2.12) is denoted by $\boldsymbol{B}_\Sigma$. The energy of this $\boldsymbol{B}$ is expressed as

$$E = \sum_j c_j^2 + \sum_{\ell=1}^{\nu} \check{c}_\ell^2. \tag{2.13}$$

The second summation, the energy of the cohomology field gives only a constant contribution. Taking $\boldsymbol{g}_\ell$ to be $\boldsymbol{h}_\ell = \boldsymbol{\nabla} \times \boldsymbol{g}_l$, the vector potential of $\boldsymbol{B}$ is

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_j \frac{c_j}{\lambda_j}\boldsymbol{\varphi}_j(\boldsymbol{x}) + \sum_{l=1}^{\nu} \check{c}_l\boldsymbol{g}_l(\boldsymbol{x}). \tag{2.14}$$

We can add any function in $L^2_H(\Omega)$ to vector potentials of the cohmology field, $\sum_l \check{c}_l\boldsymbol{g}_l(\boldsymbol{x})$. This corresponds to the gauge degree of freedom.

The relative helicity is defined by

$$\int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}_\Sigma dx = \sum_j \left(\frac{c_j^2}{\lambda_j} + L_j c_j\right), \tag{2.15}$$

where

$$L_j = \sum_{\ell=1}^{\nu} \check{c}_\ell \Delta_{j,\ell} \tag{2.16}$$

and

$$\Delta_{j,\ell} = (\boldsymbol{\varphi}_j, \boldsymbol{g}_\ell). \tag{2.17}$$

The $\Delta_{j,\ell}$ may be called the cohomology-helicity coupling constant. The L_j is named the cohomology coefficient. The difference between the relative helicity and the helicity is a constant determined only by the cohomology field and its vector potential, so we can neglect it. It should be remarked that the relative helicity (2.15) is gauge invariant because of the perfectly conductive boundary condition. In the following, we will only use this gauge-invariant quantity for the helicity and call it simply the helicity.

III. SOLUTION OF THE VARIATIONAL PROBLEM

Now we can solve the variational problem (1.1). By using the expansion Eq. (2.12), this problem becomes

$$\begin{aligned}0 &= \delta_{\{c_j\}} \sum_j \left[\left(1 - \frac{\lambda}{\lambda_j}\right) c_j^2 - \lambda L_j c_j\right] \\ &= \delta_{\{c_j\}} \sum_j \left[\left(1 - \frac{\lambda}{\lambda_j}\right)\left(c_j - \frac{\lambda\lambda_j L_j}{2(\lambda_j - \lambda)}\right)^2 \right. \\ &\quad \left. - \frac{\lambda^2\lambda_j L_j^2}{4(\lambda_j - \lambda)}\right].\end{aligned} \tag{3.1}$$

For $0 < \lambda < \min_j |\lambda_j|$, the solution is

$$c_j^0 = \frac{\lambda\lambda_j L_j}{2(\lambda_j - \lambda)} \quad (\forall j). \tag{3.2}$$

The L_j will decay algebraically in terms of j for large $|j|$. The eigenvalue λ_j will be distributed uniformly for large $|j|$. And we can expect that the summation $\sum_j c_j^0 \boldsymbol{\varphi}_j$ converges uniformly and absolutely. But we cannot always expect such convergence for the termwisely differentiated series $\sum_j c_j^0 \lambda_j \boldsymbol{\varphi}_j$. The energy and the helicity are expressed as

$$E = \sum_j \frac{\lambda^2\lambda_j^2 L_j^2}{4(\lambda_j - \lambda)^2} \tag{3.3}$$

and

$$H = \sum_j \frac{\lambda\lambda_j L_j^2}{4} \frac{2\lambda_j - \lambda}{(\lambda_j - \lambda)^2}. \quad (3.4)$$

L_j is expected to show algebraic decay in terms of $1/|j|$. For example, the decay speed is expected to be the order of $1/|j|^{3/2}$ when Ω is inside of a torus (T^2). For large $|j|$, j can be regarded as the wave number k in the Fourier analysis case except a constant with the dimensionality of $(\text{length})^{-1}$, because the local feature of the sufficiently high mode will not be sensitive to the boundary condition.

IV. ENSEMBLE

MHD fluid relaxes to a kind of steady state after it starts to develop from a given initial condition. During this relaxation, the change of the magnetic helicity is slow, and it can be neglected. The energy of the magnetic field, however, dissipates largely in the early state and finally its change is also negligible in the steady state [2]. The variational principle (1.1) determines the structure of such a steady state [we should say that the validity of the expression "steady state" comes out of the success of Eq. (1.1)] and our purpose is to propose a microscopic model, which we call a "statistical mechanics of MHD," to reproduce this principle. In our terminology, the thermodynamics of MHD, Eq. (1.1), suggests that the energy E and helicity H are the relevant state variables. H is easily controlled by external condition, but E is not as we described above. So the parameter $1/\lambda$ works like a chemical potential of the grand canonical ensemble. The limitation of this chemical potential interpretation is that these E and H are defined in the same phase space.

These E and H are additive quantities in the relaxed state. So the possible distribution consistent with Eq. (1.1) is determined by specifying the information measure. When we use the Shannon entropy, $S(p) = -\sum p \ln p$, the Boltzmann distribution form in terms of these quantities appears:

$$P(E,H) \propto \exp(-\alpha H - \beta E), \quad (4.1)$$

where α and β are constants, and these E and H are microscopically defined dynamical quantities, not macroscopic. This expression is equivalent to

$$P(E,H) \propto \exp[-\beta(E - \lambda H)]. \quad (4.2)$$

This λ is adjusted to the notation in Eq. (1.1). The β is interpreted as an inverse temperature of the magnetic field, and Eq. (1.1) corresponds to the case of large β.

A more general information measure is the Rényi entropy [10],

$$S_q(\{p_i\}) = \frac{1}{1-q} \ln\left(\sum_i p_i^q\right), \quad (4.3)$$

where q is a positive parameter. The Shannon entropy is included in the Rényi entropy in the limit of $q \to 1$. The canonical distribution based on this entropy, that is, the Tsallis distribution [12], also produces a similar result to the Boltzmann distribution as is shown below.

Although the helicity H is introduced in the Boltzmann distribution function, this naive classical statistical mechanics causes the same kind of catastrophe as what we meet in the classical treatment for the blackbody radiation.

A. Classical statistics

In the preceding section, it is proved that the volume element $\prod_j dc_j$ is temporally conserved when the flow velocity field is prescribed. The proof of this, moreover, shows that each dc_j is conserved. So we concentrate on a single mode, denoted by j, first. The helicity and the energy of this mode are $c_j^2/\lambda_j + L_j c_j$ and c_j^2, respectively. The Boltzmann distribution for this amplitude c_j is

$$P_j(c) \propto \exp\left[-\beta\left(c_j^2 - \frac{\lambda}{\lambda_j}c_j^2 - \lambda L_j c_j\right)\right]. \quad (4.4)$$

In the variational principle, Eq. (1.1), we can assume the condition $0 < \lambda < \min_j |\lambda_j|$. Assuming that β is positive, the distribution function is

$$P_j(c)dc = \sqrt{\frac{\lambda_j}{\pi\beta(\lambda_j - \lambda)}} \times \exp\left[-\beta\left(1 - \frac{\lambda}{\lambda_j}\right)(c_j - c_j^0)^2\right] dc, \quad (4.5)$$

where c_j^0 is defined in Eq. (3.2) as the solution of Eq. (1.1). In the following, the ensemble averaged value is denoted by $\langle\,\rangle$. The expectation value of the energy for this mode is

$$\langle c_j^2\rangle = \frac{\lambda_j}{2\beta(\lambda_j - \lambda)} + (c_j^0)^2 = \frac{\lambda_j}{2\beta(\lambda_j - \lambda)} + \frac{\lambda^2\lambda_j^2 L_j^2}{4(\lambda_j - \lambda)^2}. \quad (4.6)$$

The helicity is

$$\left\langle \frac{c_j^2}{\lambda_j} + L_j c_j \right\rangle = \frac{1}{2\beta(\lambda_j - \lambda)} + \frac{(c_j^0)^2}{\lambda_j} + L_j c_j^0 = \frac{1}{2\beta(\lambda_j - \lambda)} + \frac{\lambda\lambda_j L_j^2}{4}\frac{2\lambda_j - \lambda}{(\lambda_j - \lambda)^2}. \quad (4.7)$$

The distribution over our phase space, $\{c_j\}$, is simply the product for each P_j. So the average of E and H should be also simply obtained by summing up over all modes:

$$\langle E\rangle = \sum_j \left[\frac{\lambda_j}{2\beta(\lambda_j - \lambda)} + \frac{\lambda^2\lambda_j^2 L_j^2}{4(\lambda_j - \lambda)^2}\right] \quad (4.8)$$

and

$$\langle H\rangle = \sum_j \left[\frac{1}{2\beta(\lambda_j - \lambda)} + \frac{\lambda\lambda_j L_j^2}{4}\frac{2\lambda_j - \lambda}{(\lambda_j - \lambda)^2}\right]. \quad (4.9)$$

In these summations, the summations of the "cohomological terms,"

$$\sum_j \frac{\lambda^2\lambda_j^2 L_j^2}{4(\lambda_j-\lambda)^2} \quad \text{and} \quad \sum_j \frac{\lambda\lambda_j L_j^2}{4}\frac{2\lambda_j-\lambda}{(\lambda_j-\lambda)^2} \tag{4.10}$$

are expected to converge, because $|L_j|$ decays faster than $1/|j|$. The summation of the first terms in the helicity may be interpreted to converge by taking the limit in the form of

$$\lim_{J\to\infty}\sum_{j=-J}^{J}\frac{1}{2\beta(\lambda_j-\lambda)}. \tag{4.11}$$

The first term in the energy, however, diverges. This term expresses the equipartition of the energy for every mode.

The Tsallis distribution for c_j is

$$P_j(c) \propto \left[1-\beta(q-1)\left(1-\frac{\lambda}{\lambda_j}\right)(c-c_j^0)^2\right]^{1/(q-1)}, \tag{4.12}$$

and the range of c is limited to $c_j^0 - c_j^{\max} \le c_j \le c_j^0 + c_j^{\max}$, where $1/c_j^{\max} = \sqrt{\beta(1-\lambda/\lambda_j)}$. Including the normalization factor, it becomes

$$P_j(c)dc = \frac{1}{c_j^{\max}B(1/2,q/(q-1))}\left[1-\beta(q-1)\left(1-\frac{\lambda}{\lambda_j}\right)\times(c-c_j^0)^2\right]^{1/(q-1)} dc, \tag{4.13}$$

where $B(\,)$ denotes the beta function. So the energy for each mode is calculated to be

$$\langle c_j^2\rangle = \frac{\lambda_j}{(3q-1)\beta(\lambda_j-\lambda)} + (c_j^0)^2 \tag{4.14}$$

and the helicity to be

$$\left\langle \frac{c_j^2}{\lambda_j} + L_j c_j \right\rangle = \frac{1}{(3q-1)\beta(\lambda_j-\lambda)} + \frac{(c_j^0)^2}{\lambda_j} + L_j c_j^0. \tag{4.15}$$

So the difference between the Boltzmann and Tsallis distributions reduces to a factor in front of β. Therefore the choice of the entropy is irrelevant.

B. Bose-Einstein statistics

The statistical mechanics of the blackbody radiation suggests that the quantization of the field is necessary to avoid the divergence of energy we met above. But it has not succeeded yet in our MHD equation case. This difficulty can be observed in Eq. (2.6). This evolution equation is linear in field $\boldsymbol{f}$ but the velocity field of the plasma flow, $\boldsymbol{v}$, will also evolve with the same time scale. And its evolution obeys a complicated nonlinear equation, for example, the Euler equation (A1) even within the incompressible approximation.

In spite of such difficulty for the legitimate approach to the second quantization, the magnetic field of a steady state in self-organized plasma is determined by the variational principle, (1.1), in which the flow does not appear. And the purpose of the present study is to make up a statistical mechanical formulation which reproduces this variational principle in a limit. Following is one of the simplest formulations to avoid the Rayleigh-Jeans-like catastrophe

The exponentiated factor, $(1-\lambda/\lambda_j)c_j^2 - \lambda L_j c_j$, is regarded as a transformed expression of a kind of effective Hamiltonian by replacing the canonical momentum with the canonical coordinate, c_j, using an unknown effective temporal evolution equation. We do not know which part of the c_j^2 comes from the momentum, or additional momentum contribution may be hidden. We introduce an angular frequency ω_j of this jth mode. New variables, d_j, are introduced to shift the average to zero and to be normalized, that is,

$$d_j = \frac{1}{\sqrt{\omega_j}}\left[c_j - \frac{1}{2}\frac{\lambda\lambda_j L_j}{\lambda_j-\lambda}\right]. \tag{4.16}$$

The factor $1/\sqrt{\omega_j}$ is a naive normalization factor which provides the unit of the field quantum. Then we assume that this d_j is the bosonic annihilator by charging the commutation relation, $[d_i,d_j^\dagger]=\delta_{ij}$ or $[c_i,c_j^\dagger]=\omega_j\delta_{ij}$.

The Bose-Einstein statistics gives the averaged number of these quanta in the jth mode as

$$\langle n_j\rangle = \langle d_j^\dagger d_j\rangle = \frac{1}{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1}. \tag{4.17}$$

The chemical potential is taken to be zero because we quantized the magnetic field.

The expectation values of the energy and the helicity are

$$\langle c_j^\dagger c_j\rangle = \omega_j\langle n_j\rangle + (c_j^0)^2 \quad (n_j = d_j^\dagger d_j) \tag{4.18}$$

and

$$\left\langle \frac{c_j^\dagger c_j}{\lambda_j} + \frac{1}{2}L_j(c_j^\dagger + c_j)\right\rangle = \frac{\omega_j}{\lambda_j}\langle n_j\rangle + \frac{(c_j^0)^2}{\lambda_j} + L_j c_j^0. \tag{4.19}$$

The total energy and helicity are

$$\langle E\rangle = \sum_j\left[\frac{\omega_j}{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1} + \frac{\lambda^2\lambda_j^2 L_j^2}{4(\lambda_j-\lambda)^2}\right] \tag{4.20}$$

and

$$\langle H\rangle = \sum_j\left[\frac{\omega_j/\lambda_j}{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1} + \frac{\lambda\lambda_j L_j^2}{4}\frac{2\lambda_j-\lambda}{(\lambda_j-\lambda)^2}\right]. \tag{4.21}$$

The ω_j will diverge when $|\lambda_j|$ diverges, and $|L_j|$ will decay faster than $1/|j|$. So these expressions now converge.

V. SOME IMPLICATIONS

The Bose-Einstein-type statistical mechanical formalism proposed in the preceding section is a theoretically naive and simple extension of the variational principle (1.1) so that fluctuations around the variationally determined state can be described. Experimental verification is expected and some characteristic predictions of the present formalism are shown in this section for that purpose.

The fluctuations of energy and helicity are

$$\langle(\Delta E)^2\rangle = \sum_j \omega_j^2 \langle(\Delta d_j^\dagger d_j)^2\rangle = \sum_j \omega_j^2 \frac{\exp[\beta(1-\lambda/\lambda_j)\omega_j]}{\{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1\}^2} \quad (5.1)$$

and

$$\begin{aligned}\langle(\Delta H)^2\rangle &= \sum_j \left[\frac{\omega_j^2}{\lambda_j^2}\langle(\Delta d_j^\dagger d_j)^2\rangle + \frac{L_j^2\omega_j}{4}\left(\frac{\lambda_j}{\lambda_j-\lambda}\right)^2 (2\langle d_j^\dagger d_j\rangle + 1)\right] \\ &= \sum_j \left[\frac{\omega_j^2}{\lambda_j^2}\frac{\exp[\beta(1-\lambda/\lambda_j)\omega_j]}{\{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1\}^2} + \frac{L_j^2\omega_j}{4}\left(\frac{\lambda_j}{\lambda_j-\lambda}\right)^2 \frac{\exp[\beta(1-\lambda/\lambda_j)\omega_j]+1}{\exp[\beta(1-\lambda/\lambda_j)\omega_j]-1}\right],\end{aligned} \quad (5.2)$$

respectively. The summation of this second term will converge. For a torus, for example, if ω_j does not grow faster than j^2, it converges. The energy fluctuation decays exponentially for higher modes. But the power-law spectrum is predicted for the helicity fluctuation from its second term in Eq. (5.2). The power exponent is determined from j dependence of $L_j^2\omega_j$ for large $|j|$.

Spatial correlation of the magnetic field can be derived. For example, two point correlation is expressed as

$$\langle \boldsymbol{B}(x)\boldsymbol{B}(y)\rangle = \sum_j [\omega_j\langle d_j^\dagger d_j\rangle + (c_j^0)^2](\varphi(x), \varphi(y)). \quad (5.3)$$

It is straightforward (but tedious) to get more explicit expression of this kind of spatial correlations.

The power-law behavior of the helicity fluctuation with its exponent is simple and characteristic in the present quantal statistical mechanics.

VI. SUMMARY AND DISCUSSION

A statistical mechanical formulation for the self-organized MHD fluid is proposed. It is a naive extension of the variational principle (1.1) which suggests that the structure of the magnetic field is relevant to the steady state structure of such MHD fluid. So the velocity field is neglected in the present formalism.

It is shown that the eigenfunctions of curl span a convenient phase space when the system is bounded. For a given velocity field, the volume of the expansion coefficients is temporally invariant and this corresponds to Liouville's theorem in the classical Hamilton mechanics. For a cylindrical system, this has been already shown by Turner [9] and our present proof applies in a very general situation. Our domain covers not only a simply connected domain, but also a multiply connected domain. Furthermore, the same functional analytic space turns out to be a good phase space with invariant measure even for the incompressible flow (see the Appendix) and this fact may grow to one step of the statistical mechanical theory for the turbulent flow. Previous attempts mostly used plane wave to make phase spaces [7], and met some difficulties to reproduce the power-law spectrum.

In this phase space, the energy and helicity are used as additive conserving quantities to translate the variational principle (1.1) to the ensemble and introduce fluctuations. But the simplest classical statistics leads to the divergence of the expectation values. So some more assumptions are necessary to make a finite theory. One is to restrict the relevant modes to finite as was proposed in previous formalism [9], but the solution of the original variational principle (1.1) itself requires an infinite number of modes to reproduce its solution with the eigenfunctions of curl operator, as we have seen in the third section.

Our present formalism uses the quantal statistics by charging second quantization. The relevant functional $E-\lambda H$ including a chemical-potential-like parameter λ is interpreted as a transcription of an effective Hamiltonian of the system. The frequency of each mode is introduced artificially. The fluctuation currently relevant is, however, not large, that is, the temperature $1/\beta$ is small. So a linear dispersion approximation, $\omega_j = \gamma\lambda_j$, will be good. The implication of the present formulation for general geometry is stressed here: the ground-state structure has power-law spectrum in energy and helicity and it is also the case for the helicity "thermal" fluctuation in our statistical mechanical sense. This power-law behavior stems in the tangling of the dynamical magnetic field with the cohomological magnetic field. Now for the torus or cylinder, the power exponents are predicted to be three for the ground-state energy and helicity, and using the above linear approximation, the exponent for the thermal fluctuation of the helicity is two. The experimental observation of these power exponents will be a good test of the present formulation.

At nonzero temperature in the present sense, the variational principle will be modified to that for the thermodynamic free energy as

$$\delta(E - \lambda H - TS) = 0, \quad (6.1)$$

where T is $1/\beta$ in the present formalism. S denotes the entropy which may not have been observed because the temperature T has been small. But the measurement for the helicity fluctuation will reveal its statistical nature.

Purely theoretically, even if we accept the existence of the statistical mechanics for our problem, a very different first step is possible. For example, in the present paper, we assume only the Shannon entropy to select the distribution. But the steady state of the MHD system may reject to measure our knowledge to its subsystem. In such a case, we have to use Rényi entropy and a different distribution function [12] from the current Boltzmann type. Before going into the complicated forests, we now propose a familiar extension in this paper. The experimental verification is now expected.

ACKNOWLEDGMENTS

The authors thank Professor A. Hasegawa and Professor T. Tokihiro for fruitful and encouraging discussion. This study was initiated at the workshop "Various Approaches to Complex Systems" held at the International Institute for Advanced Studies in Kyoto.

APPENDIX: INVARIANT MEASURE OF INCOMPRESSIBLE FLOW

Using the eigenfunction expansion associated with the curl operator, we also obtain an invariant measure of incompressible ideal flow. Let $\boldsymbol{u}$ be a three-dimensional flow in a bounded domain Ω, which satisfies

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \boldsymbol{F} - \nabla p, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{A1}$$

where p is the pressure, and $\boldsymbol{F}$ is a force (for example $\boldsymbol{F} = \boldsymbol{j}\times\boldsymbol{B}$). We assume that $\boldsymbol{F}$ is not an explicit function of $\boldsymbol{u}$. The mass density is normalized to 1. The boundary condition is $\boldsymbol{n}\cdot\boldsymbol{u} = 0$ on $\partial\Omega$. Using $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \nabla(u^2/2)$, we may write (A1) as

$$\partial_t \boldsymbol{u} = -(\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \boldsymbol{F} - \nabla\tilde{p}, \tag{A2}$$

where $\tilde{p} = p + (u^2/2)$. Let us expand

$$\boldsymbol{u} = \sum_j v_j \boldsymbol{\varphi}_j + \sum_{\ell=1}^{\nu} \hat{v}_\ell \boldsymbol{h}_\ell, \tag{A3}$$

cf. Lemma 1. We easily verify $(\nabla\tilde{p}, \boldsymbol{\varphi}_j) \equiv 0$ $(\forall j)$ and $(\nabla\tilde{p}, \boldsymbol{h}_\ell) \equiv 0$ $(\forall \ell)$. We denote $F_j = (\boldsymbol{F}, \boldsymbol{\varphi}_j)$. By (A2) and $\nabla\times\boldsymbol{\varphi}_j = \lambda_j \boldsymbol{\varphi}_j$, we observe

$$\begin{aligned}\frac{d}{dt}v_j &= -\left(\left(\sum_m \lambda_m v_m \boldsymbol{\varphi}_m\right)\times\left(\sum_n v_n \boldsymbol{\varphi}_n\right), \boldsymbol{\varphi}_j\right) + F_j \\ &= -\sum_m\sum_n \lambda_m v_m v_n (\boldsymbol{\varphi}_m\times\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_j) + F_j \\ &= -\sum_{m\neq j}\sum_{n\neq j} \lambda_m v_m v_n (\boldsymbol{\varphi}_m\times\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_j) + F_j.\end{aligned} \tag{A4}$$

We thus have $\partial(dv_j/dt)/\partial v_j = 0$ $(\forall j)$. Similarly $d\check{v}_\ell$ is invariant.

The complete set of ideal incompressible MHD equations consists of (A1) and (2.6) with $\boldsymbol{f} = \boldsymbol{B}$, $\boldsymbol{v} = \boldsymbol{u}$, and $\boldsymbol{F} = \mu_0^{-1}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$. One thus obtains a higher dimensional invariant measure such as $\prod_j dc_j \prod_j dv_j$. In the present theory, however, we do not invoke the statistical distribution with respect to $\prod_j dv_j$. This is due to the semiempirical assertion that a finite (but small) resistivity and viscosity violate the invariance of $\prod_j dv_j$ largely, while $\prod_j dc_j$ remains almost invariant. This fact is relevant to the hypothesis of the selective conservation of the helicity in the MHD turbulence [11].

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To appear in Physica D

Statistical Mechanics of Three-Dimensional Magnetohydrodynamics in a Multiply Connected Domain

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Abstract

We study the statistical mechanics of three-dimensional magnetohydrodynamics in a multiply connected domain by constructing a Gibbs ensemble that accounts for the three rugged invariants of the ideal dynamics. The phase space we work with is defined by the eigenfunctions of the curl operator on the space of real three-dimensional solenoidal vector fields. The dynamics in this phase space satisfies the essential Liouville property. The theory predicts the appearance of a steady mean magnetic field–velocity field pair coupled with random fluctuations. It is shown that this mean field–flow satisfies the variational principle $\delta(E - \zeta H - \xi K) = 0$, where E is the energy, H is the magnetic helicity, and K is the cross-helicity. We obtain a meaningful continuum limit in which the magnetic field and velocity field exhibit finite amplitude local fluctuations, while the fluctuations of the vector potential and the velocity stream function vanish. In this limit, the energy and cross–helicity are divided among the mean field–flow and the fluctuations, whereas the fluctuation component of the helicity vanishes, so that the helicity is determined entirely by the mean field. It is shown that, in the continuum limit, the Gibbs ensemble is equivalent to the microcanonical ensemble associated with the conservation of the rugged invariants.

pacs numbers 05.20.-y, 47.27.Eq, 47.65.+a, 52.30.-q.

keywords magnetohydrodynamics, turbulence, coherent structures, statistical equilibria.

1 Introduction

An important characteristic of a magnetofluid, which distinguishes it from a nonmagnetic fluid, is the induction effect. This effect brings about the coupling of the electromagnetic field and the velocity field. As a result of the inclusion of the magnetic field, the equations of magnetohydrodynamics (MHD) are considerably more complicated than those of ordinary hydrodynamics. Surprisingly, however, an incompressible three–dimensional (3d) magnetofluid exhibits, in a certain sense, a greater degree of regularity than does an ordinary incompressible 3d fluid. Observations of space and laboratory plasmas alike reveal that the magnetic field of the plasma tends to self–organize through a turbulent phase of relaxation into a simple spiral configuration [1]. The recent numerical simulations of Politano *et al.* [2] demonstrate that the velocity field plays an important role in the self–organization process of the magnetofluid. While coherent structures are also observed in 2d MHD [3, 4] and in 2d hydrodynamics [5], there is not such an obvious propensity toward self–organization in 3d hydrodynamics. Traditionally, there have been two popular analytical approaches to characterizing the macroscopic organized states in fluid and plasma turbulence, which we shall now briefly review.

One such method, which we shall call the selective decay theory, models the organized state as a minimizer of some integral invariant of the ideal (nondissipative) dynamics subject to the constraint that certain other ideal invariants remain fixed. The conserved quantity that is minimized is usually the one that decays most rapidly in the presence of dissipation. This procedure yields a deterministic field equation, whose solutions correspond to steady solutions of the relevant dynamical equations. In 2d hydrodynamics, for example, the organized structure, according to the selective decay formalism, should correspond to a minimizer of the quadratic enstrophy subject to the condition of constant energy [1, 6]. In 3d (respectively, 2d) MHD, under the assumption that the magnetic field gives the dominant contribution to the energy, an organized magnetic structure is thought to be described as a minimizer of the magnetic energy subject to the constraint that the magnetic helicity (respectively, the integral of the square of the vector potential) remain fixed [7, 8, 9]. In general, the velocity field can not be ignored in MHD, and the kinetic energy should be accounted for in this formulation, as should the cross-helicity, which is also conserved by the ideal dynamics. A detailed discussion of the selective decay theory may be found in [1].

Another approach to studying the phenomenon of self-organization in fluid and plasma turbulence is, as we shall refer to it, the statistical equilibrium method. This theory, which originated with the work of Lee [10], was subsequently developed by, among others, Kraichnan [11, 12], Frisch *et al.* [13], and Fyfe and Montgomery [14], and Montgomery *et al.* [15]. An exhaustive review is given in [16]. This method is based upon a Gibbs ensemble for a truncated spectral representation of the ideal dynamics. The Gibbs distribution assumes the form

$$P = Z^{-1}\exp(-\alpha_1 F_1 - \alpha_2 F_2 - \cdots),$$

where $F_1, F_2, \cdots$ are the rugged (i.e., quadratic) global invariants for the nondissipative dynamics (eg; energy, enstrophy, cross-helicity, etc.), expressed in spectral form, and the α_i play the role of inverse temperatures. The partition function Z ensures that P has a total mass of 1. Alternatively, the Gibbs distribution may be obtained as a maximizer of the Gibbs–Boltzmann entropy functional, subject to constraints on the ensemble averaged values of these conserved quantities [17, 15]. In that case, the α_i act as Lagrange multipliers to enforce the given constraints. This Gibbs ensemble is supposed to represent a state toward which the ideal system tries to relax. The statistical equilibrium theory has met with much success in describing certain long–time properties of inviscid fluid and plasma systems, and it provides a basis for many qualitative predictions and assertions concerning spectral cascades of the rugged invariants in large Reynolds number turbulence. Of particular importance in this regard is the prediction of inverse cascades and normal cascades of the various global invariants when finite dissipation is introduced into the system. A quantity whose spectrum is peaked at large wavenumbers will decay more rapidly in the presence of dissipation than one which is peaked at smaller wavenumbers, because dissipation becomes more effective with increasing wavenumber [15]. Thus, an ideal global invariant whose spectrum is peaked at high modes will exhibit a "normal" cascade to larger wavenumbers, while one whose spectrum is more peaked at the lower modes will follow an "inverse" cascade to the smaller wavenumbers. An inverse cascade indicates condensation of the quantity in question at the longest wavelengths, and a normal cascade of an ideal invariant to short wavelengths points to the selective dissipation of that invariant. The process of self–organization may, therefore, be conceptualized as resulting from a normal cascade of some global invariant(s) and an inverse cascade of the other(s). The preceding discussion strongly suggests that the existence of rugged, or quadratic, ideal invariants, such that one exhibits a normal cascade and the others an inverse cascade, appears to be essential for the self–organization phenomenon to occur.

In the present paper, we shall follow a statistical equilibrium approach to study the self-organization of a 3d magnetofluid. As we have noted above, there have been many previous statistical equilibrium analyses of 3d MHD [13, 16, 15, 18]. Therefore, we would like to make prominent here the novel aspects of the present investigation. The previous studies focused on mathematically convenient, but physically restrictive, geometries such as a periodic box or a cylindrical domain, whereas our theory will apply to a wide class of bounded three-dimensional domains, including multiply connected domains, such as a torus, for example. When the domain is multiply connected and we assume that the boundary is perfectly conducting, the magnetic field may be expressed as the orthogonal sum of a nonvanishing harmonic field, and a field which is a linear combination of eigenfunctions of the curl operator. The same is true for the velocity field (see [19, 20] and Lemma 1 below). The expansion coefficients of the field and the flow with respect to this orthogonal decomposition satisfy a Liouville property, and therefore define a convenient phase space for a Gibbs–Boltzmann statistical analysis of the MHD system. Montgomery *et al.* [15] and Turner [21] have used the phase space associated with the eigenvalues of curl to construct a Gibbs statistical theory for MHD for the particular case of a cylindrical domain. Our analysis will demonstrate that the presence of the nonvanishing harmonic component of the magnetic field when the domain is multiply connected has important ramifications for the theory. This invariant harmonic field plays the role of an externally applied symmetry breaking, and it leads to a Gibbs ensemble which contains a nontrivial mean magnetic field-velocity field pair. This mean represents a coherent magnetic-kinetic structure to which the MHD system is expected to relax, and fluctuations about this structure are described by the Gibbs ensemble. In [15] it was claimed that the Gibbs ensemble based on the usual quadratic invariants for ideal 3d MHD yields a zero mean state when the spatial domain is a periodic cylinder. The reason for this erroneous assertion is that the authors failed to properly take into account the presence of the nonvanishing harmonic field.

The mean field–flow predicted by our model corresponds to a steady solution of the ideal MHD equations, and it satisfies the variational principle $\delta(E - \zeta H - \xi K) = 0$, with E the total energy (magnetic plus kinetic), H the helicity, and K the cross-helicity. The parameters ζ and ξ are related to the inverse temperatures in the Gibbs distribution. This variational principle is an extension of the classical Woltjer–Taylor variational formula, $\delta(E_{mag} - \zeta H) = 0$, for the relaxed state of a plasma [7, 8, 9, 22], in that it includes the effects of the velocity field in addition to those of the magnetic field. The inclusion of the velocity field leads to a competition between the tendency of the mean magnetic field to approach a minimum energy state and the tendency of alignment (or anti-alignment) of the field and the flow. Superimposed on the steady mean field–flow are finite amplitude local fluctuations, which are reminiscent of the so-called Alfven wave fluctuations [23]. Our model also predicts that, in statistical equilibrium, the ratio of kinetic energy to magnetic energy is less than one, regardless of the initial value of this ratio, and that the mean flow is dominated by the mean field. These predictions are particularly interesting in view of the recent result of Moffatt and Vladimirov that a necessary condition for the stability of an MHD equilibrium state is that the magnetic energy be larger than the kinetic energy [24, 25, 26].

Another new feature of our approach is that, when the spatial domain is multiply connected, we are able to obtain a meaningful continuum limit in which the energy, cross-helicity and helicity remain finite, and in which our Gibbs ensemble is equivalent to the microcanonical ensemble defined by constraints on the rugged invariants. In the above-mentioned theories, the ensemble averaged quantities diverge as the number of spectral modes is taken to infinity, and the equivalence of ensembles breaks down [27]. We take a simple approach to avoiding the ultraviolet catastrophe: we multiply the standard Gibbs Boltzmann entropy by the factor $1/N$, where N is the number of dynamical modes. This procedure, which amounts to scaling linearly with N the inverse temperatures in the standard Gibbs ensemble (i.e., the Gibbs ensemble corresponding to the standard Gibbs–Boltzmann entropy), is ultimately justified by the asymptotic equivalence of ensembles, which is established in section 6 below.

Interestingly, the process of rescaling the inverse temperatures with the number of modes *does not* yield a meaningful continuum limit if the spatial domain is simply connected. The presence of the symmetry breaking harmonic field associated with the multiple–connectedness of the domain is directly responsible for the existence of a well-defined continuum limit. When this symmetry breaking is not present (i.e., when the domain is simply connected), the ultraviolet catastrophe and the related breakdown of the equivalence of ensembles can not be overcome simply by rescaling the inverse temperatures in the Gibbs ensemble. This point will be made clear in section 5.

The paper is organized as follows. In section 2, we present the equations of 3d MHD, and list the rugged invariants of the dynamics. In section 3, the phase space is introduced, an the essential Liouville property is stated. The Gibbs ensemble is constructed and analyzed in section 4; the continuum limit is considered in section 5, and the asymptotic equivalence of the Gibbs ensemble with the microcanonical ensemble in established in section 6. We offer some concluding remarks in section 7. In Appendix 1, we provide a brief review of the mathematical analysis of the curl operator and its spectral resolution, and in Appendix 2, we analyze the eigenvalues and eigenfunctions of the curl for the special case of a periodic cylindrical domain.

2 Ideal magnetohydrodynamics

We consider an incompressible magnetofluid of constant density occupying a smoothly bounded and connected domain $\Omega \subset R^3$. Special emphasis will be placed on the case in which the domain is multiply connected. For example, Ω could be the inside of a torus, which is a geometry of particular interest in the field of fusion research (eg., the reverse–field pinch) [23]. We will assume that the magnetofluid is ideal, in the sense that the fluid viscosity and the electrical resistivity are negligible. The governing dynamical equations, expressed in Alfven speed units, are [23]

$$\frac{\partial B}{\partial t} = \nabla \times (V \times B), \tag{1}$$

$$\frac{\partial V}{\partial t} = -(V \cdot \nabla)V + (\nabla \times B) \times B - \nabla p, \tag{2}$$

$$\nabla \cdot B = 0, \qquad \nabla \cdot V = 0. \tag{3}$$

Here, B is the magnetic field, V is the fluid velocity, and p is the pressure. The pressure p is determined by B and V in response to the incompressibility constraint $\nabla \cdot V = 0$. The induced electric field is given by $\mathcal{E} = -V \times B$. This relation, together with Faraday's law, yields (1).

We assume that the boundary $\partial\Omega$ is perfectly conducting, so that the appropriate boundary conditions are

$$n \times (V \times B) = 0\,, \quad n \cdot B = 0\,, \quad n \cdot V = 0 \quad \text{on } \partial\Omega\,, \tag{4}$$

where n is the normal to the boundary.

The equations (1)-(4) possess three global quadratic (rugged) invariants [13, 28]. They are the energy

$$E = \frac{1}{2}\int_\Omega (B^2 + V^2)\, dx\,, \tag{5}$$

the cross–helicity

$$K = \int_\Omega B \cdot V\, dx\,, \tag{6}$$

and the helicity

$$H = \int_\Omega B \cdot A\, dx\,, \tag{7}$$

where A is the vector potential associated with the field B: $B = \nabla \times A$. The cross–helicity may be thought of as a measure of the correlation of the magnetic field and the velocity field [13], while the helicity is an indicator of the degree of twisting or tangling of the magnetic field lines [8, 28]. The conserved quantities (5)–(7) will play a fundamental role in the statistical theory that we develop below.

3 Phase space

We shall now introduce the phase space and the corresponding invariant measure, which are essential for our statistical treatment. The phase space that we consider here was studied by Ito and Yoshida in [20] for a general bounded three–dimensional domain, and earlier by Montgomery *et al.* [15] and Turner [21] in the case of a cylindrical domain. Let us consider the function space $L^2_\sigma(\Omega)$ of real three–dimensional solenoidal vector fields in the multiply connected domain Ω:

$$L^2_\sigma(\Omega) = \left\{ u \in L^2(\Omega) \,:\, \nabla \cdot u = 0 \text{ in } \Omega\,,\ u \cdot n = 0 \text{ on } \partial\Omega \right\}\,, \tag{8}$$

where $L^2(\Omega)$ is the Hilbert space of real square–integrable three–dimensional vector fields defined on Ω, with the inner product

$$(u, w) = \int_\Omega u \cdot w\, dx\,. \tag{9}$$

The subspace $L^2_\sigma(\Omega)$ is also a Hilbert space when endowed with this inner product. The space $L^2_H(\Omega)$ of harmonic vector fields, which represents the cohomology class, is defined by

$$L^2_H(\Omega) = \left\{ u \in L^2_\sigma(\Omega) \,:\, \nabla \times u = 0 \text{ in } \Omega \right\}\,. \tag{10}$$

It is a finite dimensional subspace of $L^2_\sigma(\Omega)$, whose dimension m is equal to the first Betti number of Ω [29]. We denote by $L^2_\Sigma(\Omega)$ the orthogonal complement of $L^2_H(\Omega)$ in $L^2_\sigma(\Omega)$; we then have the orthogonal decomposition

$$L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega)\,. \tag{11}$$

The space L^2_H is empty if and only if Ω is simply connected, in which case the spaces L^2_σ and L^2_Σ are identical. We are primarily concerned in the present paper with the case in which L^2_H has nonzero dimension. Under that circumstance, L^2_Σ is strictly smaller than L^2_σ.

We now consider the eigenvalue problem

$$\nabla \times u = \lambda u \,, \qquad u \in L^2_\Sigma(\Omega) \,. \tag{12}$$

The following result is established in [19] (see also Appendices 1 and 2 of the present paper for a further discussion of the properties of the curl operator and its spectrum).

LEMMA 1 *The eigenvalues, $\lambda_j, j = \pm 1, \pm 2, \cdots$, corresponding to (12) are real, nonzero, discrete and unbounded. The associated eigenfunctions, $\phi_j, j = \pm 1, \pm 2, \cdots$, form a complete orthogonal basis for the space $L^2_\Sigma(\Omega)$.*

We may assume that the eigenvalues are numbered in increasing order:

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots \,.$$

Note that $\lambda_j \to \pm\infty$ when $j \to \pm\infty$. Owing to Lemma 1 and the orthogonal decomposition (11), we see that any $u \in L^2_\sigma(\Omega)$ may be expressed uniquely (up to an L^2 equivalence class) as

$$u = \sum_j u_j \phi_j + \sum_{l=1}^{m} \tilde{u}_l h_l \,, \tag{13}$$

where $h_l, l = 1, \cdots, m$ is the orthogonal basis for $L^2_H(\Omega)$, and u_j and $\tilde{u}_l$ are the expansion coefficients for u: $u_j = (u, \phi_j)$ and $\tilde{u}_l = (u, h_l)$, where $(\cdot, \cdot)$ is the L^2 inner product defined by (9). The h_l may be regarded as eigenfunctions of curl corresponding to the zero eigenvalue.

Now, let $B(x,t)$ and $V(x,t)$ be the magnetic field and the velocity field of the ideal magnetofluid. Because of equations (3)–(4), and because the energy (5) is finite for all time, these fields belong to $L^2_\sigma(\Omega)$. Thus, they each have expansions of the form (13). That is,

$$B(x,t) = \sum_j b_j(t)\phi_j(x) + \sum_{l=1}^{m} \tilde{b}_l(t) h_l(x) \,, \tag{14}$$

and

$$V(x,t) = \sum_j v_j(t)\phi_j(x) + \sum_{l=1}^{m} \tilde{v}_l(t) h_l(x) \,. \tag{15}$$

The proof of the following Lemma, which is tantamount to the essential Liouville property, is provided in [20].

LEMMA 2 *The measure*

$$d\mathcal{M} = d\tilde{b}_1 \cdots d\tilde{b}_m d\tilde{v}_1 \cdots d\tilde{v}_m \prod_j db_j dv_j \,,$$

is invariant under the equations of ideal MHD.

In fact, the proof that is given in [20] demonstrates that each individual $db_j, dv_j, d\tilde{b}_l$, and $d\tilde{v}_l$ is invariant. As a result of Lemma 2, we see that the $(\tilde{b}_l, b_j, \tilde{v}_l, v_j)$ constitute an appropriate set of phase space coordinates, in the statistical mechanical sense.

In terms of these coordinates, the energy and cross-helicity may be expressed as

$$E = \frac{1}{2}\sum_j (b_j^2 + v_j^2) + \frac{1}{2}\sum_{l=1}^{m}(\tilde{b}_l^2 + \tilde{v}_l^2), \tag{16}$$

$$K = \sum_j b_j v_j + \sum_{l=1}^{m} \tilde{b}_l \tilde{v}_l . \tag{17}$$

Upon defining the function g_l by $h_l = \nabla \times g_l$, the vector potential A can be expanded as

$$A(x) = \sum_j \frac{b_j}{\lambda_j}\phi_j(x) + \sum_{l=1}^{m} \tilde{b}_l g_l(x) . \tag{18}$$

We then define the gauge-invariant relative helicity

$$H_\Sigma = \int_\Omega B_\Sigma \cdot A\, dx = \sum_j (\frac{b_j^2}{\lambda_j} + \Theta_j b_j), \tag{19}$$

where B_Σ is the projection of B onto $L^2_\Sigma(\Omega)$, and

$$\Theta_j = \sum_{l=1}^{m} \tilde{b}_l\,(g_l, \phi_j) .$$

The Θ_j are referred to as the cohomology coefficients. Note that all the Θ_j vanish when the domain Ω is simply connected.

It can be shown that the sum $(1/2)\sum_{l=1}^{m} \tilde{b}_l^2$ in (16) is actually constant, since the cohomology field $\sum_{l=1}^{m} \tilde{b}_l h_l$ is the same for any magnetic field B that satisfies the ideal MHD equations, as required by the perfectly conducting boundary conditions (see [19] for a demonstration of this fact). In effect, only the b_j, v_j and $\tilde{v}_l$ are dynamic variables. As we have indicated above, the reduced measure $d\tilde{v}_1 \cdots d\tilde{v}_m \prod_j db_j dv_j$ is also invariant. From now on, therefore, we will omit the sum $(1/2)\sum_{l=1}^{m} \tilde{b}_l^2$ from the expression (16) for the energy. Furthermore, the relative helicity H_Σ given by (19) differs from the actual helicity H defined by (7) by a constant which depends only on the cohomology magnetic field and its associated vector potential [20]. Consequently, we will consider in the sequel only the relative helicity, which, for simplicity, we will denote by H and refer to as the helicity.

4 The Gibbs Ensemble

In building a statistical theory of 3d MHD, we are immediately confronted with the fundamental difficulty that the ideal MHD system is an infinite-dimensional dynamical system. In order to construct a meaningful statistical mechanics, therefore, we begin by considering, together with the $\tilde{v}_l$, only a finite number of the modes b_j, v_j, $j = \pm 1, \cdots, \pm N/2$ for N an even positive integer. We could write down an associated truncated dynamics, but that will not be necessary for our program. The truncated energy, cross-helicity and helicity are given by expressions (16), (17) and (19), respectively, but with the summations over j now running from $-N/2$ to $N/2$. (Recall that we have dropped the term resulting from the constant cohomology magnetic field in (16), and that we are now referring to the relative helicity as the helicity.) The truncated invariant measure is $d\mathcal{M} = \prod_l d\tilde{v}_l \prod_j db_j dv_j$, where the index l ranges from 1 to m and the index j ranges from $-N/2$ to $N/2$.

In accordance with standard statistical mechanical principles [17, 30], we define the Gibbs ensemble as that probability density ρ_G on reduced phase space which maximizes the Gibbs–Boltzmann entropy functional

$$S(\rho) = -\frac{1}{N} \int \rho \log \rho \, d\mathcal{M} \,, \tag{20}$$

subject to the constraints

$$\langle E \rangle = E^0 \,, \quad \langle K \rangle = K^0 \,, \quad \langle H \rangle = H^0 \,, \tag{21}$$

where $\langle \ \rangle$ denotes expectation with respect to the ensemble ρ, and where E^0, K^0, and H^0 are given values of energy, cross-helicity and helicity, respectively. Notice that we have included the factor $1/N$ in our definition of the Gibbs-Boltzmann entropy functional. We have done so in order to overcome the well-known Jeans ultraviolet catastrophe when we pass to the continuum limit $N \to \infty$. Let us emphasize that previous statistical theories of 3d MHD all suffer from this nonphysical divergence effect [13, 16, 18]. Our simple procedure of including the factor $1/N$ in the definition of the entropy, which actually amounts to rescaling linearly with N the inverse temperatures in the usual Gibbs ensemble, leads to the convergence of the ensemble–averaged energy, cross–helicity and helicity to their prescribed finite values, provided that the domain Ω is multiply connected. By the "usual" Gibbs ensemble we mean the density that maximizes NS, with S defined by (20), subject to the constraints (21). It has been recognized already, in the case of 2d MHD, that the inverse temperatures in the Gibbs ensemble must be appropriately scaled with the number of statistical modes in order to avert the ultraviolet catastrophe upon passing to the continuum limit [27, 31]. Our analysis below shows that this is the case as well for 3d MHD in a multiply connected domain.

When the spatial domain is multiply connected, the Gibbs ensemble ρ_G, which results from maximizing the functional (20) subject to the constraints (21), becomes equivalent in the limit $N \to \infty$ to the microcanonical ensemble, which is the measure concentrated on the manifold defined by these constraints. This result is established in section 6. The asymptotic equivalence with the microcanonical ensemble is at the heart of the acceptance of the Gibbs density as a meaningful description of the statistical equilibrium state. Thus the equivalence of ensembles result provides a strong theoretical justification for our theory. In previous statistical models for MHD [13, 14, 16, 18], this asymptotic equivalence, although implicitly assumed, actually failed to be met. This would seem to render these theories logically inconsistent. In the case of 2d MHD, this defect has been removed in the recent statistical treatments of Isichenko and Gruzinov [31] and Jordan and Turkington [27]. A detailed discussion of these issues can be found in [27].

Notice that we have been careful to emphasize that the meaningful continuum limit and the equivalence of ensembles property are obtained if the domain Ω is *multiply connected*. In fact, when Ω is simply connected, the procedure of multiplying entropy by the factor $1/N$ is not sufficient to produce a well-defined continuum limit. The reason for this will become evident in section 5 below.

We now solve the variational principle $S(\rho) \to$ max subject to the constraints (21) to obtain the Gibbs density ρ_G. By the Lagrange multiplier rule for constrained optimization, we know that ρ_G satisfies

$$\delta S = \beta\delta\langle E\rangle + \alpha\delta\langle H\rangle + \mu\delta\langle K\rangle \,, \tag{22}$$

where δ denotes variation with respect to the density ρ, and β, α, and μ are the Lagrange multipliers corresponding to the constraints on energy, helicity, and cross-helicity, respectively. These multipliers are analogous to the usual inverse temperature parameters. A straightforward, but tedious, calculation yields the following expression for ρ_G:

$$\rho_G = \prod_l \tilde{\rho}_l \prod_j \rho_j \,, \tag{23}$$

with

$$\tilde{\rho}_l(\tilde{v}_l) = \sqrt{\frac{N\beta}{2\pi}} \exp\left\{-\frac{N\beta}{2}(\tilde{v}_l + \frac{\mu}{\beta}\tilde{b}_l)^2\right\} , \tag{24}$$

and

$$\rho_j(b_j, v_j) = C_j \exp\left\{-\frac{N\,((\beta^2-\mu^2)\lambda_j + 2\alpha\beta)}{2\beta\lambda_j}(b_j - \langle b_j\rangle)^2 - \frac{N\beta}{2}(v_j + \frac{\mu}{\beta}b_j)^2\right\} . \tag{25}$$

Here,

$$C_j = \frac{N}{2\pi}\sqrt{\frac{(\beta^2-\mu^2)\lambda_j + 2\alpha\beta}{\lambda_j}} \,, \tag{26}$$

is a normalization factor which enforces the condition $\int \rho_j(b_j, v_j)\, db_j dv_j = 1$. The term $\langle b_j\rangle$ in (25) is given by

$$\langle b_j\rangle = -\frac{\alpha\beta\lambda_j\Theta_j}{(\beta^2-\mu^2)\lambda_j + 2\alpha\beta} \,. \tag{27}$$

Let us remark that in order for the formulas (23)-(27) to make sense, it must be that

$$\beta > 0 \,, \quad \frac{\lambda_j}{(\beta^2-\mu^2)\lambda_j + 2\alpha\beta} > 0 \ \text{ for all } j \,. \tag{28}$$

Note that because $|\lambda_j| \to \infty$ as $|j| \to \infty$, (28) implies that if N is sufficiently large, then $\beta^2 > \mu^2$, and (28b) will hold as long as

$$\lambda_{-1} < -\frac{2\alpha\beta}{\beta^2-\mu^2} < \lambda_1 \,. \tag{29}$$

We will see shortly that condition (29) is related to a stability criterion for the mean field–flow.

While the formulas (23)-(27) for the Gibbs density seem quite complicated at first glance, the situation is not so bad. In fact, we recognize right away that $\tilde{\rho}_l(\tilde{v}_l)$ is a Gaussian density on the real line, so that it is determined entirely by its mean and variance. Also, we see that ρ_j is a joint Gaussian density in (b_j, v_j), and is therefore characterized completely by its mean vector and its covariance matrix. Let us denote by var $X = \langle (X - \langle X \rangle)^2 \rangle$ the variance of a random variable X, and by cov $(X, Y) = \langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle$ the covariance of the random pair (X, Y). Then by inspection of (24), we find that

$$\langle \tilde{v}_l \rangle = -\frac{\mu}{\beta} \tilde{b}_l , \quad \text{var } \tilde{v}_l = \frac{1}{N\beta} . \tag{30}$$

We have already given the expression (27) for $\langle b_j \rangle = \int b_j \rho_j \, db_j dv_j$. Direct, but cumbersome, calculations reveal that

$$\text{var } b_j = \frac{1}{N} \frac{\beta \lambda_j}{(\beta^2 - \mu^2)\lambda_j + 2\alpha\beta} , \tag{31}$$

$$\langle v_j \rangle = -\frac{\mu}{\beta} \langle b_j \rangle , \quad \text{var } v_j = \frac{1}{N\beta} + \frac{\mu^2}{\beta^2} \text{var } b_j , \tag{32}$$

$$\text{cov}\,(b_j, v_j) = -\frac{1}{N} \frac{\lambda_j \mu}{(\beta^2 - \mu^2)\lambda_j + 2\alpha\beta} . \tag{33}$$

Notice that all the modes $\tilde{v}_l$ are statistically independent of the (b_j, v_j), and that the variance of the $\tilde{v}_l$ is the same for each index l.

What is especially noteworthy here is the appearance of the nontrivial mean field–flow $(\langle b_j \rangle, \langle v_j \rangle), \langle \tilde{v}_l \rangle$. Indeed, the classical statistical mechanics for 3d MHD in a periodic domain, which is based upon the Gibbs–Boltzmann statistics for a truncated Fourier series approximation of the ideal MHD system, yields an ensemble with vanishing mean field–flow [13, 18]. Furthermore, for a simply connected domain, the harmonic contribution to the magnetic field vanishes, and consequently, so does the cohomology coefficient Θ_j in equation (19) for the relative helicity. As it is precisely this term that gives rise to the nontrivial mean in ρ_G, we see that our theory, if applied to 3d MHD in a simply connected domain, would lead to an ensemble for which the mean field–flow vanishes. Thus, the occurrence of the nonzero mean in the present situation is a direct consequence of symmetry breaking Θ_j associated with the multiple connectedness of the domain Ω. This nontrivial mean field $\langle b_j \rangle$ stems from the tangling of the dynamical magnetic field with the cohomological magnetic field, and the nonzero mean flow $\langle v_j \rangle, \langle \tilde{v}_l \rangle$ results from the interaction of the magnetic field and the velocity field. It is necessary to include the cross–helicity in the statistical mechanics to arrive at the prediction of a nonvanishing mean velocity field. Another very interesting conclusion, which is summarized in the following lemma, is that the $(\langle b_j \rangle, \langle v_j \rangle), \langle \tilde{v}_l \rangle$ satisfy a certain variational principle, which happens to characterize a particular steady solution of the ideal MHD equations.

LEMMA 3 *When the domain* Ω is multiply connected, we obtain a nonzero mean field–flow $(\langle b \rangle, \langle v \rangle) = (\{(\langle b_j \rangle, \langle v_j \rangle)\}_j, \{\langle \tilde{v}_l \rangle\}_l)$ *which is a critical point of the function*

$$\mathcal{F} = E - \zeta H - \xi K$$

for some multipliers ζ *and* ξ, *where* E, K, *and* H *are given by equations (16), (17) and (19), respectively (with the sums over* j *truncated at* $\pm\frac{N}{2}$*).*

Indeed, we calculate the first variation of $\mathcal{F}$ with respect to the b_j, v_j and $\tilde{v}_l$ and set it equal to zero. This procedure yields the necessary equations for a critical point, which are

$$v_j = \xi b_j\,, \quad \tilde{v}_l = \xi \tilde{b}_l\,, \quad b_j = \zeta(\frac{2b_j}{\lambda_j} + \Theta_j) + \xi v_j\,. \tag{34}$$

Upon choosing $\xi = -\mu/\beta$ and $\zeta = -\alpha/\beta$ in equations (34), we arrive at equations (27), (30a) and (32a). Note that $\langle \tilde{b}_l \rangle = \tilde{b}_l$ because the $\tilde{b}_l$ are treated as nonrandom. This establishes Lemma 3.

We now recognize that the $(\langle b_j \rangle, \langle v_j \rangle), \langle \tilde{v}_l \rangle$ satisfy the variational principle

$$\delta \left\{ \frac{1}{2} \int_\Omega (B^2 + V^2)\, dx - \zeta \int_\Omega A \cdot B\, dx - \xi \int_\Omega V \cdot B\, dx \right\} = 0\,, \tag{35}$$

over $\delta B \in W_N \equiv \text{span}\ \{\phi_j : -N/2 \le j \le N/2\}, \delta V \in W_N \oplus L^2_H$. This variational principle has solutions of the form

$$(\nabla \times B - 2\zeta B - \xi \nabla \times V\,, \varphi) = 0\,, \quad (V - \xi B\,, \varphi) = 0 \quad \forall \varphi \in W_N\,. \tag{36}$$

Let us note that B and V satisfying (36) constitute a stationary solution of the ideal MHD equations (1)-(4), in the limit $N \to \infty$. In fact, it follows from (36) that (in the limit)

$$(1 - \xi^2)\nabla \times B = 2\zeta B\,, \tag{37}$$

which implies that

$$(\nabla \times B) \times B = 0\,.$$

Here, ζ and ξ are the limiting values of the corresponding multipliers in (36). Now, any vector field V satisfies the identity

$$(V \cdot \nabla)V - \nabla(\frac{1}{2}V^2) = (\nabla \times V) \times V\,.$$

But, for V satisfying (36), there holds

$$(1 - \xi^2)\nabla \times V = 2\zeta V\,,$$

which implies that

$$(\nabla \times V) \times V = 0\,.$$

Thus, upon choosing p such that $p + (1/2)V^2$ is a constant, the right hand side of equation (2) vanishes. It is also obvious that for B and V satisfying (36), we have $V \times B = 0$, so that the right hand side of equation (1) is equal to zero, as well. We have shown that equation (36) does indeed give a steady solution of the ideal MHD equations in the limit $N \to \infty$, and, hence, that the mean $(\langle b \rangle, \langle v \rangle)$ of ρ_G corresponds to the truncation of a steady solution of the MHD equations.

Now, according to (36)-(37), we have

$$\nabla \times B = \frac{2\zeta}{1 - \xi^2} B\,, \quad V = \xi B\,.$$

Thus the magnetic field satisfies the Beltrami equation (i.e., an equation of the form $\nabla \times B = \lambda B$), and the velocity field is parallel to the magnetic field. As discussed in [32], the solution B of this Beltrami equation is stable as long as $\lambda_{-1} < 2\zeta/(1-\xi^2) < \lambda_1$. Recalling the definitions of ζ and ξ, this criterion is seen to be the limit version of the condition (29), which guarantees the existence of the Gibbs ensemble ρ_G.

A straightforward application of the Gibbs-Boltzmann statistical theory has led to the conclusion that the statistical equilibrium state of the ideal MHD system in a multiply connected domain should consist of a steady mean field-flow coupled with turbulent fluctuations about this steady state, as described by the Gibbs distribution ρ_G. These predictions are very reminiscent of those of the recent statistical mechanical theories of 2d MHD as set forth by Isichenko and Gruzinov [31, 33] and Jordan and Turkington [27, 34]. Both of these theories yield a statistical equilibrium state consisting of a mean magnetic field–velocity field pair, which is a steady solution of the ideal 2d MHD equations, together with turbulent fluctuations of the field and the flow. However, these models require special constructions, which go beyond the realm of the standard Gibbs–Boltzmann methodology, to arrive at an equilibrium state with nontrivial mean field–flow. In the former case, the authors assume the presence of a steady mean, and then build fluctuations about this state using a formal asymptotic analysis, while in the latter theory, the steady mean field–flow results from the special way in which the conserved flux and cross-helicity integrals are expressed in terms of the mean vector potential. This formulation of the constraints relies on a separation of scales hypothesis, which causes the mean to concentrate in the low wavenumbers, and the fluctuations to spread out to the high wavenumbers. We find it interesting, therefore, that no special hypotheses, and no extensions to the standard Gibbs–Boltzmann framework, are necessary here for us to obtain a nontrivial mean field–flow, provided that the spatial domain is multiply connected.

The expressions for the energy, helicity, and cross-helicity, according to the Gibbs density ρ_G are

$$E^0 = \frac{1}{2N\beta}\sum_j\left(1+\frac{(1+\xi^2)\lambda_j}{(1-\xi^2)\lambda_j-2\zeta}\right) + \frac{1}{2}\sum_j\frac{(1+\xi^2)\lambda_j^2\Theta_j^2}{((1-\xi^2)\lambda_j-2\zeta)^2} + \frac{m}{2N\beta}+\frac{1}{2}\sum_l\xi^2\tilde{b}_l^2, \tag{38}$$

$$H^0 = \frac{1}{N\beta}\sum_j\frac{1}{(1-\xi^2)\lambda_j-2\zeta}+\sum_j\frac{\zeta\lambda_j\Theta_j^2\left((1-\xi^2)\lambda_j)-\zeta\right)}{((1-\xi^2)\lambda_j-2\zeta)^2}, \tag{39}$$

$$K^0 = \frac{1}{N\beta}\sum_j\frac{\xi\lambda_j}{(1-\xi^2)\lambda_j-2\zeta}+\sum_j\frac{\xi\zeta^2\lambda_j^2\Theta_j^2}{((1-\xi^2)\lambda_j-2\zeta)^2}+\sum_l\xi\tilde{b}_l^2. \tag{40}$$

As before, $\zeta = -\alpha/\beta$ and $\xi = -\mu/\beta$, with β, α, and μ the Lagrange multipliers corresponding to the constraints on energy, helicity and cross-helicity, respectively. We see that the energy, helicity and cross-helicity are divided into mean and fluctuation parts. The second and fourth terms on the right hand side of (38) represent the contribution of the mean field–flow to the total energy. Similarly, the second sum in (39), and the second and third sums in (40) give the contributions of the mean field–flow to the helicity and cross-helicity. The difference between the given values E^0, H^0, and K^0 of these quantities and the contribution to them by the mean is taken up by the Gaussian fluctuations of the b_j, v_j and $\tilde{v}_l$. To arrive at the expressions (38)–(40), we have made use of the particular Gaussian structure of the Gibbs density.

The special case of vanishing cross-helicty (i.e.. $K^0 = 0$) is also interesting to investigate, and is easier to analyze. This amounts to setting $\mu = 0$, or equivalently, $\xi = 0$ in the preceding analysis. The Gibbs density reduces in this case to

$$\rho_G = \prod_l \tilde{\rho}_l \prod_j \rho_j \, , \tag{41}$$

where

$$\tilde{\rho}_l(\tilde{v}_l) = \sqrt{\frac{N\beta}{2\pi}} \exp\{-\frac{N\beta}{2}\tilde{v}_l^2\} \, , \tag{42}$$

and

$$\rho_j(b_j, v_j) = \frac{N\beta}{2\pi}\sqrt{\frac{\lambda_j - 2\zeta}{\lambda_j}} \exp\left\{-\frac{N\beta}{2}\left(\frac{\lambda_j - 2\zeta}{\lambda_j}\right)(b_j - \langle b_j\rangle)^2 - \frac{N\beta}{2}v_j^2\right\} . \tag{43}$$

Here,

$$\langle b_j\rangle = \frac{\zeta\lambda_j\Theta_j}{\lambda_j - 2\zeta} \, , \tag{44}$$

and $\zeta = -\alpha/\beta$, where α is the helicity multiplier and β is the energy multiplier. We find that the mean velocity vanishes: $\langle v_j\rangle = 0, \langle\tilde{v}_l\rangle = 0$, and that

$$\text{var } b_j = \frac{\lambda_j}{N\beta(\lambda_j - 2\zeta)} \, , \quad \text{var } v_j = \text{var } \tilde{v}_l = \frac{1}{N\beta} \, , \quad \text{cov } (b_j, v_j) = 0 \, . \tag{45}$$

Thus, b_j and v_j are statistically independent when $K^0 = 0$. Note that we must have $\beta > 0$, and $\lambda_j/(\lambda_j - 2\zeta) > 0$ for all j in order for ρ_G to be well-defined. This condition will be met as long as $\lambda_{-1} < 2\zeta < \lambda_1$.

The expressions for the energy and the helicity now become

$$E^0 = \frac{1}{2N\beta}\sum_j\left(1 + \frac{\lambda_j}{\lambda_j - 2\zeta}\right) + \frac{1}{2}\sum_j\frac{\zeta^2\Theta_j^2\lambda_j^2}{(\lambda_j - 2\zeta)^2} + \frac{m}{2N\beta} \, , \tag{46}$$

$$H^0 = \frac{1}{N\beta}\sum_j\frac{1}{\lambda_j - 2\zeta} + \sum_j\frac{\zeta\lambda_j\Theta_j^2(\lambda_j - \zeta)}{(\lambda_j - 2\zeta)^2} \, . \tag{47}$$

As in the case of nonzero cross-helicity, the conserved quantities are divided among the mean and the fluctuations, with the second sums in each of (46) and (47) representing the contribution from the mean, and the remaining terms in these expressions representing the contribution from the fluctuations. Furthermore, the mean magnetic field $\{\langle b_j\rangle\}_j$ is easily seen to be a critical point of the functional $E_{mag} - \zeta H$, with $E_{mag} = (1/2)\sum_j b_j^2$ the (deterministic) magnetic energy, and $H = \sum_j((b_j^2/\lambda_j) + b_j\Theta_j)$ the (deterministic) relative helicity. Therefore, the $\langle b_j\rangle$ correspond to the truncation of a solution to the Beltrami equation $\nabla \times B = 2\zeta B$. It has been argued by several authors that the Beltrami equation, which implies the force-free condition $(\nabla \times B) \times B = 0$, should be satisfied by the relaxed magnetic field in a plasma [1, 7, 8, 9, 22]. It has been conjectured by Taylor [8, 9], Hasegawa [1], and others that such a force-free final state results from a process of selected dissipation of the magnetic energy with respect to the helicity. In our statistical model, it is the mean magnetic field that satisfies the force-free condition. While the total energy must remain constant under the ideal dynamics, the statistical theory points to a kind of *effective dissipation.* The energy of the mean can, in fact, decay to some lower level. The "lost" energy is transfered to the local fluctuations. This sort of effective dissipation has also been predicted in statistical models of 2d MHD [27, 31, 33, 34] and 2d hydrodynamics [35, 36, 37, 38].

5 The continuum limit

We now wish to examine the limit $N \to \infty$, paying special attention to what becomes of the contributions of the fluctuations and the mean field-flow to the conserved quantities. For the remainder of the text, we will assume, unless otherwise stated, that the domain Ω is multiply connected (so that, in particular, the cohomology coefficients Θ_j are not identically zero). We begin with the case $K^0 = 0$ of vanishing cross-helicity. The expressions (46) and (47) for the energy and the helicity depend on the parameters β and ζ, which must be determined such that the entropy $S(\rho_G)$ is maximum among all densities ρ that satisfy the constraints (21). To solve for β and ζ would seem to require a numerical method, which we will not pursue here. It is clear that β and ζ depend on N. ζ will approach a finite limit as $N \to \infty$, as required by equations (46)-(47) and the condition $\lambda_{-1} < 2\zeta < \lambda_1$. The requirement that the energy and the helicity remain equal to E^0 and H^0, respectively, guarantees that β stays bounded away from 0 and that 2ζ stays bounded away from λ_1 and λ_{-1} as $N \to \infty$. We will assume, for now, that β approaches a finite limit β^* as $N \to \infty$. It will become clear from the ensuing analysis that this is indeed the case, except in the event that the total energy E^0 is exactly equal to the limiting value $\bar{E}$ of the mean-field energy. Thus, in retrospect, we could just prescribe E^0 to be larger than $\bar{E}$ to guarantee that β remains finite.

To proceed, we need some results concerning the asymptotic behavior of the eigenvalues λ_j and the cohomology coefficients Θ_j as $|j| \to \infty$. In Appendix 2, we study the asymptotic distribution of the eigenvalues and the decay properties of the cohomology coefficients for the special case in which the domain Ω is a periodic cylinder (so that Ω is topologically equivalent to a torus). There, we find that for large $|j|$, $\lambda_j \sim c_1 j + c_2$ for some constants c_1 and c_2, and that $\Theta_j \sim |j|^{-1}$. Intuitively, we expect that the asymptotic properties of Θ_j and λ_j should not be highly sensitive to the particular geometry of the domain Ω, but we have not been able to produce such estimates for an arbitrary domain. However, we show in Appendix 1 that

$$\text{the infinite series } \sum_j \Theta_j^2 \text{ converges.} \tag{48}$$

This result and the fact (stated in section 3 and proved in Appendix 1) that

$$\lim_{j \to \infty} |\lambda_j| = \infty \,, \tag{49}$$

are sufficient to guarantee the validity of our arguments in this section and the next section.

For the convenience of the reader, we state two elementary results from the theory of infinite series to which we will refer repeatedly throughout the subsequent analysis. The proofs of these results can be found, for example, in [39].

LEMMA 4 *Let $a_k, k = 1, 2, \cdots$, be a sequence of real numbers such that $\lim_{k\to\infty} a_k = a$ for some real number a; then $\lim_{N\to\infty} N^{-1}\sum_{k=1}^{N} a_k = a$.*

LEMMA 5 Assume that the series $\sum_{k=1}^{\infty} a_k$ *is absolutely convergent. If the sequence $b_k, k = 1, 2, \cdots$, converges to a finite limit, then the series $\sum_{k=1}^{\infty} b_k a_k$ is also absolutely convergent.*

We shall now establish that, in the limit $N \to \infty$, the mean field and the fluctuations both make nonzero contributions to the total energy of the system. We rewrite the first expression on the right hand side of (46) as

$$\frac{1}{2N\beta}\sum_j \left(1 + \frac{\lambda_j}{\lambda_j - 2\zeta}\right) = \frac{1}{2\beta} + \frac{1}{2N\beta}\sum_j \frac{\lambda_j}{\lambda_j - 2\zeta} \tag{50}$$

The first term on the right hand side of (50), which represents the contribution to the energy from the fluctuations of the v_j, converges to $1/(2\beta^*)$ when $N \to \infty$, where β^* is the limiting value of β. (Recall that in this analysis we are assuming that β^* is finite. Later in this section, we will discuss the special case $\beta^* = \infty$.) The second expression on the right hand side of (50) is the energy arising from the fluctuations of the b_j. Now, by (49) and the fact that ζ stays bounded away from $\lambda_{\pm 1}$ as $N \to \infty$, we can find sequences a_j and b_j (independent of N) such that $a_j \le \lambda_j/(\lambda_j - 2\zeta) \le b_j$ for all j, N, and such that $\lim_{|j|\to\infty} a_j = \lim_{|j|\to\infty} b_j = 1$. Appealing to Lemma 4, we deduce that each of the expressions $(2N\beta)^{-1}\sum_{1\le|j|\le N/2} a_j$ and $(2N\beta)^{-1}\sum_{1\le|j|\le N/2} b_j$ converges to $1/(2\beta^*)$ when $N \to \infty$. As $\lambda_j/(\lambda_j - 2\zeta)$ is sandwiched between a_j and b_j, we deduce that the term representing the contribution to the energy from the the fluctuations of the b_j converges to $1/(2\beta^*)$, as well. Notice, however, that the energy from the fluctuations of cohomological velocity, which is given by $\sum_l \text{var}\ \tilde{v}_l = m/(2N\beta)$, disappears when $N \to \infty$. Hence, the energy arising from the fluctuations is divided evenly into kinetic and magnetic components in the continuum limit, with each component contributing the amount $1/(2\beta^*)$. The remaining term on the right hand side of (46), i.e.,

$$\frac{1}{2}\sum_j \frac{\zeta^2\Theta_j^2\lambda_j^2}{(\lambda_j - 2\zeta)^2},$$

represents the contribution of the mean field to the energy. Now, because Θ_j satisfies (48), and because $(\lambda_j/(\lambda_j - 2\zeta))^2$ can be bounded independently of N by a sequence that converges to finite values as $|j| \to \infty$, we conclude from Lemma 5 that this sum will converge to a nonzero, positive value as $N \to \infty$. In other words, the mean field contributes a positive amount to the total energy of the system in the continuum limit.

The situation for the helicity is different. The contribution of the fluctuations to the helicity, which is given by the first term on the right hand side of equation (47), vanishes in the continuum limit. This is established readily from (49), the fact that ζ remains bounded away from $\lambda_{\pm 1}$, and Lemma 4. Next, by an argument analogous to the one that was used to demonstrate that the mean field component of the energy converges to a finite value, we may show that the second term on the right hand side of (47), which is the helicity due to the mean field, approaches a finite limit as $N \to \infty$. Evidently, it must converge to H^0. The important conclusion is that the helicity is determined entirely by the mean magnetic field in the continuum limit. The preceding analysis points to a cascade of energy to small spatial scales, where the fluctuations reside, and a cascade of helicity to large spatial scales, where the mean field resides.

We remark that the limiting expression for the mean field helicity is

$$\bar{H}(\zeta^*) = \sum_{|j|\geq 1} \frac{\zeta^* \lambda_j \Theta_j^2 (\lambda_j - \zeta^*)}{(\lambda_j - 2\zeta^*)^2},$$

where $\lambda_{-1} < \zeta^* < \lambda_1$ and ζ^* is the limiting value of ζ, when $N \to \infty$. It can be shown that for any H^0, there is a unique $\zeta^* \in (\lambda_{-1}, \lambda_1)$ such that $\bar{H}(\zeta^*) = H^0$ [32]. The mean field energy in the limit is given by

$$\bar{E}(\zeta^*) = \frac{1}{2} \sum_{|j|\geq 1} \frac{(\zeta^*)^2 \Theta_j^2 \lambda_j^2}{(\lambda_j - 2\zeta^*)^2}.$$

Hence, the limiting mean field energy $\bar{E}$ is uniquely determined by the given value of H^0. In addition, since $E^0 = 1/\beta^* + \bar{E}$ in the continuum limit, we see that the total energy E^0 is required to be at least as large as $\bar{E}$ for there to be a well-defined continuum limit. It follows that $\beta^* = \infty$ if and only if $E^0 = \bar{E}$, and in that case the energy is given entirely by the mean field in the limit. Otherwise, as demonstrated above, both the fluctuations and the mean field contribute to the energy in the limit.

We may also show that the fluctuations of the vector potential A vanish as $N \to \infty$. Indeed, since the expansion coefficients of the vector potential are related to those of the magnetic field via the equation $a_j = b_j/\lambda_j$, we have

$$\begin{aligned} \sum_j \text{var } a_j &= \sum_j \frac{\text{var } b_j}{\lambda_j^2} \\ &= \frac{1}{N\beta} \sum_j \frac{1}{\lambda_j(\lambda_j - 2\zeta)} \\ &\to 0 \text{ as } N \to \infty. \end{aligned}$$

To obtain the second line of the calculation, we have used equation (45a), which gives the expression for the variance of b_j. The third line of this display follows from an argument similar to the one that was used to show that the contribution of the fluctuations to the helicity vanishes in the limit. That the fluctuations of the vector potential should disappear, while those of the magnetic field survive makes sense intuitively; A is obtained by "integrating" the magnetic field B, and integration tends to smooth out the fluctuations.

The model predicts, in addition, that in statistical equilibrium, the total magnetic energy is larger than the total kinetic energy, regardless of the initial ratio of these quantities. The fluctuation magnetic and kinetic energy are equal, but the mean magnetic field contributes a positive amount to the energy, while the energy of the mean velocity field is identically zero. This result is suggestive of the so-called dynamo effect [40], whereby kinetic energy is transferred to magnetic energy as the result of the generation of a large-scale magnetic field. The relaxation of the ratio of kinetic to magnetic energy to a value less than 1 is also a feature of 2d MHD, where it is observed in direct numerical simulations of 2d magnetofluid turbulence [3], and predicted by statistical equilibrium models [27, 34].

The analysis and conclusions for the case $K^0 \neq 0$ are not much different than for the case of zero cross-helicity, so we will not present them in detail. We shall merely list the most important conclusions. As in the case of zero cross-helicity the contribution of the fluctuations to the helicity vanishes in the continuum limit, while there is a nonzero contribution of the fluctuations to the energy in the limit. In addition, both the contribution of the mean field-flow and the contribution of the fluctuations to the cross-helicity survive in the continuum limit. Recall, that there is a nonzero mean velocity field present when the cross-helicity is different from zero. It is still possible to show that the fluctuations of the vector potential will die out when $N \to \infty$, and that in this limit, there is an equipartition of the fluctuation magnetic and kinetic energy, and the ratio of kinetic to magnetic energy is always less than 1. The latter result is true because $\langle v_j \rangle = \xi \langle b_j \rangle, \langle \tilde{v}_l \rangle = \xi \bar{b}_l$, and $|\xi| < 1$. That the kinetic energy is less than the magnetic energy in statistical equilibrium may be tied to a recent MHD equilibrium stability criterion of Moffatt and Vladimirov [24, 25].

We close this section with a cursory account of the difficulties that arise when the spatial domain Ω is simply connected. We concentrate on the case $K^0 = 0$ for simplicity. The cohomology coefficients Θ_j are all equal to 0 when the domain is simply connected, and, as a result, the mean field vanishes. The expressions (46)-(47) for the energy and helicity reduce to

$$E^0 = \frac{1}{2\beta} + \frac{1}{2N\beta} \sum_j \frac{\lambda_j}{\lambda_j - 2\zeta}, \quad H^0 = \frac{1}{N\beta} \sum_j \frac{1}{\lambda_j - 2\zeta}.$$

Assuming that E^0 is larger than the minimum value of energy allowed by the helicity constraint, the only way that both of the sums will simultaneously converge as $N \to \infty$ to their prescribed (nonzero) finite values is if β remains finite and 2ζ approaches either λ_1 or λ_{-1}, depending on the sign of H^0, in such such a way that $N(\lambda_{\pm 1} - 2\zeta) = O(1)$. Thus, according to our canonical ensemble, the energy splits into a finite part that resides in fluctuations at one of the lowest modes, and a part that is divided among the other modes. But the fact that the fluctuations at a lowest mode persist when N increases implies that the canonical ensemble can not be equivalent with the microcanonical ensemble in the limit. In addition, the prediction of a zero mean state in the canonical ensemble is inconsistent with the microcanonical ensemble. The latter produces, when E^0 is close to the minimum energy allowed by the constraint on helicity, a nonzero mean state with small fluctuations in each mode. The upshot is that, when the spatial domain is simply connected, we can not obtain, within the present framework, a meaningful continuum limit in which the crucial equivalence of ensembles property holds.

6 Equivalence with the microcanonical ensemble

In order to justify our statistical model, and in particular to substantiate the factor $1/N$ in our expression (20) for the Gibbs-Boltzmann entropy, we establish the asymptotic equivalence of the Gibbs density ρ_G with the microcanonical ensemble, which is the measure concentrated on the manifold defined by the dynamical constraints on energy, helicity and cross-helicity. There are two basic hypotheses underlying this analysis. One hypothesis is that the energy, cross-helicity and helicity, as defined by (5)-(7), represent a complete set of relevant additive dynamical invariants, in the sense that they serve to characterize fully the equilibrium state. The other hypothesis is that the dynamics is mixing, or ergodic, on the manifold defined by these invariants. Not only is the assumption of ergodicity reasonable, but it is also necessary in any statistical mechanical theory of hydrodynamic or magnetohydrodynamic flow. The assumption concerning the relevant set of invariants, however, is more subtle, as it is well known that ideal MHD actually conserves infinitely many integrals [8, 9]. However, the importance of these additional invariants is debatable, as we shall discuss in the next section. The previous statistical theories mentioned above [13, 16, 15, 21, 18] also included only the quadratic invariants. However, until the issue concerning the relevance of the other invariants is settled, we would like to consider the restriction to only the rugged invariants (5)-(7) as a convenient first approximation. The accuracy of this approximation may be tested by comparing the predictions of the model with numerical and experimental observations.

Let us now demonstrate the asserted asymptotic compatibility with the microcanonical ensemble. Specifically, we will show that

$$\text{var } E_N \to 0\,, \quad \text{var } K_N \to 0\,, \quad \text{var } H_N \to 0\,, \quad \text{as } N \to \infty\,. \tag{51}$$

The expressions for E_N, K_N and H_N, which, being functions of the b_j, v_j and $\tilde{v}_l$, may be thought of as random variables, are given by equations (16), (17) and (19), respectively, with the index j ranging from $-N/2$ to $N/2$. Since $\langle E_N\rangle = E^0, \langle K_N\rangle = K^0$ and $\langle H_N\rangle = H^0$, the equations (51) clearly imply the desired asymptotic equivalence of ρ_G with the microcanonical ensemble.

To keep the analysis simple, we will demonstrate the equivalence only for the special case $K^0 = 0$, but the result may be established for general K^0 with a little extra effort. In this argument, we will use at several the mutual independence of the b_j, v_j and $\tilde{v}_l$. Notice that b_j and v_j are statistically independent only when $K^0 = 0$. Thus, a slightly refined argument is needed for the case $K^0 \neq 0$.

Using (16), we calculate var E_N as follows:

$$\text{var } E_N = \frac{1}{4}\sum_j(\text{var }(b_j^2) + \text{var }(v_j^2)) + \frac{1}{4}\sum_l \text{var }(\tilde{v}_l^2)\,.$$

Now, we use (44), (45) and the fact the mean flow vanishes to obtain, after some algebraic manipulations, the identity

$$\text{var } E_N = \frac{1}{2N^2\beta^2}\sum_j \frac{\lambda_j^2}{(\lambda_j - 2\zeta)^2} + \frac{\zeta^2}{N\beta}\sum_j \frac{\Theta_j^2\lambda_j^3}{(\lambda_j - 2\zeta)^3} + \frac{N+m}{2N^2\beta^2}\,. \tag{52}$$

Next, we calculate var K_N using (17). This gives

$$\begin{aligned}\text{var } K_N &= \sum_j \text{var }(b_j v_j) + \sum_l \tilde{b}_l^2 \text{var } \tilde{v}_l \\ &= \sum_j \langle b_j^2\rangle\langle v_j^2\rangle + \sum_l \tilde{b}_l^2 \text{var } \tilde{v}_l\,.\end{aligned}$$

¿From this result and equations (45) and (44), we arrive at

$$\text{var } K_N = \frac{\zeta^2}{N\beta}\sum_j \frac{\lambda_j^2\Theta_j^2}{(\lambda_j - 2\zeta)^2} + \frac{1}{N^2\beta^2}\sum_j \frac{\lambda_j}{\lambda_j - 2\zeta} + \frac{1}{N\beta}\sum_l \bar{b}_l^2 . \tag{53}$$

Finally, referring to (19), we find that

$$\text{var } H_N = \sum_j \frac{\text{var } b_j^2}{\lambda_j^2} + \sum_j \Theta_j^2 \text{var } b_j .$$

This result, together with (44) and (45), leads to

$$\text{var } H_N = \frac{2}{N^2\beta^2}\sum_j \frac{1}{(\lambda_j - 2\zeta)^2} + \frac{4\zeta}{N\beta}\sum_j \frac{\Theta_j}{(\lambda_j - 2\zeta)^2} + \frac{1}{N\beta}\sum_j \frac{\Theta_j^2\lambda_j}{\lambda_j - 2\zeta} . \tag{54}$$

Now, to establish the equivalence of ensembles property (51), we must show that each of the terms in the equations (52), (53), and (54) vanishes in the limit $N \to \infty$. This is readily established using (48)-(49), Lemmas 4 and 5, and arguments similar to those that were constructed in section 5 to analyze continuum limit. Since the analysis is completely straightforward and almost identical to that of the previous section, we will not present it here. Let us simply point out that a simple analysis reveals the asymptotic estimates var $E_N = O(1/N)$ and var $K_N = O(1/N)$. In addition, the third term in the expression for var H_N can be shown to be $O(1/N)$. However, as we do not have, in general, an estimate of the rate at which the eigenvalues λ_j diverge as $|j| \to \infty$, we do not know the rate at which the first two terms on the right hand side of (54) decay. If the sum of $(\lambda_j - 2\zeta)^{-2}$ converges, as it does when Ω is a periodic cylinder, then we know that these expressions approach 0 at least as quickly as N^{-2}.

7 Concluding Remarks

We have investigated the Gibbs–Boltzmann statistical mechanics of an ideal three-dimensional magnetofluid in a multiply connected domain. The particular phase space that we work with is associated with the eigenfunctions of the curl operator. The use of this phase space is justified by a Liouville property, which implies that the volume of the expansion coefficients is temporally invariant. There are two particularly interesting aspects of our theory. One is the prediction of a nontrivial mean field–flow which is a steady solution of the ideal MHD equations. This steady state, which represents a macroscopic coherent structure that is expected to emerge during the evolution of the magnetofluid, has no counterpart in the classical Gibbs–Boltzmann statistical theory for 3d MHD in a periodic domain [13, 18]. As explained above, it is precisely the multiple connectedness of the domain that gives rise to the nontrivial steady mean. The other especially noteworthy result of our model is the prediction that the fluctuations of the field and the flow survive in the continuum limit, while those of the vector potential disappear in the limit. As a result, the energy and the cross-helicity are divided into mean and fluctuation parts, while the helicity is determined entirely by the mean field. Similar conclusions have been reached in recent statistical theories of ideal MHD turbulence in two dimensions [27, 31, 33, 34]. There, the magnetic field and velocity field exhibit finite amplitude turbulent fluctuations in the continuum limit, while the fluctuations of the magnetic vector potential vanish.

An essential assumption in our theory is that the three so-called rugged invariants serve to completely characterize the coarse-grained state of the magnetofluid, while, in fact, ideal 3d MHD has infinitely many conserved integrals [8, 9, 22]. In particular, if U is any volume bounded by magnetic field lines, then the integral

$$H_U = \int_U B \cdot A \, dx \,, \tag{55}$$

is temporally invariant. The conservation of the integrals (55) for any such volume U reflects the fact that the field lines must maintain their topological properties under the ideal dynamics [28]. For example, if two closed field lines are linked n times initially, then they must remain so for all time. It has been argued by Taylor, however, that if there is *any* departure from infinite conductivity, as there surely must be for any real plasma, then these topological constraints are broken: field lines may break and coalesce [8]. In this case, it would be unreasonable to expect that $\int B \cdot A$ should be conserved for each line of force. On the other hand, changes in field topology should not result in significant changes in the field itself; so if the conductivity is sufficiently large, the total helicity $\int_\Omega B \cdot A \, dx$, which is the sum of $\int B \cdot A$ over all field lines, remains a good invariant. Hence, if we are interested in the long-time behavior of a highly, but not infinitely, conducting plasma, then there is some justification for incorporating the helicity (7) into the statistical mechanics, while ignoring all the other integrals of the form (55).

Acknowledgments

The authors wish to thank the referees for many helpful comments and suggestions, which led to substantial improvements in the manuscript. The research of R. J. was partially supported through a National Science Foundation Postdoctoral Research Fellowship and a University of Michigan Rackham Faculty Fellowship.

Appendix 1

In this appendix, we give a brief review of the function spaces of vector fields associated with the curl operator and its spectral resolution. We also provide proofs of (48) and (49), which are used repeatedly in sections 5 and 6. For a more detailed analysis of the curl operator, the reader is referred to [19] and references therein.

Let $\Omega \subset R^3$ be a bounded domain with a smooth boundary $\partial\Omega = \cup_{i=1}^{n} \Gamma_i$ (Γ_i is a connected surface). We consider cuts of the domain Ω. Let $\Sigma_1, \cdots, \Sigma_m$ ($m \geq 0$) be cuts such that $\Sigma_i \cap \Sigma_j = \emptyset$ ($i \neq j$), and such that $\Omega \setminus (\cup_{i=1}^{m} \Sigma_i)$ becomes a simply connected domain. The number m of such cuts is called the first Betti number of Ω. When $m > 0$, the domain Ω is multiply connected, and then for any vector field u we can define the flux through each cut by

$$\Phi_{\Sigma_i}(u) = \int_{\Sigma_i} n \cdot u ds \qquad (i = 1, 2, \cdots, m),$$

where n is the unit normal vector onto Σ_i with an appropriate orientation. By Gauss's formula, $\Phi_{\Sigma_i}(u)$ is independent of the location of the cut Σ_i, provided that $\nabla \cdot u = 0$ in Ω and $n \cdot u = 0$ on $\partial\Omega$.

We denote $L^2(\Omega)$ the Hilbert space of square-integrable (complex) vector fields in Ω, which is endowed with the standard inner product $(a, b) = \int_\Omega a \cdot \overline{b}\, dx$. We define the following subspaces of $L^2(\Omega)$;

$$\begin{aligned} L^2_\Sigma(\Omega) &= \{w;\ \nabla \cdot w = 0 \text{ in } \Omega,\ n \cdot w = 0 \text{ on } \partial\Omega,\ \Phi_{\Sigma_i}(w) = 0\ (i = 1, \cdots, m)\}, \\ L^2_H(\Omega) &= \{h;\ \nabla \cdot h = 0, \nabla \times h = 0 \text{ in } \Omega,\ n \cdot h = 0 \text{ on } \partial\Omega\}, \\ L^2_G(\Omega) &= \{\nabla\phi;\ \Delta\phi = 0 \text{ in } \Omega\}, \\ L^2_F(\Omega) &= \{\nabla\phi;\ \phi = c_i\ (\in \mathbf{C}) \text{ on } \Gamma_i\ (i = 1, \cdots, n)\} \end{aligned}$$

The following orthogonal decomposition holds [19, 41]

$$L^2(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega) \oplus L^2_G(\Omega) \oplus L^2_F(\Omega).$$

The space of solenoidal vector fields with vanishing normal component on $\partial\Omega$ is defined by

$$L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega).$$

The subspace $L^2_H(\Omega)$ corresponds to the cohomology class, whose member is a harmonic vector field and $\dim L^2_H(\Omega) = m$ (the first Betti number of Ω). When Ω is simply connected, we have $m = 0$ and $L^2_H(\Omega) = \{0\}$. In a multiply connected domain ($m > 0$), the harmonic field equations

$$\begin{aligned} \nabla \cdot h &= 0, \quad \nabla \times h = 0 \quad \text{in } \Omega, \\ n \cdot h &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

have nontrivial solutions which have nonvanishing fluxes. Physically, these harmonic fields represent the vacuum magnetic field rooted outside Ω. Under the perfectly conducting boundary condition, every flux Φ_{Σ_i} is conserved, and hence every h_ℓ ($\in L^2_H(\Omega)$ is conserved.

Another decomposition of $L^2(\Omega)$ can be given:

$$L^2(\Omega) = L^2_\sigma(\Omega) \oplus \{\nabla\phi;\ \phi \in H^1(\Omega)\},$$

This relation, which is known as the Weyl decomposition, implies that $L^2_\sigma(\Omega)$ is the orthogonal complement of the space of potential flows. The gauge-invariance of the relative helicity H_Σ defined by (19) follows from this orthogonality.

The following theorem and lemmas concerning spectral resolution of the curl operator lead to Lemma 1 of Sec. 3 and the results (48)-(49) stated in section 5. The theorem is proved in [19].

THEOREM *Let $\Omega \subset R^3$ be a smoothly bounded domain. Define a curl operator $\mathcal{S}$ on the Hilbert space $L^2_\Sigma(\Omega)$ by*

$$\mathcal{S}u = \nabla \times u, \quad D(\mathcal{S}) = \{u \in L^2_\Sigma(\Omega)\ ;\ \nabla \times u \in L^2_\Sigma(\Omega)\},$$

where $D(\mathcal{S})$ is the domain of the operator $\mathcal{S}$. Then $\mathcal{S}$ is a self-adjoint operator. The spectrum of $\mathcal{S}$ consists of only point spectra $\sigma_p(\mathcal{S})$, which is a discrete set of nonzero real numbers.

The space $L^2_\sigma(\Omega)$ of solenoidal vector fields is spanned by the eigenfunctions of $\mathcal{S}$ together with $h_\ell \in L^2_H(\Omega)$ [19, 42].

For an eigenfunction ϕ_j of $\mathcal{S}$, its vector potential, in the Coulomb gauge, is given by ϕ_j/λ_j, where λ_j is the corresponding eigenvalue. We arrange the eigenvalues in increasing order:

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots .$$

The vector potential of the harmonic field $h_\ell \in L^2_H(\Omega)$ can also be found in the space $L^2_\Sigma(\Omega)$ (see Proposition 1 of Ref. [19]). For $g_\ell \in L^2_\Sigma(\Omega)$ such that $\nabla \times g_\ell = h_\ell$, the inner product (g_ℓ, ϕ_j) yields the cohomology coefficient Θ_j defined in Sec. 3.

LEMMA A1.1 *The eigenvalues of the operator $\mathcal{S}$ satisfy*

$$\lim_{|j|\to\infty} |\lambda_j| = \infty.$$

PROOF: The self-adjoint operator $\mathcal{S}$ has a compact inverse $\mathcal{S}^{-1}$. By the Hilbert-Schmidt theorem [43], the spectrum of $\mathcal{S}^{-1}$,

$$\{\cdots, \lambda_{-2}^{-1}, \lambda_{-1}^{-1}, \lambda_1^{-1}, \lambda_2^{-1}, \cdots\},$$

can accumulate only to zero. Hence, $\lim_{|j|\to\infty} |\lambda_j| = \infty$.

LEMMA A1.2 *The cohomology coefficients Θ_j are square summable. That is, the infinite series $\sum_j \Theta_j^2$ converges.*

PROOF: Let $A_h \in L^2_\Sigma(\Omega)$ be the vector potential of the harmonic component of the magnetic field $B_h = \sum_{l=1}^m \tilde{b}_l(t) h_l(x)$, i.e., $\nabla \times A_h = B_h$ (with notation as introduced in Sec. 3). Since the domain Ω is bounded, we have a Poincaré type estimate

$$\|A_h\|_{L^2} \leq C\|\nabla \times A_h\|_{L^2} = C\|B_h\|_{L^2}, \tag{56}$$

where C is a positive constant depending only on the domain Ω. Using the expansion of A_h in terms of the eigenfunctions ϕ_j of the self-adjoint curl operator $\mathcal{S}$, we observe that

$$\|A_h\|^2_{L^2} = \sum_j |(A_h, \phi_j)|^2 = \sum_j \Theta_j^2. \tag{57}$$

Combining (56) and (57), we obtain

$$\sum_j \Theta_j^2 \leq (C\|B_h\|)^2,$$

which establishes the lemma.

Appendix 2

For purpose of illustration, we consider now an explicit example of the eigenfunction expansion, and study the asymptotic behavior of Θ_j. Let the domain Ω be a periodic cylindrical domain of unit radius. The eigenfunctions of the curl operator are given by the "Chandrasekhar-Kendall functions" [44]: in the (r, φ, z) cylindrical coordinates,

$$u = \lambda(\nabla\psi \times \nabla z) + \nabla \times (\nabla\psi \times \nabla z), \tag{58}$$

where

$$\lambda = \pm(\mu^2 + k^2)^{1/2}, \quad \psi = J_p(\mu r)e^{i(m\varphi - kz)}, \tag{59}$$

and J_p is the p^{th} order Bessel function. Here z is normalized by $L/2\pi$ (L is the length of the periodic cylinder). We easily find that u is an eigenfunction of the curl operator corresponding to the eigenvalue $\lambda \in R$. The eigenvalue is determined by the requirement that the normal component of u should vanish on the surface of the cylindrical domain. This condition becomes trivial when $k = p = 0$. Such eigenfunctions are written explicitly as

$$\alpha \begin{pmatrix} 0 \\ J_1(\mu r) \\ J_0(\mu r) \end{pmatrix},$$

where α is the normalization factor. For these axisymmetric functions, we apply the zero-flux condition $\Phi_\Sigma = 0$ (Σ is the cross-section of the cylinder), which reads

$$2\pi \int J_0(\mu r)r\, dr = 2\pi\mu^{-1}J_1(\mu) = 0.$$

Hence, the eigenvalue $\lambda = \mu$ corresponding to the axisymmetric eigenfunction is given by the zero point of $J_1(x)$.

The harmonic field in the cylindrical domain is ∇z, and its vector potential is given by $g = (r^2/2)\nabla\varphi$. Since g is an axisymmetric function, only axisymmetric eigenfunctions contribute to the cohomology coefficients Θ_j. We obtain

$$\Theta_j = \pi\alpha_j \int_0^1 J_1(\mu_j r)r^2\, dr = \pi\alpha_j\mu_j^{-1}J_2(\mu_j),$$

where α_j and μ_j are, respectively, the normalization factor and the eigenvalue of the j-th axisymmetric eigenfunction. For large μ_j, which is the j-th zero-point of $J_1(x)$, we use Hankel's asymptotic expression,

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\cos(x - \pi(2\nu + 1)/4) \quad (|x| \gg 1),$$

to obtain the asymptotic estimate $\mu_j \approx \pi(j + 5/4)$. From this we find that $\alpha_j \sim \sqrt{\mu_j}$ for large j, and hence, $\Theta_j \sim |j|^{-1}$ for large $|j|$.

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