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SHEAR FLOW SURPRISES

2. Applications

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Coupling of sound and internal waves in shear flows

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Gravity waves in the parallel shear flow of a continuously stratified compressible fluid are considered. It is demonstrated that the shear induces a coupling between the sound waves and the internal gravity waves. The conditions for the effectiveness of the coupling are defined. It is also shown that, under suitable conditions, beat waves can be generated. [S1063-651X(96)12012-2]

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It is well known that in a medium with a gravity-induced stratification the *buoyancy forces* tend to excite *internal gravity waves* originating from a balance between the fluid inertia and the gravitational restoring force [1,2]. The internal gravity waves (hereafter referred to as IGW's), propagating in a differentially moving fluid—that is, in a *shear flow* with continuous, gravity-induced stratification, display a rich and complex structure.

In order to study this problem it is very convenient to employ the scheme where a moving coordinate system is used and the temporal problem is examined directly. The method can, in principle, be used for any velocity profiles but it is mostly useful for ones that are piecewise linear [3–5]. Going to the moving frame mitigates the need for a Laplace transform [5–7] and greatly simplifies the solution of the initial value problem.

The problem of the evolution of IGW in an *incompressible* parallel shear flow with linear velocity profile was recently considered by Chagelishvili [8]. In that study, nonmodal algebraically growing solutions, indicating the possibility of anomalous amplification of IGW in shear flows, were readily found. This paper deals with the same problem for a compressible, unbounded, parallel flow with a uniform (linear) shear.

In [9], where the evolution of two-dimensional (2D) perturbations in a compressible, plane Couette flow was considered the mechanism of the energy exchange between the mean flow and sound-type perturbations was discovered. A *linear* mechanism of mutual transformation of waves, and a corresponding energy transfer induced by the existence of the velocity shear was found in [10] for the 2D waves in an unbounded, parallel *hydromagnetic* flow (see also [11]). It seems likely that analogous mechanisms will be operative in other kinds of parallel shear flows, where conditions for the excitation of several (more than one) wave modes exist.

Since we are dealing with the shear flow in which sound waves (SW) and IGW may be simultaneously excited, it is reasonable to expect that these modes may become effectively coupled implying a linear mutual transformation with corresponding energy transfer between the modes.

Let us consider the evolution of two-dimensional perturbations in a compressible, unbounded shear flow with a steady unidirectional mean velocity (parallel flow) that varies linearly with height. Let us choose the coordinate axes such that the regular velocity vector $\mathbf{U}_0 \equiv (Ay,0)$, is along x, and the acceleration due to gravity $\mathbf{g}=(0,-g_0)$ is along negative y. The basic system of linearized equations, describing the evolution of the small-scale, 2D perturbations in this flow, takes the form

$$D_t\rho' + \rho_0(\partial_x u_x + \partial_y u_y) + (\partial_y \rho_0)u_y = 0, \tag{1}$$

$$D_t S' + (\partial_y S_0)u_y = 0, \tag{2}$$

$$D_t u_x + A u_y = -\frac{1}{\rho_0}\partial_x P', \tag{3}$$

$$D_t u_y = -\frac{1}{\rho_0}\partial_y P' + \frac{\rho'}{\rho_0^2}\partial_y P_0, \tag{4}$$

$$\rho' = \left(\frac{\partial \rho_0}{\partial S_0}\right)_{P_0} S' + \frac{P'}{c_s^2}, \tag{5}$$

where $c_s \equiv [(\partial P_0/\partial \rho_0)]^{1/2}$, and $D_t \equiv \partial_t + Ay\partial_x$. Making use of the equilibrium condition $\partial_y P_0 = -\rho_0 g_0$, it is straightforward to eliminate ρ' from (1) and (4) to yield

$$D_t P' + \rho_0 c_s^2(\partial_x u_x + \partial_y u_y) - \rho_0 g_0 u_y = 0, \tag{6}$$

$$D_t u_y = -\frac{1}{\rho_0}\partial_y P' - \frac{g_0}{\rho_0}\left(\frac{\partial \rho_0}{\partial S_0}\right)_{P_0} S' - \frac{g_0}{\rho_0 c_s^2} P'. \tag{7}$$

To "set up" the analysis, we affect the transformation, $x_1 = x - Ayt$; $y_1 = y$; $t_1 = t$, $(D_t \to \partial_{t_1}$; $\partial_y \to \partial_{y_1} - At_1\partial_{x_1})$, which effectively takes us from the laboratory to the local rest frame of the basic flow [5,9–12]. In new coordinates, where the initial inhomogeneity in space (y) has been exchanged for a new inhomogeneity in time, we may expand the perturbations as

$F=\int dk_{x_1}dk_{y_1}\hat{F}(k_{x_1},k_{y_1},t_1)\exp[i(k_{x_1}x_1+k_{y_1}y_1)]$, and convert Eqs. (6), (2), (3), and (7) to a set of first order, ordinary differential equations for $\hat{F}(k_{x_1},k_{y_1},t_1)$, which will be hereafter referred to as spatial Fourier harmonics (SFH) [9–12]. It is convenient to write these equations in dimensionless notation: $R\equiv A/c_s k_{x_1}$, $T\equiv c_s k_{x_1}t_1$, $\beta_0\equiv k_{y_1}/k_{x_1}$, $\beta(T)\equiv\beta_0-RT$, $v_{x,y}\equiv\hat{u}_{x,y}/c_s$, $e\equiv -k_{x_1}\hat{S}'/(\partial_y S_0)$, $f\equiv\hat{P}'/P_0$, $\alpha\equiv P_0/\rho_0 c_s^2$, and $\xi\equiv g_0/k_{x_1}c_s^2$. We also note that the dimensionless measure of the characteristic frequency of pure internal gravitational waves, $\omega_0^2\equiv -(g_0/\rho_0)(\partial\rho_0/\partial S_0)_{P_0}(\partial_y S_0)$, can be readily defined as $W^2\equiv(\omega_0/c_s k_{x_1})^2$.

In this notation, the set of equations reduces to

$$\alpha\partial_T f=-i[v_x+\beta(T)v_y]+\xi v_y, \tag{8}$$

$$\partial_T e=v_y, \tag{9}$$

$$\partial_T v_x=-Rv_y-i\alpha f, \tag{10}$$

$$\partial_T v_y=-i\alpha\beta(T)f-W^2e+(1-\alpha)\xi f. \tag{11}$$

When gravity is absent ($W^2=\xi=0$) these equations [without Eq. (9)] reduce to the system describing plain sound waves in free shear flows [9]. Note that the IGW can be retained in the system by assuming a nonzero W^2. Furthermore, the coupling between IGW and SW will be nonzero even if the gravity-induced coupling (ξ) is small and negligible. Thus, without any fear of losing basic physics, we go ahead and neglect ξ everywhere, and find the simplified system of equations [$F=i\alpha f$],

$$\partial_T F=v_x+\beta(T)v_y, \tag{12}$$

$$\partial_T e=v_y, \tag{13}$$

$$\partial_T v_x=-Rv_y-F, \tag{14}$$

$$\partial_T v_y=-\beta(T)F-W^2e. \tag{15}$$

The spectral energy density of the SFH may be defined as $E\equiv(v_x^2+v_y^2)/2+F^2/2+W^2e^2/2$, where the three terms correspond, respectively, to the fluid kinetic energy, the acoustic potential energy, and the internal-wave potential energy. The spectral energy density $E(T)$ satisfies the differential equation $\partial_T E=-Rv_x v_y$. When $R=0$ (the fluid at rest), $E(T)$ is conserved as expected.

In terms of a new variable $\psi(T)\equiv F-\beta(T)e[\partial_T\psi=v_x+Re]$, it is easy to transform Eqs. (12)–(15) into the following pair of second order differential equations:

$$\partial_{TT}\psi+\psi+\beta(T)e=0, \tag{16}$$

$$\partial_{TT}e+[W^2+\beta^2(T)]e+\beta(T)\psi=0, \tag{17}$$

representing two oscillators coupled through $\beta(T)$ [13], with $\omega_1\equiv 1$ and $\omega_2(T)\equiv\sqrt{W^2+\beta^2(T)}$ as their respective eigenfrequencies. The presence of shear in the flow ($R\neq 0$) ensures temporal variability of one of the uncoupled eigenfrequencies [$\omega_2(T)$] and of the *coupling coefficient* $\beta(T)$. Note that the time dependence of these quantities may be considered *adiabatic* when $R\ll 1$ [9,10].

Fundamental vibrational frequencies of the coupled oscillators Eqs. (16) and (17) are equal to [13]

$$\Omega_\pm^2=\tfrac{1}{2}[\omega_1^2+\omega_2^2\pm\sqrt{(\omega_1^2-\omega_2^2)^2+4\beta^2}]. \tag{18}$$

Note that in the absence of gravity ($g_0=W=0$) $\Omega_+(T)\approx\sqrt{1+\beta^2(T)}$ reduces to the plain sound mode [9], while $\Omega_-(T)=0$, as it, certainly, should be.

Since the oscillation system, described by Eqs. (16) and (17), has two degrees of freedom its behavior may be determined by two functions $\psi(T)$ and $e(T)$. Note that all other physical quantities may be explicitly expressed in terms of ψ, e, and their first derivatives: $F=\psi+\beta(T)e$, $v_x=\partial_T\psi-Re$, and $v_y=\partial_T e$.

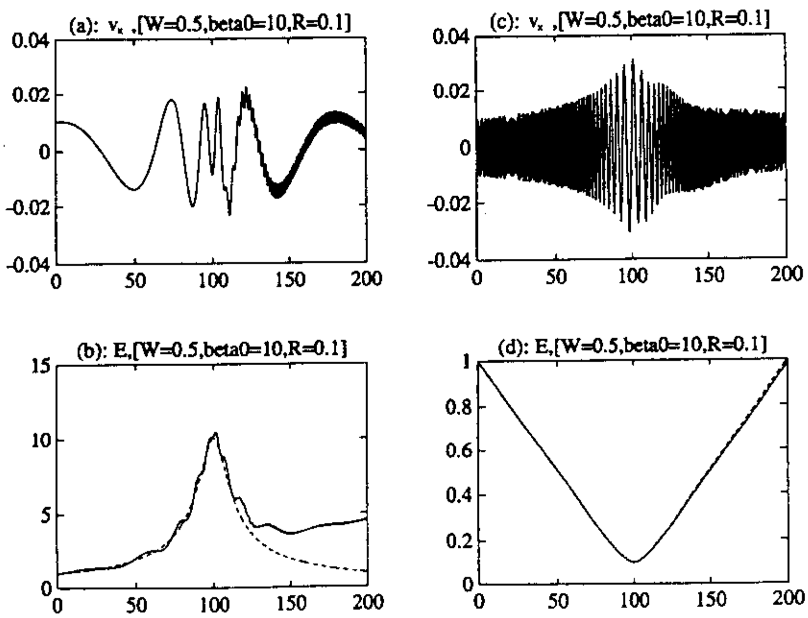

FIG. 1. The temporal evolution of the velocity $v_x(T)$ and energy $E(T)/E(0)$, respectively, for an initially pure IGW [(a) and (b)] and SW [(c) and (d)] modes. Dashed lines in (b) and (d) represent the $\Omega_-(T)/\Omega_-(0)$ (IGW) and $\Omega_+(T)/\Omega_+(0)$ (SW) curves, respectively, for initially excited modes. $\beta_0=10$, $R=0.1$, and $W=0.5$.

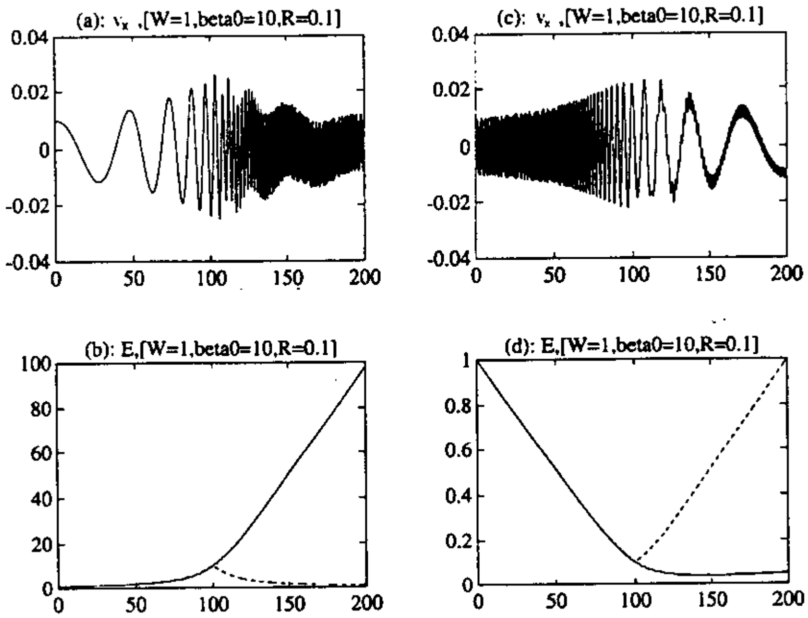

FIG. 2. The temporal evolution of the velocity $v_x(T)$ and energy $E(T)$, respectively, for an initially pure IGW [(a) and (b)] and SW [(c) and (d)] modes. Dashed lines in (b) and (d) represent the $\Omega_-(T)/\Omega_-(0)$ (IGW) and $\Omega_+(T)/\Omega_+(0)$ (SW) curves, respectively, for initially excited modes. $\beta_0=10$, $R=0.1$, and $W=1$.

The necessary conditions [14,10] for an effective energy exchange between two weakly coupled oscillators are the existence of a so-called "degeneracy region," (DR) where $|\omega_1^2-\omega_2^2|\leq|\beta(T)|$, and that the DR should be "passed" slowly — the traversal time should be much greater than the period of the beats $|\partial_T\omega_2(T)|\ll|\beta(T)|$. The degeneracy region is in the neighborhood of $T_*\equiv\beta_0/R$, and $W=1$ leads to the most efficient mode coupling, and hence to the possibility of mutual transformation of the modes. It is straightforward to see that for the current problem, the existence of DR is ensured if $|\beta(T)|<1$. As regards the condition for $|\partial_T\omega_2(T)|$, in our case it reduces to the inequality $R\ll\sqrt{W^2+\beta^2(T)}$, which is true for all T if $R\ll W$. Since $R\ll 1$, it is clear that for $W=1$, the conditon is always satisfied.

Regarding the "adiabatic behavior" of the modes, we should expect that the modes should normally follow the dispersion curves of their own: spectral energy density of either IGW $[E_-(T)]$ or SW $[E_+(T)]$ should be proportional to its corresponding frequency: $E_\pm\sim\Omega_\pm$ [9]. This mode of energy evolution, however, will not pertain in DR, where efficient transformation of one wave into the other occurs for $W=1$. For instance, the energy of an initially excited IGW mode increases approximately by the $E_-(T)\sim\Omega_-(T)$ law up to the vicinity of the point T_*, where it is partially transformed into SW. Afterwards, its energy evolution would still proceed adiabatically, but now according to the law $E_+(T)\sim\Omega_+(T)$.

One more, quite impressive, evolution regime can be realized when $R\ll\beta_0\ll 1$. In this particular case (with $W=1$), normal frequencies of ψ and e "oscillators" $[\Omega_+(T)$ and $\Omega_-(T)]$ are almost equal to each other and the coupling is inherently efficient. In this case beat modes will result: if initially, only one, say, the "e oscillator" (i.e.,

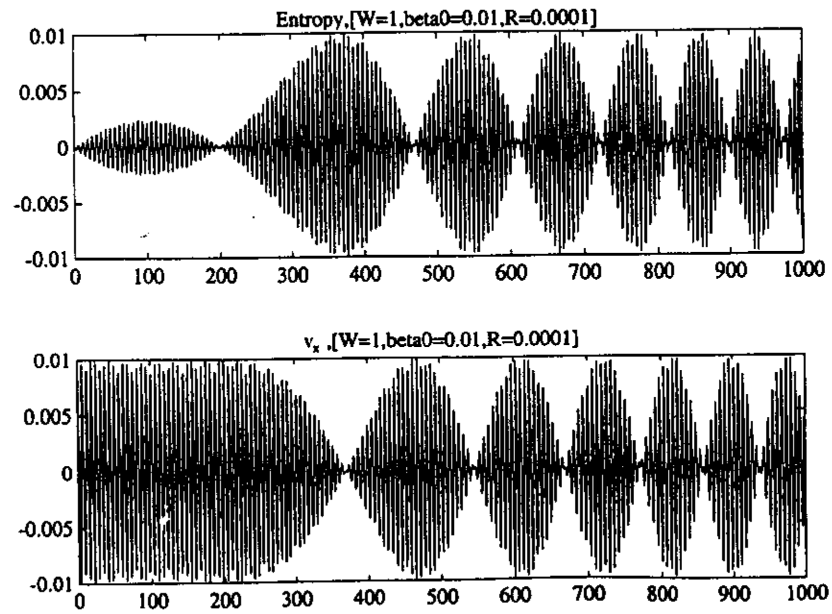

FIG. 3. Beat waves, displayed for $e(T)$ and $v_x(T)$ when $\beta_0=10^{-2}$, $R=10^{-4}$, and $W=1$.

$v_{x_0}=v_{y_0}=F_0=0$, and $e_0\neq 0$) is excited, beat waves with frequency $\Omega_b\equiv\Omega_+(T)-\Omega_-(T)$ will appear in time. Notice that the frequency is variable, and gets smaller and smaller after T exceeds T_*.

In order to demonstrate the mutual transformation of IGW and SW with corresponding energy transfer between the modes, it is essential to choose initial conditions in such a way that at $T=0$, only one of the two modes is nonzero. Originally, we must calculate for $T=0$ the auxiliary quantities [13]: $\sigma_\pm=(\Omega_\pm^2-\omega_1^2)/\beta_0$.

For exciting pure IGW (Ω_- mode), we should choose $e_0=\sigma_-\psi_0$, and $\partial_T e_0=\sigma_-\partial_T\psi_0$. Recalling that $\psi_0=F_0-\beta_0 e_0$, $\partial_T\psi_0=v_{x_0}+Re_0$, and $\partial_T e_0=v_{y_0}$, we can simply take $F_0=e_0=0$ and an arbitrary v_{x_0}, and $v_{y0}=\sigma_- v_{x0}$. In exactly the same fashion we can excite pure SW with $F_0=e_0=0$, and $v_{y0}=\sigma_+ v_{x0}$.

The results of numerical calculations are partly presented in Figs. 1–3. They are in almost complete agreement with qualitative expectations.

In Figs. 1(a) [1(c)] and 11(b) [1(d)], we display the temporal evolution of the velocity $v_x(T)$ and energy $E(T)/E(0)$, respectively, for a pure IGW [SW] initial condition, and with the initial data $\beta_0=10$, $R=0.1$, and $W=0.5$. It is clearly seen that IGW [SW] evolves in the usual manner [following adiabatically the corresponding $\Omega(T)/\Omega(0)$ curve, presented by the dashed line] until it reaches DR ($T_*=100$ here), where a small portion of the other wave appears.

Figure 2 is a repetition of Fig. 1 with the notable difference that the resonant value of $W=1$ is taken. The mutual transformation of modes is now especially effective. The graphs show that there occurs almost complete transformation of IGW into SW and *vice versa*.

Finally, we display in Fig. 3 the results of numerical calculations for $e(T)$ and $v_x(T)$ for $\beta_0=10^{-2}$, $R=10^{-4}$, and $W=1$ chosen to favor beat wave generation. The graphs unambiguously show pronounced beat waves with a continuous back and forth energy transfer between the physical variables.

By studying a highly simple 2D model of a stratified fluid, we have explored the consequences of the shear-induced coupling between the internal gravity and the sound wave that leads to the mutual modal transformation, and to a corresponding energy transfer. Apart from the concrete novelty of the results obtained in this paper, the *main message* of this work is that the velocity shear may act like an effective "mixer" of the different waves sustainable in shear flows of arbitrary origin and constitution.

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Velocity shear-induced effects on electrostatic ion perturbations

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Linear evolution of electrostatic perturbations in an unmagnetized electron–ion plasma shear flow is studied. New physical effects, arising due to the non-normality of linear dynamics are disclosed. A new class of *nonperiodic collective mode* with vortical motion of ions, characterized by intense energy exchange with the mean flow, is found. It is also shown that the velocity shear induces extraction of the mean flow energy by ion-sound waves and that during the shear-induced evolution the ion-sound waves turn eventually into ion plasma oscillations. © *1997 American Institute of Physics.* [S1070-664X(97)00112-2]

I. INTRODUCTION

The classical stability theory of continuous media motion (normal mode approach) has been successful in explaining how different kinds of shear flows become unstable. However, in some quite simple and important kinds of shear flows, (e.g., plane Couette and Poiseuille, or pipe Poiseuille flows) the approach has serious problems, evoked by the non-self-adjoint character of the governing equations.[1,2] That is why the predictions of the traditional stability approach fail to match the results of most experiments with these kinds of flows.[1]

An alternative approach to the problem is that of Kelvin,[3] which implies the change of independent variables from a laboratory to a moving frame and the study of the nonexponential temporal evolution of the *spatial Fourier harmonics* (SFH) of perturbations. The method is operative for any smooth mean velocity profile, but it is most manageable in the ones that are linear, or piecewise linear.[4,5] The effectiveness of the method has been repeatedly proved, for example, in helping to obtain unlooked-for results on the dynamics of the perturbations in hydrodynamic[6–10] and hydromagnetic[11–19] shear flows.

In Ref. 9, the evolution of two-dimensional (2-D) SFH in a compressible, plane hydrodynamic Couette flow was considered. The analysis, which involved the nonmodal approach, revealed the existence of a new mechanism of energy exchange between the mean flow and sound-type perturbations. In particular, it was shown that the energy of the SFH may grow linearly in time—perturbations extract the energy from the mean shear flow. This process appears to be quite universal and one should expect that it may also be influential in a wide variety of continuous media with analogous kinematics.

In this *paper* we shall examine the case of electron–ion plasma shear flow and show that the process of velocity shear-induced energy transfer from the mean flow to the collective modes exists, and can be quite efficient in this case, too. Moreover, as we shall see, the peculiarity of the plasma state of the medium plays a distinctive role and leads to a whole group of interesting new effects.

The most interesting new effect of velocity shear, which deserves special attention, is that it induces excitation of a completely *new class* of nonperiodic, electrostatic perturbations with vortical motion of the plasma ion component. These perturbations are able to effectively exchange their energy with the mean flow and under certain conditions may play a *dominant* role in the behavior of the plasma flow.

The presence of the velocity shear crucially affects, also, the behavior of familiar ion electrostatic *wave* modes. As is well known, in the collisionless unmagnetized plasma with $T_i \ll T_e$ (where T_e and T_i are electron and ion temperatures, respectively) there exists a weakly damped low-frequency, electrostatic, ion mode. When its wavelength greatly exceeds the electron Debye length $\lambda_{De} \equiv (T_e/4\pi e^2 n_0)^{1/2}$, then this low-frequency mode represents the *ion-sound wave* with constant phase velocity $C_s \equiv (T_e/M)^{1/2}$ (M is the ion mass, while by m we shall denote the electron mass). However, the velocity shear induces a "linear drift" of SFH (a process well acknowledged in the literature exploring the nonmodal approach) that, mathematically, is exposed in the temporal variation of the wave number vector $\mathbf{k}(t)$. It means that the influence of the *dispersion* of the ion mode, arising as the result of the violation of the quasineutrality for the perturbations, which, in other words, is related to the finiteness of $k\lambda_{De}$, should be taken into due account. The reason is simple: if initially $|k\lambda_{De}| \ll 1$, the linear drift process will, eventually, transfer the SFH to the region of $\mathbf{k}$ space, where the latter condition does not hold. In physical terms it means that under certain circumstances the ion-sound waves, drawing energy from the mean shear flow, subsequently turn into ion plasma oscillations. The latter collective mode is weakly damped if $\lambda \gg \lambda_{Di} \equiv (T_i/4\pi e^2 n_0)^{1/2}$.

The paper is organized in the following way. In the next section the main consideration is presented and the mathematical background of the problem is duly outlined. In the

Other physical variables of the problem may be expressed by Ψ and $\Psi^{(1)}$ in the following way:

$$N_i=\mathcal{K}\Psi, \tag{16}$$

$$N_e=\mathcal{K}\Psi/(1+\xi^2\mathcal{K}^2), \tag{17}$$

$$v_y=\frac{1}{\mathcal{K}}[\beta\mathcal{K}^2\Psi^{(1)}+R(1+\gamma^2)\Psi+\text{const}\times\mathcal{K}], \tag{18}$$

$$v_x+\gamma v_z=\frac{1}{\mathcal{K}^3}[(1+\gamma^2)\mathcal{K}^2\Psi^{(1)}-R\beta(1+\gamma^2+\mathcal{K}^2)\Psi - \text{const}\times\beta\mathcal{K}]. \tag{19}$$

Note that (16)–(19) are exact expressions, valid for *arbitrary* values of the shear parameter R. They may be used for reproduction of the variables through Ψ and $\Psi^{(1)}$, obtained by the direct numerical solution of Eq. (14).

III. DISCUSSION AND CONCLUSIONS

The general solution of (14) is the sum of its *special* solution and the *general* solution of the corresponding homogeneous (const=0) equation: $\Psi=\Psi_h+\Psi_s$. When $\Omega(\tau)$ depends on τ adiabatically, implying[21,9,19]

$$|\Omega(\tau)^{(1)}|\ll\Omega^2(\tau), \tag{20}$$

then the homogeneous equation can be solved approximately.

For the flows with $R\ll1$ the condition (20) holds for a wide range of possible values of $|\beta(\tau)|$. In other words, since the temporal variability of $|\beta(\tau)|$ is determined by the "linear drift" of the SFH, (20) is valid at all stages of the evolution of the SFH. When the condition (20) holds, the approximate expression for Ψ_h may be written in the following way:

$$\Psi_h(\tau)\approx\frac{C}{\sqrt{\Omega(\tau)}}\exp[i(\varphi(\tau)+\varphi_0)], \tag{21}$$

where $\varphi(\tau)=\int_0^\tau\Omega(\tau')d\tau'$.

The special solution of the inhomogeneous equation (14) deserves particular attention, because, as we shall see later, it describes a *new* class of nonperiodic, electrostatic ion perturbations. The solution is derived owing to the smallness of the R parameter. It may be expressed by the following series:[22,9,19]

$$\Psi_s(\tau)=\text{const}\times\sum_{n=0}^{\infty}R^{2n}y_n(\tau), \tag{22a}$$

$$y_0(\tau)=f(\tau)/\Omega^2(\tau), \tag{22b}$$

$$y_n(\tau)=-\frac{1}{\Omega^2(\tau)}\frac{\partial^2 y_{n-1}}{\partial\beta^2}. \tag{22c}$$

Since $R\ll1$, the terms with higher powers of R are negligible and the special *approximate* solution of the inhomogeneous equation (14) may be written explicitly as

$$\Psi_s(\tau)\approx\frac{\text{const}\times f(\tau)}{\Omega^2(\tau)}. \tag{23}$$

Below we shall focus our attention on the behavior of 2-D perturbations in the XOY plane ($\gamma=0$). This case admits simple analytical examination and exposes soundly the qualitative novelty of the problem. Using Eqs. (11) and (16)–(19), we get the following simple expression for the spectral energy density:

$$\mathcal{E}(\tau)\simeq\frac{1}{2}\left[C^2\Omega(\tau)+\left(\frac{\text{const}}{\mathcal{K}(\tau)}\right)^2\right]. \tag{24}$$

When $C/\text{const}\ll1$ the SFH may be treated as mainly incompressible and vortical perturbation, while when $C/\text{const}\gg1$ it is mainly of the sound type.

The spatial characteristics of the SFH $[k_{x_1},k_y(\tau)]$ and the value of the shear parameter R manage the evolution of the frequency of oscillations and the actual intensity of the energy exchange between the SFH and the background flow. In particular, the temporal variability of these processes is essentially induced by the "linear drift" of the SFH in the **k** space.[9,19]

For the sound-type ($C\neq0$, const=0) perturbations, as it is evident from (15a), the frequency of oscillations varies with the variation of $\mathcal{K}(\tau)$. Originally, at moderate values of $\mathcal{K}(\tau)$, due to the smallness of ξ, the oscillation mode may be treated as an ion-sound wave $[\Omega(\tau)\sim\mathcal{K}(\tau)]$. Afterward, when $\mathcal{K}(\tau)$ reaches large enough values, the dispersing influence of the denominator in the first term of (15a) becomes more and more imperative and, when $\xi^2\mathcal{K}^2(\tau)\gg1$, the frequency $\Omega(\tau)$ already exhibits ion plasma oscillations. Following, according to (24), the evolution of energy of this mode $[\mathcal{E}(\tau)\sim\Omega(\tau)]$, we find that initially, for $\beta_0>0$, at $0<\tau<\tau_*\equiv\beta_0/R$, the energy decreases and reaches its minimum at $\tau=\tau_*$. A while later, it begins to increase at $\tau_*<\tau<\infty$, when the SFH "emerges" into the area of **k** space in which $k_y(\tau)k_{x_1}<0$ (the "growth area" for the sound-type perturbations[9]). If the SFH is in the "growth area" from the beginning ($\beta_0<0$), its energy increases monotonously. When $\xi\mathcal{K}(\tau)\gg1$ the rate of energy increase becomes less and less and the energy asymptotically tends to a constant value.

When $C=0$ and $\neq0$, the SFH may be treated as mainly incompressible and vortical perturbations. In this case $\Psi\simeq\text{const}\times f(\tau)/\Omega^2(\tau)$, while $v_y\simeq\text{const}/\mathcal{K}^2(\tau)$ and $v_x\simeq-\text{const}\times\beta/\mathcal{K}^2(\tau)$. The energy of the SFH varies as $\mathcal{E}(\tau)\simeq\mathcal{K}^{-2}(\tau)$ and reduces to the well-known expression, describing the "transient" growth of the energy of SFH.[9,12] A transient increase of the energy takes place if initially $k_{y_1}/k_{x_1}>0(\beta_0>0)$ and occurs near the $\tau_*\equiv\beta_0/R$ moment of time, when $\beta(\tau)$ tends to zero and $\mathcal{K}(\tau)$ attains its minimum value. That is the behavior of 2-D vortical perturbations. One should expect that the evolution of three-dimensional (3-D) vortical perturbations should be similar to the behavior of the analogous structures in incompressible, inviscid fluids, extensively studied in Ref. 19. It was shown that the energy of 3-D vortical perturbations also grows nonexponentially, but unlike transiently growing 2-D perturbations, the energy of 3-D perturbations saturates, attaining in asymptotics some constant value.

Some similarity of the vortexes in plasmas and neutral fluids were noticed and productively exploited from the beginning of the 1980s (cf. Refs. 23 and 24). Up to now vortexes were associated with quite complicated plasma systems. In our study we have found the new class of vortexes in one of the most simple kinds of plasma problem. The likeness of these vortical patterns with the analogous structures in usual fluids[9,19] holds only in the low-R ($R \ll 1$) range. For larger values of the R parameter ($R \simeq 1$) one should expect notable differences between the behavior of vortical perturbations in neutral fluids and electron–ion plasma.

Certainly, in the general case ($C \simeq \mathrm{const}$), the "vortical" and the "sound-type" evolution of perturbations are superimposed on one another.

Finally, we would like to recall, again, those remarkable "shear effects" upon electrostatic ion perturbations, which may be recognized as the main outcome of this study.

(1) A new class of nonperiodic collective modes with vortical motion of ions is discovered. These "shear vortexes" are characterized by an intense energy exchange with the mean flow.

(2) It is found that ion-sound waves, through the agency of the velocity shear, become able to extract the mean flow energy. This process has much in common with the analogous process in the hydrodynamics of classical (neutral) fluids.[9,19]

(3) It is shown that the ion-sound waves, in the course of the velocity shear-induced evolution, may turn eventually into ion plasma oscillations. This effect arises due to the violation of quasineutrality for perturbations and due to the shear-induced "linear drift" of SFH.

It seems tempting and reasonable to speculate that a part of the mean flow energy, acquired by perturbations through the above found channels of energy transfer, may be considered as credible sources for the eventual onset of plasma turbulence in such flows.

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Velocity shear generated Alfvén waves in electron–positron plasmas

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Linear magnetohydrodynamic (MHD) modes in a cold, nonrelativistic electron–positron plasma shear flow are considered. The general set of differential equations, describing the evolution of perturbations in the framework of the nonmodal approach is derived. It is found, that under certain circumstances, the compressional and shear Alfvén perturbations may exhibit large transient growth fueled by the mean kinetic energy of the shear flow. The velocity shear also induces mode coupling, allowing the exchange of energy as well as the possibility of a strong mutual transformation of these modes into each other. The compressional Alfvén mode may extract the energy of the mean flow and transfer it to the shear Alfvén mode via this coupling. The relevance of these new physical effects to provide a better understanding of the laboratory e^+e^- plasmas is emphasized. It is speculated that the shear-induced effects in the electron–positron plasmas could also help solve some astrophysical puzzles (e.g., the generation of pulsar radio emission). Since most astrophysical plasmas are relativistic, it is shown that the major results of the study remain valid for weakly sheared relativistic plasmas. © *1996 American Institute of Physics.* [S1070-664X(96)04210-3]

I. INTRODUCTION

It is commonly recognized that electron–position (henceforth referred to as e^+e^-) plasmas are created in a variety of astrophysical situations. A well-known example is the pulsar magnetosphere, where in the superstrong magnetic fields $B \sim 10^8$ T (10^{12} G), gamma rays, with energy greater than twice the rest energy of the electron, decay into (e^+e^-) pairs: $\gamma + B \rightarrow e^- + e^+ + B$. The components of these (*primary*) pairs are accelerated to very high energies by parallel electric fields, and emit gamma rays, triggering, in turn, a *pair cascade*.[1] As a result of this process a *secondary* pair plasma with the mean Lorentz factor $\Gamma \sim 10^2$–10^3 and the multiplicity factor (the ratio of the number of secondaries to the number of primaries) $\mathscr{M} \sim 10^3$–10^5 is formed.[2]

The e^+e^- plasmas are also likely to be found in the bipolar outflows (jets) in Active Galactic Nuclei (AGN),[3] and at the center of our own Galaxy.[4] In AGNs, the observations of superluminal motion are commonly attributed to the expansion of e^+e^- relativistic beams pervading a subrelativistic medium. This model implies a copious production of e^+e^- pairs via γ–γ interactions creating an e^+e^- atmosphere around the source. The actual production of e^+e^- pairs due to photon–photon interactions occurs in the coronas of AGN accretion disks, which upscatter the soft photons emitted by the accretion disks by inverse Compton scattering.

The presence of e^+e^- plasma is also argued in the MeV epoch of the early Universe.[5] In the standard cosmological model, temperatures in the MeV range ($T \sim 10^{10}$ K–1 MeV) prevail up to times $t \simeq 1$ s after the Big Bang. In this epoch, the main constituent of the Universe is an e^+e^- plasma in equilibrium with photons, neutrinos, antineutrinos, and a minority population of heavier ions.

Contemporary progress in the production of pure positron plasmas[6] has also made it possible to create *nonrelativistic* e^+e^- plasmas in the laboratory by a number of different experimental approaches (see Refs. 7 and 8 and references therein). The condition for the plasma collective effects to be important is that the annihilation (e.g., via positronium atom formation, or two-body collisions) time scale t_a should be much longer than the time scale for plasma effects ($t_p \sim 1/\omega_p$). When this criterion ($t_a \gg t_p$) holds, experimental observation of the collective phenomena becomes possible.[8]

From the theoretical point of view the e^+e^- plasma, being a subclass of equal-mass plasmas, may display physical processes and properties quite different from those of a conventional ion–electron plasma. In the latter case, the smallness of the m_e/m_i ratio is exploited to an extensive degree and is responsible for certain well-known properties of such media. While in the former case, with equal absolute charge to mass ratio for both of the constituents, important symmetries should appear; these can lead to considerable simplification in the mathematical description of the collective phenomena in a e^+e^- plasma.[8] Several novel features can also emerge.

During the last few years a considerable amount of work has been done in the analysis of linear and nonlinear Alfvén wave propagation in e^+e^- plasmas. It is contended that these weakly damped waves (in contrast to, e.g., Langmuir or magnetoacoustic waves) may be the source of the observed electromagnetic emission from the pulsars and the

e^+e^- jets. According to Mikhailovsky *et al.*,[9] among all the possible low-frequency modes in the e^+e^- plasma, the Alfvén waves are the most likely candidate to escape the pulsar magnetosphere.

Several linear and nonlinear processes have been proposed for the generation of Alfvén waves in an e^+e^- plasma. For instance, a *linear* effect, which can lead to the excitation of Alfvén waves, is the Čerenkov interaction with plasma particles. However, this process requires an inversion in the distribution function, and, could not quite explain high levels of the observed radiation.[10] A further search for universal amplification effects, which can transfer the energy stored in an e^+e^- plasma into Alfvén wave energy, therefore, is of principal importance in this context.

The aim of this paper is precisely to look for such a mechanism. We consider the problem of linear excitation, and the subsequent evolution of the Alfvén waves in an e^+e^- plasma flow with *velocity shear*. Note that the shear flow possesses a considerable amount of kinetic energy, and that the associated velocity vector field is spatially inhomogeneous. The resulting velocity gradients may play quite an unexpected and sometimes even crucial role in the overall dynamics of wave processes occurring in such flows. This conjecture was soundly confirmed recently; the use of an effective *nonmodal approach* to the study of physical processes in shear flows[11] reveals a whole branch of new physical phenomena provoked by the velocity shear in various kinds of hydrodynamic and hydromagnetic flows.[11–15] A particularly interesting example is the discovery of a new energy exchange mechanism between the mean flow and sound-type perturbations in a two-dimensional (2-D) compressible, plane Couette flow.[12] It was shown that the perturbations, *extracting* energy from the mean shear flow, may grow linearly in time.

Another new effect—the *linear* coupling and mutual transformation of waves with a corresponding energy transfer induced by the velocity shear—was found in Ref. 13. Originally, the effect was demonstrated for the simplest example, i.e., of the 2-D waves in an unbounded, parallel *hydromagnetic* flow with uniform velocity shear. It was subsequently shown that an analogous mechanism is operative in other kinds of parallel shear flows as long as the system can naturally sustain several (more than one) modes.[14,15]

In this paper we explore the possibility of these effects in the shear flow of a magnetized, cold nonrelativistic e^+e^- plasma. It is natural to expect that the shear-induced effects will lead to interesting consequences for these flows. Qualitatively similar behavior should pertain for warm/hot and relativistic flows.

Before giving the plan of the paper, we would like to place this work in perspective. Since the typical transient phenomena induced by the velocity shear in different systems are rather similar in character, there is no new "fundamental" physics unearthed in this paper. The novelty, however, lies in (1) the choice of a system that is of great astrophysical and cosmological significance, and (2) studying the interaction of the shear and compression Alfvén waves in a three-dimensional flow, an investigation that could possibly help us in understanding the nature of the pulsar radiation. Toward this end, a more realistic relativistic theory is being developed. The two-fluid model used in this paper further allows us to investigate mode coupling mediated by the finite skin-depth effects, a process not available in magnetohydrodynamics (MHD).

The paper is organized in the following way: In Sec. II, we present the general formalism: a universal set of linearized equations, describing the evolution of perturbations in a cold nonrelativistic sheared e^+e^- plasma flow, are derived. (The detailed derivation of the *induction equation* is presented separately in Appendix A.) In Sec. III, a model flow, with uniform shear, is investigated in considerable detail. Extremely interesting phenomena like the large transient amplifications of the Alfvén waves, and energy exchange between the "shear" and the compressional Alfvén waves are appropriately demonstrated. The final section is devoted to a discussion of the possible applications of this new physics to the theory of pulsar radio emission, to the e^+e^- jet outflows in AGNs, and related subjects.

Since most astrophysical flows tend to be relativistic, it is shown in Appendix B that the mathematical structure, and hence the basic physical results of the nonrelativistic approach, remain essentially unchanged for a weakly sheared relativistic flow.

II. GENERAL FORMALISM

The basic set of two-fluid MHD equations for the e^+e^- plasma consists of the mass and momentum conservation equations, supplemented by Maxwell's equations:

$$\partial_t n^{\pm} + \nabla \cdot (n^{\pm} \mathbf{V}^{\pm}) = 0, \tag{1}$$

$$mn^{\pm}[\partial_t + (\mathbf{V}^{\pm}, \nabla)]\mathbf{V}^{\pm} = \pm en^{\pm}\left(\mathbf{E} + \frac{1}{c}\mathbf{V}^{\pm} \times \bar{\mathbf{B}}\right) - \nabla P^{\pm}, \tag{2}$$

$$\nabla \cdot \mathbf{E} = 4\pi e(n^+ - n^-). \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\partial_t \bar{\mathbf{B}}, \tag{4}$$

$$\nabla \cdot \bar{\mathbf{B}} = 0. \tag{5}$$

$$\nabla \times \bar{\mathbf{B}} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\partial_t \mathbf{E}, \tag{6}$$

where $\mathbf{J}$ is the current density vector,

$$\mathbf{J} \equiv e(n^+\mathbf{V}^+ - n^-\mathbf{V}^-) \tag{7}$$

and $n^{\pm}$, $\mathbf{V}^{\pm}$, and $P^{\pm}$ are, respectively, the number density, velocity, and pressure of the appropriate species.

Let us consider an e^+e^- plasma embedded in an external uniform magnetic field $\mathbf{B}_0 = (B_0, 0, 0)$ along the x axis. The plasma is characterized by a sheared bulk flow velocity $\mathbf{U}_0$. The instantaneous values of velocity components for each species in the plasma may be decomposed into their mean and perturbed components:

$$\mathbf{V}^{\pm} \equiv \mathbf{U}_0^{\pm} + \mathbf{u}^{\pm} = \mathbf{U}_0 + \mathbf{u}^{\pm}. \tag{8}$$

Similarly, writing $\mathbf{B}=\mathbf{B}_0+\mathbf{B}$, we can derive the linearized versions of (1)–(6). Equations (3)–(5) remain the same, while the rest take the form

$$D_t n^{\pm}+n_0\nabla\cdot\mathbf{u}^{\pm}=0, \tag{9}$$

$$D_t\mathbf{u}^{\pm}+(\mathbf{u}^{\pm}\cdot\nabla)\mathbf{U}_0=-\left(\frac{C_s^2}{n_0}\right)\nabla n^{\pm}\pm\frac{e}{m}\left(\mathbf{E}+\frac{\mathbf{U}_0}{c}\times\mathbf{B}+\frac{\mathbf{u}^{\pm}}{c}\times\mathbf{B}_0\right), \tag{10}$$

$$\nabla\times\mathbf{B}=\frac{4\pi e}{c}\left[(n_+-n_-)\mathbf{U}_0+n_0(\mathbf{u}_+-\mathbf{u}_-)\right]+\frac{1}{c}\,\partial_t\mathbf{E}, \tag{11}$$

where $D_t\equiv\partial_t+(\mathbf{U}_0,\nabla)$ is the *convective* derivative over the velocity vector field $\mathbf{U}_0$, and $C_s^{\pm}\equiv[\partial P^{\pm}/\partial(mn^{\pm})]^{1/2}$ is the sound speed for each species. We also assumed that in equilibrium $n_0^+=n_0^-\equiv n_0$, and the two sound speeds are equal for the mean flow.

Let us now define the following set of *one-fluid* variables:

$$\rho_0\equiv 2mn_0, \tag{12a}$$

$$\rho\equiv m(n_++n_-), \tag{12b}$$

$$\rho_e\equiv e(n_+-n_-), \tag{12c}$$

$$\mathbf{j}\equiv en_0(\mathbf{u}_+-\mathbf{u}_-), \tag{12d}$$

$$\mathbf{v}\equiv(\mathbf{u}_++\mathbf{u}_-)/2. \tag{12e}$$

In these variables [note that $\rho_0\mathbf{v}=mn_0(\mathbf{u}_++\mathbf{u}_-)$], the linearized *two-fluid* equations become the following set of *one-fluid* equations:

$$D_t\rho+\rho_0\nabla\cdot\mathbf{v}=0, \tag{13}$$

$$D_t\rho_e+\nabla\cdot\mathbf{j}=0, \tag{14}$$

$$\rho_0[D_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{U}_0]=-C_s^2\,\nabla\rho+\frac{1}{c}\,\mathbf{j}\times\mathbf{B}_0, \tag{15}$$

$$D_t\mathbf{j}+(\mathbf{j}\cdot\nabla)\mathbf{U}_0=-C_s^2\,\nabla\rho_e+\left(\frac{e}{m}\right)^2\rho_0\left(\mathbf{E}+\frac{1}{c}\,(\mathbf{U}_0\times\mathbf{B}+\mathbf{v}\times\mathbf{B}_0)\right), \tag{16}$$

$$\nabla\cdot\mathbf{E}=4\pi\rho_e, \tag{17}$$

$$c\nabla\times\mathbf{E}=-\partial_t\mathbf{B}, \tag{18}$$

$$\nabla\cdot\mathbf{B}=0, \tag{19}$$

$$c\nabla\times\mathbf{B}=4\pi\rho_e\mathbf{U}_0+4\pi\mathbf{j}+\partial_t\mathbf{E}. \tag{20}$$

Hereafter the plasma will be assumed to be (a) cold ($C_s=0$) and (b) *nonrelativistic*. The latter assumption ($\partial_t\mathbf{E}\simeq0$) simplifies Eq. (20) to

$$\mathbf{j}\approx(c/4\pi)\nabla\times\mathbf{B}-\rho_e\mathbf{U}_0, \tag{21}$$

implying ($\mathbf{U}_0\times\mathbf{B}_0=0$)

$$\mathbf{j}\times\mathbf{B}_0=-(c/4\pi)[\nabla(\mathbf{B},\mathbf{B}_0)-(\mathbf{B}_0,\nabla)\mathbf{B}], \tag{22}$$

which, in turn, immediately converts the equation of motion (15) into its standard form relating the variables $\mathbf{v}$ and $\mathbf{B}$,

$$D_t\mathbf{v}+(\mathbf{v},\nabla)\mathbf{U}_0=(v_A^2/B_0)(\partial_x\mathbf{B}-\nabla B_x), \tag{23}$$

where $v_A^2\equiv B_0^2/4\pi\rho_0=B_0^2/8\pi mn_0$.

Derivation of the *induction equation* for the magnetic field—another vector equation that links the magnetic field and velocity perturbations—is straightforward but tedious. A very general form [Eq. (A8)] is derived in Appendix A. For the Alfvén wave physics, it is quite adequate to assume that the plasma is quasineutral, i.e., $\rho_e=0$. Further simplification results if $\mathbf{U}_0$ is a *linear* function of coordinates; the last term in (A8) then, is zero, and the induction equation reduces to

$$D_t(\mathbf{B}-\lambda^2\,\Delta\mathbf{B})=(\mathbf{B}-\lambda^2\,\Delta\mathbf{B},\nabla)\mathbf{U}_0+(\mathbf{B}_0,\nabla)\mathbf{v}-\mathbf{B}_0(\nabla\cdot\mathbf{v})+\lambda^2(\nabla\times\mathbf{U}_0,\nabla)(\nabla\times\mathbf{B}). \tag{24}$$

Note that when the collisionless skin depth λ is small enough, Eq. (24) further reduces to the more familiar form used in standard MHD,

$$D_t\mathbf{B}=(\mathbf{B},\nabla)\mathbf{U}_0+(\mathbf{B}_0,\nabla)\mathbf{v}-\mathbf{B}_0(\nabla\cdot\mathbf{v}). \tag{25}$$

In the next section, we shall use the closed set of Eqs. (13), (23), and (24) along with the no monopole condition Eq. (19).

III. VELOCITY SHEAR AND ALFVÉN WAVES

In this section, we solve a model problem to illustrate a variety of shear-induced physical effects. We consider a simple unidirectional mean flow $\mathbf{U}_0\equiv(Ay,0,0)$, with a linear shear profile along the Y axis. For this case, $D_t\equiv\partial_t+Ay\partial_x$, and $\nabla\times\mathbf{U}_0=-A\hat{\mathbf{e}}_z$. The system of relevant equations can now be written in the explicit form

$$(\partial_t+Ay\partial_x)\rho+\rho_0(\partial_xv_x+\partial_yv_y+\partial_zv_z)=0, \tag{26}$$

$$\partial_xB_x+\partial_yB_y+\partial_zB_z=0, \tag{27}$$

$$(\partial_t+Ay\partial_x)v_x+Av_y=0, \tag{28}$$

$$(\partial_t+Ay\partial_x)v_y=(v_A^2/B_0)[\partial_xB_y-\partial_yB_x], \tag{29}$$

$$(\partial_t+Ay\partial_x)v_z=(v_A^2/B_0)[\partial_xB_z-\partial_zB_x], \tag{30}$$

$$(\partial_t+Ay\partial_x)[(1-\lambda^2\Delta)B_y]=B_0\,\partial_xv_y-A\lambda^2\partial_z(\partial_zB_x-\partial_xB_z), \tag{31}$$

$$(\partial_t+Ay\partial_x)[(1-\lambda^2\Delta)B_z]=B_0\,\partial_xv_z-A\lambda^2\partial_z(\partial_xB_y-\partial_yB_x). \tag{32}$$

Equations (26)–(32) define a model linear problem. Conventional "stability" analysis of this system will revolve around an eigenmode analysis—the eigenmodes being the time asymptotic modes of oscillation that this system can sustain. In this generally valid and powerful approach the transient behavior of the perturbations never figures, because as time goes to infinity, only normal modes propagate. It turns out, however, that for a class of linear operators,[11] the conventional normal mode analysis may not reveal the entire richness of the dynamics; in fact, extremely crucial aspects of the time evolution of the perturbations may be altogether missed. For instance, a very large transient amplification could drive a system to a nonlinear turbulent state while the normal mode analysis for the same system would predict complete stability and hence no possible transition to turbulence.

Recent investigations show that shear flows (the system under discussion) display interesting dynamical behavior, which will be completely missed in a normal mode analysis (Refs. 11–15). These aspects, generally pertaining to the details of transient dynamics, are much better elucidated by solving an initial value problem. This approach, called the nonmodal analysis, will be followed in this paper.

To "set up" the nonmodal analysis, we make the following transformation of variables:

$$x_1=x-Ayt; \quad y_1=y; \quad z_1=z; \quad t_1=t. \tag{33}$$

It is well known[11] that this transformation, a change from the Eulerian to the Lagrangian frame, leads to immense simplification in the solution of the initial-value problem.

In these new coordinates, the relevant equations take the form

$$\partial_{t_1}\rho+\rho_0[\partial_{x_1}v_x+(\partial_{y_1}-At_1\partial_{x_1})v_y+\partial_{z_1}v_z]=0, \tag{34}$$

$$\partial_{x_1}B_x+(\partial_{y_1}-At_1\partial_{x_1})B_y+\partial_{z_1}B_z=0, \tag{35}$$

$$\partial_{t_1}v_x+Av_y=0, \tag{36}$$

$$\partial_{t_1}v_y=(v_A^2/B_0)[\partial_{x_1}B_y-(\partial_{y_1}-At_1\partial_{x_1})B_x], \tag{37}$$

$$\partial_{t_1}v_z=(v_A^2/B_0)[\partial_{x_1}B_z-\partial_{z_1}B_x], \tag{38}$$

$$\partial_{t_1}[(1-\lambda^2\Delta)B_y]=B_0\,\partial_{x_1}v_y-A\lambda^2\partial_{z_1}[\partial_{z_1}B_x-\partial_{x_1}B_z], \tag{39}$$

$$\partial_{t_1}[(1-\lambda^2\Delta)B_z]=B_0\,\partial_{x_1}v_z-A\lambda^2\partial_{z_1}[\partial_xB_y-(\partial_{y_1}-At_1\partial_{x_1})B_x], \tag{40}$$

where $\Delta=\partial_{x_1}^2+(\partial_{y_1}-At_1\partial_{x_1})^2+\partial_{z_1}^2$.

We may further expand the perturbations as

$$F=\int dk_{x_1}\,dk_{y_1}\,dk_{z_1}\,\hat{F}(k_{x_1},k_{y_1},k_{z_1},t_1)\times\exp[i(k_{x_1}x_1+k_{y_1}y_1+k_{z_1}z_1)], \tag{41}$$

and introduce the following dimensionless quantities: $R\equiv A/v_Ak_{x_1}$, $\tau\equiv v_Ak_{x_1}t_1$, $\beta_0\equiv k_{y_1}/k_{x_1}$, $\beta(\tau)\equiv\beta_0-R\tau$, $\gamma\equiv k_{z_1}/k_{x_1}$, $v_{x,y,z}\equiv\hat{v}_{x,y,z}/v_A$, $b_{x,y,z}\equiv i\hat{B}_{x,y,z}/B_0$, $\epsilon^2\equiv\lambda^2k_{x_1}^2$, and $M^2(\tau)\equiv1+\beta^2(\tau)+\gamma^2$.

In the new notation, (35) gives an algebraic relation between the dimensionless components of the magnetic field perturbation,

$$b_x=-\beta b_y-\gamma b_z. \tag{42}$$

This relation, in turn, allows us to convert (37)–(40) into a closed set of first-order ordinary differential equations (ODEs) for the "transverse" (with respect to $\mathbf{B}_0$) variables v_y, v_z, b_y, and b_z:

$$\partial_\tau v_y=[1+\beta^2]b_y+\gamma\beta b_z, \tag{43}$$

$$\partial_\tau v_z=[1+\gamma^2]b_z+\gamma\beta b_y, \tag{44}$$

$$\partial_\tau[(1+\epsilon^2M^2)b_y]=-v_y-R\epsilon^2\gamma[(1+\gamma^2)b_z+\gamma\beta b_y], \tag{45}$$

$$\partial_\tau[(1+\epsilon^2M^2)b_z]=-v_z+R\epsilon^2\gamma[(1+\beta^2)b_y+\gamma\beta b_z]. \tag{46}$$

These equations may be rearranged in the following way:

$$\partial_\tau[(1+\epsilon^2M^2)b_y+R\epsilon^2\gamma v_z]=-v_y, \tag{47}$$

$$\partial_\tau[(1+\epsilon^2M^2)b_z-R\epsilon^2\gamma v_y]=-v_z. \tag{48}$$

Notice that the remaining variables v_x and $D\equiv i\hat{\rho}/\rho_0$, can be determined through [see Eqs. (34) and (36)]

$$\partial_\tau D=v_x+\beta v_y+\gamma v_z, \tag{49}$$

$$\partial_\tau v_x=-Rv_y, \tag{50}$$

once the "transverse" variables are known. In addition, Eqs. (47) and (50) connect v_x with b_y and v_z by the following algebraic relation:

$$R(1+\epsilon^2M^2)b_y-v_x+R^2\epsilon^2\gamma v_z=\text{const.} \tag{51}$$

We now introduce an appropriate "measure" of the spectral energy density of spatial Fourier harmonics,

$$\mathscr{E}(\tau)\equiv\frac{|v_x|^2+|v_y|^2+|v_z|^2}{2(1+\epsilon^2M^2)}+\frac{|b_x|^2+|b_y|^2+|b_z|^2}{2}, \tag{52}$$

which includes the kinetic energy of the plasma, the energy of the magnetic field, and also the energy of the electric field. The latter evokes the factor $(1+\epsilon^2M^2)$ in the denominator of the first term. Clearly, when $\epsilon\ll1$, the energy of the electric field is $(v/c)^2$ times less than the fluctuating magnetic field energy (as in usual MHD) and (52) reduces to a sum of only the kinetic and magnetic energies. The spectral energy density $\mathscr{E}(\tau)$ is defined in such a way that for the "no shear" ($R=0$) limit, it is a conserved quantity:

In terms of the auxiliary quantities,

$$A_y\equiv(1+\epsilon^2M^2)b_y, \tag{53a}$$

$$A_z\equiv(1+\epsilon^2M^2)b_z, \tag{53b}$$

Eqs. (43)–(46) may be written as an equivalent system,

$$\partial_\tau v_y=\frac{1}{1+\epsilon^2M^2}[(1+\beta^2)A_y+\gamma\beta A_z], \tag{54}$$

$$\partial_\tau v_z=\frac{1}{1+\epsilon^2M^2}[(1+\gamma^2)A_z+\gamma\beta A_y], \tag{55}$$

$$\partial_\tau A_y=-v_y-\frac{R\epsilon^2\gamma}{1+\epsilon^2M^2}[(1+\gamma^2)A_z+\gamma\beta A_y], \tag{56}$$

$$\partial_\tau A_z=-v_z+\frac{R\epsilon^2\gamma}{1+\epsilon^2M^2}[(1+\beta^2)A_y+\gamma\beta A_z]. \tag{57}$$

Before proceeding to the next section where we solve various special cases of this general problem [Eqs. (43)–(46) or (54)–(57)], it is interesting to realize that one can readily eliminate v_y and v_z in Eqs. (43)–(46) to obtain the following coupled pair of second-order ODEs for the transverse components of magnetic field perturbations (b_y and b_z):

$$\partial_\tau^2 b_y - \left(\frac{\epsilon^2 R\beta(\gamma^2-4)}{1+\epsilon^2 M^2}\right)\partial_\tau b_y + \left(\frac{(1+\beta^2)+\epsilon^2 R^2(2-\gamma^2)}{1+\epsilon^2 M^2}\right) b_y$$

$$= -\left(\frac{\gamma\beta}{1+\epsilon^2 M^2}\right) b_z - \left(\frac{R\epsilon^2\gamma(1+\gamma^2)}{1+\epsilon^2 M^2}\right)\partial_\tau b_z, \tag{58}$$

$$\partial_\tau^2 b_z - \left(\frac{\epsilon^2 R\beta(\gamma^2+4)}{1+\epsilon^2 M^2}\right)\partial_\tau b_z + \left(\frac{(1+\gamma^2)+\epsilon^2 R^2(2+\gamma^2)}{1+\epsilon^2 M^2}\right) b_z$$

$$= -\left(\gamma\beta(1+2R^2)\,\frac{\epsilon^2}{1+\epsilon^2 M^2}\right) b_y$$

$$+\left(\frac{R\epsilon^2\gamma(1+\beta^2)}{1+\epsilon^2 M^2}\right)\partial_\tau b_y. \tag{59}$$

A. "Zero shear" ($R=0$) case

In spite of the relative simplicity of the system (two-coupled second-order ODEs), analytical progress requires further simplifying assumptions. We begin by studying the well-known shearless flow, $R=0$.

In this case the system of equations simplify enormously. From (58) and (59) one gets

$$\partial_\tau^2 b_y + \omega_1^2 b_y + \alpha b_z = 0, \tag{60}$$

$$\partial_\tau^2 b_z + \omega_2^2 b_z + \alpha b_y = 0, \tag{61}$$

where $\omega_1^2 \equiv (1+\beta_0^2)/(1+\epsilon^2 M_0^2)$, $\omega_2^2 \equiv (1+\gamma^2)/(1+\epsilon^2 M_0^2)$, and $\alpha \equiv \gamma\beta_0/(1+\epsilon^2 M_0^2)$.

The normal frequencies of these oscillations, calculated by the standard formula

$$\Omega_\pm^2 \equiv \frac{1}{2}\left[(\omega_1^2+\omega_2^2) \pm \sqrt{(\omega_1^2-\omega_2^2)^2+4\alpha^2}\right], \tag{62}$$

are equal to

$$\Omega_c^2 \equiv \Omega_+^2 = \frac{1+\gamma^2+\beta_0^2}{1+\epsilon^2 M_0^2}, \tag{63}$$

$$\Omega_s^2 \equiv \Omega_-^2 = \frac{1}{1+\epsilon^2 M_0^2}, \tag{64}$$

and may easily be identified respectively with *compressional* and *shear Alfvén wave* frequencies.[16,17] In dimensional notation, the frequencies take the familiar form

$$\bar{\Omega}_c^2 \equiv k_{x_1}^2 v_A^2 \Omega_c^2 = \frac{v_A^2 k^2}{1+\lambda^2 k^2}, \tag{65}$$

$$\bar{\Omega}_s^2 \equiv k_{x_1}^2 v_A^2 \Omega_s^2 = \frac{v_A^2 k_{x_1}^2}{1+\lambda^2 k^2}. \tag{66}$$

It is easy to show that b_x and $\psi \equiv \gamma b_y - \beta_0 b_z$ obey

$$\partial_\tau^2 b_x + \Omega_c^2 b_x = 0, \tag{67}$$

$$\partial_\tau^2 \psi + \Omega_s^2 \psi = 0, \tag{68}$$

which are linearly independent. As expected, the shearless system is characterized by two fundamental normal modes with well-known eigenfrequencies Ω_c and Ω_s.

B. Two-dimensional waves ($\gamma=0$)

The shearless case is standard and relatively uninteresting. Let us now turn the shear on ($R\neq 0$), but consider another simple case for which the perturbations are two dimensional (2-D), i.e., $k_{z_1}=0$ ($\gamma=0$). In this case $M^2(\tau)=1+\beta^2(\tau)$, and (54)–(57) can be manipulated to give

$$\partial_\tau^2 A_y + \left(\frac{1+\beta^2}{1+\epsilon^2(1+\beta^2)}\right) A_y = 0 \tag{69}$$

and

$$\partial_\tau^2 A_z + \left(\frac{1}{1+\epsilon^2(1+\beta^2)}\right) A_z = 0. \tag{70}$$

These equations describe "oscillations" with variable frequencies $\omega_c^2(\tau) \equiv (1+\beta^2)/[1+\epsilon^2(1+\beta^2)]$ and $\omega_s^2(\tau) \equiv 1/[1+\epsilon^2(1+\beta^2)]$. These can be (for 2-D perturbations) again viewed as the compressional and shear Alfvén frequencies, respectively. For sufficiently small values of R, these frequencies vary slowly (adiabatically) and the approximate analytic solutions of (69) and (70) may be written as[12]

$$A_y(\tau) = \frac{C_1}{\sqrt{\omega_c(\tau)}} \exp\{i[\phi_c(\tau)+\phi_{c0}]\}, \tag{71a}$$

$$A_z(\tau) = \frac{C_2}{\sqrt{\omega_s(\tau)}} \exp\{i[\phi_s(\tau)+\phi_{s0}]\}, \tag{71b}$$

where the amplitudes of these modes are determined through the corresponding adiabatic invariants: $C_1 \equiv a_c^2(\tau)\omega_c(\tau)$ and $C_2 \equiv a_s^2(\tau)\omega_s(\tau)$; C_1 and C_2, though products of two time-varying quantities, are constants with their values determined by the initial conditions. The phases are given by

$$\phi_{c,s}(\tau) \equiv \int \omega_{c,s}(\tau)\,d\tau. \tag{72}$$

It is easy to find that the amplitudes of all physical variables may be expressed through the amplitudes of A_y and A_z. In particular, $|v_x| = R|A_y|$, $|v_y| \simeq \omega_c|A_y|$, $|v_z| \simeq \omega_s|A_z|$, $|b_x| = |\beta|\omega_s^2|A_y|$, $|b_y| = \omega_s^2|A_y|$, and $|b_z| = \omega_s^2|A_z|$. Using these expressions, together with (52), (71a), and (71b), we can derive the following important *analytic* expression for the spectral energy density:

$$\mathcal{E}(\tau) \simeq \omega_s^2(\tau)[C_1^2\omega_c(\tau) + C_2^2\omega_s(\tau)]. \tag{73}$$

This is an approximate equation, but it is found to work excellently when $R \ll 1$. In Figs. 1 and 2 we present results of a direct numerical integration of the general, unsimplified defining equations for the following set of parameters: $\beta_0=10$, $R=0.1$, and $\epsilon=0.5$. The initial perturbation, corresponding to Fig. 1, consists of a pure compressional Alfvén mode. The frequency of the perturbation is given by $\omega_c(\tau)$, while the corresponding amplitude (envelope function) is $a_c(\tau) \equiv C_1/\sqrt{\omega_c(\tau)}$. Figure 1(a) shows that in time periods of interest, the amplitude of $b_y(\tau)$ increases by well over an order of magnitude. The corresponding graph for $\mathcal{E}(\tau)$ is presented in Fig. 1(b). The solid line displays the results of the numerically calculated $\mathcal{E}(\tau)$, while the circles represent

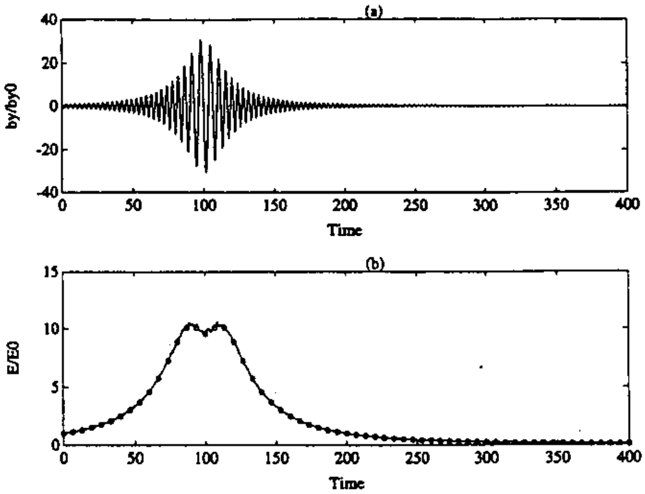

FIG. 1. Here $b_y(\tau)/b_y(0)$ (a) and $\mathcal{E}(\tau)/\mathcal{E}(0)$ (b) vs τ for 2-D ($\gamma=0$) spatial Fourier harmonics (SFH), for an initial pure compressional Alfvén perturbation. Other parameters of the system are $\beta_0=10$, $R=0.1$, and $\epsilon=0.5$. The solid line in (b) displays numerically calculated $\mathcal{E}(\tau)$, while the circles represent $\mathcal{E}(\tau)$ calculated by the approximate equation (73).

$\mathcal{E}(\tau)$ calculated by the approximate equation (73). The excellent agreement between the exact and approximate solutions is evident, and it pertains even when the initial conditions are changed from a pure compressional to a pure shear Alfvén mode [see Figs. 2(a) and 2(b)] or to some admixture of these two modes.

The dip at $\tau=100$ in Fig. 1(b) is just a consequence of the detailed time dependence of $\omega_s^2(\tau)\omega_c(\tau)$ [which determines $\mathcal{E}(\tau)$ for $C_2=0$], and has no other physical significance. Another manifestation of the time dependence of $\omega_s^2(\tau)=[1+\epsilon^2[1+(B_0-R\tau)^2]]^{-1}$, which, for $\beta_0,R_0>0$, has a maximum at $\tau=\tau_*\equiv\beta_0/R$, is displayed in Fig. 2(a); the oscillations are most rapid at $\tau=\tau_*$ (~ 100 for the example shown) and slow down on either side of τ_*.

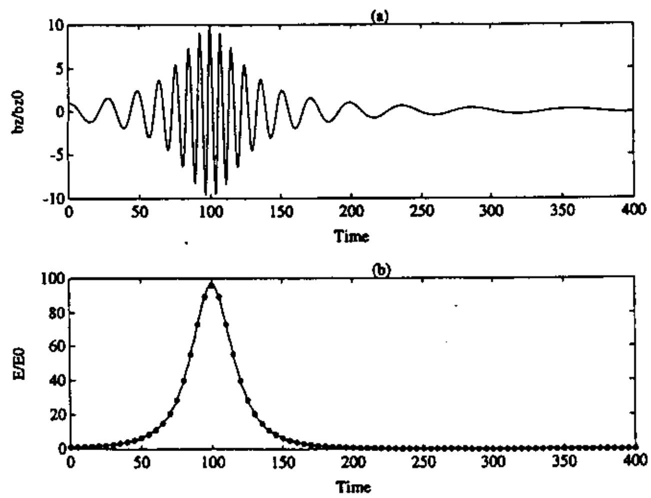

FIG. 2. Here $b_z(\tau)/b_z(0)$ (a) and $\mathcal{E}(\tau)/\mathcal{E}(0)$ (b) vs τ for 2-D ($\gamma=0$) SFH for an initial pure shear Alfvén perturbation. Other parameters of the system are $\beta_0=10$, $R=0.1$, and $\epsilon=0.5$. The solid line in (b) displays numerically calculated $\mathcal{E}(\tau)$, while the circles represent $\mathcal{E}(\tau)$ calculated by the approximate equation (73).

Results of the numerical calculations, as well as those of direct evaluations by the approximate formula (73), show that 2-D perturbations for sufficiently large values of ϵ ($\epsilon \gtrsim 0.1$), exhibit quite strong (up to several orders of magnitude) *transient amplifications*. In fact, the ratio of the spectral energy at $\tau=\tau_*$ [τ_* corresponds to the moment, when the time dependent component of the wave number vector $k_y(\tau)=k_{y_1}-At_1k_{x_1}$ changes its sign; because of the asymmetry introduced by shear, the direction of k becomes significant] to the initial energy (at $\tau=0$), may be readily found from (73) to be

$$\frac{\mathcal{E}(\tau_*)}{\mathcal{E}(0)}=\left(\frac{1+\epsilon^2(1+\beta_0^2)}{1+\epsilon^2}\right)^{3/2}\frac{C_1^2+C_2^2}{C_1^2\sqrt{1+\beta_0^2}+C_2^2}. \tag{74}$$

This expression clearly shows that the substantial transient growth of the perturbation energy is a strong function of the initial orientation of the **k** vector (value of β_0), and of the value of the dimensionless skin depth parameter ϵ. For large enough values of $\epsilon_0\beta_0$, Eq. (74) predicts large amplification factors. For $C_1=0$, $\epsilon=0.25$, $\beta_0=10$, for example, $\mathcal{E}(T_*)/\mathcal{E}(0)\sim 26\sqrt{26}\sim 100$. On the other hand, the amplification factor does not depend directly on the strength of the ambient magnetic field $\mathbf{B}_0$.

Note that for large enough values of τ ($\tau \gg \tau_*$) the shear Alfvén wave frequency ω_s tends to zero, while ω_c asymptotically approaches the cyclotron frequency for the e^+e^- plasma. In other words, the shear-induced "linear drift" of the perturbations[11–15] leads to the asymptotic transition of the compressional Alfvén waves into the cyclotron waves. Going back to the specified evolution equations for the physical variables, we can easily show that in this asymptotic regime, amplitudes of all components of the magnetic field perturbations and v_z tend to zero, while the amplitudes of v_x and v_y attain constant values.

For the shear Alfvén mode the phase integral (72) can be evaluated in terms of elementary functions leading to the explicit solution

$$A_z(\tau)=A_z(0)\left(\frac{1+\epsilon^2(1+\beta^2)}{1+\epsilon^2(1+\beta_0^2)}\right)^{1/4}$$
$$\times\cos\left(\frac{1}{\epsilon R}\ln\left(\frac{\epsilon\beta_0+\sqrt{1+\epsilon^2(1+\beta_0^2)}}{\epsilon\beta+\sqrt{1+\epsilon^2(1+\beta^2)}}\right)\right). \tag{75}$$

The analytical results are, of course, meaningful only when the frequencies vary slowly (adiabatically) with time. Applicability of the adiabatic approximation is governed by the conditions

$$|\partial_\tau\omega_{c,s}(\tau)|\ll\omega_{c,s}^2(\tau), \tag{76a}$$

which hold for small enough values of the R parameter. For the case of the shear Alfvén mode, condition (76a) implies

$$\epsilon^2R|\beta|\ll[1+\epsilon^2(1+\beta^2)]^{1/2}, \tag{76b}$$

while for the compressional Alfvén mode it reads as

$$R|\beta|\ll(1+\beta^2)^{3/2}[1+\epsilon^2(1+\beta^2)]^{1/2}. \tag{76c}$$

As an aside, it is worth mentioning that in terms of the new independent variable,

$$z \equiv \epsilon^2(\beta_0 - R\tau)^2/(1+\epsilon^2), \tag{77}$$

we can reduce (70) to the *Gauss Hypergeometric* equation:[18]

$$4z(1-z)\partial_T^2 A_z + 2(1-z)\partial_T A_z - CA_z = 0, \tag{78}$$

where $C \equiv 1/\epsilon^2 R^2$.

C. Three-dimensional waves with $\lambda \ll 1$

For high-density plasmas, there exists a broad range of wave numbers for which $|\lambda^2\Delta| \ll 1$, and can be neglected. In this case, (58)–(59) reduces to the following pair of equations:

$$\partial_\tau^2 b_y + (1+\beta^2)b_y + \gamma\beta b_z = 0, \tag{79}$$

$$\partial_\tau^2 b_z + (1+\gamma^2)b_z + \gamma\beta b_y = 0, \tag{80}$$

which are similar to (60)–(61). The essential difference here is that now we have $\beta(\tau)$'s instead of β_0's and, hence, the coupling coefficient and one of the eigenfrequencies are *time dependent*: $\omega_1^2(\tau) \equiv 1+\beta^2$, $\omega_2^2 \equiv 1+\gamma^2$, $\alpha(\tau) \equiv \gamma\beta$. The "normal frequencies" of these oscillations, calculated again by (62), are

$$\Omega_1^2(\tau) \equiv \Omega_+^2 = 1+\gamma^2+\beta^2, \tag{81}$$

$$\Omega_2^2 \equiv \Omega_-^2 = 1, \tag{82}$$

and may easily be termed as *compressional*, and *shear Alfvén wave* frequencies, respectively. However, this time, the frequency of the compressional Alfvén wave is *time dependent* and only when $R \ll 1$, it would vary adiabatically.

The system under investigation is mathematically equivalent to a pair of linear pendulums, connected by a spring with a varying stiffness coefficient $[\alpha(\tau) = \gamma\beta(\tau)]$. The length of one of these pendula also varies in time. Strictly speaking, due to this variation, the canonical theory of coupled oscillations is no longer valid. However, when $\omega_1(\tau)$ and $\alpha(\tau)$ vary slowly (adiabatically), as they do when $R \ll 1$, the standard theory of coupled oscillations may serve as a useful guide in understanding and interpreting the inherent physical processes.

A rather similar mechanical problem was investigated in Ref. 19. It was pointed out that for an effective energy exchange to occur between two weakly coupled pendulums, the following conditions must be satisfied.

(a) There should exist a so-called "degeneracy region" (DR), where

$$|\Omega_+^2 - \Omega_-^2| \leq |\alpha(\tau)|. \tag{83}$$

(b) The DR should be "passed" slowly—the traversal time should be much greater than the period of the beats:

$$|\partial_\tau \Omega_+(\tau)| \ll |\alpha(\tau)|. \tag{84}$$

It is easy to notice that in our case the difference $\Omega_+(\tau) - \Omega_-(\tau)$ attains its minimum value at $\tau = \tau_*(=\beta_0/R)$. It is, therefore, evident that the DR is in the neighborhood of τ_* [at times, when $0 < |\beta(\tau)| < 1$]. In the vicinity of $\tau = \tau_*$, $\gamma < 1$ leads to the most efficient mode coupling, and hence to the possibility of mutual transformation of the modes. It can be readily seen that condition (84) holds in DR when $R \ll 1$.

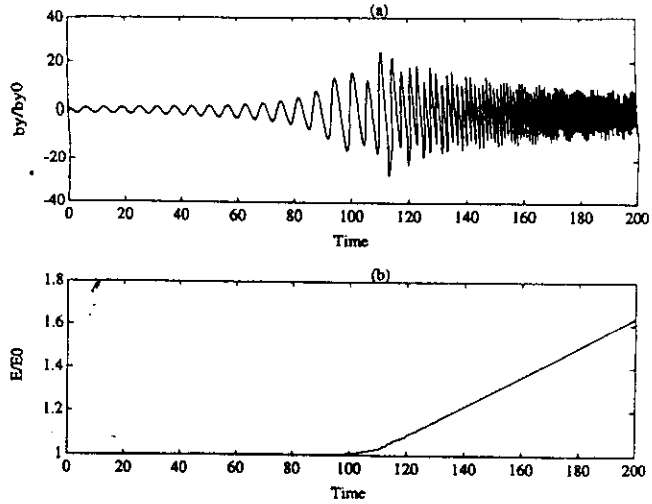

FIG. 3. The temporal evolution of 3-D SFH $[b_y(\tau)/b_y(0)$ (a) and $\mathcal{E}(\tau)/\mathcal{E}(0)$ (b)] for an initially pure shear Alfvén mode with $\beta_0 = 10$, $R = 0.1$, $\epsilon = 5\times10^{-3}$, and $\gamma = 0.1$.

The "adiabatic behavior" of the modes implies that they should normally follow the dispersion curves of their own: spectral energy density (52) of either the shear Alfvén mode $[\mathcal{E}_-(\tau)]$ or the compressional Alfvén mode $[\mathcal{E}_+(\tau)]$ should be proportional to its corresponding frequency $\mathcal{E}_\pm \sim \Omega_\pm$.[19,13,15] This mode of energy evolution, however, will not pertain in DR, where efficient transformation of one wave into the other occurs. For instance, as we see from Figs. 3 and 4, the energy of an initially excited shear Alfvén mode increases approximately by the $\mathcal{E}_-(\tau) \sim \Omega_-(\tau)$ law up to the vicinity of the point τ_*, where it is partially transformed into a compressional Alfvén mode. Afterward, its energy evolution would still proceed adiabatically, but now according to the law $\mathcal{E}_+(\tau) \sim \Omega_+(\tau)$.

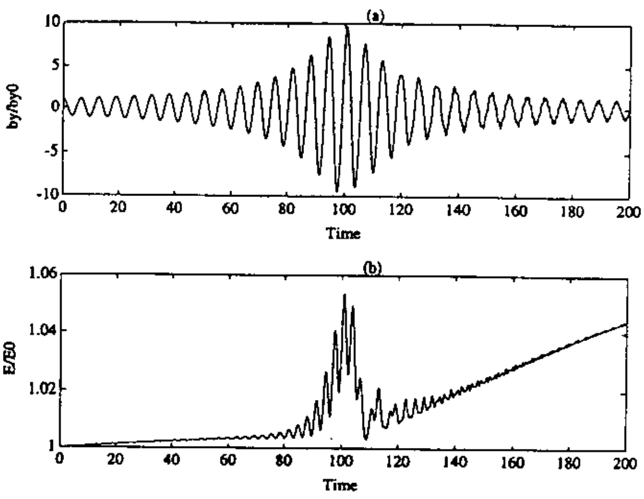

FIG. 4. The temporal evolution of 3-D SFH $[b_y(\tau)/b_y(0)$ (a) and $\mathcal{E}(\tau)/\mathcal{E}(0)$ (b)] for an initially pure shear Alfvén mode with $\beta_0 = 10$, $R = 0.1$, $\epsilon = 5\times10^{-3}$, and $\gamma = 1.0$.

The salient features of the richness of the structures shown in Figs. 3–4 [$\beta_0=10$, $R=0.1$, $\epsilon=5\times10^{-3}$, and $\gamma=0.1(\gamma=1)$] can be readily understood by examining the approximate analytical expressions for the frequencies and the amplitudes. In both these cases, an initial shear wave ($\Omega_s^2=1$), at times $\tau\sim\tau_*$ ($=100$), begins transforming into a compressional wave whose frequency $\Omega_c=[1+r^2+(\beta-R_0\tau)^2]^{1/2}=[1+r^2+R_0^2(\tau_*-\tau)^2]^{1/2}$ tends to increase as $\tau>\tau_*$, and can become much greater than $|\Omega_s|$ for large enough times. The compressional wave, generated by the shear induced coupling, becomes the dominant mode after $\tau>120$ for Fig. 3(a).

The only difference between the two cases is the magnitude of γ, which determines the efficiency of coupling. It is clear that the transformation of the shear into the compressional mode is visibly more pronounced and occurs earlier for $\gamma=0.1$, as compared to $\gamma=1$. That is precisely the reason why Fig. 3(a) is so asymmetric in time—the initial shear wave for $\tau<\tau_*$ is almost fully converted into a compressional wave for $\tau>\tau_*$. The plot in Fig. 4(a), on the other hand, retains the symmetry of Fig. 2(a) because, due to poor conversion efficiency, the initial shear wave gets contaminated by only a small admixture of compressional wave, even for times in excess of τ_*.

It is again the difference in conversion efficiency that accounts for the obvious differences in the energy plots given in Figs. 3(b) and 4(b), respectively. For Fig. 3(b), at times $\tau\gtrsim\tau_*$, the conversion to the compressional wave is almost complete. Afterward the energy increases rapidly because $E(\tau)$ for this mode scales with $\Omega_c(\tau)$, which increases almost linearly with time for $\tau>\tau_*$. The behavior of the $E(\tau)$ shown in Fig. 4(b) is a little more complicated. The sudden increase in energy at $\tau=\tau_*$ is the usual increase for the energy of the shear wave as a function of time [see Fig. 2(b), for comparison]. The gradual overall rise after $\tau=\tau_*$, however, is due to the energy contributed by the newly generated small-amplitude compressional wave. The involved structure of $E(\tau)$ near $\tau=\tau_*$ is due to the fact that in this region (DR, the region of mode conversion), the energy associated with a mode is not merely proportional to the mode frequency.

It should be noted that for the case, considered in this section, b_x and $\psi\equiv\gamma b_y-\beta b_z$ obey coupled equations,

$$\partial_\tau^2 b_x+\Omega_c^2 b_x=-2Rv_y, \tag{85}$$

$$\partial_\tau^2\psi+\Omega_s^2\psi=-2Rv_z, \tag{86}$$

which reduce to (67)–(68) for $R=0$. Equations (85)–(86) clearly show that in the nonzero shear case, the compressional and the shear Alfvén modes are intrinsically coupled; the strength of coupling is measured by R.

It is, therefore, evident that under certain circumstances (existence of the "degeneracy region" and the satisfaction of the "slow pass" conditions) the modes may effectively transform into each other with a corresponding energy transfer. The strong interaction of the modes is ensured by sufficiently large values of β_0 (of k_y) and ϵ. The transient amplication is further enhanced by a smaller γ (smaller k_z). For a given k_x, larger k_y and smaller k_z lead to the most spectacular results. The compressional mode is able to extract energy from the mean shear flow continuously (at $\tau>\tau_*$) and in this feature it closely resembles the plain sound wave whose evolution has been studied earlier.[12]

IV. DISCUSSION AND SPECULATIONS OF ASTROPHYSICAL INTEREST

The results presented in the previous section show that even in the simplest e^+e^- plasma shear flow—the parallel flow of a cold and nonrelativistic plasma—the presence of the shear leads to new physical processes notably changing the evolution of oscillation modes in the plasma, and causing their interaction with each other and with the mean (bulk) flow. Main results of our investigation are the following.

(i) Two-dimensional (2-D) perturbations ($\gamma=0$), localized in the X-Y plane, exhibit adiabatic evolution (for $R\ll1$) when their amplitude and frequency (phase) characteristics vary slowly (adiabatically) in time. Under certain conditions (determined, for example, by the initial orientation of the **k** vector and by the value of skin depth parameter ϵ), large transient growths, up to several order of magnitude, are possible.

(ii) Three-dimensional (3-D) perturbations, (with $\gamma\neq0$) are physically coupled with one another via the shear parameter R. The coupling remains strong, even for perturbations with a wavelength much longer than the collisionless skin depth. This coupling leads to a mutual transformation of the shear Alfvén and the compressional Alfvén waves in a cold nonrelativistic plasma. Transient amplification of the modes is also found. The efficiency of mutual transformation is greater for small γ.

(iii) For the general case, when no constraints are put onto the flow and perturbation parameters, one may expect the complex interplay of the effects mentioned in the previous two paragraphs.

These features of the wave dynamics in the e^+e^- plasma may prove to be quite significant in advancing our understanding of the processes taking place in the pulsar magnetosphere and the pulsar wind.

To appreciate this connection, we make a short digression. It is generally agreed that the processes of radio emission from the pulsars are still poorly understood (see, for the most recent review, Ref. 2 and references therein): there does not exist a widely accepted theoretical model. Although an e^+e^- plasma is thought to be a possible candidate for the radio emission, not too much is known either about the location of the assumed sources of the e^+e^- plasma ("gaps") or about the details of the plasma production processes. Furthermore, it is impossible to predict the velocity distribution of particles; a knowledge that is critical for the identification of a "workable" emission mechanism.

However, there are some general aspects of the phenomenon, which can be specified with some certainty. In particular, it *is* known that the e^+e^- plasma is produced in the polar cap regions at some height above the surface of the neutron star and, that the secondary plasma involves electrons and positrons flowing outward along the open field lines. It is also known that the pulsar radio emission is characterized by prominently high brightness temperatures ($T_b\sim10^{25}$–10^{30}

Ko, thus requiring some sort of a *coherent* emission mechanism.

The broad literature on pulsar radio emission problem contains several proposed models based, generally, on three kinds of plasma processes:[20,2] antenna mechanisms, reactive instabilities, and maser mechanisms. Currently the most preferred emission mechanism for pulsars is the maser mechanism, or the relativistic plasma emission. The mechanism operates in two stages: (1) an instability that generates Langmuir-like or Alfvén-type waves that cannot escape to infinity and (2) some kind of *nonlinear* "conversion process" that transforms a part of the energy of these waves into the escaping radiation.[21,2]

There are several empirical observations that any proposed mechanism must respect.[2]

(i) The mechanism should not be strongly dependent on the strength of the pulsar magnetic field B: it should apply with equal success to both weak-B (millisecond) and strong-B (young, fast) pulsars.

(ii) Coherent emission should occur in many localized, *transient* subsources. It implies that the optimum model should describe the origin and characteristics of these subsources.

(iii) The mechanism should contain a guaranteed "feedback." In the case of the relativistic plasma emission, for instance, a continuous "pump" (overtaking of slower particles by faster particles) is needed for the maintenance of those features of the particle distribution function, which are responsible for the maser process.

It would, thus, appear that in this connection the shear-induced processes that ensure natural, safe, and effective transfer of the mean e^+e^- plasma flow energy into the energy of the excited linear waves, may be worth exploring. We have demonstrated that the evolution of these waves is strongly influenced by the shear forces: coupling of various wave modes, their resulting mutual transformation, and corresponding energy exchange allows the flow energy to be eventually converted into a mode of choice that can escape as radiation.

A most notable characteristic of these processes, in the context of their possible relevance to pulsar plasma physics, is that they are quite insensitive to the strength of the ambient magnetic field $\mathbf{B}_0$: efficiency of the processes described in this study mostly depend on the features of the flow, and of the nature of perturbations.

A caveat is in order here. In the present paper, we have studied the highly idealized model of a cold, nonrelativistic parallel shear flow of an e^+e^- plasma. Our main focus was on the delineation of the physics relating to the effects of velocity shear on the wave dynamics. The results of this study, therefore, have only a limited direct applicability; they could, for example, be applied to the investigation of the physics of the *nonrelativistic* e^+e^- laboratory plasma flows.[8] We have also generalized the basic theory to cold relativistic flows. It turns out that the basic results derived in this paper survive wholly intact for cold relativistic flows that are weakly sheared. Since the relativistic calculation is quite straightforward, it is presented in Appendix B.

We are fully aware that a quantitative determination of the velocity shear-induced effects to the theory of pulsar radio emission requires the extension of this simple model to include other important physical effects, such as real geometry and kinematics of the e^+e^- shear flows, plasma temperature, other plasma inhomogeneities, etc. Postponing the detailed analysis of these effects to future work, we would like to stress that the velocity shear-induced phenomena are interesting, and may be quite relevant to the problem of pulsar radio emission. It is conceivable that the pulsar radiation, as well as the radiation of e^+e^- jets in AGNs, is energetically fueled by the huge amount of rotational energy in the "central engines" liberated partially in the plasma outflows. We argue that, afterward, this energy is transferred to the excited wave-like perturbations via the "shear channel."

APPENDIX A: DERIVATION OF THE INDUCTION EQUATION

In order to derive the *induction equation* for the magnetic field perturbation $\mathbf{B}$ we, first of all, determine the electric field vector $\mathbf{E}$ through the generalized Ohm's law [Eq. (16)]:

$$\mathbf{E}=-\left(\frac{\mathbf{U}_0}{c}\right)\times\mathbf{B}-\left(\frac{\mathbf{v}}{c}\right)\times\mathbf{B}_0+\left(\frac{m^2}{e^2\rho_0}\right)[D_t\mathbf{j}+(\mathbf{j},\nabla)\mathbf{U}_0]. \tag{A1}$$

Next, we calculate $\nabla\times\mathbf{E}$, and insert it into Faraday's law to get the induction equation. Using Eqs. (21) and (A1), we find

$$D_t\mathbf{j}+(\mathbf{j},\nabla)\mathbf{U}_0=(c/4\pi)[D_t(\nabla\times\mathbf{B})+(\nabla\times\mathbf{B},\nabla)\mathbf{U}_0]-\mathbf{U}_0\,D_t\rho_e. \tag{A2}$$

It is easy to derive the following vector identities ($\nabla\cdot\mathbf{B}=0$, $\mathbf{B}_0$ is uniform):

$$\nabla\times(\mathbf{U}_0\times\mathbf{B})=(\mathbf{B},\nabla)\mathbf{U}_0-(\mathbf{U}_0,\nabla)\mathbf{B}, \tag{A3}$$

$$\nabla\times(\mathbf{v}\times\mathbf{B}_0)=(\mathbf{B}_0,\nabla)\mathbf{v}-\mathbf{B}_0(\nabla\cdot\mathbf{v}), \tag{A4}$$

$$\nabla\times[(\mathbf{U}_0,\nabla)(\nabla\times\mathbf{B})+(\nabla\times\mathbf{B},\nabla)\mathbf{U}_0] -\nabla\times\{\mathbf{U}_0\times[\nabla\times(\nabla\times\mathbf{B})]+(\nabla\times\mathbf{B})\times(\nabla\times\mathbf{U}_0)\}, \tag{A5}$$

$$\nabla\times\{\mathbf{U}_0\times[\nabla\times(\nabla\times\mathbf{B})]\}=(\Delta\mathbf{B},\nabla)\mathbf{U}_0-(\mathbf{U}_0,\nabla)\Delta\mathbf{B}, \tag{A6}$$

$$\nabla\times[(\nabla\times\mathbf{B})\times(\nabla\times\mathbf{U})]=(\nabla\times\mathbf{U}_0,\nabla)(\nabla\times\mathbf{B})-(\nabla\times\mathbf{B},\nabla)\times(\nabla\times\mathbf{U}_0). \tag{A7}$$

All these expressions may be combined, and after obvious rearranging of terms we finally get the following explicit form of the *induction equation*:

$$D_t[\mathbf{B}-\lambda^2\,\Delta\mathbf{B}-(m^2/e^2\rho_0)\nabla\times(\rho_e\mathbf{U}_0)] =(\mathbf{B}-\lambda^2\,\Delta\mathbf{B},\nabla)\mathbf{U}_0-\mathbf{B}_0(\nabla\cdot\mathbf{v})+(\mathbf{B}_0,\nabla)\mathbf{v} +\lambda^2[(\nabla\times\mathbf{U}_0,\nabla)(\nabla\times\mathbf{B})-(\nabla\times\mathbf{B},\nabla)(\nabla\times\mathbf{U}_0)], \tag{A8}$$

where $\lambda\equiv(mc^2/4\pi e^2n_0)^{1/2}$ is the collisionless skin depth, and $\Delta\equiv\partial_x^2+\partial_y^2+\partial_z^2$ is the usual spatial Laplace operator.

APPENDIX B: WEAKLY SHEARED RELATIVISTIC FLOW

In this appendix, we show that the problem of a weakly sheared cold relativistic flow is mathematically entirely equivalent to the problem studied in the main text of this paper. For the sake of simplicity, we shall deal with the case when the wavelength of the modes is much larger than the collisionless skin depth, i.e., $\lambda|k|\ll 1$.

Let us assume that the ordered ambient flow is relativistic, and is characterized by $\mathbf{B}_0=\hat{\mathbf{e}}_x B_0$, and the momentum

$$\mathbf{P}_0=\hat{e}_x P_0(y). \tag{B1}$$

Naturally, this $\mathbf{P}_0$ has an associated flow velocity,

$$\mathbf{U}_0=\hat{\mathbf{e}}_x U_0(y)=\hat{e}_x \frac{P_0(y)}{m\gamma_0(y)}, \tag{B2}$$

where $\gamma_0=(1+P_0^2/m^2c^2)^{1/2}$ is the standard relativistic factor. In this section, it is more convenient to write the equations of motion in terms of the canonical momenta $\mathbf{p}^{\pm}$. These are

$$\mathscr{L}\mathbf{p}^{\pm}\equiv[\partial_t+U_0(y)\partial_x]\mathbf{p}^{\pm}=\pm e\left(\mathbf{E}+\frac{\mathbf{U}_0\times\mathbf{B}}{c}+\frac{\mathbf{u}^{\pm}\times\mathbf{B}_0}{c}\right), \tag{B3}$$

which are readily converted into one-fluid equations in the variables $\hat{\mathbf{P}}=(\mathbf{p}^{+}+\mathbf{p}^{-})/2$, $\mathbf{U}=(\mathbf{u}^{+}+\mathbf{u}^{-})/2$, the current $\mathbf{J}$, the density $\rho_0=2mn_0$, etc. The one-fluid equations from (B3) take the form

$$\mathscr{L}\hat{P}_y=\frac{mB_0}{4\pi\rho_0}[\partial_x B_y-\partial_y B_x], \tag{B4}$$

$$\mathscr{L}\hat{P}_z=\frac{mB_0}{4\pi\rho_0}[\partial_x B_z-\partial_z B_x], \tag{B5}$$

which, apart from a slightly complicated form of $\mathscr{L}$, are similar to (29)–(30). It is quite straightforward to evaluate the current in terms of $\mathbf{P}$'s, and then we can obtain the other two relevant equations,

$$\mathscr{L}B_y=\frac{B_0}{m\gamma_0}\partial_x\hat{P}_y, \tag{B6}$$

$$\mathscr{L}B_z=\frac{B_0}{m\gamma_0}\partial_x\hat{P}_z, \tag{B7}$$

exactly similar in form to the $\lambda=0$ version of (31) and (32); only γ_0 has made its appearance. Equations (B4)–(B7), along with $\nabla\cdot\mathbf{B}=0$, form a closed linear system.

The prescription of nonmodal analysis is to make the following transformation of variables:

$$x_1=x-U_0(y)t,\quad y_1=y,\quad z_1=z,\quad t_1=t, \tag{B8}$$

with the derivatives $\partial_{x_1}=\partial_x$, $\partial_{z_1}=\partial_z$.

$$\mathscr{L}=\partial_t+U_0(y)\partial_x=\partial_{t_1}, \tag{B9}$$

and

$$\partial_y=\partial_{y_1}-t_1[\partial_y U_0(y)]\partial_{x_1}. \tag{B10}$$

Since the flow is supposed to be weakly sheared, we could assume

$$U_0(y)=a+Ay, \tag{B11}$$

$Ad\ll a$, where d is the characteristic width of the flow, i.e., the flow has a strong steady directed component plus a weak varying part. Using (B10), one finds

$$\partial_y=\partial_{y_1}-At_1\partial_{x_1}, \tag{B12}$$

and one also approximates

$$\gamma_0=\left(1+\frac{P_0^2}{m^2c^2}\right)^{1/2}=\left(1-\frac{U_0^2}{c^2}\right)^{-1/2}\simeq\left(1-\frac{a^2}{c^2}\right)^{-1/2}. \tag{B13}$$

Thus, we see that the relativistic set of equations will have exactly the same mathematical form as the $\lambda=0$ version of (43)–(46): the only difference is that the Alfvén speed is modified due to a constant γ_0, $v_A^2=B_0^2/4\pi\rho_0\gamma_0$. The structural equivalence proves that all the later results are valid for the relativistic weakly sheared flows.

In this derivation, we have still neglected the displacement current. This assumption is justified as long as $v_A/c\ll 1$. For a strongly magnetized, relativistic e-p plasma, however, the displacement current must be retained for a proper description. This will, indeed, be done in the forthcoming work.

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ESCAPING RADIO EMISSION FROM PULSARS: POSSIBLE ROLE OF VELOCITY SHEAR

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ABSTRACT

It is demonstrated that the velocity shear, intrinsic to the e^+e^- plasma present in the pulsar magnetosphere, can efficiently convert the nonescaping longitudinal Langmuir waves (produced by some kind of a beam or stream instability) into propagating (escaping) electromagnetic waves. It is suggested that this shear-induced transformation may be the basic mechanism needed for the eventual generation of the observed pulsar radio emission.
Subject headings: plasmas — pulsars: general — waves

It is generally believed that the radio emission from a pulsar has its origin in the processes occurring in its magnetospheric plasma, which has two main constituents: an ultrarelativistic (primary) beam, and a relativistic (secondary) e^+e^- plasma, created via the *pair cascade process* (Sturrock 1971). These processes (dependent, perhaps, on the differential dynamics of these constituents) generate a variety of waves, some of which propagate out of the magnetosphere, travel through the interstellar medium, and are seen as radio emission by a distant observer (Ginzburg & Zheleznyakov 1970). Over the years, several different models for the pulsar radio emission (Ginzburg & Zheleznyakov 1975; for the most recent and comprehensive review, see, e.g., Melrose 1995) have been suggested, and certain aspects of the phenomenon, like the polarization properties of the emission, are rather well understood (Kazbegi et al. 1991; Kazbegi, Machabeli, & Melikidze 1991; Kazbegi et al. 1996). However, there are still many unanswered questions. One of the most significant and puzzling problems is the delineation of a satisfactory mechanism for the conversion of potential waves (like the Langmuir waves), readily generated in the magnetosphere, into escaping radio waves. In this Letter we propose that the velocity shear inherent in the magnetospheric e^+e^- plasma can provide the desired conversion mechanism; this may lead to a more comprehensive theory for the generation of the observed radio emission.

The first step in this process, perhaps, is the excitation of Langmuir waves by some kind of a beam or two-stream instability (Ruderman & Sutherland 1975; Cheng & Ruderman 1977; Asseo, Pellat, & Rosado 1980; Asseo, Pellat, & Sol 1983). Initially, the instability was attributed to the primary ultrarelativistic electron or positron beam. However, the beam has too low a density and too large a Lorentz factor, so that the characteristic growth time turns out to be a few times more than the time needed for the beam particles to escape the pulsar magnetosphere (Benford & Buschauer 1977; Egorenkov, Lominadze, & Mamradze 1983). In order to overcome this difficulty, Usov (1987) (see also Ursov & Usov 1988) suggested the interesting idea of a *nonstationary plasma flow*. According to this model, clouds of e^+e^- plasma are injected into the pulsar magnetosphere from time to time (with small enough intervals). Fast particles from the following clump overtake slower ones from the preceding clump, creating favorable conditions for the development of a two-stream instability, leading to the generation of Langmuir waves propagating along the magnetic field lines. In this model the instability is attributed to the dense and low Lorentz factor e^+e^- plasma, and its growth rate is found to be large enough. Thus it appears that either by Usov's or through an alternative mechanism, it should be possible to produce Langmuir waves of sufficient intensity.

The second crucial step in the development of a model is to pinpoint a mechanism(s) that will convert the energy "accumulated" in the Langmuir waves into the energy of such waves that can escape out of a pulsar magnetosphere.

There seem to be a variety of physical processes that could mediate mode conversion: induced wave scattering (Machabeli 1983), wave-wave interaction (Gedalin & Machabeli 1983; Mamradze, Machabeli, & Melikidze 1980), and *mode couplings due to some kind of a plasma inhomogeneity* (Melrose 1995). In the latter class, however, an extremely important inhomogeneity, i.e., the inhomogeneity of the velocity field (velocity *shear*), has, until recently, attracted very little attention (Scharlemann, Arons, & Fawley 1978; Arons & Scharlemann 1979; Kaladze & Mamradze 1984) in spite of the fact that Arons & Smith (1979) had, long ago, outlined a basic mechanism of an electrostatic instability of a sheared stream of charged particles flowing along a strong magnetic field. They conjectured that the energy may be liberated indirectly "through coupling of electrostatic modes generated by the instability to propagating electromagnetic modes" (Arons & Smith 1979, p. 728).

In this Letter we intend to prove that this hypothesis of Arons and Smith is highly plausible. In particular, we shall demonstrate that the velocity shear of the relativistic e^+e^- plasma flow can mediate an efficient conversion of the longitudinal, nonescaping waves (Langmuir waves) into the desired electromagnetic waves, which can propagate outward. Notice

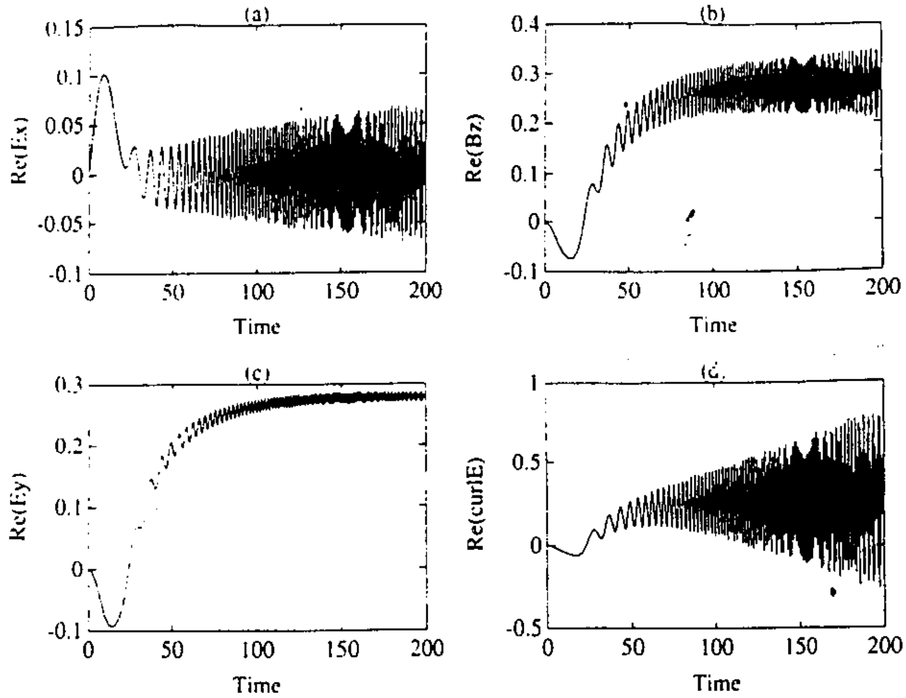

FIG. 1.—Evolutionary plots ("one-stream case") for real parts of (*a*) $e_x(\tau)$, (*b*) $b_z(\tau)$, (*c*) $e_y(\tau)$, and (*d*) $e_y(\tau) - R\tau e_x(\tau)$. Time is measured in dimensionless units $\tau = k_x t$. $R = 4 \times 10^{-3}$, $\alpha = 1$, and $\gamma = 10$.

that shear-induced mode conversion and energy exchange is known to be an efficient and widespread phenomenon (see, e.g., Chagelishvili, Rogava, & Tsiklauri 1996; Chagelishvili & Chkhetiani 1995; Rogava & Mahajan 1996; Rogava, Mahajan, & Berezhiani 1996).

We consider a collisionless, viscosity-free, and *cold* e^-e^+ plasma. Following Arons & Smith (1979), we neglect the plasma pressure and model the flow by the following, relativistic two-fluid equations:

$$\partial_t n^{\pm} + \nabla\cdot(n^{\pm}V^{\pm}) = 0, \quad (1)$$

$$[\partial_t + (V^{\pm}\cdot\nabla)]P^{\pm} = \pm e(E + V^{\pm} \times B), \quad (2)$$

$$\nabla\cdot E = 4\pi e[n^{+} - n^{-}], \quad (3)$$

$$\nabla \times E = -\partial_t B, \quad (4)$$

$$\nabla \times B = 4\pi e[n^{+}V^{+} - n^{-}V^{-}] + \partial_t E, \quad (5)$$

where the notation is standard with the speed of light taken to be unity. The equilibrium velocity of electrons and positrons in the sheared stream will be modeled by $V_0^{\pm} \equiv V_0 = \{U_0 + Ay, 0, 0\}$, where A measures the strength of the shear. It will be assumed that the stream is weakly sheared, in the sense that Ay is much smaller than the average part U_0. The resulting momentum becomes $P_{0x}(y) \simeq P_0 + ay$, where $a \equiv mA\gamma_0^3$, $P_0 \equiv m\gamma_0 U_0$, and $\gamma_0 \equiv (1 - U_0^2)^{-1/2}$ is the average Lorentz factor. In the first part of our model, the equilibrium velocities of electrons and positrons are equally sheared ($A^{+} = A^{-} = A$), and there is no mutual streaming of the two species ($\gamma_0^{+} = \gamma_0^{-}$).

In order to delineate the basic features of shear-induced mode conversion, we make the following simplifying assumptions: (1) the plasma is quasi-neutral ($n_0^{\pm} \equiv n_0$) with an equilibrium (one fluid) mass density $\rho_0 \equiv 2mn_0$; (2) the magnetic field $B_0 = \text{const.} = B_0$ tends to infinity, restricting the e^+e^- plasma motion to the x-axis (a quasi-one-dimensional system): thus the perpendicular dynamics will be altogether neglected; and (3) the perturbation wavevectors lie in the $X - Y$ plane (defined by B_0 [U_0] and by the direction of the velocity shear). We consider, from now, the *lt waves*, for which the electric field vector E lies in the $X - Y$ plane, and the magnetic field perturbation $B = \{0, 0, B_z\}$ is along the z-axis.

With these simplifications, the magnetospheric plasma can be described by the following set of linearized equations:

$$D_t\rho_q + \partial_x J_x = 0, \quad (6)$$

$$D_t J_x = (\omega_p^2/4\pi\gamma_0^3)E_x, \quad (7)$$

$$\partial_x E_x + \partial_y E_y = 4\pi\rho_q, \quad (8)$$

$$\partial_t B_z = \partial_y E_x - \partial_x E_y, \quad (9)$$

$$\partial_t E_x = -\partial_x B_z, \quad (10)$$

where $D_t \equiv \partial_t + (U_0 + Ay)\partial_x$, $\omega_p^2 \equiv 8\pi e^2 n_0/m$, and we have used the *one-fluid* variables: $\rho_q \equiv e(n^+ - n^-)$, the perturbed charge density, and $J \equiv en_0(u^+ - u^-)$, the perturbed current density.

Note that equations (9) and (10) contain the usual time derivative, while in equations (6) and (7) we have the convective derivative D_t. Since it is assumed that $Ay \ll U_0$, we can approximate $\partial_t \simeq D_t - U_0\partial_x$. The advantage of the resulting system is that it may be handled by the standard method of "Kelvin modes" (see, e.g., Marcus & Press 1977; Criminale & Drazin 1990). This method requires the change of variables—$x_1 = x - (U_0 + Ay)t$, $y_1 = y$, $t_1 = t$—that leads to a substantial simplification in the solution of the initial-value problem.

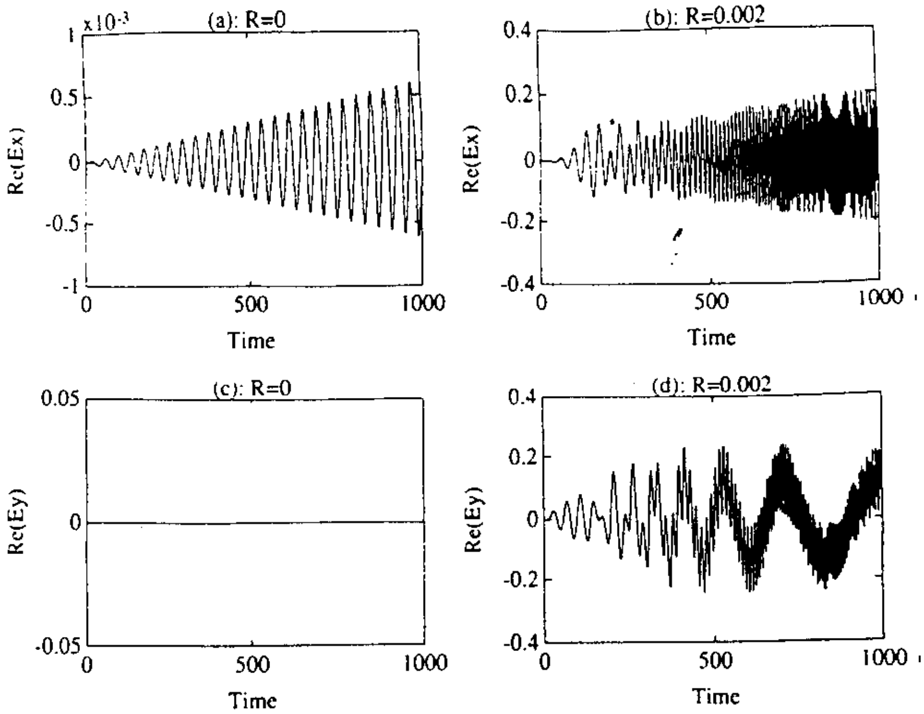

FIG. 2.—Evolutionary plots [real parts of $e_x(\tau)$ and $e_y(\tau)$] for two streams of e^-e^+ plasma with average Lorentz factors $\gamma_1 = 10$ and $\gamma_2 = 10^4$ ($\sigma = 1$). (*a*) and (*c*) are plotted for the "zero shear" ($R = 0$) case, while for (*b*) and (*d*), $R = 2\times10^{-3}$.

The differential operators, appearing in the above equations, are so transformed—$D \Rightarrow \partial_t - (U_0 - Ay)\partial_x \Rightarrow \partial_\tau$, $\partial_x \Rightarrow \partial_x$, $\partial_y \Rightarrow \partial_y - At\partial_x$—that an initial inhomogeneity in space (y) is exchanged for a new inhomogeneity in time. The Fourier transform in the new spatial variables converts equations (6) and (7) and equations (9) and (10) to a set of first-order, ordinary differential equations (ODEs) for the evolution of the spatial Fourier harmonics (SFH) (see, e.g., Chagelishvili, Rogava, & Segal 1994). The wavevector components may also be written in the original (x, y, t) coordinates: $k_x = k_x$ and $k_y(t) = k_y - Atk_x$. It is of principal importance to note that the velocity shear induces linear drifts of SFH so that initially longitudinal modes can become eventually oblique.

By introducing the dimensionless quantities—$\rho \equiv \hat{\rho}/en_0$, $J \equiv \hat{J}_x/en_0$, $e_{x,y} \equiv (k_x/en_0)\hat{E}_{x,y}$, $b_z \equiv (k_x/en_0)\hat{B}_z$, $\sigma \equiv \omega_p^2/4\pi k_x^2$, $\tau \equiv k_x t$, $R \equiv A/k_x$, $\beta_0 \equiv k_y/k_x$, $\beta(\tau) \equiv \beta_0 - R\tau$—we can reduce the original system to the following complete set of dimensionless equations:

$$\partial_\tau \rho = -iJ, \qquad (11)$$

$$\partial_\tau J = -(2\sigma/\gamma_0^3)[4i\pi\rho - \beta(\tau)e_y], \qquad (12)$$

$$(\partial_\tau - iU_0)e_y = -ib_z, \qquad (13)$$

$$(\partial_\tau - iU_0)b_z = -i[1 + \beta^2(\tau)]e_y + 4\pi\beta(\tau)\rho. \qquad (14)$$

In this Letter we shall investigate the evolution of those modes for which the initial perturbations are purely longitudinal ($k_y = k_x$ and $\beta_0 = 0$). This is indeed the most important case because purely longitudinal Langmuir waves are the easiest to excite in a pulsar magnetosphere. For further analysis, it is convenient and revealing to combine equations (11)–(14) to obtain two equations for the variables $E(\tau) \equiv e^{-iU_0\tau}e_y$ and $D(\tau) \equiv -4\pi i\rho$:

$$\partial_\tau^2 D + W^2 D = -W^2 R\tau e^{iU_0\tau}E, \qquad (15)$$

$$\partial_\tau^2 E + (1 + R^2\tau^2)E = -R\tau e^{-iU_0\tau}D, \qquad (16)$$

where $W^2 \equiv 8\pi\sigma/\gamma_0^3$.

Equations (15) and (16) clearly reveal that shear ($R \neq 0$) is responsible for the mutual coupling of the purely potential, longitudinal Langmuir oscillations (with phase velocity $\omega/k = W$) and the purely transverse electromagnetic waves (with phase velocity $\omega/k = 1$). The entire time dependence (of the coupling terms as well as of the effective frequency) is due to the nonzero shear and will be slow or *adiabatic* for $R \ll 1$. For the problem at hand, the effective shear parameter indeed turns out to be small: it is typically a few orders of magnitude smaller than unity. The detailed estimates will be given in a later, larger paper.

Though the physical meaning of equations (15) and (16) is transparent enough, it is instructive to look at some representative solutions. In Figure 1, we plot the functions $e_x(\tau)$, $e_y(\tau)$, $b_z(\tau)$, and $e_y + R\tau e_x$ (the latter function measures in dimensionless notations value of $\nabla \times E$) obtained by a numerical integration of the defining equations. For this case, the values of parameters are $R = 4 \times 10^{-2}$, $\sigma = 1$, $\gamma_0 = 10$, and the initial perturbation is taken to be a pure longitudinal Langmuir wave [$e_x(0) \neq 0$, while $e_y = b_z = 0$]. We see that as time progresses, the fields $e_y(\tau)$ and $b_z(\tau)$ are excited and the wave becomes more and more nonpotential ($e_y + R\tau e_x$ is increasing): the initial perturbation (longitudinal and purely potential Langmuir wave) begins to acquire transverse "features" as it evolves.

It would seem that we have now identified both pieces of the puzzle: (1) a reasonable mechanism (some kind of a beam or stream instability) for generating longitudinal, potential Langmuir waves [with $k(0)\|B_0$] in the e^-e^- plasma in the pulsar magnetosphere; and (2) an effective shear-induced coupling to transform these nonescaping waves into the longitudinal-transversal, nonpotential waves that are perfectly capable of escaping the stellar environment.

We now propose a comprehensive model. We do this by incorporating Usov's (1987) nonstationary injection hypothesis into our model. Let us now consider two streams of e^-e^- plasma with average Lorentz factors γ_1 and γ_2 ($\gamma_2 > \gamma_1$). The resulting equations,

$$\partial_\tau^2 D_1 + W_1^2(D_1 + D_2) = -W_1^2 R\tau e^{iU_1\tau}E, \quad (17)$$

$$(\partial_\tau + i\Delta U)^2 D_2 + W_2^2(D_1 + D_2) = -W_2^2 R\tau e^{iU_1\tau}E. \quad (18)$$

$$\partial_\tau^2 E + (1 + R^2\tau^2)E = -R\tau e^{-iU_1\tau}(D_1 + D_2). \quad (19)$$

where $\Delta U \equiv U_2 - U_1$, $W_1^2 \equiv 8\pi\sigma/\gamma_1^3$, and $W_2^2 \equiv 8\pi\sigma/\gamma_2^3$, explicitly encompass both of the essential processes leading to the pulsar radio emission: the onset and amplification of Langmuir oscillations due to a built-in two-stream instability, and the subsequent conversion of these oscillations into escaping radiation. Corresponding plots are presented on Figure 2 for two streams with $\gamma_1 = 10$ and $\gamma_2 = 10^2$ ($\sigma = 1$, as above). Figures 2*a* and 2*c* represent the zero shear case ($R = 0$), while for Figures 2*b* and 2*d*, $R = 2 \times 10^{-3}$. In the former case, the two-stream instability is "switched on," and the amplitude $e_x(\tau)$ increases with time. But $e_y(\tau) = 0$ for all times, and the wave remains potential. In the latter case, however, the presence of nonzero shear changes the situation drastically: the wave becomes nonpotential, and the electromagnetic component $e_y(\tau)$ is strongly excited.

The transformation of purely longitudinal, nonpropagating modes into electromagnetic waves is just one of the many mode transformation processes that can actually happen in the magnetospheric plasma (see, e.g., Rogava et al. 1996 for Alfvén modes). A comprehensive paper dealing with shear-mediated interactions of various linear waves sustained by an e^-e^- plasma (see, for review, Volokitin, Krasnoselskikh, & Machabeli 1983; Lominadze et al. 1986; Lyubarsky 1995) is under preparation.

In summary, we have demonstrated in this Letter that the mode coupling induced by velocity shear could be a vital link in the chain of processes that must be invoked in order to solve the puzzle behind the pulsar radio emission. We must also remember that one of the most severe criteria imposed on possible pulsar radio emission models is that the bona fide mechanism must apply to both the weak-B_0 (millisecond) and the strong-B_0 (young, fast) pulsars (Melrose 1995). In other words, this criterion demands that the true mechanism must apply in a range of 4–5 orders of magnitude in B_0. The velocity-shear–based mechanism of mode conversion seems to be tailor-made to satisfy this requirement.

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