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SHEAR FLOW SURPRISES
1. General Effects

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These are lecture notes, intended for distribution to participants.

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Abstract

In this lecture I would like to speak about some recent advancements in the physics of shear flows. In particular my aim is to present to the audience some surprisingly simple new effects in the linear theory of perturbation evolution in parallel shear flows. These effects were disclosed in last few years owing to the usage of so called "nonmodal approach" having its roots in very old, well forgotten paper by Lord Kelvin (1887) [1]. The effects, which will be discussed are: (a) Appearance of an unusual class of algebraically growing vortical solutions (otherwise known as "transient solutions", or "Kelvin modes"); (b) effect of the mean flow energy extraction by compressible disturbances in shear flows; (c) effect of shear-induced wave coupling, mutual transformation of waves and energy exchange between the modes.

An emphasize will be made on those difficulties in mathematical description of parallel shear flows, which make necessary the change of paradigm from conventional normal mode approach to Kelvin approach. I'll describe basic details of the latter method and will derive all above cited effects by means of this method.

1 What is shear flow?

To the best of my knowledge there is no rigorous and standard definition of the shear flow concept in a relevant literature. In the framework of the present lecture I shall call **shear flows** all those flows of continuous media (neutral fluids and/or plasmas) that have *spatially inhomogeneous* mean velocity fields.

$$\mathcal{V}\equiv\mathcal{V}(x,y,z;t). \tag{1}$$

2 Are shear flows of a wide occurence?

The answer is YES: almost all known examples of laboratory, terrestrial (atmospheric and/or ocean) and astrophysical flows are shear flows.

Examples of *astrophysical shear flows* include:

- Planet atmospheres and interiors,
- Stars (Solar differential rotation),
- Stellar (Solar, pulsar, etc.) winds,
- Pulsar magnetosphere,
- Accretion columns in X-ray pulsars,
- Accretion disks,
- Jets (bipolar and unipolar outflows) in quasars and AGN's.

3 What are elementary examples [2] of *parallel* shear flows?

3.1 Plane Couette flow:

A flow enclosed between two parallel planes with a constant relative velocity U_0.

$$\mathcal{V} = \big(Ay,\ 0,\ 0\big), \tag{2a}$$

$$A \equiv U_0/L, \tag{2b}$$

3.2 Plane Poiseuille flow:

A flow between two fixed parallel planes in the presence of a pressure gradient.

$$\mathcal{V} = \big(\alpha[h^2 - (2y-h)^2],\ 0,\ 0\big), \tag{3a}$$

$$\alpha \equiv -(1/8\eta)\partial_x P, \tag{3b}$$

3.3 Pipe Poiseuille flow:

A flow in a circular cross-section pipe of a length l with ΔP pressure difference between the ends of the pipe.

$$\mathcal{V} = \big(\alpha(R^2 - r^2),\ 0,\ 0\big), \tag{4a}$$

$$\alpha \equiv \Delta P/4\eta l, \tag{4b}$$

Parallel shear flows are important, because on length-scales much smaller in comparison with flow spatial dimensions arbitrary piecewise linear smooth velocity profiles may be treated as Couette-like shear flows.

4 How to deal with shear flows?

Linear stability of shear flows: the standard approach is the method of *normal modes* or *eigenvalue analysis*, which proceeds in two stages:

- Linearize about the laminar solution,
- Look for *unstable eigenvalues* of the linearized problem

An "unstable eigenvalue" is an eigenvalue in the complex upper half-plane, corresponding to an eigenmode of the linearized problem that grow exponentially as a function of time. It is natural to expect that a shear flow will behave unstably if and only if there exists such a growing eigenmode.

5 Are there any problems with shear flows?

(Un)fortunately, the answer is YES.

5.1 Experimental difficulties:

For some flows (e.g. Rayleigh-Bénard convection flow, or rotating Couette (Taylor-Couette) flow) conclusions of traditional approach well match with laboratory studies.

BUT for the other kinds of hydrodynamic flows, especially those driven predominantly by *shear* forces, the predictions of the normal modes approach fail to match most experiments. In particular [3]:

- For *Poiseuille* flow, eigenvalue analysis predicts a critical Reynolds number $R = 5772$ at which instability should first occur. While in the laboratory, transition to turbulence is observed at Reynolds numbers as low as $R \simeq 1000$.
- For *Couette* flow, eigenvalue analysis predicts stability **for all** R, while in reality transition is observed for Reynolds numbers as low as $R \simeq 350$.

Traditionally, this anomaly ("subcritical transition to turbulence") was recognised as a failure of linearization and was attributed to nonlinear effects.

5.2 Theoretical difficulties:

Linear algebra says that even if all of the eigenvalues of a linear system are distinct and lie inside the lower half-plane, inputs to that system may be amplified by arbitrarily large factors if the eigenfunctions *are not orthogonal* to one another [4]. An opertor whose eigenfunctions are mutually orthogonal is said to be "normal," and the linear operators that arise in the Bénard and Taylor-Couette problems fall in this category.

Reddy et al. discovered [5] (surprisingly it happened only a few years ago) that the operators that arise in Poiseuille and Couette flows are non-normal. At the same time it was shown that small perturbations to these flows may be amplified by factors of many *thousands* even when all the eigenvalues are in the lower half-plane [6–8].

6 Is eigenvalue analysis the *only* tool to deal with shear flows?

Fortunately the answer is NO.

7 What is an alternative paradigm for shear flows?

The answer is: **Kelvin's approach.**

Lord Kelvin (W. Thomson) in his 1887 paper [1], entitled "Stability of fluid motion: rectilinear motion of viscous fluid between two parallel plates" [Philos. Mag. **24**, 188 (1887).] has given an explicit analytical solution for linearized disturbances in unbounded parallel viscous flow with uniform shear. Later little attention was paid to this solution. Strikingly enough in most accounts of hydrodynamic stability theory (for example Lin [9]; Betchov & Criminale [10]; Drazin & Reid [11]) it is not even mentioned! In astrophysics Kelvin-like method and corresponding solutions were introduced by Goldreich & Lynden-Bell [12] and later the method (so called "shearing sheet approximation") was successfully used by various authors [13–18] in a number of astrophysical situations, especially in accretion disk physics.

8 What is Kelvin's method and solution?

Consider two-dimensional unbounded plane Couette flow of an incompressible viscous fluid:

$$\nabla \cdot \mathbf{V} = 0, \tag{5}$$

$$\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \Delta \mathbf{V}, \tag{6}$$

with background velocity field, given by (2). We let $V_x \equiv \partial_y \Psi$, $V_y \equiv -\partial_x \Psi$, so that $\Omega \equiv (\nabla \times \mathbf{V})_z = -\Delta \Psi$. Now setting $\Psi = \Psi_0 + \Psi_1$ and $\Omega = \Omega_0 + \Omega_1$, we get:

$$(\partial_t + Ay\partial_x)\Omega_1 = \nu \Delta \Omega_1, \tag{7}$$

8.1 Normal mode approach

It separates variables by $\Psi_1 = \phi(y) exp(ik_x - i\omega t)$, which implies

$$\Omega_1(x, y, t) = -[\phi''(y) - k^2 \phi(y)] exp(ikx - i\omega t), \tag{8}$$

and transforms (7) to **Orr-Sommerfeld** equation:

$$(Aky - \omega)(\phi'' - k^2 \phi) = -i\nu(\phi^{IV} - 2k^2 \phi'' + k^4 \phi). \tag{8}$$

Due to Marcus and Press [19]: *There is not much doubt that viscous plane Couette flow is always stable to small disturbances, ones which satisfy the linear Orr-Sommerfeld equation. Nevertheless, no direct proof of stability is known.* Analytic treatments revealed variety of different asymptotic regimes in the wavenumber/Reynolds number plane. While numerical studies face embarrasing sensitivity of the equation to different truncation errors.

So Marcus and Press conclude: *Since the Orr-Sommerfeld equation is so uncooperative, it would seem reasonable to investigate how far one can proceed without it.*

8.2 "Kelvin's cold trail" [19]:

Kelvin noticed that (7) possess a symmetry associated with y direction: a combination of y translation and an velocity boost to a moving frame, or (equivalently) a pure translation in the *Lagrangian* coordinates

$$y_1 \equiv y, \qquad x_1 \equiv x - Ayt. \tag{10}$$

So that an alternative ansatz to (8) is the following separation of variables:

$$\Omega_1(x,y,t) = g(t)exp(ik_{x_1}x + ik_{y_1}y_1) = g(t)exp[ik_{x_1}x_1 + i(k_{y_1} - Ak_{x_1}t)y]. \quad (11)$$

This trick gives ODE for $g(t)$. In contrast with Orr-Sommerfeld equation this one is soluble by inspection and the solution is [19]:

$$g(t) = Gexp(-\nu t[k_{x_1}^2 + k_{y_1}^2/4 + (1/3)(Ak_{x_1}t - (3/2)k_{y_1})^2]). \quad (12)$$

One can see already that *asymptotic* stability is likely, since at late times

$$\Omega_1 \approx exp^{-\frac{1}{3}A^2\nu k_{x_1}^2 t^3}. \quad (13)$$

However, the main advantage of Kelvin's approach is that it allows to trace not only asymptotic behaviour of the solutions but gives a chance to look at the temporal evolution of perturbations **at early times**. In order to expose this more clearly let us consider inviscid $\nu = 0$ case and replay Kelvin's analysis for the linearized equations, derived directly from continuity and Euler equations:

$$\partial_x u_x + \partial_y u_y = 0, \quad (14)$$

$$(\partial_t + Ay\partial_x)u_x + Au_y = -c_s^2\partial_x d, \quad (15)$$

$$(\partial_t + Ay\partial_x)u_y = -c_s^2\partial_y d, \quad (16)$$

Applying again Kelvin's substitution of variables we convert the set to

$$\partial_{x_1} u_x + (\partial_{y_1} - At_1\partial_{x_1})u_y = 0, \quad (17)$$

$$\partial_{t_1} u_x + Au_y = -c_s^2\partial_{x_1} d, \quad (18)$$

$$\partial_{t_1} u_y = -c_s^2(\partial_{y_1} - At_1\partial_{x_1})d. \quad (19)$$

The coefficients of the initial system were spatially inhomogeneous—they depend on the spatial coordinate y. In new variables this inhomogeneity is exchanged onto the temporal inhomogeneity so that Kelvin's separation of variables (11) reduces the equations to the following simple set:

$$v_x + \beta(\tau)v_y = 0, \quad (20)$$

$$v_x^{(1)} = -Rv_y - D, \quad (21)$$

$$v_y^{(1)} = -\beta(\tau)D, \quad (22)$$

where hereafter $F^{(n)}$ will denote the n-th order time derivative of F and: $D\equiv i\hat{d}$, $R\equiv A/(c_s k_x)$, $\tau\equiv c_s k_x t_1$, $\beta(\tau)\equiv k_y(0)/k_x - R\tau\equiv\beta_0 - R\tau$, $v_x\equiv\hat{u}_x/c_s$, $v_y\equiv\hat{u}_y/c_s$.

Eqs.(20–22) imply that $v_y - \beta(\tau)v_x\equiv\mathcal{C} = const$ and the problem has exact analytic solution which is:

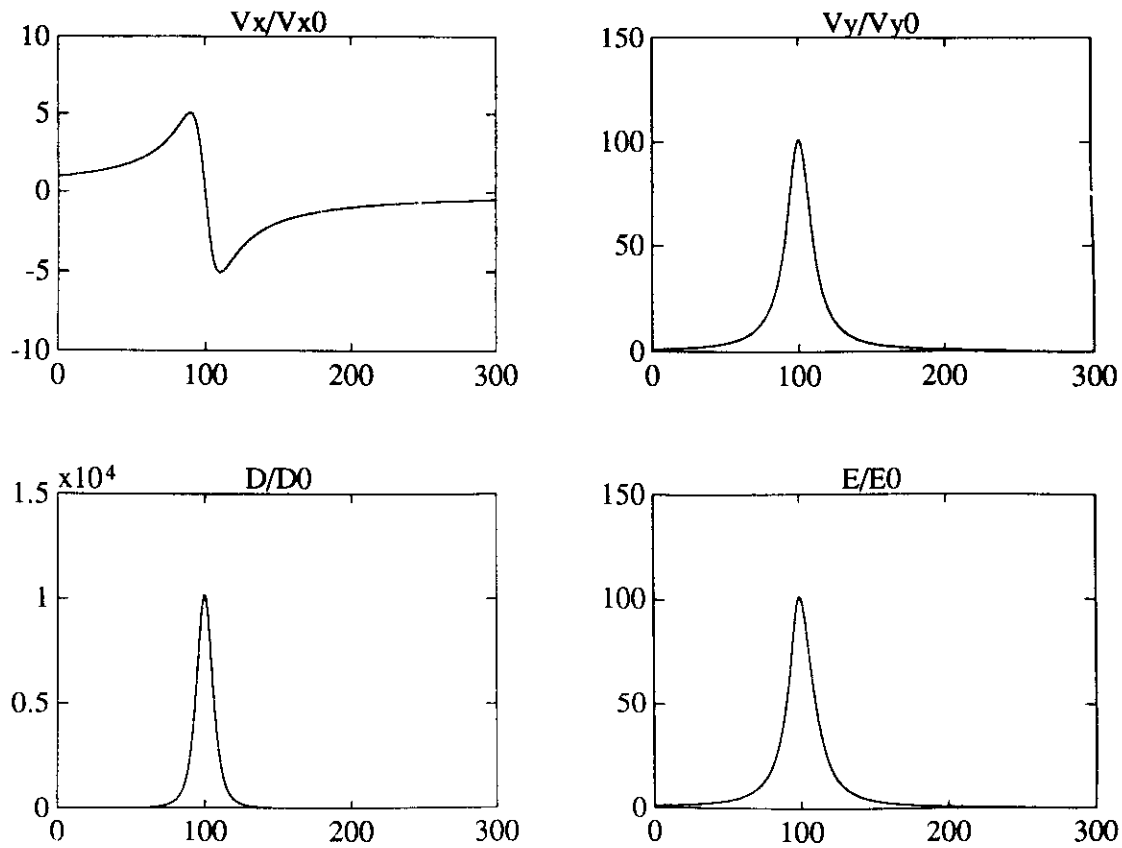
$$v_x(\tau) = -\frac{\mathcal{C}\beta(\tau)}{1+\beta^2(\tau)}, \tag{23}$$

$$v_y(\tau) = \frac{\mathcal{C}}{1+\beta^2(\tau)}, \tag{24}$$

$$D(\tau) = -\frac{2R\mathcal{C}}{[1+\beta^2(\tau)]^2}, \tag{25}$$

$$E(\tau)\equiv\frac{v_x^2+v_y^2}{2} = \frac{\mathcal{C}^2}{2[1+\beta^2(\tau)]}, \tag{26}$$

These solutions exhibit notable transient growth at early times, when $k_{x_1}k_{y_1} > 0$. Energy of perturbation increases up to the moment of time $\tau_*\equiv\beta_0/R$ and afterwards begins to decrease tending to zero at asymptotics. Notice that $E(\tau_*)/E(0)\sim 1+\beta_0^2$, i.e., we have substantial transient increase when $k_{y_1}/k_{x_1}\gg 1$.



9 What is surprising with shear flows?

9.1 Role of compressibility [20]: shear-induced energy extraction

In the same setup but taking into account compressibility:

$$(\partial_t + Ay\partial_x)d + \partial_x u_x + \partial_y u_y = 0, \tag{27}$$

$$(\partial_t + Ay\partial_x)u_x + Au_y = -c_s^2\partial_x d, \tag{28}$$

$$(\partial_t + Ay\partial_x)u_y = -c_s^2\partial_y d, \tag{29}$$

and repeating Kelvin's standard procedure we get:

$$D^{(1)} = v_x + \beta(\tau)v_y, \tag{30}$$

$$v_x^{(1)} = -Rv_y - D, \tag{31}$$

$$v_y^{(1)} = -\beta(\tau)D, \tag{32}$$

The total energy density of the perturbations

$$E \equiv (|v_x|^2 + |v_y|^2)/2 + |D|^2/2, \tag{33}$$

obeys the following ODE

$$E^{(1)} = -Rv_x v_y, \tag{34}$$

which clearly implies that temporal evolution of perturbations, whatever it will be, is evoked by the presence of nonzero shear $R \neq 0$.

Evaluating the expression for the R parameter we find that $R = (V_0/c_s)(l_x/L)$. Since we are considering only small-scale perturbations ($l_x \equiv 1/k_x \ll L$), it is clear that if we consider the *subsonic* flow ($V_0 < c_s$), then $R \ll 1$.

There is an important *algebraic* relation between the perturbation functions:

$$v_y - \beta(\tau)v_x - RD = \mathcal{C}, \tag{35}$$

which helps to derive from (30–32) the following second-order explicit ODE:

$$v_x^{(2)} + [1 + \beta^2(\tau)]v_x = -\mathcal{C}\beta(\tau), \tag{36}$$

General solution of (36) is the sum of the *special* solution of this equation and the *general* solution of the corresponding homogeneous equation.

$$v_x^{(2)} + [1 + \beta^2(\tau)]v_x = 0. \tag{37}$$

The latter solution is readily obtained owing to the smallness of R parameter, when $\omega(\tau)$ depends on τ adiabatically. Mathematically, this condition may be written simply as $|\omega(\tau)^{(1)}| \ll \omega^2(\tau)$ or, taking into account the definition of $\omega(\tau)$, as:

$$R|\beta(\tau)| \ll \omega^3(\tau) = \left[1 + \beta^2(\tau)\right]^{3/2}. \tag{38}$$

For subsonic Couette flows $R \ll 1$ and the condition (38) holds for all possible values of $|\beta(\tau)|$. In other words, since the temporal variability of $|\beta(\tau)|$ is determined by the "linear drift" of SFH, (38) is valid at all stages of the evolution of the SFH and the approximate solution of the homogeneous equation may be written as

$$\hat{v}_x(\tau) = \frac{C}{\sqrt{\omega(\tau)}} sin[\varphi(\tau) + \varphi_0], \tag{39a}$$

$$\varphi(\tau) = \int \omega(\tau) d\tau = -\frac{1}{2R}\left[\beta(\tau)\omega(\tau) + ln|\beta(\tau) + \omega(\tau)|\right]. \tag{39b}$$

As regards *special* solution of the inhomogeneous equation (36) it may be derived owing to the smallness of the R parameter too. In particular, the solution may be expressed by the following series:

$$\bar{v}_x(\tau) = \mathcal{C} \sum_{n=0}^{\infty} R^{2n} y_n(\tau), \tag{40a}$$

$$y_0(\tau) = -\beta(\tau)/\omega^2(\tau), \tag{40b}$$

$$y_n(\tau) = -\frac{1}{\omega^2(\tau)} \frac{\partial^2 y_{n-1}}{\partial \beta^2}. \tag{40c}$$

Since $R \ll 1$, the terms with higher powers of R are negligible and the general solution of the inhomogeneous equation (36) may be written approximately as:

$$v_x(\tau) \simeq \frac{C}{\sqrt{\omega(\tau)}} sin(\varphi(\tau) + \varphi_0) - \frac{\mathcal{C}\beta(\tau)}{\omega^2(\tau)}. \tag{41}$$

The total energy density of the perturbations:

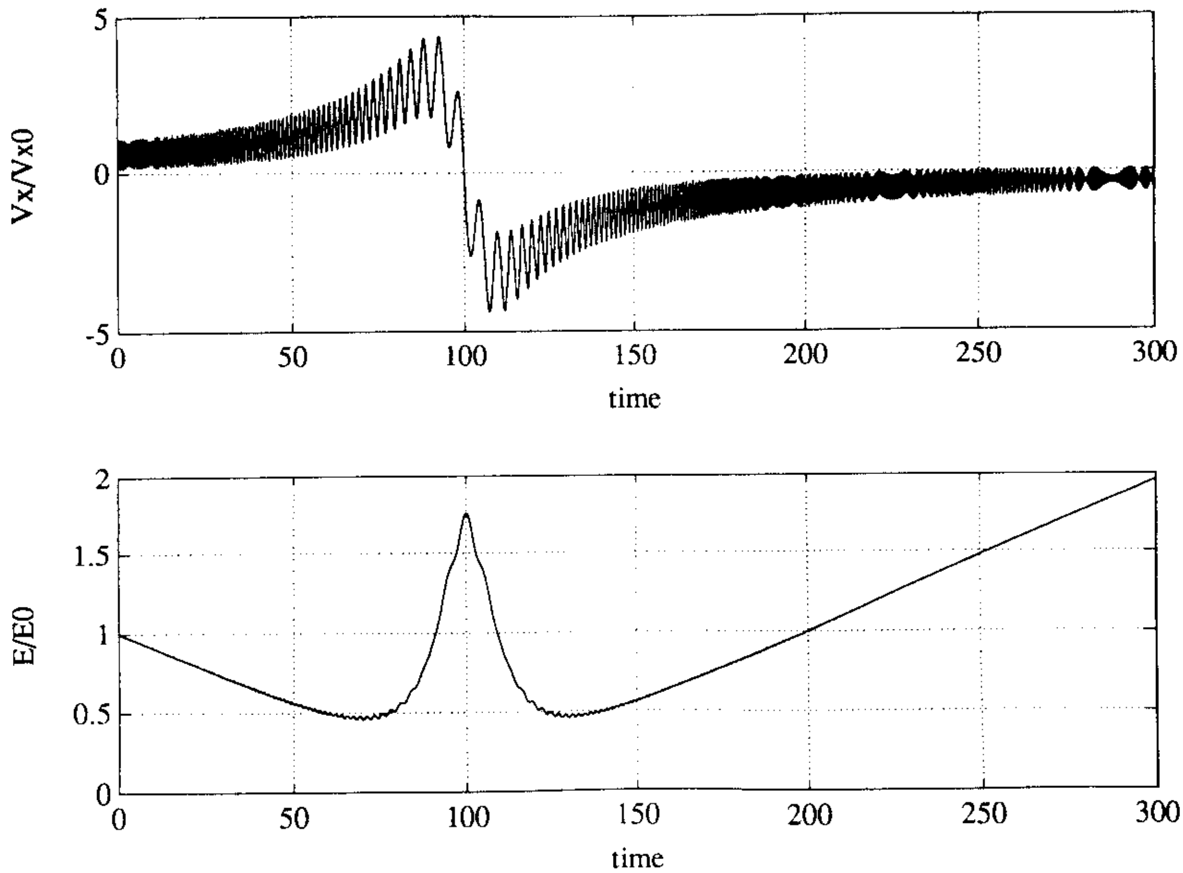
$$E(\tau) \approx \frac{1}{2}\left[C\omega(\tau) + \left(\frac{\mathcal{C}}{\omega(\tau)}\right)^2\right]. \tag{42}$$

- Clearly, when $C/\mathcal{C} \ll 1$ the SFH may be treated as mainly incompressible and vortical perturbation and the solution reduces to above described Kelvin's incompressible transients.

- When $C/\mathcal{C}\gg 1$ perturbation becomes mainly of the sound-type. $E\sim\omega(\tau) = \sqrt{1+(\beta_0 - R\tau)^2}$. Initially, for $k_y(0)k_x > 0$ ($\beta_0 > 0$), at $0 < \tau < \tau_*$, the energy decreases and reaches its minimum at $\tau = \tau_*$. Afterwards, it begins to increase at $\tau_* < \tau < \infty$, when the SFH "emerges" into the area of **k**-plane in which $k_y(\tau)k_x < 0$ ("growth area" for the sound-type perturbations). If the SFH is in the "growth area" from the beginning ($\beta_0 < 0$), its energy increases monotonously.

- In the general case (see Fig.2) the "transient growth" and the "sound-type" evolution are superimposed on each other.

Thus we see that velocity shear ensures existence of a surprising new effect in this simple system: **compressible, 2D perturbations are able to extract effectively the energy of the mean shear flow.**



9.2 Multiwave case ($n > 1$ modes): shear-induced wave transformations [21].

Let us complicate further our setup and consider 2D, compressible, magnetized, unbounded parallel flow with uniform velocity shear (plane, magnetized Couette flow). External regular magnetic field is supposed to be uniform and $\mathbf{B_0} \| \mathbf{U_0}$.

The basic system of linearized equations governing the evolution of the small-scale, 2D perturbations in this flow is:

$$(\partial_t + Ay\partial_x)d + \partial_x u_x + \partial_y u_y = 0, \tag{43}$$

$$(\partial_t + Ay\partial_x)u_x + Au_y = -c_s^2 \partial_x d, \tag{44}$$

$$(\partial_t + Ay\partial_x)u_y = -c_s^2 \partial_y d + c_A^2 [\partial_x b_y - \partial_y b_x], \tag{45}$$

$$(\partial_t + Ay\partial_x)b_y = \partial_x u_y, \tag{46}$$

$$\partial_x b_x + \partial_y b_y = 0, \tag{47}$$

repeating once again standard Kelvin procedure we get:

$$D^{(1)} = v_x + \beta(\tau) v_y, \tag{48}$$

$$v_x^{(1)} = -Rv_y - D, \tag{49}$$

$$v_y^{(1)} = -\beta(\tau)D + \sigma^2(1 + \beta(\tau)^2)b, \tag{50}$$

$$b^{(1)} = -v_y, \tag{51}$$

where all notataions are the same as above and in addition $\mathbf{b} \equiv \mathbf{B}'/|\mathbf{B_0}|$, c_A is an Alfvén velocity, $b \equiv i\hat{b_y}$, and $\sigma^2 \equiv (c_A/c_s)^2$.

The dimensionless total energy density of the perturbations in the **k**-space we define as:

$$E \equiv (|v_x|^2 + |v_y|^2)/2 + |D|^2/2 + \sigma^2(|b_x|^2 + |b_y|^2)/2. \tag{52}$$

If we introduce a new variable: $\psi \equiv D + \beta(\tau)b$ we can reduce the system (48–51) to the pair of the ordinary differential equations of the second order:

$$\psi^{(2)} + {\omega_1}^2 \psi - \beta(\tau) b = 0, \tag{53}$$

$$b^{(2)} + {\omega_2}^2 b - \beta(\tau)\psi = 0. \tag{54}$$

The equations of this type are well-known in the general theory of oscillations. They describe coupled oscillations with two degrees of freedom. In particular, uncoupled eigenfrequencies appearing in (56–57) are: $\omega_1 \equiv 1$ and $\omega_2(\tau) \equiv \sqrt{\sigma^2 + (1+\sigma^2)\beta(\tau)^2}$, while the coupling coefficient is $k(\tau) \equiv -\beta(\tau)$. The presence of shear in the flow ($R \neq 0$) ensures temporal variability of one of the uncoupled eigenfrequencies ($\omega_2(\tau)$) and the coupling coefficient $k(\tau)$. Note that a dependence of these quantities on time may be considered as adiabatic when $R \ll 1$.

The most suitable condition for the transformation of the MHD waves and corresponding energy transfer is when $\sigma^2 \simeq 1$ (i. e. $c_A \simeq c_s$ — equipartition between magnetic and thermal energies) and the transformation occurs nearby $\tau = \tau_*$. Numerical simulations show that for certain values of R there happens almost complete transformation of SMW into FMW (if, initially, was excited SMW) and *vice versa.* Fig.3 illustrates SMW-FMW transformation process. It is clearly seen that at $0 < \tau < \tau_*$ the SMW energy remains almost constant, as it should be, since the energy varies in the interval adiabatically $E(\tau) \sim \Omega_1$ and Ω_1 is almost constant there. For $\tau > \tau_*$, where the wave has been already transformed into the FMW, $E(\tau) \sim \Omega_2$ and increases quasi-linearly with the increase of τ. Thus, if initially we had a wave (SMW) which did not exchange an energy with the mean flow, after the transformation appears a wave (FMW) which effectively extracts the shear energy from the regular flow. It is clear that this kind of transformation process may change radically the behavior of the flow.

Fig.3(a)

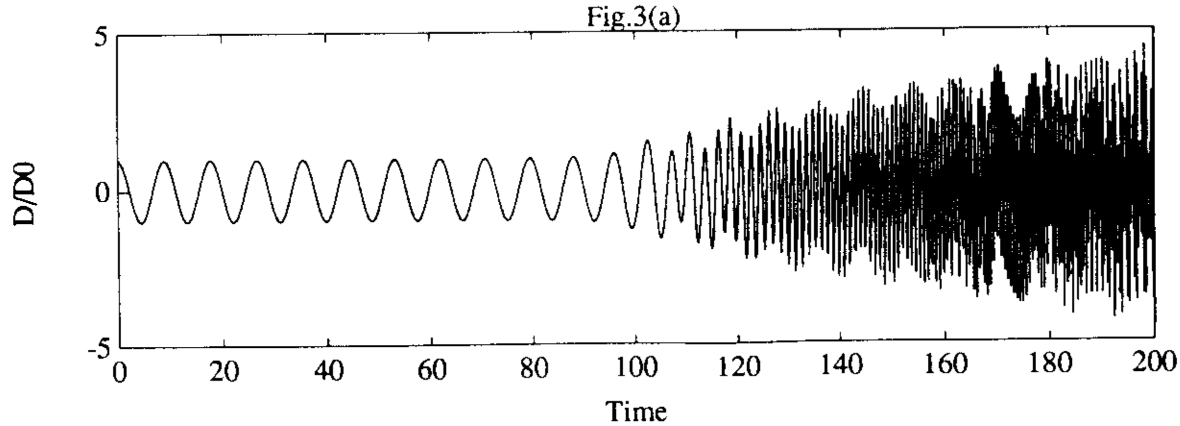
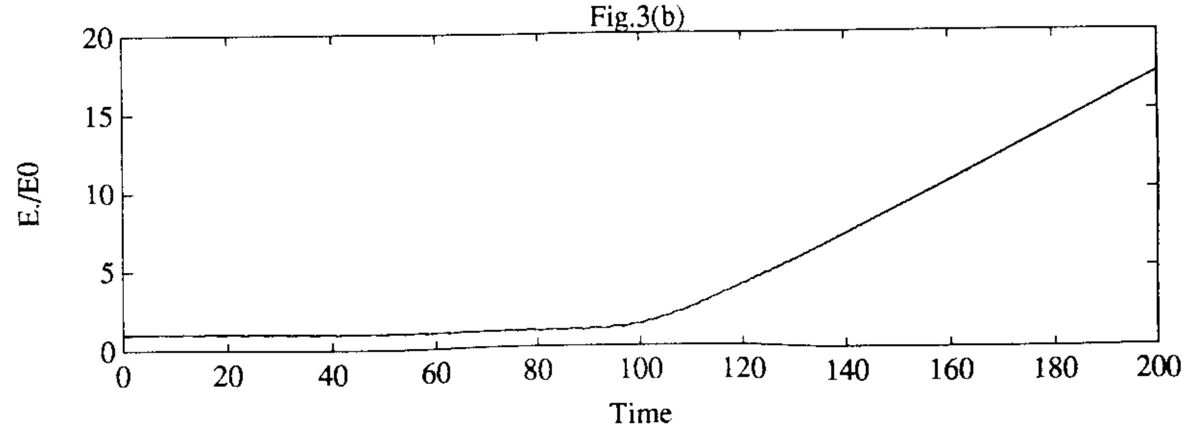


Fig.3(b)



Thus we see that velocity shear induces yet another surprising effect: **linear mechanism of mutual wave transformations with corresponding energy transfer between them.** The nature of the wave transformation effect, discussed in this paper, qualitatively differs from the already known linear transformation mechanisms. Density inhomogeneity induced mode transformation occurs permanently in the limited spatial area (across the density inhomogeneity), while in our case transformation of linear waves occurs in the whole volume, filled by the flow, in the limited time interval.

10 What is the main message of this lecture?

Velocity shear is an important physical aspect, which may considerably influence physical processes in moving continuous media. This aspect up to recent times was either ignored and/or inadequately understood. My main message is to point at the importance and universality of shear effects, to emphasize and advertise their attractive simplicity and, thus, to encourage my colleagues to look for shear effects in their specific fields of interest.

11 Next lecture?

In the next lecture I will speak about some applications of above discussed general shear effects in a few specific hydrodynamic and plasma problems. In particular, the following items will be discussed:

- Coupling of sound and internal gravity waves in shear flows [22];
- Velocity shear induced effects on electrostatic ion perturbations [23];
- Velocity shear generated Alfvén waves in e^+e^- plasmas [24];
- Escaping radio emission from pulsars: possible role of velocity shear [25].

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12 Appendix A: Adiabatic oscillations

Linear, second order ODE

$$F^{(2)} + \omega^2(\tau)F = 0. \qquad (A.1)$$

describes linear oscillations of mathematical pendulum (or, more generally, some linear oscillator) with variable (time dependent) parameters (i.e., length).

An arbitrary Function $G(\tau)$ is said to vary *slowly*, or *adiabatically* if its relative variation during the period of the oscillations is small. Or, in other words, if:

$$P(\tau)|G^{(1)}(\tau)| \ll |G(\tau)| \quad \Leftrightarrow \quad 2\pi|G^{(1)}(\tau)| \ll \omega(\tau)|G(\tau)|, \qquad (A.2)$$

For the function $\omega(t)$ by itself the condition (A.2) reduces to

$$|\omega^{(1)}(\tau)| \ll \omega^2(\tau). \qquad (A.3)$$

If we seek for the solution of equation (A.1) in the following form:

$$F(\tau) = a(\tau)e^{i\phi(\tau)}, \qquad (A.4)$$

$$\phi(\tau) \equiv \int_0^\tau \omega(\tau')d\tau', \qquad (A.5)$$

then for the derivative of $F(\tau)$ we have:

$$F^{(2)} = a^{(2)}e^{i\phi} + i(2a^{(1)}\omega + a\omega^{(1)})e^{i\phi} - a\omega^2 e^{i\phi},$$

and equation (A.1) reduces to:

$$a^{(2)}e^{i\phi} + i(2a^{(1)}\omega + a\omega^{(1)})e^{i\phi} = 0. \qquad (A.6)$$

The adiabatic character of the $\Omega(t)$ variation ensures that the first term on the left-hand side in (A.6) is much less than the second and the third ones. It means

$$2a^{(1)}\omega \simeq -a\omega$$

and, consequently,

$$C \equiv a(\tau)^2\omega(\tau). \qquad (A.7)$$

A quantity C is called an *adiabatic invariant* of equation (A.1). The solution of this equation may be written explicitly as:

$$F(\tau) = \frac{C}{\sqrt{\omega(\tau)}} exp\left[i\left(\int_0^\tau \omega(\tau')d\tau' + \phi_0\right)\right]. \qquad (A.8)$$

13 Appendix B: Coupled oscillations

The mathematical description of the motion of two coupled linear oscillators leads to the following pair of second order ODE's:

$$F_1^{(2)} + \omega_1^2 F_1 + \mathcal{C} F_2 = 0, \tag{B.1}$$

$$F_2^{(2)} + \omega_2^2 F_2 + \mathcal{C} F_1 = 0, \tag{B.2}$$

where the *eigenfrequencies* of the F_1 and F_2 oscillators are ω_1 and ω_2, respectively, and $\mathcal{C}$ is the corresponding *coupling coefficient*. For **constant** eigenfrequencies and coupling coefficient the general solution of the system can always be represented as a combination of the *normal modes*:

$$F_1 = F_+ \cos(\Omega_+ t - \phi_+) + F_- \cos(\Omega_- t - \phi_-), \tag{B.3}$$

$$F_2 = \sigma_+ F_+ \cos(\Omega_+ t - \phi_+) + \sigma_- F_- \cos(\Omega_- t - \phi_-), \tag{B.4}$$

where the *fundamental* or *normal frequencies* of the coupled oscillations, $\Omega_\pm$, are determined by

$$\Omega_\pm^2 \equiv \frac{1}{2}\left[(\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4\mathcal{C}^2}\right]. \tag{B.5}$$

The auxiliary quantities $\sigma_\pm$ in (B.4) relate oscillation amplitudes of the two normal modes to each other:

$$\sigma_\pm \equiv \frac{\Omega_\pm^2 - \omega_1^2}{\mathcal{C}} = \frac{\mathcal{C}}{\Omega_\pm^2 - \omega_2^2}, \tag{B.6}$$

while the $\phi_\pm$ are the initial phases of the coupled oscillators.

In a coupled system described by (B.1)-(B.2) it is always possible (with properly chosen initial conditions) to excite a simple harmonic motion in which *both* oscillators have *the same* frequency, viz. one of the fundamental frequencies Ω_+ or Ω_-. From (B.3)-(B.4) it is easily seen that this regime is established when either F_+ or F_- is equal to zero. It immediately follows that

- $F_+ \neq 0$ and $F_- = 0$, when $F_{20} = \sigma_+ F_{10}$, and $\partial_t F_{20} = \sigma_+ \partial_t F_{10}$
- $F_- \neq 0$ and $F_+ = 0$, when $F_{20} = \sigma_- F_{10}$, and $\partial_t F_{20} = \sigma_- \partial_t F_{10}$.

When eigenfrequencies and/or coupling coefficient of a coupled oscillating system vary in time and when the variation is slow or *adiabatic*, then the system exhibits notable mutual transformations of normal oscillations with corresponding energy transfer between them. The *mechanical* example of the oscillatory system, governed by this kind of equations, is the system of two coupled pendulums with slowly (adiabatically) variable lengths (i.e., eigenfrequencies) and the interpendulum coupling coefficient. There are two necessary conditions for the effectiveness of the energy exchange between the weakly coupled pendulums:

- (A) There should exist a so called "degeneration region," (DR) where $|\Omega_+^2 - \Omega_-^2| \leq |\mathcal{C}(\tau)|$. In other words, in the case of weak coupling this condition implies that $\Omega_- \approx \Omega_+$, which means that the maximum energy exchange between the pendulums occurs when they have approximately the same length.
- (B) the DR should be "passed" slowly – in time interval sufficiently exceeding the beating period: $|\Omega_\pm^{(1)}(\tau)| \ll |\mathcal{C}(\tau)|$.

Literature Cited:

1. Lord Kelvin (W. Thomson), Phil. Mag. **24**, Ser. **5**, 188 (1887).
2. L. D. Landau and E. M. Lifshitz *Fluid Mechanics* Vol.6 of Course of Theoretical Physics (London, Pergamon Press, 1959).
3. L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, Science **261**, 578 (1993).
4. T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, New York 1976).
5. S. C. Reddy, P. J. Schmid, and D. S. Hennigson, SIAM J. Appl. Math. **53**, 15 (1993).
6. L. H. Gustavsson, J. Fluid mech. **224**, 241 (1991).
7. D. S. Hennigson, A. Lundbladh, A. V. Johansson, J. Fluid Mech. **250**, 169 (1993).
8. K. M. Butler and B. F. Farrell, Phys. Fluids A **4**, 1637 (1992).
9. C. C. Lin, *The Theory of Hydrodynamic Stability* (Cambridge University Press, Cambridge 1955).
10. R. Betchov and W. O. Criminale, *Stability of Parallel Flows* (Academic Press, New York 1967).
11. P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge 1981).
12. P. Goldreich and D. Lyndenn-Bell, Mon. Not. R. Astron. Soc. **130**, 125 (1965).
13. A. Toomre, Astrophys. J. **158**, 899 (1969)
14. P. Goldreich and S. Tremaine, Astrophys. J. **222**, 850 (1978)
15. L. O'C. Drury, Mon. Not. R. Astron. Soc. **193**, 337 (1980)
16. J. G. Lominadze, G. D. Chagelishvili, and R. A. Chanishvili, Pis'ma Astron. Zh. **14**, 856 (1988) [Sov. Astron. Lett. **14**, 364 (1988)].

17. S. A. Balbus and J. F. Hawley, Astrophys. J. **400**, 610 (1992)

18. S. H. Lubow, and H. C. Spruit, Astrophys. J. **445**, 337 (1995).

19. P. Marcus and W. H. Press, J. Fluid Mech. **79**, 525 (1977)

20. G. D. Chagelishvili, A. D. Rogava, and I. N. Segal, Phys. Rev. (E) **50**, 4283 (1994).

21. G. D. Chagelishvili, A. D. Rogava, and D. G. Tsiklauri, Phys. Rev. (E) **53**, 6028 (1996).

22. A. D. Rogava and S. M. Mahajan, Phys. Rev. (E) **55**, 1185 (1997).

23. A. D. Rogava, G. D. Chagelishvili, and V. I. Berezhiani, Phys. Plasmas, **4** (12) p.?? (1997).

24. A. D. Rogava, S. M. Mahajan, and V. I. Berezhiani, Phys. Plasmas **3**, 3545 (1996).

25. S. M. Mahajan, G. Z. Machabeli, and A. D. Rogava, Ap. J. Lett **479**, L129 (1997).