INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P, P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

H4-SMR 1012 - 16

AUTUMN COLLEGE ON PLASMA PHYSICS

13 October - 7 November 1997

Magnetic Discontinuities (Current Sheets) in Evolving Magnetic Fields

E.N. PARKER

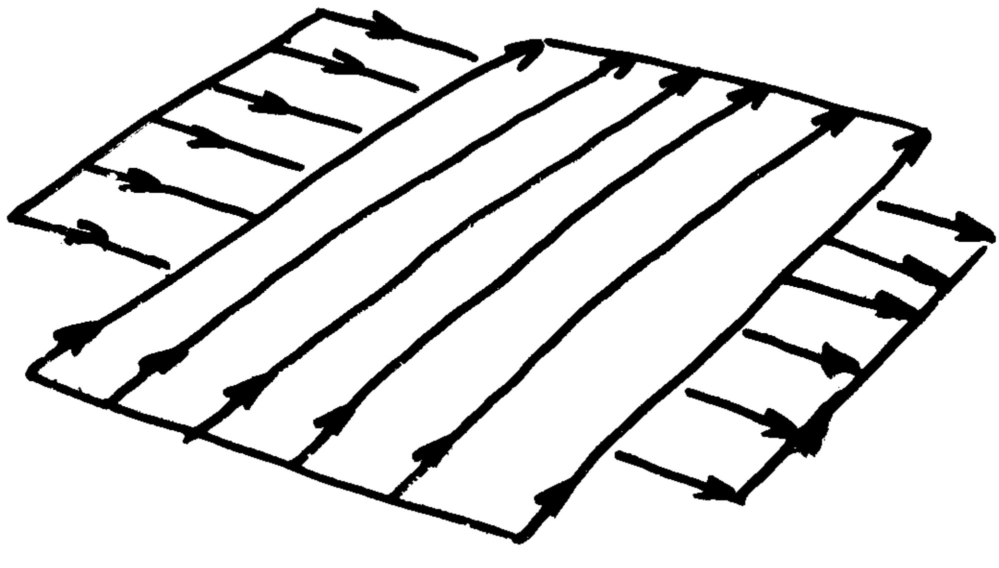
University of Chicago, U.S.A

These are lecture notes, intended for distribution to participants.

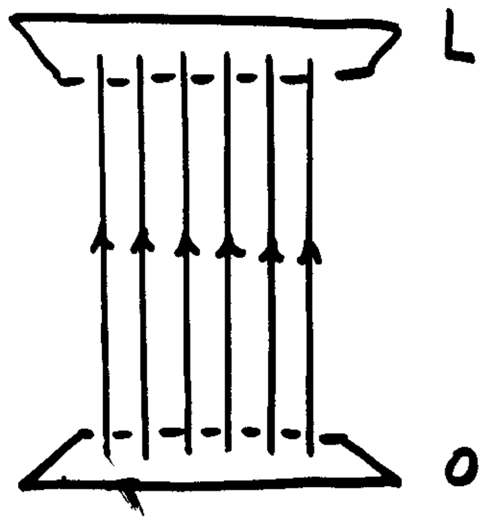
MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224463 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 224559 TELEX 460392

BASIC MAGNETOSTATIC THEOREM

<u>Almost all field line topologies have tangential discontinuities (current sheets) as an intrinsic part of magnetostatic equilibrium</u>.
(assuming an infinitely conducting fluid).
A tangential discontinuity is a rotational (directional) discontinuity, with continuous field magnitude.



To be clear as to the meaning of these words, consider the simple case where an initially uniform field B_0 extends through a region of infinitely conducting fluid $0 < z < L$. Then introduce the fluid motion



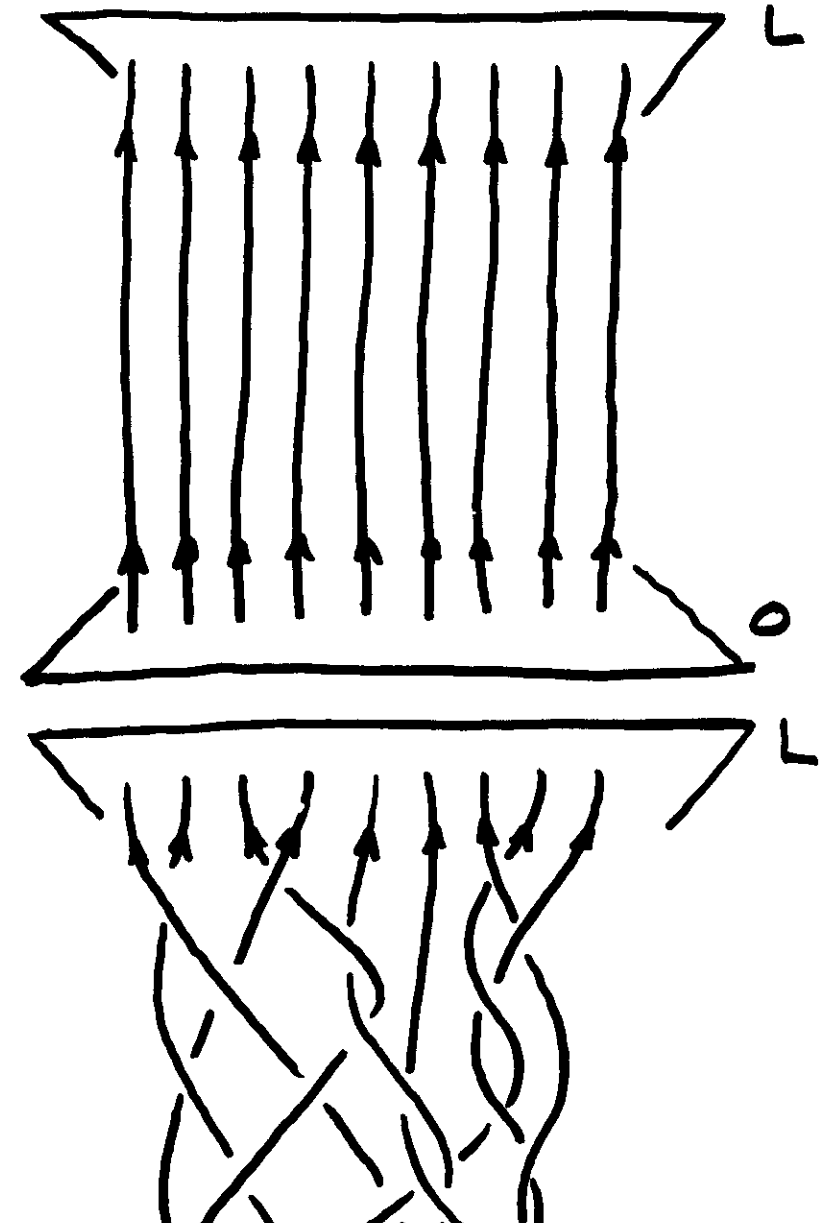
$$v_x = +kz\frac{\partial\psi}{\partial y}, \quad v_y = -kz\frac{\partial\psi}{\partial x}, \quad v_z = 0$$

$$\psi = \psi(x, y, kzt)$$

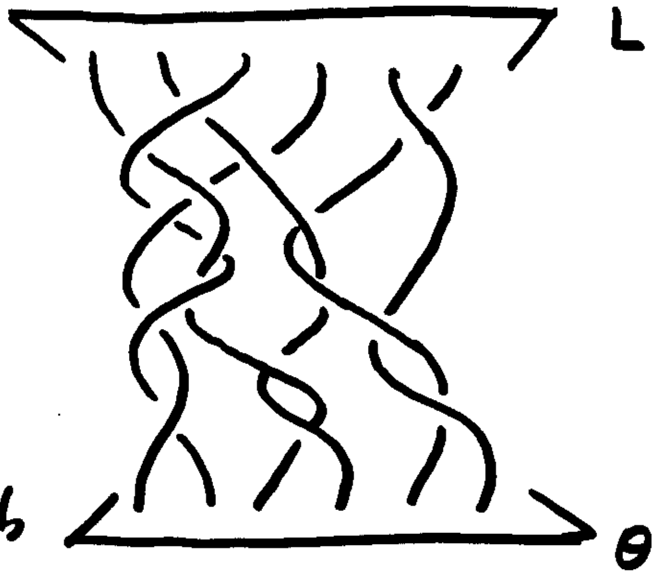
which leaves the field fixed at $z = 0$ and winds and interweaves the field lines by any <u>continuous</u> mixing ψ introduced at $z = L$. The field becomes

$$B_x = +kt\frac{\partial\psi}{\partial y}, \quad B_y = -kt\frac{\partial\psi}{\partial y}, \quad B_z = B_0$$

after a time t.


L
0
L
0

Since $\psi(x, y, kzt)$ is a continuous bounded, well behaved function of x, y, kzt, it follows that B_i is continuous, bounded, etc.

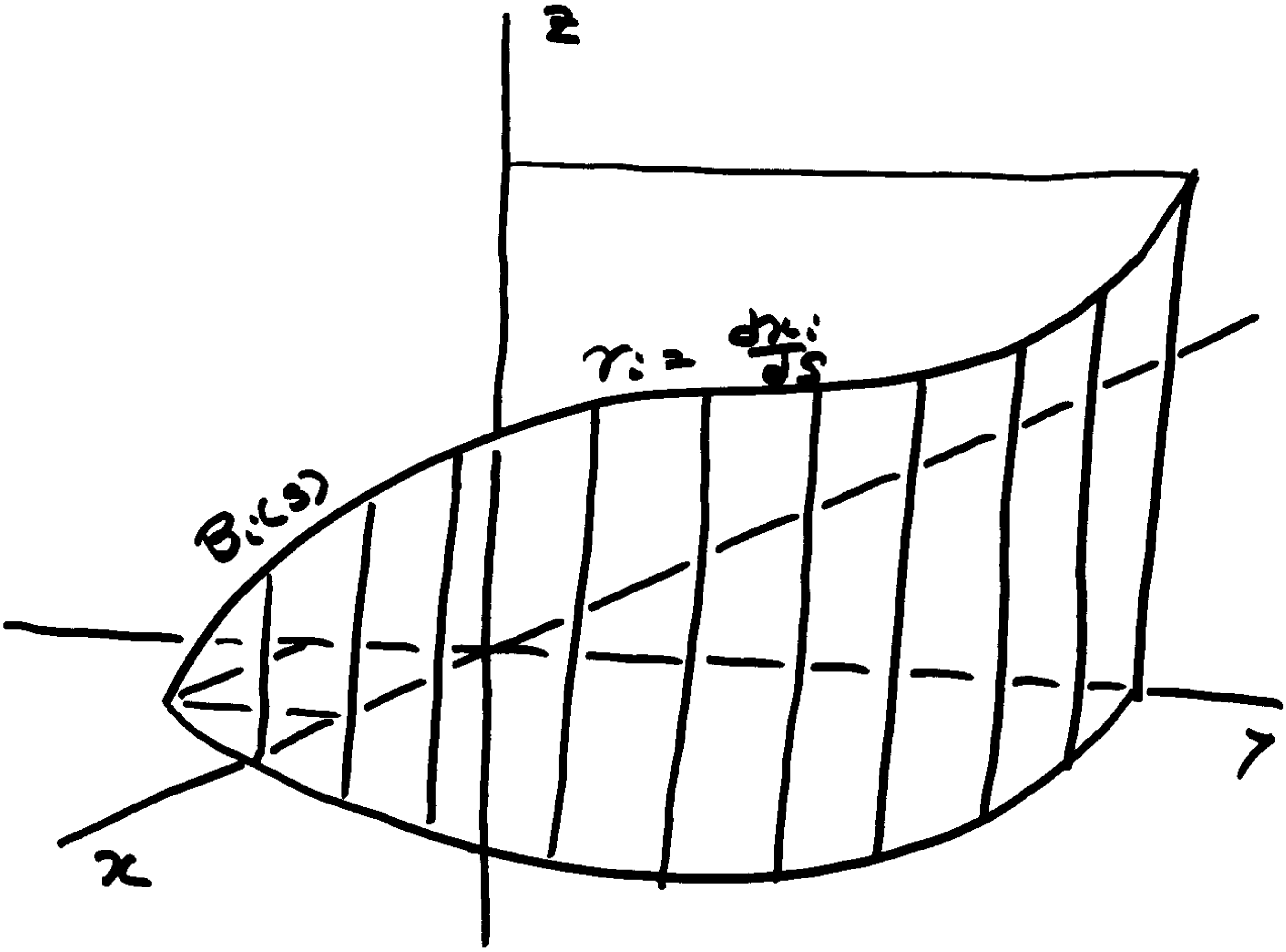


Fix the footpoints of the field at $t = \tau$ $z = 0, L$, and then release the fluid so that the field relaxes to static equilibrium. This is the lowest energy state available for the given topology.

~~[illegible]~~

The essential point is that the magnetic field develops tangential discontinuities as it relaxes to equilibrium, in a finite time, or infinite time.

In the real world, $\eta \neq 0$, Magnetic energy is dissipated at the "discontinuities", which have finite thickness so that they are never in internal static equilibrium



$$B_i(s) + \delta B_i = B_i(s) + \frac{\partial B_i}{\partial x_j}\,\delta x_j + \ldots$$

All nine $\partial B_i / \partial x_j$ must be available.

$$\frac{dB_i}{ds} = r_j \frac{\partial B_i}{\partial x_j} \quad , \quad B_j \left(\frac{\partial B_i}{\partial x_j} - \frac{\partial B_j}{\partial x_i} \right) = 0$$

6 eqns linear in $\partial B_i / \partial x_j$

Specify $B_i(s)$ and, e.g. $\partial B_i / \partial z$.
Then compute the remaining six.

The remaining six components of $\partial B_i / \partial x_j$ cannot be computed if the det $= 0$.

$$\begin{vmatrix} 0 & -B_y & +B_y & 0 & B_z & 0 \\ 0 & +B_x & -B_x & 0 & 0 & B_z \\ 0 & 0 & 0 & 0 & -B_x & -B_y \\ 1 & 0 & 0 & 1 & 0 & 0 \\ \gamma_x & \gamma_y & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_x & \gamma_y & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_x & \gamma_y \end{vmatrix} = 0$$

$$(\gamma_x^2 + \gamma_y^2)(\gamma_x B_y - \gamma_y B_x) = 0$$

$$\gamma_x = \pm i \gamma_y, \qquad \frac{\gamma_x}{\gamma_y} = \frac{B_x}{B_y}$$

$$\frac{dx}{B_x} = \frac{dy}{B_y}$$

for the force-free magnetic field.

In formal mathematical terms note that the curl and divergence of $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ yield

$$\nabla^2 \mathbf{B} + \alpha^2 \mathbf{B} + \nabla\alpha \times \mathbf{B} = 0,$$

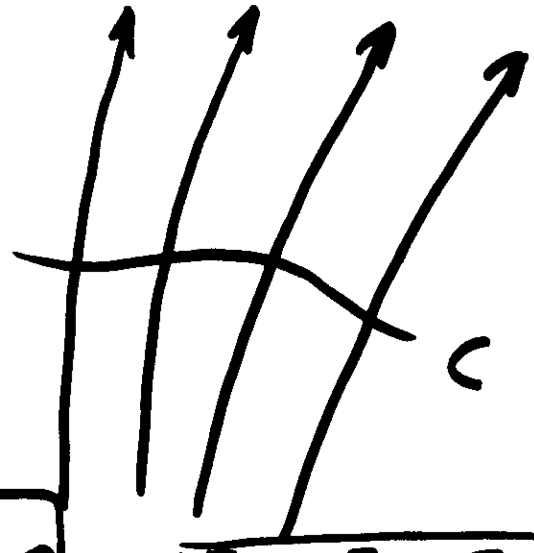
and

$$\mathbf{B} \cdot \nabla\alpha = 0,$$

respectively. The first of these seems elliptic (imaginery characteristics), providing continuous solutions that are uniquely determined by $\mathbf{B}$ at the boundaries. But the second equation, for the torsion coefficient α, shows real characteristics, which are just the field lines of $\mathbf{B}$. This permits TD's along the field lines, thereby avoiding the uniqueness of the continuous solutions.

OPTICAL ANALOGY

Flux surface defined by the lines of force across curve C.



$\boxed{\oint ds \cdot B = 0}$

Force-Free Field
$\nabla \times B = \alpha B$

Note that $B = -\nabla\phi$ in any flux surface. Lines of force described by

$$B h_1 \frac{dx_1}{ds} = -\frac{1}{h_1}\frac{\partial\phi}{\partial x_1}, \quad B h_2 \frac{dx_2}{ds} = -\frac{1}{h_2}\frac{\partial\phi}{\partial x_2}.$$

Same form as equations for optical ray path, with eikonal ϕ, in index of refraction B

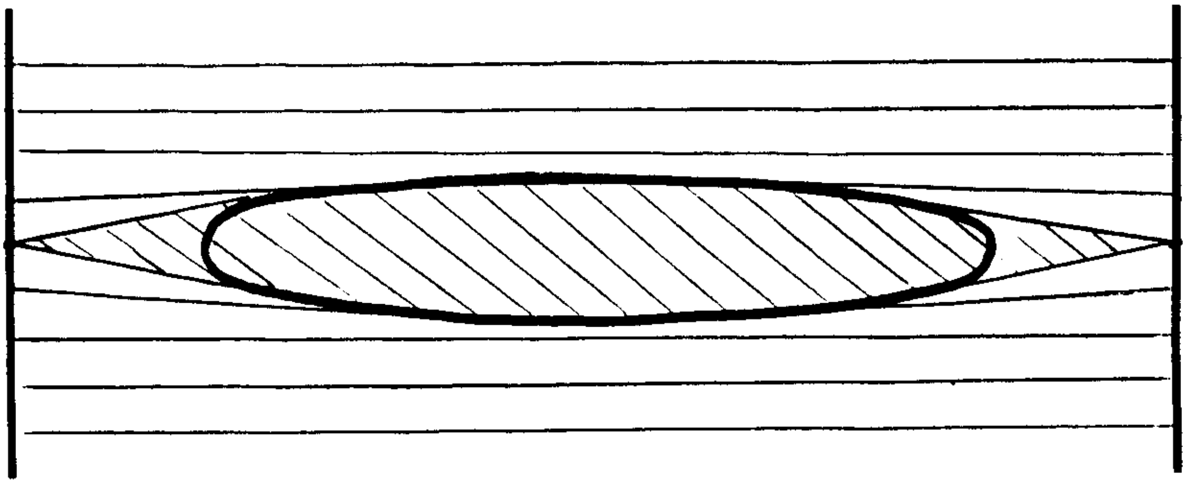
Apply Fermat's principle

$$\delta \int_{P_1}^{P_2} ds\, B = 0$$

Use Euler equations to compute lines of force across a region with specified applied pressure

Regions of enhance pressure tend to exclude lines of force.

A local pressure maximum may create a gap in a flux surface.



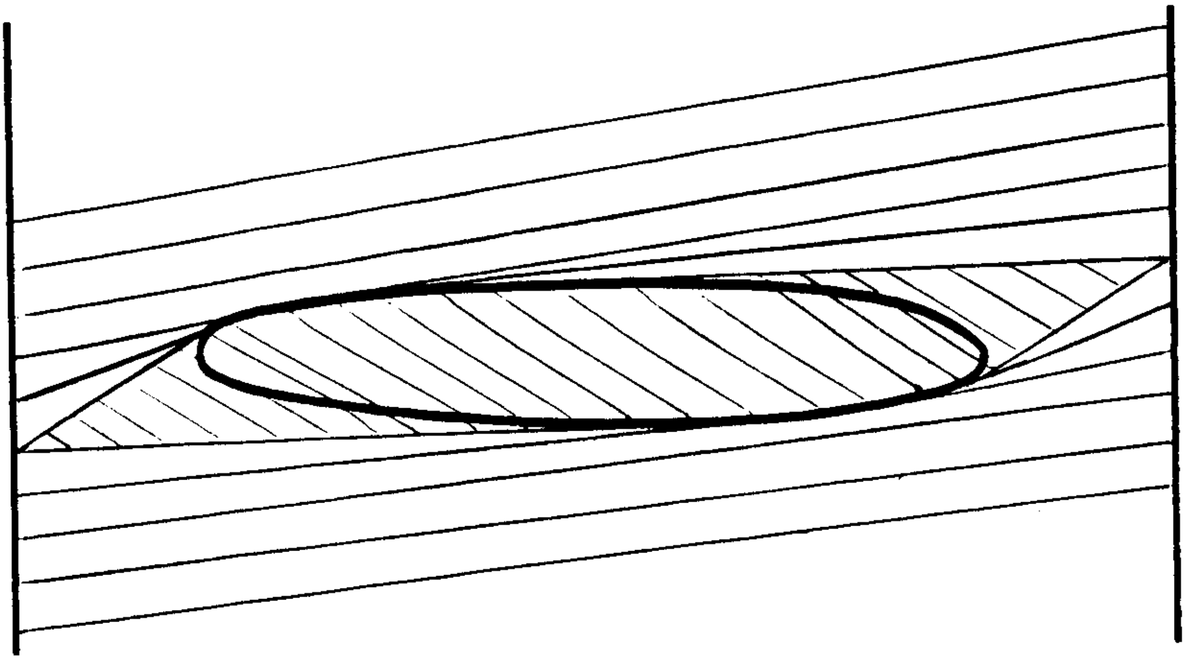


Fig 10

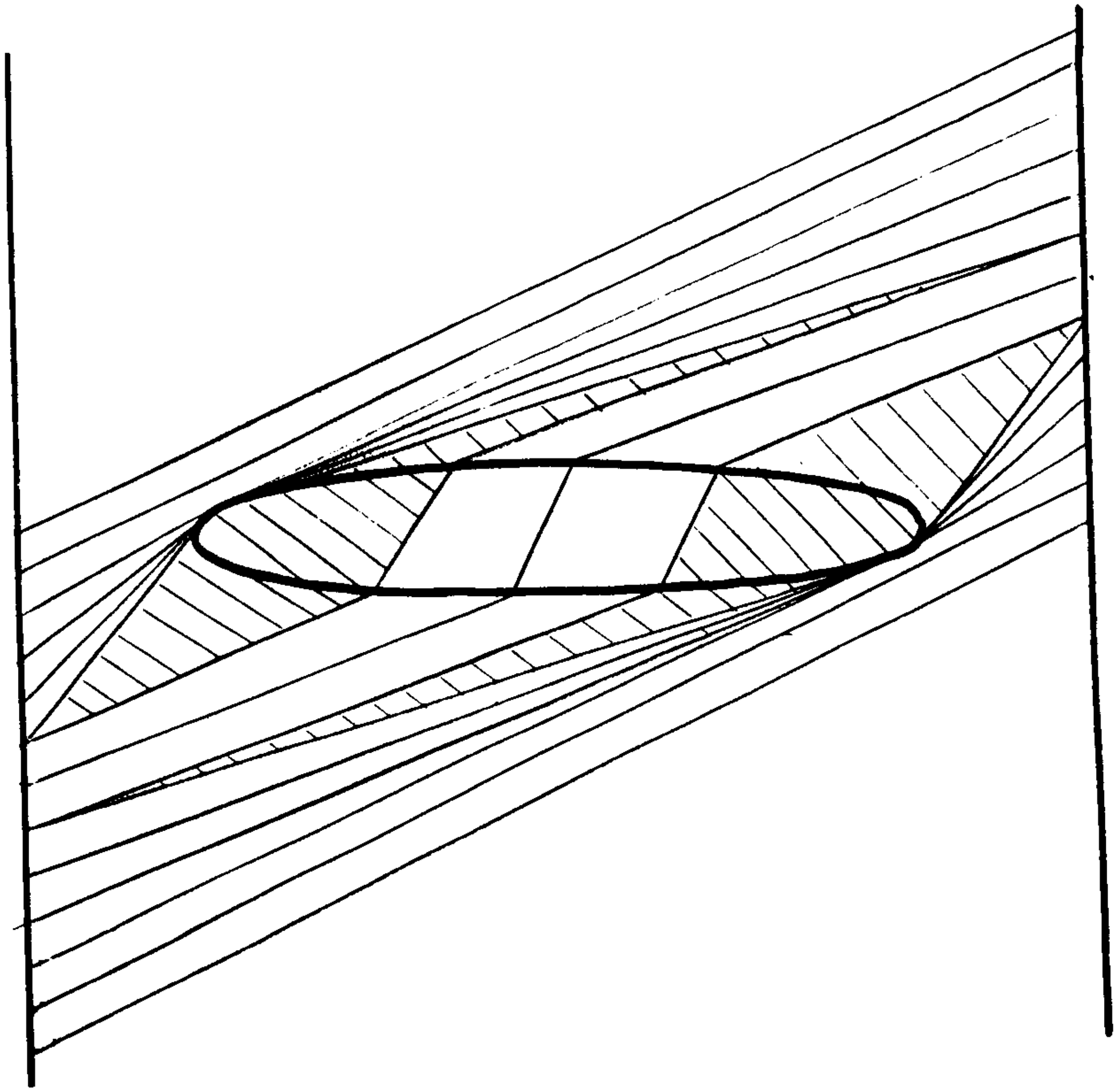
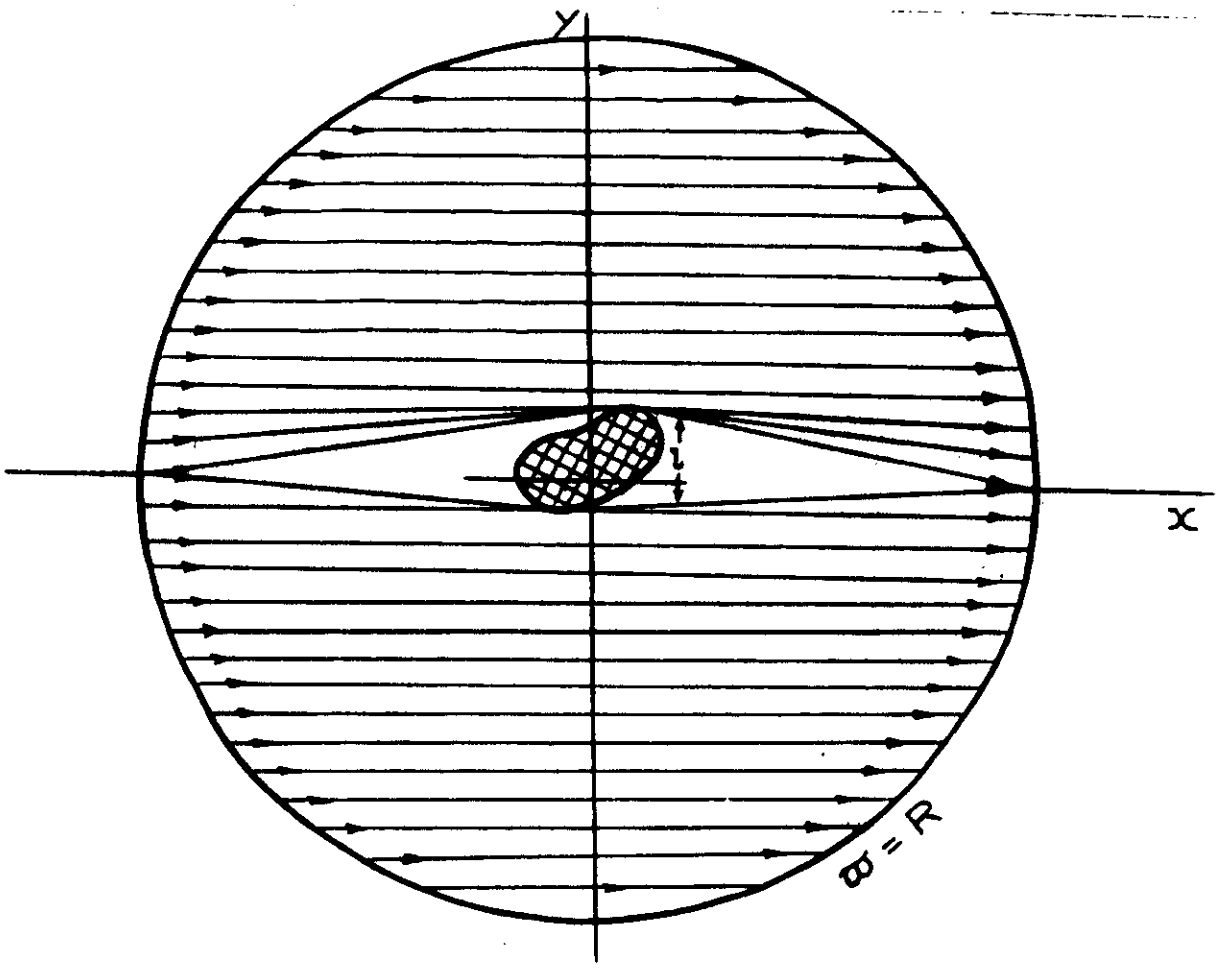
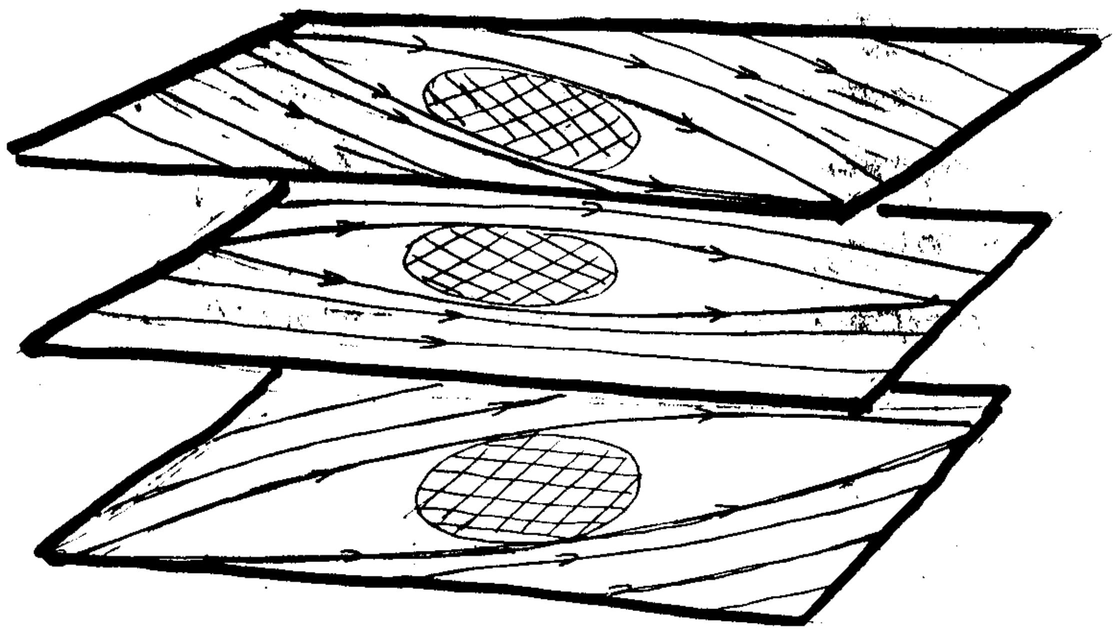


Fig 2c



The essential point is that the variations in the magnetic pressure in an interwoven force-free field produce gaps in the local flux surfaces.

It follows that field lines are refracted around local regions of enhance magnetic pressure $B^2/8\pi$, creating a gap in the flux surface. The fields on either



side of S are generally not parallel where they meet each other in the gap.

See <u>SPONTANEOUS CURRENT SHEETS IN MAGNETIC FIELDS</u>, Oxford 1994 for illustrative examples.

A gap extending through a finite stack of flux surfaces allows otherwise separate fields to come into contact.

The separate fields are not precisely parallel where they meet, in almost all cases. The result is a tangential discontinuity where they meet in the gap.

B^2 is continuous across the TD.

The field direction is discontinuous

$$J = (cB/2\pi)\sin\tfrac{1}{2}\theta.$$

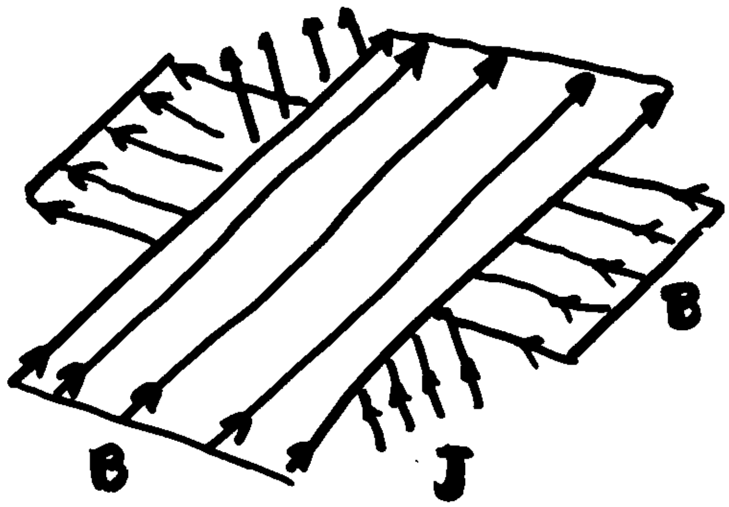
If $\mathbf{B}_+$ and $\mathbf{B}_-$ represent the fields on opposite sides of a surface of TD, the surface current density is

$$\mathbf{J} = \frac{c}{4\pi}(\mathbf{B}_+ + \mathbf{B}_-)\tan\theta \quad , \quad B_+^2 = B_-^2 ,$$

where 2θ is the angle between $\mathbf{B}_+$ and $\mathbf{B}_-$. With $B = |\mathbf{B}_\pm|$,

$$J = \frac{c}{2\pi} B \sin\theta .$$

Note that in static equilibrium, then, $\mathbf{J}$ is parallel to <u>neither</u> $\mathbf{B}_+$ <u>nor</u> $\mathbf{B}_-$.



Each surface of TD represents the <u>contact</u> between regions of continuous $\mathbb{B}$ wherein $\mathbb{B}\cdot\nabla\alpha = 0$.

If we think of the TD as a "delta fn" in α, the TD does not violate $\mathbb{B}\cdot\nabla\alpha = 0$ because there is no magnetic flux in the TD.

A TD may form after only a limited interweaving of $\mathbb{B}$. A couple of careless twists and a TD develops.

What happens when a small resistivity is present? The TD is a current sheet, which immediately broadens.

$$j = \frac{1}{t^{1/2}} \exp\left(-\frac{\xi^2}{4\eta t}\right)$$

However, the topology of the field is such that a TD is essential for static equilibrium. So the Maxwell stress begins to drive the field and fluid back toward a TD, <u>which it cannot achieve.</u>

The result is a thin layer of dynamical nonequilibrium and resistive dissipation, which continues until the diffusive reconnection of the field reduces the topology to such a level that a TD is not required for equilibrium. This dynamical state is the familiar "rapid reconnection" or "neutral point reconnection". It arises spontaneously in any magnetic field subject to convective interweaving.

Bursts of rapid reconnection limit the degree of interweaving of the field in the bipolar magnetic active regions on the Sun.

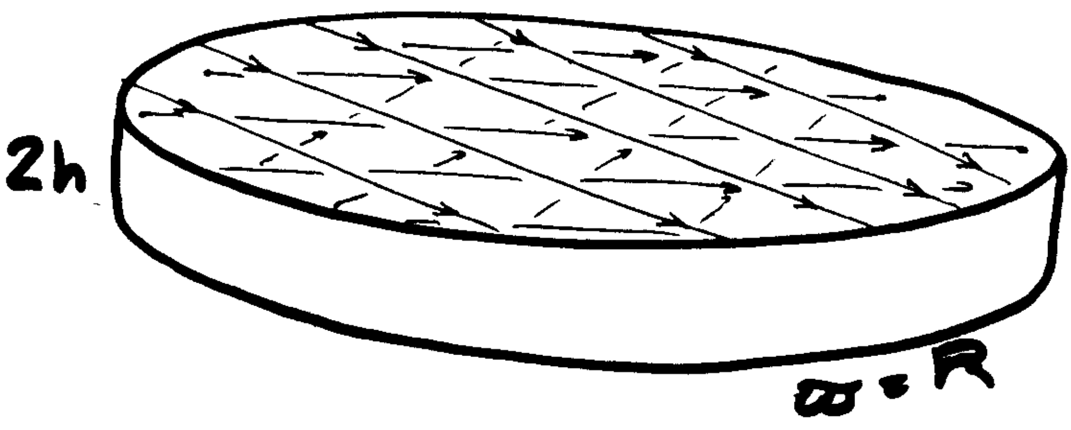
Discontinuities can be demonstrated in many ways.

IV As a formal example of the tangential discontinuity, consider the linear laminar force-free field. $\alpha = q =$ constant

$$B_x = B_0 \cos qz,\ B_y = -B_0 \sin qz,\ B_z = 0,$$

$$B_\varpi = B_0 \cos(qz + \varphi),\ B_\varphi = -B_0 \sin(qz+\varphi).$$

confined to the thin layer $-h < z < +h$ ($hq = O(1)$) and fixed in the circular infinitely conducting rigid boundary $\varpi = R \gg h$.



Squeeze the layer of field gently, smoothly in a neighborhood l of the origin.

The result is the creation of a tangential discontinuity in the field. $l = O(h)$

Work in the limit of large R.

The squeezed field becomes

$B_\varpi + b_\varpi,\ B_\varphi + b_\varphi,\ b_z$

and the torsion coefficient $\alpha = q$ is written

$$\alpha = q(1 + A).$$

The force-free field equation reduces to

$$\frac{1}{\varpi}\frac{\partial b_z}{\partial \varphi} - \frac{\partial b_\varphi}{\partial z} = q A B_\varpi , \qquad (1)$$

$$\frac{\partial b_z}{\partial \varpi} = q(1+A)\, b_\varphi + q A B_\varphi , \qquad (2)$$

$$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\varpi b_\varphi = q(1+A)\, b_z . \qquad (3)$$

If $b_\varphi \sim \varpi^{-s}$ for $\varpi \gg l$, then eqn. (3) states that $b_z \sim \varpi^{-s-1}$. With $\partial/\partial z = O(q)$, $\partial/\partial\varphi = O(1)$, $ql = O(1)$, eqns. (1) and (2) reduce to

$$\frac{1}{q}\frac{\partial\, b_\varphi/B_0}{\partial z} + A\cos u = O\left(\frac{b_\varphi}{B_0}\frac{1}{q^2 R^2}\right)$$

$$\frac{b_\varphi}{B_0}(1+A) - A\sin u = O\left(\frac{b_\varphi}{B_0}\frac{1}{q^2 R^2}\right)$$

<u>at $\varpi \cong R$</u>, where $B_\varpi = B_0 \cos(qz+\varphi)$
$b_\varpi = 0$, $u = qz + \varphi$

Neglecting the right hand sides, integration yields

$$A = -1 \pm \sin u/(\sin^2 u - c)^{\frac{1}{2}}$$

$$b_\varphi/B_0 = \sin u \mp (\sin^2 u - c)^{\frac{1}{2}}$$

where $u = qz + \psi$. c is a constant.

If $c < 0$, the radicals have no zeros and so cannot change sign. So if A and b_φ/B_0 are small at $u = \pi/2$, they are $O(1)$ at $-\pi/2$. With $c > 0$ both A and b_φ/B_0 are small $O(c^{1/2})$ as required by the physics, except that A diverges as $\sin^2 u$ declines to c.

Since b_φ / B_0 is not larger than the ϖ^{-2} of a 2-D dipole, it follows that $C = O(1/q^2R^2)$. The torsion A is then of the form of a delta fn,

$$A = 2C^{1/2} \sum_m \delta(u - m\pi) + O(1/q^2R^2)$$

and

$$\alpha = q(1+A) = q\left[1 + 2C^{1/2} \sum_m \delta(u - m\pi)\right].$$

There is a tangential discontinuity at $u = 0, \pm\pi$, etc. The discontinuity in field direction is $O(1/qR)$

The same result is given by the formal optical analogy, where gaps appear in the local flux surfaces in response to pressure inhomogeneities.

Note that apart from the singular surfaces, $\alpha = q$, neglecting terms $O(1/q^2R^2)$.

Hence

$$\nabla^2 B_i + q^2 B_i = 0$$

except at singularities.

Suppose there were no singularities. Then complete general solution can be constructed. Solution is unique, given $B_\varpi = B_0 \cos(qz+\varphi)$ at $\varpi = R$.

No solutions with field connecting straight across $\varpi = R$, except original.

$\therefore$ Continuity is unphysical.