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CURRENT-VORTEX FILAMENTS IN A MAGNETIZED PLASMA

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These are lecture notes, intended for distribution to participants.

Current-vortex filaments

in a magnetized plasma

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- Introduction
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- Hamiltonian structure
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- Vortex collapse
- Distributed vortices
- Drift vortices (Hasegawa-Mima)
- Conclusions

1 Introduction

• The 2D Euler equation for an ordinary fluid has solutions in the form of point vortices. This means that a continuous fluid is approximated by a discrete system. The important point is that the point-vortex system has the same constants of the motion as the original one.

Do plasma equations have solutions that are similar to point vortices in Euler's equation?

• A two-fluid plasma model is used to analyse the existence of current-vorticity filaments ('point-vortices') in magnetized plasmas.

The model can be viewed as R(educed) MHD extended with a generalized Ohm's law.

The set of equations involves spatial scale lengths which extend from the global MHD scale-lengths down to the electron inertia skin depth.

• The dynamical equations can be written in Lagrangian form for three fields that are advected with different velocities.

This system is a generalization and extension of Euler's equation for an ordinary fluid.

$$\frac{\partial G_\alpha}{\partial \tau} + \mathbf{v}_\alpha \cdot \nabla G_\alpha = 0, \qquad \mathbf{v}_\alpha = \mathbf{e}_z \times \nabla \Phi_\alpha, \qquad G_\alpha = G_\alpha(\Phi_\alpha).$$

• The equations are Hamiltonian with non-canonical Poisson brackets, the energy functional is the Hamiltonian.

The system possesses three infinite sets of invariants (Casimirs) that arise from the structure of the equations.

Topological invariants, related to the plasma motion and magnetic field, constrain the plasma dynamics and provide an analytical tool that may prove itself powerful in decrypting the complexity of the plasma nonlinear behaviour.

• The equations describe

- current-vorticity filaments,
- stationary propagating, distributed drift-Alfvén vortices and magnetic islands,
- collisionless magnetic reconnection where magnetic flux is converted into electron momentum and ion vorticity.

Derivation of the two-fluid model

Electron Equations

The electron momentum balance and continuity equation:

$$m_e n\Big(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\Big) = -en\Big(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\Big) - \nabla nT - \nabla\cdot\overleftrightarrow{\Pi}$$

and

$$\frac{\partial n}{\partial t} + \nabla\cdot n\vec{v} = 0.$$

Neglecting perpendicular inertia (and resistivity), one obtains from the momentum balance the velocity in the (x, y)-plane,

$$\mathbf{v}_\perp = \frac{c}{B_0}\mathbf{e}_z\times\nabla\phi - \frac{c}{eB_0}\mathbf{e}_z\times\frac{\nabla nT}{n} + \frac{v_z}{B_0}\nabla\mathcal{A}\times\mathbf{e}_z.$$

The leading term is the $\mathbf{E}\times\mathbf{B}$-drift.

Substitute $\mathbf{v}_\perp$. The contribution from the stress tensor $\overleftrightarrow{\Pi}$ to the parallel momentum balance

$$(\nabla\cdot\overleftrightarrow{\Pi})_z = -\frac{cm_e}{eB_0}\nabla nT\times\mathbf{e}_z\cdot\nabla v_z$$

cancels the pressure gradient contribution to $\mathbf{v}_\perp\cdot\nabla v_z$ in the inertia term.

Density: $\quad n(\vec{x},t) = n_0(x)[1+\tilde{n}(\vec{x},t)].$

$n_0(x)$ is the density of the background plasma. $\tilde{n}$ represents the density fluctuations. For the plasma motions under consideration $\tilde{n}$ remains small but

$$\nabla\tilde{n} \sim \nabla n_0(\mathbf{x}).$$

Ion Response

Nonlinear ion response to motions with characteristic velocities along the magnetic field that are larger than the ion thermal velocity. The derivation is analogous, but

- neglect ion pressure
- neglect parallel ion motion
- take into account perpendicular ion inertia.

The ion response in this cold ion approximation ($T_i \ll T_e$):

$$\frac{\partial}{\partial t}(\ln\frac{n}{n_0} - {\rho_s}^2\nabla_\perp^2\Phi) + \frac{cT_e}{eB_0}[\Phi,\ \ln\frac{n}{n_0} - {\rho_s}^2\nabla_\perp^2\Phi] = 0,$$

with $\Phi = e\phi/T_e$ and ${\rho_s}^2 = T_e/m_i\omega_i^2$.

2 Generalized two-fluid model

Low-β plasma with electric and magnetic fields

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathcal{A}}{\partial t}\mathbf{e}_z, \qquad \mathbf{B} = B_0\mathbf{e}_z + \nabla\mathcal{A}\times\mathbf{e}_z,$$

with $\phi = \phi_0(\mathbf{x}) + \tilde{\phi}(\mathbf{x},t)$ and $\mathcal{A} = \mathcal{A}_0(\mathbf{x}) + \tilde{\mathcal{A}}(\mathbf{x},t)$.

Electron fluid: parallel component of the momentum balance and continuity equation

$$m_e n\frac{\mathrm{d}v_z}{\mathrm{d}t} = -enE_\parallel - \nabla_\parallel p_e, \qquad \frac{\mathrm{d}n}{\mathrm{d}t} + \nabla_\parallel n v_z = 0.$$

Time derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{B_0}\mathbf{e}_z\cdot\nabla\phi\times\nabla,$$

derivative along the magnetic field:

$$\nabla_\parallel = \partial/\partial z - \mathbf{e}_z\cdot\nabla\mathcal{A}\times\nabla.$$

The parallel electron momentum is transported with the $E\times B$ velocity which is due to cancellation of the gyro-viscosity stress tensor with the contribution of the pressure gradient drift to the inertia $\mathbf{v}_\perp\cdot\nabla$ term.

The parallel ion velocity is much smaller than the electron velocity

$$v_z \approx -\frac{J_z}{en_0} = \frac{c}{4\pi e n_0}\nabla_\perp^2 A,$$

Electron equations:

$$\frac{\partial}{\partial \tau}(A - \lambda_e^2\nabla_\perp^2 A) \;+\; [\Phi - \ln\frac{n}{n_0},\; A - \lambda_e^2\nabla_\perp^2 A] \;-\; [\ln\frac{n}{n_0},\; \lambda_e^2\nabla_\perp^2 A] = -\frac{\partial}{\partial z}(\Phi - \ln\frac{n}{n_0}),$$

$$\frac{\partial}{\partial \tau}\ln\frac{n}{n_0} + [\Phi,\; \ln\frac{n}{n_0}] - [A,\; \nabla_\perp^2 A] = -\frac{\partial}{\partial z}\nabla_\perp^2 A.$$

with

$$\lambda_e = \frac{d_e}{\rho_s},$$

- $d_e = c/\omega_{pe}$ electron inertial skin depth,
- $\rho_s = \sqrt{T_e/m_i\omega_i^2}$ ion gyroradius at the electron temperature.

brackets: $[f,\; g] = \mathbf{e}_z \cdot \nabla f \times \nabla g.$

Normalized variables

$$\Phi = \frac{e\phi}{T_e},\; A = (\frac{ec_A}{cT_e})\mathcal{A},\; \tau = \omega_i t,\; (\frac{x}{\rho_s}, \frac{y}{\rho_s}) \rightarrow (x, y),\; \frac{\omega_i}{c_A}z \rightarrow z.$$

Quasi-neutrality condition $\ln n/n_0 = \ln n_i/n_0$.
The ion response in the cold ion approximation $(T_i \ll T_e)$:

$$\frac{\partial G_I}{\partial \tau} + [\Phi_I,\ G_I] = 0,$$

where

$$G_I \equiv \nabla_\perp^2 \Phi - \ln n/n_0, \qquad \Phi_I \equiv \Phi.$$

- Equations describe phenomena with frequencies below the ion cyclotron frequency and the magnetosonic frequency, and take into account finite electron pressure, finite parallel electron inertia and drift effects related with the density gradient.

The spatial scales may range from MHD lengths to the electron inertia skin depth.

- The reduced MHD model is recovered in the limits

$$\lambda_e \to 0, \qquad \nabla^2 \ll 1, \qquad G_I = 0.$$

Consider solutions that propagate with a constant velocity u_z; all functions depend on

$$\tau - \lambda z,\ x,\ y, \qquad \lambda \equiv \frac{c_A}{u_z}.$$

The electron and ion equations can be written in Lagrangian conservation form

$$\frac{\partial G_\alpha}{\partial \tau} + [\Phi_\alpha,\ G_\alpha] = 0, \qquad \alpha = +,\ -,\ I$$

Three fields

$$\pm 2\lambda_e G_\pm \equiv -A + \lambda\Phi + (\pm\lambda_e - \lambda)(\ln\frac{n}{n_0} \pm \lambda_e \nabla_\perp^2 A),$$

$$G_I \equiv \nabla_\perp^2 \Phi - \ln n/n_0,$$

three 'velocity' potentials

$$\Phi_\pm \equiv \frac{\lambda_e \Phi \mp A}{\lambda_e \mp \lambda}, \qquad \Phi_I \equiv \Phi.$$

- All equations are in Lagrangian form and can be viewed as a generalized 2D Euler system for three fields G_α that are advected with 'velocity' fields described by potentials Φ_α.

- The scalars G_α are pointwise conserved and frozen into the velocity fields $\mathbf{e}_z \times \nabla\Phi_\alpha$.

This structure implies that any function of G_α is conserved, i.e. the system has three distinct sets of conserved quantities (Casimirs)

$$\int d^2x \ F_\alpha(G_\alpha) = \text{const}$$

where the integration is over any domain that moves with the appropriate velocity field.
This elegant structure of the equations is lost in the limit of reduced MHD.

- Either cold electrons $\quad \lambda_e/\lambda = u_z/v_{the} > 1$

  or isothermal electrons $\quad \lambda_e/\lambda < 1$

  $\lambda \approx \lambda_e \quad \rightarrow\rightarrow$ electron Landau damping.

- The strictly 2D case: $\quad \lambda \rightarrow 0$, i.e. $u_z \rightarrow \infty$.

- In the limit $\lambda \rightarrow \infty$ one obtains the Boltzmann distribution $\ln n/n_0 = \Phi$. Upon substituting this distribution into the ion equation, one obtains the Hasegawa-Mima equation.

3 Hamiltonian formulation

Energy integral

$$W = \frac{1}{2}\int_D d^2x \left\{ |\nabla_\perp A|^2 + \lambda_e^2 |\nabla_\perp^2 A|^2 + \ln^2\frac{n}{n_0} + |\nabla_\perp \Phi|^2 \right.$$
$$\left. -2\lambda(\ln\frac{n}{n_0} - \Phi)\nabla_\perp^2 A \right\}.$$

- magnetic energy,
- kinetic energy of the parallel motion of the electrons,
- internal energy of the electrons,
- kinetic energy of the $E \times B$ motion of the ions,
- time integrated divergence of the z-components of the Poynting vector and of the electron thermal energy flux.

In terms of the fields G_α and the potentials Φ_α:

$$W = \frac{1}{2}\int_D d^2x \left[-\sum_\alpha \Phi_\alpha G_\alpha + \frac{2\lambda_e}{\lambda_e - \lambda} G_+^2 + \frac{2\lambda_e}{\lambda_e + \lambda} G_-^2 \right].$$

Integrals of the quadratic terms $G_\pm^2$ are Casimirs and can be subtracted.

The conserved quantity

$$H = -\frac{1}{2}\sum_{\alpha}\int_D d^2x\ G_\alpha \Phi_\alpha.$$

is the Hamiltonian functional.

Functional (Fréchet) derivatives: $\delta H/\delta G_\alpha = -\Phi_\alpha$.

If one defines the Poisson bracket as

$$\{f, g\} = \sum_{\alpha}\int d^2x\ G_\alpha[\frac{\delta f}{\delta G_\alpha},\ \frac{\delta g}{\delta G_\alpha}]$$

then the dynamical equations take the form

$$\frac{\partial G_\alpha}{\partial t} = \{G_\alpha,\ H\}.$$

4 Current-Vortex filaments

2D Euler

$$\frac{\partial \omega}{\partial t} + [\psi,\ \omega] = 0, \quad \omega = \nabla^2 \psi.$$

Point-vortex solutions ($\mathbf{r} = (x, y)$):

$$\omega = \nabla^2 \psi = \sum_i \kappa_i \delta(\mathbf{r} - \mathbf{r}_i),$$

the filament positions evolve according to

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{e}_z \times \nabla \psi|_{\mathbf{r}_i}.$$

Homogeneous plasma: the equilibrium values of Φ, A and $\ln n/n_0$ are constant.

$$\frac{\partial G_\alpha}{\partial \tau} + [\Phi_\alpha,\ G_\alpha] = 0, \qquad \alpha = +,\ -,\ I$$

Solutions in the form of singular current-vortex filaments:

$$G_\alpha = \sum_i \kappa_{\alpha,i} \delta(\mathbf{r} - \mathbf{r}_{\alpha,i}).$$

Combine with the definitions of the G_α's:

$$\nabla^2(\Phi - \lambda A) = \sum_{\alpha,i} \kappa_{\alpha,i}\delta(\mathbf{r} - \mathbf{r}_{\alpha,i}),$$

$$\nabla^2(-A + \lambda\Phi) - \gamma^2(-A + \lambda\Phi) = -\sum_{\alpha,i} C_\alpha \kappa_{\alpha,i}\delta(\mathbf{r} - \mathbf{r}_{\alpha,i}),$$

where

$$C_\pm = \frac{1 \mp \lambda\lambda_e}{\pm\lambda_e - \lambda}, \quad C_I = -\lambda, \quad \gamma^2 = \frac{1 - \lambda^2}{\lambda_e^2 - \lambda^2}.$$

- Localized solutions require

  $\gamma^2 \geq 0$, i.e. λ to $\lambda^2 \geq \max(1, \lambda_e^2)$ or $\lambda^2 \leq \min(1, \lambda_e^2)$.

  The limit $\lambda \longrightarrow \pm 1$ corresponds to propagation at the Alfvén velocity, $u_z = \pm c_A$. In this limit $\gamma \longrightarrow 1$

- The potentials $\Phi - A/\lambda$ and $A - \Phi/\lambda$ are the electric and vector potential in the moving frame, respectively.

Solutions:

$$\Phi - \lambda A = \sum_{\alpha,i} \frac{\kappa_{\alpha,i}}{2\pi} \ln|\mathbf{r} - \mathbf{r}_{\alpha,i}|, \qquad -A + \lambda\Phi = \sum_{\alpha,i} C_\alpha \frac{\kappa_{\alpha,i}}{2\pi} K_0(\gamma|\mathbf{r} - \mathbf{r}_{\alpha,i}|),$$

velocity potentials:

$$\Phi_\beta(\mathbf{r}) = \frac{1}{2\pi(1-\lambda^2)} \sum_{\alpha,i} \kappa_{\alpha,i} \Big[\ln|\mathbf{r} - \mathbf{r}_{\alpha,i}| + C_\alpha C_\beta K_0(\gamma|\mathbf{r} - \mathbf{r}_{\alpha,i}|)\Big].$$

• *This generalizes the well-known hydrodynamical point-vortex model to magnetized plasmas.*
In contrast to hydrodynamics we have three types of filaments, each moves in a distinct velocity field:

- κ_I-filaments have singularities in the vorticity and the density.

- $\kappa_\pm$-filaments have singularities in the current and the density.

• All filaments contribute to the potentials which consist of a logarithmic part similar to hydrodynamic point-vortices, and a part that decreases exponentially with the distance from the filament.

The system of vortex filaments is a solution if and only if the filament positions evolve according to

$$\frac{d\mathbf{r}_{\alpha,i}}{d\tau} = \mathbf{e}_z \times \nabla\Phi_{\alpha,i}|_{\mathbf{r}_{\alpha,i}}$$

Like hydrodynamical vortices, the equations of motion of current-vortex filaments define a Hamiltonian dynamical system:

$$\frac{d\mathbf{r}_{\alpha,i}}{d\tau} = \{\mathbf{r}_{\alpha,i},\ \hat{H}\}$$

with the usual definition of the Poisson bracket

$$\{f, g\} = \sum_{\beta,j} \frac{1}{\kappa_{\beta,j}} \left(\frac{\partial f}{\partial x_{\beta,j}} \frac{\partial g}{\partial y_{\beta,j}} - \frac{\partial f}{\partial y_{\beta,j}} \frac{\partial g}{\partial x_{\beta,j}} \right).$$

The Hamiltonian is

$$\begin{aligned} \hat{H} &= -\frac{1}{2} \sum_{\alpha,i} \kappa_{\alpha,i} \Phi_\alpha(\mathbf{r}_{\alpha,i}) \\ &= -\frac{1}{4\pi} \frac{1}{1-\lambda^2} \sum_{\alpha,i} \sum_{\beta,j} \kappa_{\alpha,i} \kappa_{\beta,j} \left[\ln |\mathbf{r}_{\beta,j} - \mathbf{r}_{\alpha,i}| + C_\alpha C_\beta K_0(\gamma |\mathbf{r}_{\beta,j} - \mathbf{r}_{\alpha,i}|) \right] \end{aligned}$$

with $(\alpha, i) \neq (\beta, j)$ in order to exclude self-interactions.

$\hat{H}$ depends only on the distance between filaments: the system has 3 additional constants of the motion:

$$P = \sum_{\alpha,i} \kappa_{\alpha,i} x_{\alpha,i}, \quad Q = \sum_{\alpha,i} \kappa_{\alpha,i} y_{\alpha,i}, \quad I = \frac{1}{2} \sum_{\alpha,i} \kappa_{\alpha,i} (x_{\alpha,i}^2 + y_{\alpha,i}^2).$$

These invariants are not in involution:

$$\{P, Q\} = \sum_{\alpha,i} \kappa_{\alpha,i}, \quad \{P, I\} = Q, \quad \{Q, I\} = -P.$$

These equations imply: $\{P^2 + Q^2, I\} = 0,$

so that the system has three invariants in involution:

$$\hat{H}, \ I, \ P^2 + Q^2$$

- *the motion of three vortices is integrable* for any combination of vortex strengths and positions.

- $$\Sigma_{\alpha,i} \kappa_{\alpha,i} = 0, \quad P = 0, \quad Q = 0.$$

the independent integrals P, Q and I are in involution:

the four-filament problem is integrable.

5 Examples of Filament Systems

Filament pair

The interaction between two filaments:

$$\frac{d}{d\tau}(\mathbf{r}_1 - \mathbf{r}_2) = (\kappa_1 + \kappa_2)L_{1;2}\mathbf{e}_z \times (\mathbf{r}_1 - \mathbf{r}_2),$$

$L_{1,2}$ depends on $\mid \mathbf{r}_1 - \mathbf{r}_2 \mid$

$$L_{1;2} = \frac{1}{2\pi(1-\lambda^2)} \frac{1 - C_\alpha C_\beta \gamma |\mathbf{r}_1 - \mathbf{r}_2| K_1(\gamma |\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}.$$

Hence, the distance between the filaments remains constant.

Constant of the motion: $\kappa_1 \mathbf{r}_1 + \kappa_2 \mathbf{r}_2 = constant.$

The filaments rotate around this center with frequency

$$\omega = (\kappa_1 + \kappa_2)\ L_{1;2}.$$

"Neutral" pair: $\kappa_1 = -\kappa_2 \equiv \kappa \longrightarrow\longrightarrow \quad d(\mathbf{r}_1 - \mathbf{r}_2)/d\tau = 0.$
"neutral" pair moves in the direction perpendicular to the line that connects the filament positions, with velocity

$$v = \kappa L_{1;2}\mathbf{e}_z \times (\mathbf{r}_1 - \mathbf{r}_2).$$

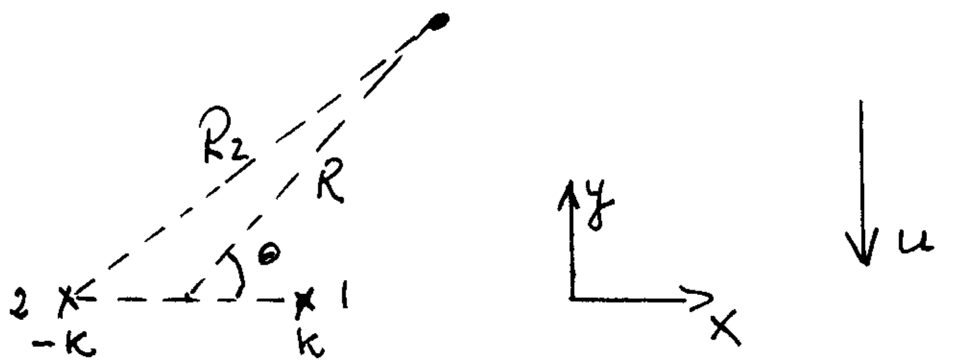
"Neutral" pair of filaments of the same type α:

$$x_1 = x_{10},\ y_1 = y_0 + u\tau; \qquad x_2 = x_{20}, y_2 = y_0 + u\tau,$$

$$u = -\frac{\kappa}{2\pi(1-\lambda^2)l}\left[1 - C_\alpha^2\gamma \mid l \mid K_1(\gamma \mid l \mid)\right], \quad l \equiv x_{10} - x_{20}$$

- Far field of this filament pair ($\mid l \mid /R \ll 1$):

$$\Phi - \lambda A = -\frac{\kappa}{2\pi}\frac{l}{R}\cos\theta, \quad -A + \lambda\Phi = \frac{\kappa}{2\pi}C_\alpha l \cos\theta\gamma K_1(\gamma R),$$



is equal to that of a distributed Alfvén vortex outside its circular separatrix $[\Phi_\alpha - ux]_{R=a} = 0$, that propagates with the velocity u along the y-axis.

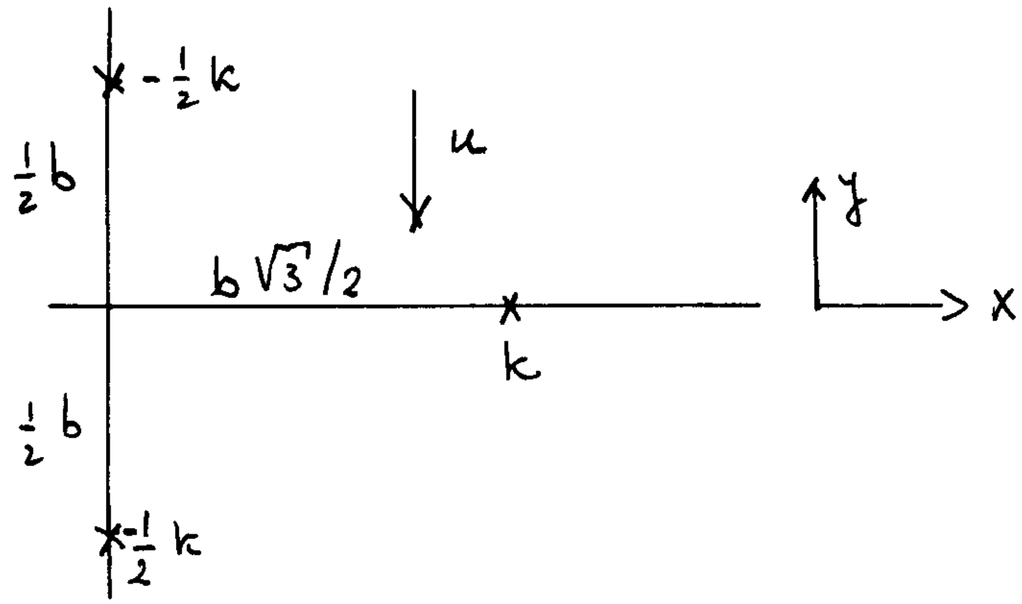
- Weak collisions between Alfvén vortices can be simulated by the interaction of filament pairs.

Three filaments

The interaction of three filaments is integrable, but extremely complicated.

Example: 3 filaments of the same type α located in the vertices of an equilateral triangle

$$(0,\ \frac{b}{2}),\ (0,\ -\frac{b}{2}),\ (b\frac{\sqrt{3}}{2},\ 0), \qquad \kappa_1 = \kappa_2 = -\frac{\kappa}{2},\ \kappa_3 = +\kappa.$$



This behaves as a "neutral" filament pair with $\pm\kappa$ at a mutual distance $l = b\sqrt{3}/2$.

This system moves in the y-direction with velocity

$$u = -\frac{\kappa\sqrt{3}}{4\pi(1-\lambda^2)b}\left[1 - C_\alpha^2 \gamma b K_1(\gamma b)\right].$$

Filament chains

Consider N filaments that all move in the y-direction with the same, constant velocity u.

$$x_{\alpha,i} = \overline{x}_{\alpha,i}, \qquad y_{\alpha,i} = \overline{y}_{\alpha,i} + u\tau.$$

The conservation of Q and I require

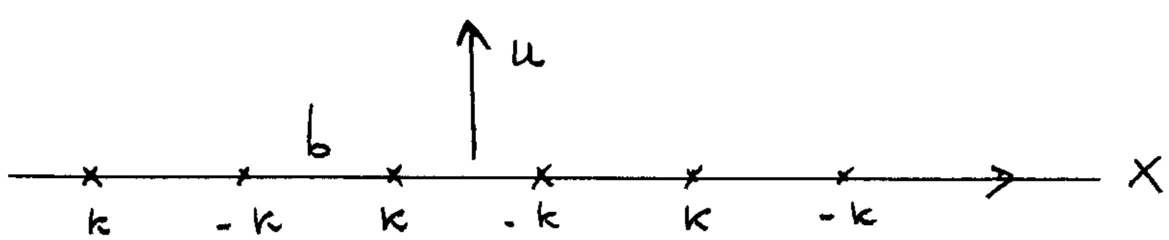
$$\sum_{\alpha,i} \kappa_{\alpha,i} = 0, \quad \sum_{\alpha,i} \kappa_{\alpha,i}\overline{y}_{\alpha,i} = 0.$$

$2N$ equations of motion:

$$0 = \sum_{\beta,j} \kappa_{\beta,j} L_{\alpha,i;\beta,j}(\overline{y}_{\alpha,i} - \overline{y}_{\beta,j}), \qquad u = \sum_{\beta,j} \kappa_{\beta,j} L_{\alpha,i;\beta,j}(\overline{x}_{\alpha,i} - \overline{x}_{\beta,j}).$$

$N \times N$ matrix $\{L_{\alpha,i;\beta,j}(\overline{x}_{\alpha,i} - \overline{x}_{\beta,j})\}$ is skew-symmetric: its determinant vanishes for odd N. Hence, N *is even.*

The first set of equations can be satisfied by placing the N filaments at equal distances along the x-axis, $\overline{y}_{\alpha,i} = \overline{y}_{\beta,j}$. This configuration forms a filament chain consisting of a single row



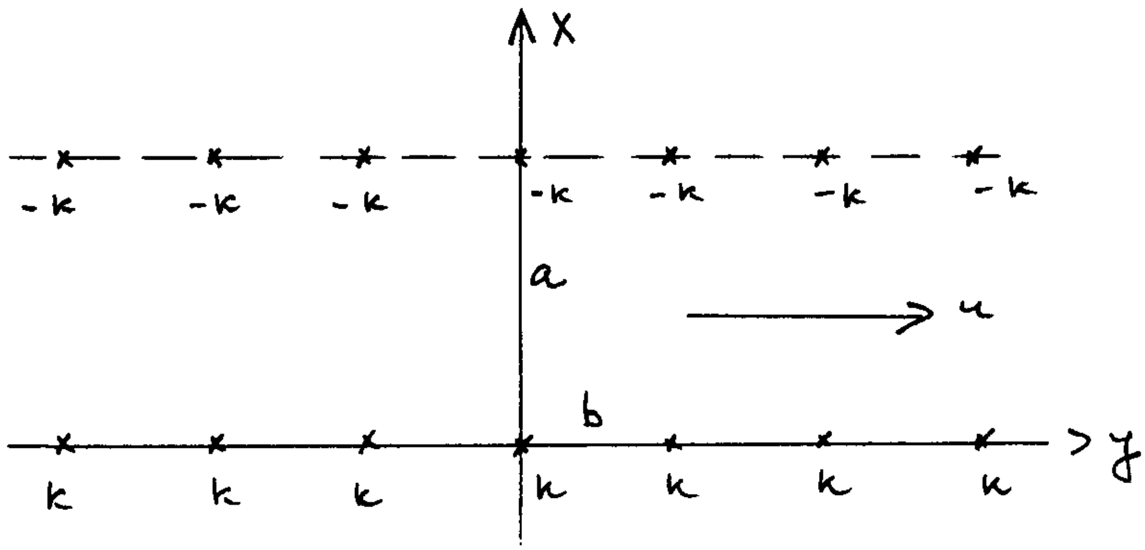
Solutions in the form of *double-row filament chains* that are similar to the von Kármán vortex streets.

Infinite number of filaments of a single type α.

Symmetrical double-row chain:

$\kappa_{0,n} = -\kappa_{1,n} = \kappa, \; n = 0, \pm 1, \pm 2, \ldots; \; m = 0, 1$

in the points $(x, y) = (am, bn)$.

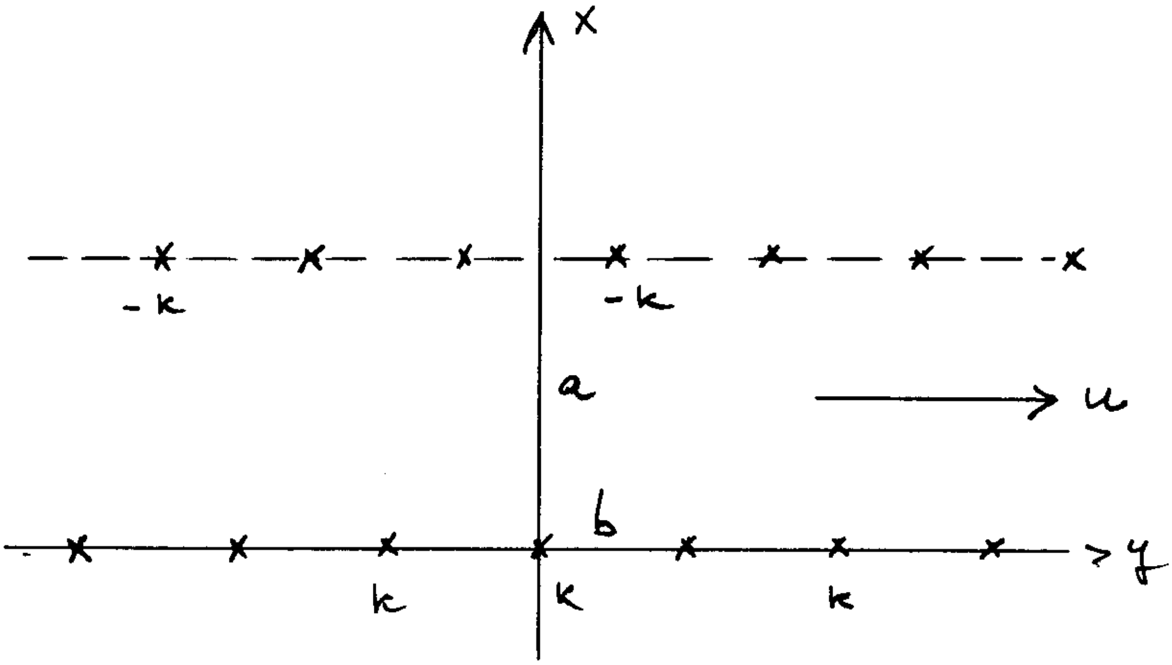


The configuration moves in the y-direction with the velocity

$$u_{sym} = \frac{\kappa a}{2\pi(1-\lambda^2)} \sum_{n=-\infty}^{\infty} \frac{1}{a^2 + b^2 n^2} \cdot \left[1 - C_\alpha^2 \gamma \sqrt{a^2 + b^2 n^2} K_1(\gamma \sqrt{a^2 + b^2 n^2})\right].$$

Staggered double chain

$$(x,\, y) = [am,\, b(n + m/2)], \qquad \kappa_{0,n} = -\kappa_{1,n} = \kappa.$$



This configuration moves along the y-axis with velocity

$$u_{st} = \frac{\kappa a}{\pi(1-\lambda^2)} \sum_{n=0}^{\infty} \frac{1}{a^2 + b^2(1/2+n)^2} \cdot \left[1 - C_\alpha^2 \gamma \sqrt{a^2 + b^2(1/2+n)^2} K_1(\gamma \sqrt{a^2 + b^2(1/2+n)^2})\right] (1.)$$

6 Vortex collapse

Three or more point vortices in ideal fluids can merge to produce a single vortex.

This vortex collapse, indicates that under some initial conditions the uniqueness of Euler's equation is lost.

A similar phenomena occurs in plasmas.

At scales below the ion-sound gyroradius or for

$\gamma \rightarrow 0$, i.e., $\lambda \rightarrow \pm 1$ propagation with the Alfvén velocity,

$$K_0(\gamma|\mathbf{r}_{\beta,j} - \mathbf{r}_{\alpha,i}|) \rightarrow -\ln|\mathbf{r}_{\beta,j} - \mathbf{r}_{\alpha,i}|.$$

Hamiltonian:

$$\hat{H} = -\frac{1}{4\pi}\frac{1}{1-\lambda^2}\sum_{\alpha,i}\sum_{\beta,j}\kappa_{\alpha,i}\kappa_{\beta,j}(1 - C_\alpha C_\beta)\ln|\mathbf{r}_{\beta,j} - \mathbf{r}_{\alpha.i}|,$$

The resulting equations of motion are scale-invariant under

$$\tau \rightarrow a^2\tau \qquad\qquad \mathbf{r} \rightarrow a\mathbf{r}.$$

Self-similar solutions

$$r_{\alpha,j}(\tau) = \sqrt{1 - \tau/\tau_*}\, r_{\alpha,j}(0), \quad \phi_{\alpha,j}(\tau) = \phi_{\alpha,j}(0) - \Omega\tau_* \ln(1-\tau/\tau_*).$$

Here, $r_{\alpha,j}$ and $\phi_{\alpha,j}$ are the polar coordinates of the filaments.

$t = 0$ indicates the initial time, Ω and t_* are constants.

All filaments merge at $r = 0$, the collapse time being τ_*.

Necessary conditions for a collapse

$$P = 0, \qquad Q = 0, \qquad I = 0,$$

while the conservation of the Hamiltonian requires

$$\sum_{\alpha,i}\sum_{\beta,j} \kappa_{\alpha,i}\kappa_{\beta,j} \frac{1 - C_\alpha C_\beta}{1 - \lambda^2} = 0, \qquad (\alpha, i) \neq (\beta, j).$$

If the sum of intensities for each type of filament $\kappa_\alpha = \sum_i \kappa_{\alpha,i}$ vanishes then the collapse results in the total annihilation of vortices.

The annihilation condition can not be satisfied by a system of filaments of a single type. For this reason annihilation is impossible in hydrodynamics. In contrast to fluids, the conservation laws in plasmas seem to allow the annihilation of vortex filaments.

Propagating, distributed vortices

Inhomogeneous plasma; localized solutions

$$\ln n/n_0 \to \ln n_0(x)/n_0 = -x/\lambda_n, \qquad A \to 0, \qquad \Phi \to 0,$$

$\lambda_n = l_n/\rho_s$ is the normalized density scale length.

Look for localized structures that propagate in the y direction with constant velocity u_y. Such structures are described by the relations

$$[G_\alpha, \ \Phi_\alpha - x/\lambda_y] = 0, \qquad \lambda_y = c_s/u_y, \ \ c_s = (T_e/m_i)^{1/2}.$$

General solutions

$$G_\alpha = F_\alpha(\Phi_\alpha - x/\lambda_y).$$

The F_α's are arbitrary functions of their arguments. They need not to be the same over all space, but may change across separatrices. which are curves $\Phi_\alpha - x/\lambda_y = constant$ that connect to the singular points where

$$\nabla_\perp(\Phi_\alpha - x/\lambda_y) = 0.$$

Boundary conditions at $|x| \to \infty$ require that the F_α's are linear in their arguments. Take these functions to be linear everywhere, so that

$$G_\alpha = F'_\alpha(\Phi_\alpha - x/\lambda_y).$$

The proportionality constants F'_α cannot be the same over all space and must be allowed to change at the separatrices.

Equations

$$\nabla^2 \begin{pmatrix} \Phi - \lambda_z A \\ -A + \lambda_z \Phi \end{pmatrix} = \begin{pmatrix} \frac{\sum F'_\alpha}{1-\lambda_z^2} & \frac{\sum F'_\alpha c_\alpha}{1-\lambda_z^2} \\ -\frac{\sum F'_\alpha c_\alpha}{1-\lambda_z^2} & \gamma^2 - \frac{\sum F'_\alpha c_\alpha^2}{1-\lambda_z^2} \end{pmatrix} \begin{pmatrix} \Phi - \lambda_z A - (1-\lambda_z^2)\frac{x}{\lambda_y} \\ -A + \lambda_z \Phi \end{pmatrix}.$$

In the outer region:

$$F'_0 = -\lambda_y/\lambda_n, \quad F'_\pm = (\lambda_y/\lambda_n)(\lambda_e \mp \lambda_z)/2\lambda_e.$$

Thus all elements of the matrix vanish except for the 22-element which is equal to the eigenvalue in the external region

$$k_e^2 = (1 - \frac{\lambda_y}{\lambda_n})\frac{1-\lambda_z^2}{\lambda_e^2 - \lambda_z^2}.$$

Localized solutions can exist if $k_e^2 > 0$. Note that $\lambda_y/\lambda_n = u_*/u_y$ with $u_* = cT_e/eB_0 l_n$ being the diamagnetic velocity.

The boundary condition at large $|x|$ implies that the solutions have the form of dipole vortices

$$\Phi_\alpha = \Phi_\alpha(r) \cos\theta$$

with r, θ being polar coordinates.

Suppose that there are two X-points and a single separatrix circle at $r = r_0$. At this circle one of the quantities $\Phi_\alpha - x/\lambda_y$ vanishes.

The eigen values $k_{1,2}$ of the matrix in the inner region are determined by matching $\Phi - \lambda_z A$ and $-A + \lambda_z \Phi$ and their first derivatives at the separatrix circle:

$$(k_1^2 - k_2^2)\hat{K}(k_e) + (k_e^2 - k_2^2)\hat{\mathcal{B}}(k_1) - (k_e^2 - k_1^2)\hat{\mathcal{B}}(k_2) = 0,$$

where

$$\hat{K}(k) = kK_2(kr_0)/K_1(kr_0), \quad \hat{\mathcal{B}}(k) = \mathrm{sgn}(k^2)|k|\mathcal{B}_2(kr_0)/\mathcal{B}_1(kr_0),$$

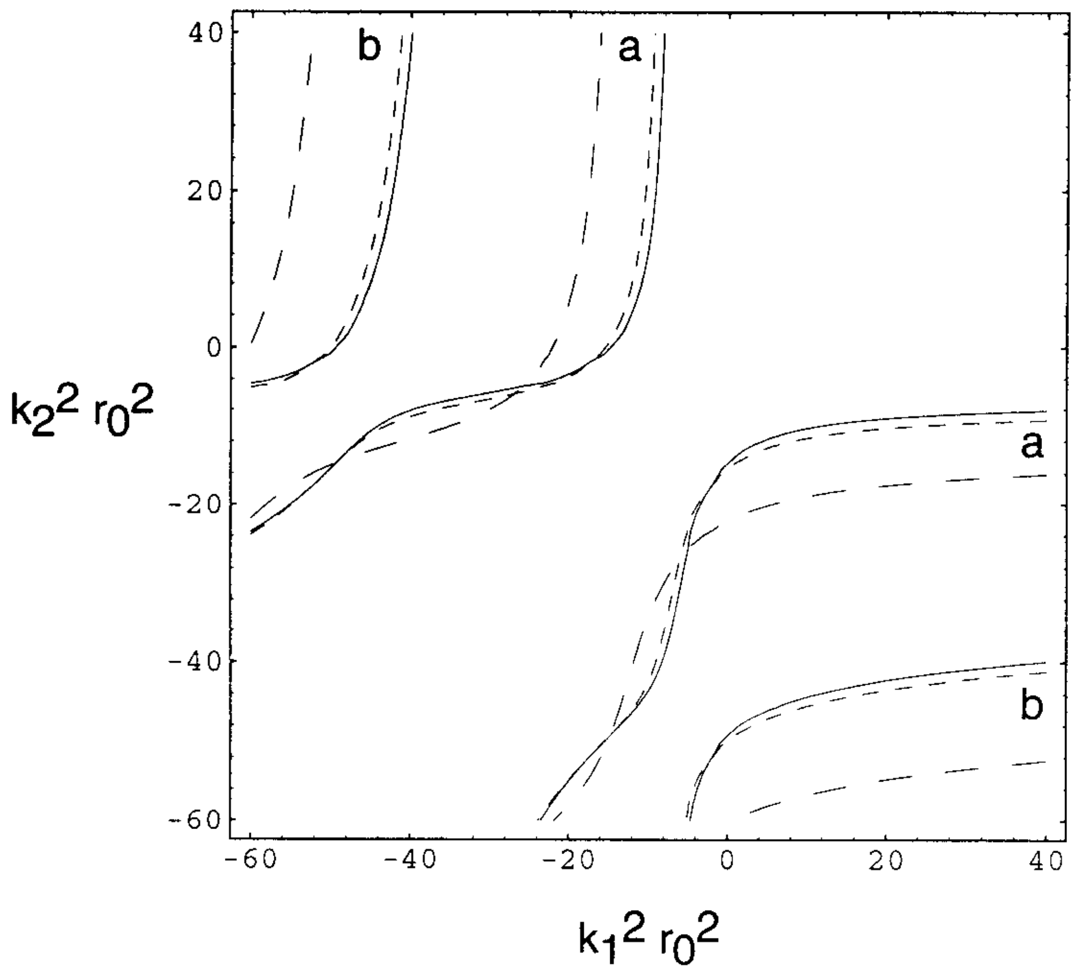
K and $\mathcal{B}$ are Bessel functions,

$$\mathcal{B}_i(k) = I_i(|k|) \qquad k^2 > 0; \qquad \mathcal{B}_i(k) = J_i(|k|) \qquad k^2 < 0.$$

A second relationship between $k_{1,2}$ and k_e follows from the matrix

$$\frac{k_1^2}{k_e^2} + \frac{k_2^2}{k_e^2} - 1 = (1 - c_\alpha^2)\frac{k_1^2 k_2^2}{k_e^4},$$

depending upon which of the $\Phi_\alpha - x/\lambda_y$ vanishes at $r = r_0$. The eigenvalues of the system are given by the intersection points of these curves with the dispersion relation.



Hasegawa- Mima equation,

drift vortices

6 Drift vortices

The set of nonlinear equations do not only have wave-like solutions that extend over complete space, but also spatially localized solutions

$$\tilde{\Phi} \to 0, \quad \tilde{\Psi} \to 0, \qquad\qquad |\vec{x}| \to \infty.$$

Electrostatic phenomena ($\Psi \to 0$) in an inhomogeneous plasma in a homogeneous magnetic field $B_0\vec{e}_z$. Neglect the ion motion along field lines: disregard coupling to propagating sound waves.

Boltzmann distribution

$$\ln \frac{\tilde{n}}{n_0(x)} = \tilde{\Phi}.$$

The electron continuity equation and the quasi-neutrality condition

$$\frac{\partial}{\partial t} \ln n + \frac{T_0}{eB_0} \vec{e}_z \times \nabla\tilde{\Phi} \cdot \nabla \ln n - \frac{1}{en} \nabla_{\parallel} J_z = 0,$$

(no normalization)

$$\frac{1}{en} \nabla_{\parallel} J_z = \rho_s^2 \left(\frac{\partial}{\partial t} + \frac{T_0}{eB_0} \vec{e}_z \times \nabla\tilde{\Phi} \cdot \nabla\right) \nabla_{\perp}^2 \tilde{\Phi}.$$

Eliminate the perturbed density $\tilde{n}$ and the current density J_z.

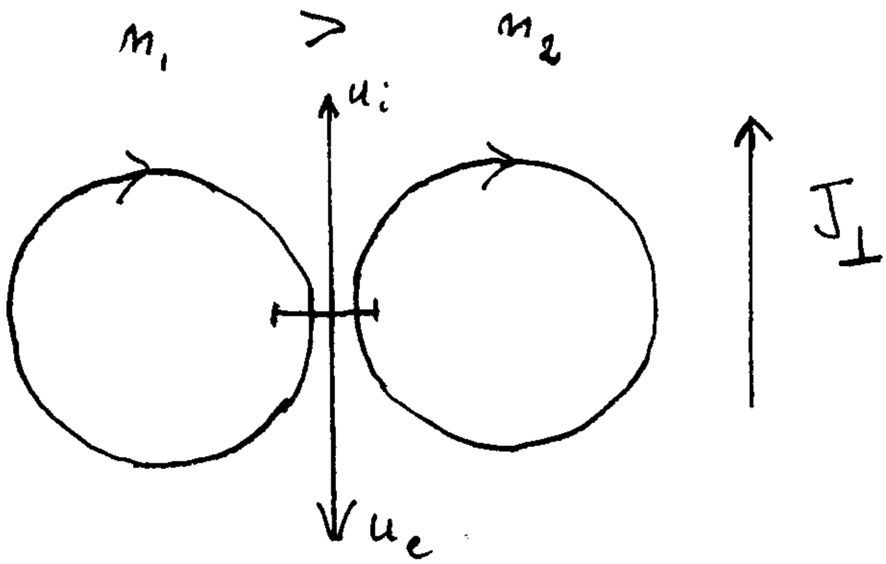
~~Introduction~~ Remark

The word 'drift' refers to the *diamagnetic drift velocities* of electron and ion fluids embedded in a strong magnetic field:

$$\vec{u}_{*(e,i)} = \frac{1}{Zen}\frac{\vec{B} \times \nabla p}{B^2} \approx \frac{T}{ZeBl_n} = \frac{\rho}{l_n}v_{th},$$

ρ is the thermal gyro-radius and l_n the characteristic scale-length of the plasma density

$$l_n = -\frac{1}{n}\frac{dn}{dx}.$$



Corresponding current density: $\vec{J}_\perp = \vec{B} \times \nabla p / B^2$.

$$\frac{1}{\omega_{ci}}\frac{\partial}{\partial t}\big[\tilde{\Phi} - \rho_s^2\nabla_\perp^2\tilde{\Phi}\big] + \rho_s^2\vec{e}_z \times \nabla\tilde{\Phi}\cdot\nabla\big[-\frac{x}{l_n} + \tilde{\Phi} - \rho_s^2\nabla_\perp^2\tilde{\Phi}\big] = 0,$$

Charney-Hasegawa-Mima equation. Its structure is analogous to that of the 2D Euler equation for the vorticity of an incompressible fluid.

- Generalized vorticity

$$\Omega = -\frac{x}{l_n} + \tilde{\Phi} - \rho_s^2\nabla_\perp^2\tilde{\Phi}$$

is pointwise conserved along the flow lines of the $E \times B$ drift. All functionals

$$\int d^2x\ G(\Omega)$$

are conserved.

- Conserved energy

$$E = \frac{1}{2}\int_D d^2x\ \big(\tilde{\Phi}^2 + \rho_s^2|\nabla\tilde{\Phi}|^2\big).$$

(Appropriate boundary conditions are assumed.)

Look for solutions that propagate with constant velocity u in the y-direction: solutions depend on $(x, \eta = y - ut)$

$$\vec{e}_z \cdot \nabla(\tilde{\Phi} - \frac{u}{u_*}\frac{x}{l_n}) \times \nabla\Big[-\frac{x}{l_n} + \tilde{\Phi} - \rho_s^2 \nabla_\perp^2 \tilde{\Phi}\Big] = 0.$$

General solution

$$-\frac{x}{l_n} + \tilde{\Phi} - \rho_s^2 \nabla_\perp^2 \tilde{\Phi} = F(\tilde{\Phi} - \frac{u}{u_*}\frac{x}{l_n}).$$

The functional form of F need not to be the same over all space. It can be different in different areas of the (x, η) plane. These areas are separated by $\tilde{\Phi} - (u/a)x = constant$ curves that connect to the singular points defined by $\nabla[\tilde{\Phi} - (u/a)x] = 0$.

Localized solutions for which $\tilde{\Phi} \to 0$ for $|\vec{x}| \to \infty$ require

$$F = \frac{u_*}{u}(\tilde{\Phi} - \frac{u}{u_*}\frac{x}{l_n})$$

in the *exterior region, i.e., outside the separatrices.*

$$\rho_s^2 \nabla_\perp^2 \tilde{\Phi} - \kappa^2 \tilde{\Phi} = 0, \qquad \kappa^2 \equiv 1 - u_*/u.$$

Localized solutions require $\kappa^2 > 0$.

$$u < 0, \qquad u > u_*.$$

These intervals are complementary to the range $0 < u < u_*$ *where electrostatic drift waves exist.*

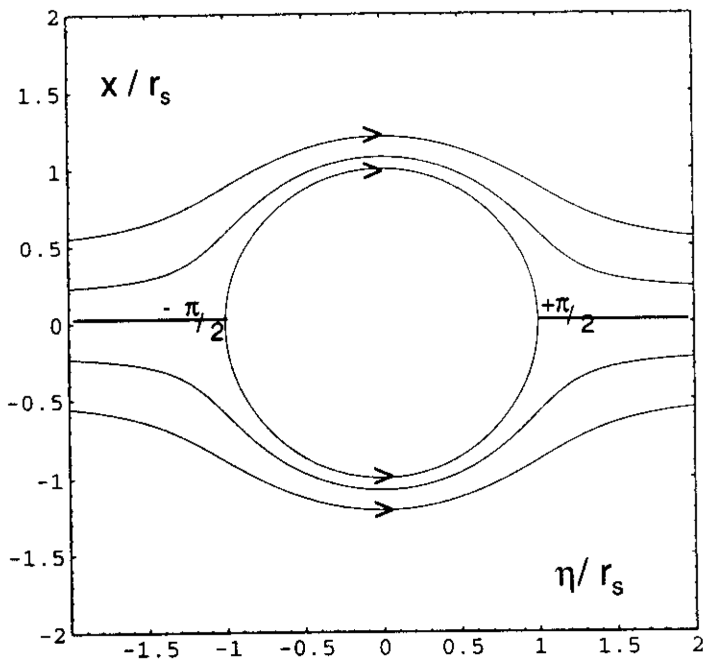
Outer solution:

$$\tilde{\Phi} = \frac{u}{u_*}\frac{\rho_s}{l_n} r_s \frac{K_1(\kappa r)}{K_1(\kappa r_s)} \cos\theta, \qquad r \geq r_s.$$

Since K_1 *is singular for* $r \to 0$*, this solution can not be valid everywhere, but only in the exterior region.*

Singular points (θ_s, r_s) of $\tilde{\Phi} - (u/u_*)x/l_n$ are points where $\nabla[\tilde{\Phi} - (u/u_*)x/l_n] = 0$,

$$\theta_s = \pm\frac{1}{2}\pi, \quad A_1 K_1(\kappa r_s) - \frac{u}{u_*}\frac{\rho_s}{l_n} r_s = 0.$$



The function $\tilde{\Phi} - (u/u_*)x/l_n$ vanishes along the separatrices, i.e., along the circle $r = r_s$ and along the positive and negative η-axis outside this circle.

Inner solution. Assume inside the separatrix circle

$$F = (\gamma^2 + 1)(\tilde{\Phi} - \frac{u}{u_*}\frac{x}{l_n})$$

with real γ^2, so that

$$\rho_s^2 \nabla_\perp^2 \tilde{\Phi} + \gamma^2 \tilde{\Phi} = [(\gamma^2 + 1)\frac{u}{u_*} - 1]\frac{x}{l_n}.$$

Calculate the regular solution in $r < r_s$ and match $\tilde{\Phi}$ and $\partial\tilde{\Phi}/\partial r$ at $r = r_s$:

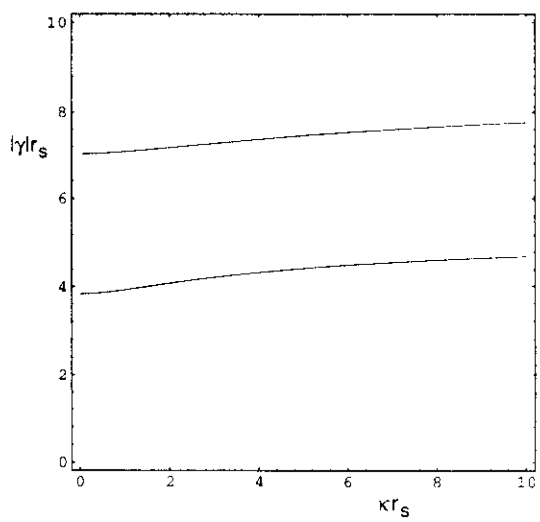
$$\tilde{\Phi} = \frac{\rho_s}{l_n}\frac{u}{u_*}[1 + \frac{\kappa^2}{\gamma^2}(\frac{u}{u_*} - 1)]r\cos\theta + CJ_1(|\gamma|r)\cos\theta,$$

with

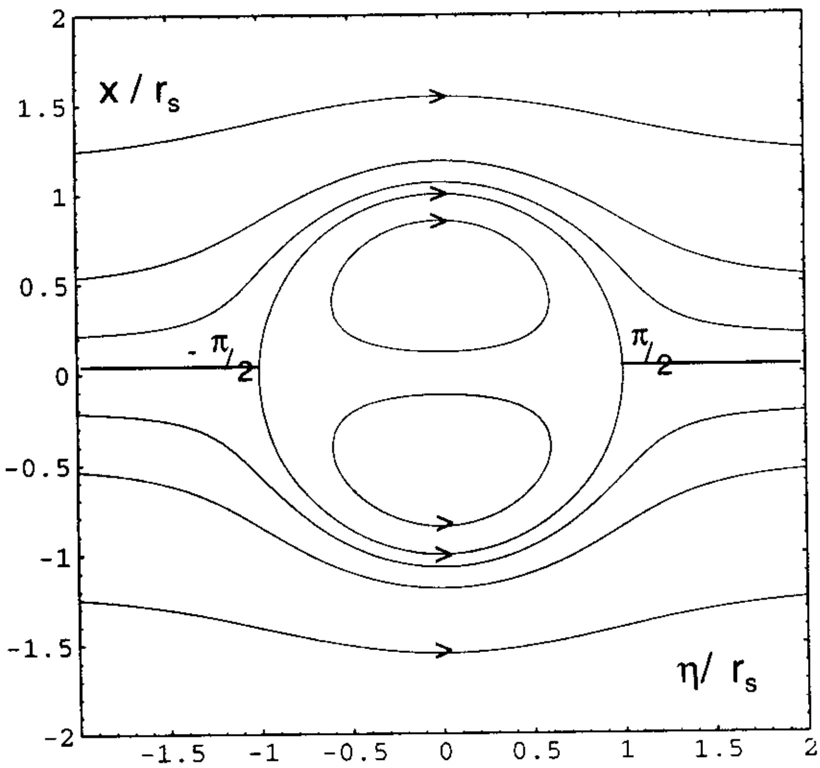
$$C = -r_s\frac{\rho_s}{l_n}\frac{u}{u_*}\frac{\kappa^2}{\gamma^2 J_1(|\gamma|r_s)}, \qquad \gamma^2 > 0.$$

Dispersion relation

$$-\frac{K_2(\kappa r_s)}{K_1(\kappa r_s)} = \frac{\kappa}{|\gamma|}\frac{J_2(|\gamma|r_s)}{J_1(|\gamma|r_s)}, \qquad \kappa^2 = 1 - u_*/u.$$



Given the parameters that define the plasma u_* and ρ_s/l_n, two free parameters r_s and u exist.



The flow lines $\Phi - (u/u_*)(x/l_n) = constant$ of the dipole vortex in the inner and outer regions.

7 Conclusions

• The dynamical equations can be written in Lagrangian form for 3 fields that are advected with different velocities.

This system is a generalization and extension of Euler's equation for an ordinary fluid.

The system has three infinite sets of invariants (Casimirs) that arise from the structure of the equations.

• Similar to the Euler case, the plasma equations have solutions in the form of 3 different types of *current-vortex filaments* that move with 3 different velocity fields.
The discrete system has the same integrals of the motion as the original one

• A number of equilibrium point-vortex distributions have been discussed.

The far field of a neutral pair of current-vortex filaments is identical to the field of a distributed, propagating dipole vortex outside its separatrix.

• The current-vortex system has a solution in the form of a collapse and even of an annihilation of the structures.