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AUTUMN COLLEGE ON PLASMA PHYSICS

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LECTURES ON BASIC PLASMA THEORY COMPLETED WITH PROBLEMS AND THEIR SOLUTIONS

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CONTENTS

ABSTRACT

These lectures are devoted to the basic plasma theory which is presented in his historical way of development. They consist of 3 parts. The first part (lectures 1 - 7) concerns the thermodynamically equilibrium and spatially unbounded plasmas. The second part (lectures 8 - 13) represents the theoretical description of stability problems for nonequilibrium plasmas - magnetically confined and spatially bounded plasma, stimulated radiation of fast charged particles and plasma-beam interaction, quasilinear theory of plasma oscillations and non-linear phenomena of wave-wave interactions in plasmas. The third part (last 2 lectures) is devoted to the concrete problems and their solutions are given. Finally, in the appendix the short history of fundamental papers on the kinetic plasma theory beginning from L. LANDAU's and of course A. VLASOV's papers are presented in the context of the present day concepts formulated primarily by N. BOGOLYUBOV in 1946.

PREFACE

These lectures were given by Professor A. RUKHADZE for the researchers of "ECOLE -POLYTECHNIQUE" in 1996.

They were written and corrected by F. AMAUDRIC du CHAFFAUT , who solved all the exercises as well.

The lectures present the basic plasma physics in its historical way of development and consist of all necessary knowledge about plasma electrodynamics. The appendix to these lectures is devoted to the history of fundamental papers on the kinetic plasma theory beginning from L. LANDAU's and of course A. VLASOV's papers and finishing by the well-known N. BOGOLYUBOV's book in which the general method of derivation of the VLASOV's kinetic equation as well as BOLTZMANN's equation was developed.

We hope that these lectures will be useful for young researchers of "ECOLE - POLYTECHNIQUE ".

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F. du CHAFFAUT A. RUKHADZE

PART 1

Basic Equations and Electromagnetic Properties of Thermodynamically Equilibrium and Unbounded Plasmas

LECTURE 1

Plasma and its Parameters. Gas Approximation

* 1 What is the "PLASMA"

The first incomplete definition of " PLASMA " was given by I. Langmuir in 1923. According to this definition " Plasma is a shine gas consisting of electrons, several types of ions and neutral atoms and molecules ". But people saw the plasma and moreover used it, more than thousand and thousand years before I. Langmuir. It is obvious that the first who saw the plasma was the God. Creating the Earth and Water and Sky, he noticed that everything was at dark and sad " let to be light." Then he noticed the Sun in the sky. Of course it was the solar plasma. But this phenomenon occurs outside of people's understanding for many thousand centuries. Moreover they did not suspect that they dealt with real plasma when they observed the lightning and even used it.

The first mention about ionized gas of particules was done by O. Heaviside when he predicted more than 100 years ago that around the Earth at the altitudes of 300-400 km there exists a layer of sufficiently high density ionized gas, which reflects the radiowaves. This layer is known as ionosphere. O. Heaviside not only predicted the existence of ionospheric F-layer but also gave explanation : "the origin of this layer is the atmospheric gas ionization by the ultraviolet Solar radiation". According to the modern representation the concentration of charged particles exceeds $(n_e \sim n_i) \sim (1 \text{ to } 3)\cdot 10^{6}\ cm^{-3}$, their temperature $T_e \sim (1\text{-}2)\cdot 10^{3}$ K and $T_i/T_e \sim 0{,}3$. At the same time the concentration of neutrals n_0 is $\sim 10^9\ cm^{-3}$ and their temperature $T_0 \sim 200K$ or the ionization degree $\sim n_e/(n_e+n_0) < 10^{-3}$ (weak ionized gas). The Earth magnetic field at this altitude is $B_0 \sim 0.5$ Gauss and therefore the pressure ratio will be $\beta = 8\pi (n_e+n_i)\, Te/ B_0^2 \sim 10^{-4} << 1$. Thus for the F-Layer the electron Langmuir frequency $\omega_{Le} = \sqrt{4\pi e^2 n_e / m} \sim 8\cdot 10^7 s^{-1}$, whereas the electron collision frequency $\nu_e \sim 3\cdot 10^3\ s^{-1}$ which provides the stable radiocommunication on the Earth in the range of radiowave lengths $20m < \lambda < 2000m$.

Inspite of very important role of the ionospheric F-Layer for mankind the regular investigations of the parameters and properties of this plasma were begun only in the 60-s when the rockets and atmospheric probes appeared. Much before, the properties of ionized gas or plasma were investigated in the laboratory experiments when the physicists tried to create artificial plasma.

The most significant achievement in this way was received by I. Langmuir at the beginning of 20-s. He introduced the conception of plasma as a gas of charged particles and neutral atoms and molecules, their concentrations n_e, n_i and n_0 and temperatures T_e, T_i and T_0. Besides he discovered in the gas discharge plasmas the high frequency oscillations with phase velocity much larger than electron thermal velocity, not depending on the masses of ions and neutrals . Moreover he measured their frequency $\omega = \omega_{Le} = \sqrt{4\pi e^2 n_e / m}$, that is known as Langmuir electron frequency.

I. Langmuir described also the low frequency oscillations in the gas discharge plasmas with linear dispersion dependence $\omega = k\cdot v_S$ (like sound waves). The phase velocity of such waves v_S is much less than electron thermal velocity and is of the order of ion thermal velocity.

I. Langmuir was sure that these waves represent the usual sound waves and used hydrodynamical expression for their description:

$$v_s = \sqrt{\gamma P / \rho} = \sqrt{\gamma (T_e + T_i) / M} \tag{1}$$

Here $\gamma = c_p / c_v$, as it was supposed by I. Langmuir. Unfortunately this assumption is incorrect and only in 1954 the correct expression for v_s was obtained by G. Gordeev, who revealed the physical sense of low frequency oscillations in this type of plasmas.

Many types of gas discharge plasmas are known today. They are created by different types of ionizing radiation: microwave and optical discharge plasmas, radiofrequency and direct current gas discharges, electron and ion beam discharges and etc. They have very large applications in physics and technology: light sources and current commutators, plasma-chemical and nuclear fusion reactors, plasma accelerators and *"thrusters"* are based on gas discharges. Therefore, the great interest of scientists and engineers in the plasma physics and technology is natural.

The parameters of gas discharge plasmas are numerous in a wide ranges. Thus in the neon lamps $n_e \sim n_i \sim 10^{11}$ to 10^{13} cm^{-3}, $n_0 \sim 3 \cdot 10^{13}$ to 10^{16} cm^{-3} (or $P_0 \sim 10^{-3}$ to 1 torr), $T_e \sim 10^4$ to 10^5K (or $T_e \sim 1$ to 10ev) and $T_i \sim 1000$K (or $T_i \sim 0.1$ ev). In other words plasma in the neon lamps usually is a low density weakly ionized and highly nonisothermal with $T_e >> T_i$. On the other hand, in the MHD energy convertors the plasma has high density and is practically completely ionized ($n_e \sim n_i \sim 10^{18}$ cm^{-3}, $n_0 \sim 10^{19}$ cm^{-3}), and very low temperature $T_e \sim T_i \sim T_0 \sim 0.2$ to 0.3 ev. At the opposite of the ionospheric plasma, which is practically unbounded, the laboratory gas discharge plasmas are essentially bounded, their sizes don't exceed several centimeters or decimeters (plasma in MHD convertors)

The thermonuclear plasma deserves a specific attention, the plasma is very hot in the thermonuclear devices. The idea of magnetic plasma confinement and initiation fusion reactions in a hot plasma was proposed at the beginning of 50-s by A. Sakharov and I. Tamm in USSR and L. Spitzer in USA. There exist different types of thermonuclear plasma reactors. The most popular are tokamaks, toroidal magnetic confinement (mirror) systems. Plasma in the thermonuclear devices must be very hot, $T_e \sim T_i \sim 10^8$ K$\sim$10 kev, and at plasma density $n \sim 10^{14}$ cm^{-3} the confinement time, according to the Lawson's criterium, is

$$n\tau > 10^{14} \tag{2}$$

which leads to $\tau > 1$s. The strength of magnetic field $B_0 \sim 40$ to 50 KGauss which provides the fulfillment of the inequality $\beta = (8\pi nT)/B_0^2 < 10^{-2} << 1$.

The very interesting alternative method of initiation fusion reactions is the so-called inertial confinement and heating of solid target (d-T tablets). The confinement time of such dense and hot plasma ($n \sim 10^{23}$ cm^{-3} ,$T \sim 10$ kev) is less than inertial time, or $\tau \leq a/v_s$, where $v_s \sim 10^8$ cm/s is the sound velocity and a is the target radius.Taking into account Lawson's criterium one can estimate $\tau \sim 10^{-9}$ s = 1 ns and $a < v_s\tau \sim 0.1$cm. The energy input necessary for heating of such a target plasma is about $Q = 4/3(\pi a^3 nT) \sim 1$ MJ and heating source power $P_W \sim 10^{15}$ W.

In 1964 Soviet physicists N. Basov and O. Krokhin proposed to use the very powerfull laser radiation as a source for target heating and initiating the thermonuclear reactions. The another method of heating was proposed in 1970 by H. Winterberg (USA) and E. Zavoskiy (USSR), they proposed to use for this aim a short pulse ($\tau \sim 10^{-7}$s) of very powerfull relativistic electron beam with $P_b \cong 10^{14}$ W and Q ~ 10MJ.

In connection with the thermonuclear plasmas it must be mentioned that the sun and stars are natural thermonuclear reactors. In the inner part of stars, plasma is very hot T ~ 10 to $1000 \cdot 10^6$ K and very dense n ~ 10^{24} to 10^{26} cm^{-3}, whereas on their surfaces T ~ 10^4 K and density 1 to 10 cm^{-3}. Investigations of the stars plasmas and inter planetary plasmas is the main goal of Astrophysics.

In conclusion let us discuss the parameters of solid state plasmas in metals and semiconductors. In solid state the real particles are placed in a periodical field of lattice (crystalline) and therefore one can say about fermion type excitation with positive (holes) or negative (electrons) carriers. There arises complicated energetical structure in which the effective masses of carriers are determined as $m_{\pm} = (\partial^2 \varepsilon_{\pm} P)/\partial P^2)^{-1}$, where $\varepsilon_{\pm}(P)$ are the energy spectrum of carriers in the conductive zones. Usually in a semiconductor $m_+ \sim$ m (is of the order of the real electron mass) whereas $m_- \sim$ 0.1 to 0.01 m. At the same time, in metals exist only negative carriers with $m_- \sim$ m, and the wide band of conducting zone is practically infinite. For description of conducting media usually, the conception of carriers is used: "electron-hole" plasma in semiconductors and purely electron plasma in metals. However it must be done very accurate.

Thus from above discussion it is seen that plasma is very wide spread in nature, more than 99% substances of the Universe exists in a plasma state. Therefore, it is natural that a plasma is very often considered as the 4-th aggregate state of matter.

* 2 Plasma as a gas of charged particles

Below we will consider plasma as a gas of charged particles. What does that mean? For clarifying this problem the interactions between the plasma particles must be considered. Let us begin from neutral particles - the problem already has been investigated by great L. Boltzmann. He understood that the interaction between them is very strong, but they interact only on very short distances. Therefore he imagined them as a hard spherical balls with radius a ~ 10^{-7} to 10^{-8} cm. The potential of neutral particles interaction then can be written as

(3)

$$U(r) = \begin{cases} \infty \text{ if } r \le a \\ 0 \text{ if } r \ge a \end{cases}$$

Inspite of very strong interaction, if the density of neutrals is sufficiently small , the following inequality takes place

(4)

$$\eta_0 = a/r_{av} = a n_0^{1/3} << 1$$

then the motion of neutral particles is practically free, they interact to each other very seldom and in the first approximation we can neglect this interaction completely.

In the second approximation we can take into account the interaction between the particles as a small correction to the free motion. Thus the inequality (4) represents a condition of validity of gas approximation for the neutral component of plasma. It is obvious that the condition (4) is valid also for the interaction of charged particles with neutrals.
Quite another physical meaning has the condition for validity of gas approximation for the interaction between the charged particles of a plasma. The Coulomb interaction is a long range one and therefore the gas approximation is valid if the potential energy of charged particles interaction is small in comparison with their kinetic energy (freedom energy). In other words, gas approximation is valid if :

$$\eta_1 = U_\alpha(r_{av})/\langle\varepsilon_\alpha\rangle \sim (e^2 n_\alpha^{1/3})/\langle\varepsilon_\alpha\rangle \ll 1 \tag{5}$$

Here $n_\alpha \sim n_{e,i}$, $T_\alpha = T_{e,i}$, $m_\alpha = m_{e,i}$ and

$$\langle\varepsilon_\alpha\rangle = \begin{cases} T_\alpha & \text{if } T_\alpha > \varepsilon_{F\alpha} = ((3\pi^2)^{2/3}\eta^2 n_\alpha^{2/3})/2m_\alpha \\ \varepsilon_{F\alpha} & \text{if } \varepsilon_{F\alpha} > T_\alpha \end{cases} \tag{6}$$

Here $n \sim n_{e,i}$, $T=T_{e,i}$, $m= m_{e,i}$.

The condition (5) was firstly formulated in 1937 by L. Landau
It must be noted that for a nondegenerate, $T_\alpha > \varepsilon_{F\alpha}$ isothermal, $T \sim T_e \sim T_i$, and neutral plasma, $n \sim n_e \sim n_i$, the conditions (4) and (5) are similar in the sense that with increasing of particles density (n_0 or $n_{e,i}$) plasma becomes more and more nonideal. At the same time, for a degenerate case, $\varepsilon_{Fe,i} > T_{e,i}$, the physical sense of (5) is opposite to (4) and corresponds to the fact that when n increases then the plasma becomes more and more ideal. This follows from dependence $\varepsilon_F \sim n^{2/3}$ which leads to $\eta_1 \sim n^{-1/3}$. Thus the more dense degenerate plasma in metals occurs to be more ideal.
Another difference between the conditions (4) and (5) follows from comparing the ratio of interaction ranges for charged and neutral particles to the average distances between them. In agreement to (4) , this ratio is small. In this case, relation (5) has quite opposite meaning. For convincing this let us consider the potential of a point charged particle q located at r=0 in the nondegenerate plasma :

$$\Delta\phi = 4\pi q\,\delta(r) + 4\pi e\{n_e e^{e\phi/T_e} - n_i e^{-e\phi/T_i}\} \tag{7}$$

For simplicity electron and ion charges in a plasma are supposed to be equal and opposite, $e_i = -e$, and consequently their densities $n_e=n_i=n$. Then from (7) under the conditions $|e\phi| \ll T_e, T_i$, follows:

$$\phi = (q e^{-r/D})/r, \quad D = \{\Sigma_{e,i}\, 4\pi e_\alpha^2 n_\alpha/T_\alpha\}^{-1/2} \text{ Debye length} \tag{8}$$

and $r_{D\alpha} = (T_\alpha/(4\pi e_\alpha^2 n_\alpha))^{1/2}$ are the Debye lengths of electrons and ions, $\alpha = e,i$.

It is easy to understand that D characterizes the Coulomb interaction range of charged particles in a plasma. Therefore for the ratio of this range to the average distance between the particles one obtains (9):

$$D n^{1/3} \sim \sqrt{T/e^2 n^{1/3}} \sim 1/\eta_1^{1/2} >> 1 \qquad (9)$$

This inequality is opposite to (4) and it means that the average distance between charged particles in gaseous plasma is much less than the interaction range, or a large number of charged particles must exist in a Debye sphere. It is easy to show that this statement takes place in the case of degenerate plasma too.

It must be noticed that a plasma can be considered not only as a simple totality of charged particles but as a medium if its size is much larger than Debye length. Moreover only under this condition the Debye length has a physical meaning.

In conclusion let us make some estimations of conditions (4) and (5) for different plasmas. First of all we must notice that for $a \sim 10^{-7}$ to 10^{-8} cm, from (4) follows the validity of gaseous approximation for neutrals $n_0 \leq 10^{21}$ to 10^{22} cm^{-3}. This means that gaseous approximation for neutrals is valid up to hundreds atmospheric pressure. It is obvious that for the usual gases this condition is satisfied with great supply.

Another situation takes place for charged components of plasmas and for condition (5). For ionospheric plasma in F-Layer where $n_{e,i} \sim 10^6$ to $10^7 cm^{-3}$, and $T_e \sim 10^4 K$, $T_i \sim 10^3 K$ we have $\eta_1 \leq 10^{-4} << 1$, or this plasma is highly ideal. Analogical situation takes place in the laboratory gas discharge plasmas with $n_e \sim 10^{11}$ to 10^{14} cm^{-3} and $T_e \sim 10^4$ to $10^5 K$ where $\eta_1 \leq 10^{-2}$ to $10^{-4} << 1$. At the same time in the high density plasmas of MHD convertors and light sources usually $n \sim 10^{13}$ to 10^{19} cm^{-3} and $T_e \leq 1$ to $5 \ 10^4$ K. Therefore $\eta_1 \sim 0.1$ to 0.5 which means that, in such plasmas, nonideal effects are essential.

For thermonuclear plasmas in the magnetic confinement devices $n \sim 10^{14}$ cm^{-3} and $T \sim 10^8 K$ what means that $\eta_1 \leq 10^{-5} << 1$, whereas for inertial fusion plasma with $n \sim 10^{23} cm^{-3}$ and $T \sim 10^8 K$ we have $\eta_1 \sim 0.01$. In the last case the slightly nonideal effects must be taken into account.

Finally we will say some words about solid states plasmas. In a good conducting metals as copper $n_e \sim 5 \cdot 10^{22}$ cm^{-3}, and therefore electrons are degenerate, $\varepsilon_{Fe} \sim 1ev$ and $\eta_1 \sim 0.2$, and they can be approximately considered as a weakly nonideal gas. But for the most metals $n_e \leq 10^{22}$ cm^{-3}, and $\eta \geq 1$, which means that the electrons in such metals represent liquid, so-called degenerate Fermi-liquid. In semiconductors, carriers parameters are varied in a very wide range and therefore different situations are possible. Below we restrict ourselves in consideration of only gaseous plasma.

LECTURE 2

The Simplest Plasma Models - The Model of Independent Particles

* 3 The single particle model - Its achievements and failures

It is obvious that the most consistent description of plasma properties was reached by using the kinetic description. But historically the early plasma models were much more simple and despite of this, they gave quite good results, in a good agreement with experiments. However sometimes such models were applied to problems outside of the frameworks of models. Then some disappointments arised, which stimulated the development and improvement of other models until the perfect kinetic description was proposed by A. Vlasov.

Below we will follow this historical process of the development of plasma physics and begin our consideration from the simplest model - the model of independent particles. This model firstly proposed by I. Langmuir consists of Newton equations of electron and ion motions which are completed with Maxwell equations. This model was very fruitfull for investigating the propagation of radiowaves through the ionospheric plasma, as it was shown by V.Ginzburg, before the second world war. The equations of motions in the model of independent particles look as

(10)

$$d\mathbf{v_e}/dt = e/m\{\mathbf{E} + 1/c[\mathbf{v_e}\times\mathbf{B}]\} - \nu_e \mathbf{v_e}$$

$$d\mathbf{v_i}/dt = + e_i/M\{\mathbf{E} + 1/c[\mathbf{v_i}\times\mathbf{B}]\} - \nu_i \mathbf{v_i}$$

Here $\mathbf{v_e}$ and $\mathbf{v_i}$ are the electron and ion velocities, $\nu_e = \nu_{ei} + \nu_{e0}$ and $\nu_i = \nu_{ie} + \nu_{i0}$ their collision frequencies, for which the following equality takes place $m\nu_{ei} = M\nu_{ie}$. The electric **E** and magnetic **B** fields must be finite because they determined the Lorentz force acting on a test charge q :

(11)

$$\mathbf{F} = q\{\mathbf{E} + 1/c[\mathbf{v}\times\mathbf{B}]\}$$

These quantities satisfy the Maxwell equations

(12)

$$\mathrm{div}\,\mathbf{E} = 4\pi\rho = \Sigma 4\pi en\,,\ \mathrm{div}\,\mathbf{B} = 0$$

$$\mathrm{rot}\,\mathbf{E} = -1/c[\partial\mathbf{B}/\partial t],\ \mathrm{rot}\,\mathbf{B} = 1/c[\partial\mathbf{E}/\partial t] + 4\pi\mathbf{j}/c = 1/c[\partial\mathbf{E}/\partial t] + 4\pi/c\,\Sigma\, en\mathbf{v}$$

moreover for each components of charged particles the continuity equation is satisfied (α= e,i)

(13)

$$\partial n_\alpha/\partial t + \mathrm{div}\, n_\alpha \mathbf{v}_\alpha = 0$$

Thus in accordance with (10) the motions of charged particles are defined by the electric and magnetic field **E** and **B** and at the same time these fields themselves are determined by the charged particles motions. Thus we have selfconsistence connection between the particles motions and electromagnetic fields. This idea of selfconsistence was proposed by I. Langmuir at the beginning of 20-s. However it has remained still misunderstandable for many scientists up to day.

From the equations of particles motions (10) only one vector quantity must be defined - the current density **j** which appears in the field equations (12) as an external source. As about charge density ρ, this quantity can be easily defined by using the continuity equation (13). Taking in consideration that the magnetic field **B** also can be expressed in terms of electric field **E** we conclude that the problem of any plasma model is the calculation of induced current density

$$j_i = \sum env_i = \sigma_{ij}(\mathbf{E})\, E_j \tag{14}$$

This relation in general represents nonlinear Ohm's law and σ_{ij} (**E**) is nonlinear operator of plasma conductivity

Instead of j and ρ one can introduce the induction vector **D** by the following relation

$$\partial \mathbf{D}/\partial t = \partial \mathbf{E}/\partial t + 4\pi \mathbf{j} \tag{15}$$

Using this relation the field equations (12) take the form

$$\mathrm{div}\, \mathbf{D} = 4\pi\rho_0, \quad \mathrm{div}\, \mathbf{B} = 0 \tag{16}$$

$$\mathrm{rot}\, \mathbf{E} = -1/c(\partial \mathbf{B}/\partial t), \quad \mathrm{rot}\, \mathbf{B} = 1/c(\partial \mathbf{D}/\partial t) + 4\pi \mathbf{j}_0/c$$

These equations differ from (12), they take into consideration not only the induced current and charges densities j and ρ but also the external sources j_0 and ρ_0. These equations in addition to the equations of motions (10) represent the complete system of the simplest plasma model - the model of independent particles. The validity limits of this model can be estimated by considering some basic linear problems. In linear approximation and in the absence of external magnetic field $\mathbf{B}_0$ and the sources j_0 and ρ_0 the solutions of the equation (10) can be presented as $e^{-(i\omega t+i\mathbf{k}\cdot\mathbf{r})}$. Then one can easily obtain

$$\mathbf{v}_e = (ie\mathbf{E})/[m(\omega+i\nu_e)], \quad \mathbf{v}_i = (ie_i\mathbf{E})/[M(\omega+i\nu_i)] \tag{17}$$

which leads to the following expression for induced current density

$$\mathbf{j} = \Sigma_\alpha e_\alpha n_\alpha \mathbf{v}_\alpha = \Sigma_\alpha (ie_\alpha^2 n_\alpha \mathbf{E})/[m_\alpha(\omega+i\nu_\alpha)] = \sigma\, \mathbf{E} \tag{18}$$

Thus for plasma conductivity we have

$$\sigma_{ij} = \sigma\delta_{ij}, \quad \sigma = \Sigma\,(ie^2 n/m[\omega+i\nu]) \tag{19}$$

(we omit the summation index α). Using the definition (15) one can introduce the dielectric permittivity

$$D_i = \varepsilon_{ij}E_j, \quad \varepsilon_{ij} = \delta_{ij} + 4\pi i\, \sigma_{ij}/\omega \tag{20}$$

For the isotropic plasma we will get the following result

$$\varepsilon_{ij} = \varepsilon(\omega)\delta_{ij}, \quad \varepsilon(\omega) = 1 + 4\pi i\sigma(\omega)/\omega = 1 - \Sigma\omega_L^2/\omega(\omega+i\nu) \tag{21}$$

where $\omega_{L\alpha} = \sqrt{4\pi e_\alpha^2 n_\alpha / m_\alpha}$ is the Langmuir frequency of charged particles of type α.

Now we can verify the validity of the model of independent particles using the relations (16) to (20). First of all let us check the static limit

$$\sigma(0) = \sum e^2 n/m\nu \approx e^2 n_e/m\nu_e, \quad \varepsilon = 1 + 4\pi i\sigma(0)/\omega \tag{22}$$

Here $\sigma(0)$ is the static conductivity of isotropic plasma. In weakly ionized plasma $\nu_e = \nu_{e0}$ and the expression becomes correct not only qualitatively but also quantitavely. At the same time, in completely ionized plasma, or more exactly when

$\nu_{e0} << \nu_{ei} = $ [illegible] (the electron-ion effective collision frequency),

the expression (21) occurs approximately 2 times less than the correct expression known as L. Spitzer's formula for static conductivity of plasma :

$$\sigma_{sp} = 1.96\, e^2 n_e / m\nu_{ei} \tag{23}$$

The very important conclusion which follows from (21) is that the independent particles model explains quantitavely the propagation of high frequency transverse electromagnetic waves in an isotropic plasma. From the Maxwell equations in the abscence of external sources j_0 and ρ_0 follows the dispersion equation for such waves :

$$k^2c^2 = \omega^2\varepsilon(\omega) \tag{24}$$

Using the relations (20)-(21) and supposing $\omega >> \nu_e$, it's easy to find the solution of this equation in the high frequency limit [$\omega \rightarrow \omega + i\delta$]

$$\omega^2 = \omega_{Le}^2 + k^2c^2, \quad \delta = -\nu_e\omega_{Le}^2/\omega^2 \tag{25}$$

With regard to the low frequency limit ($\omega << \nu_e$) from (24) it follows the well-known expression for the penetration depth of the normalous skin-effect.

$$\lambda_{sk} \approx c/\sqrt{(2\pi\sigma\omega)} \tag{26}$$

This expression is correct for completely ionized plasma as well as for weakly ionized. Of course this statement must take into account the above remark about the static conductivity of plasma (see (22) and (23)).

More essential seems to be that the formula (26) is correct only if $\nu_e > v_{Te}/\lambda_{sk}$. In the opposite limit as it was shown by A. Pippard in 1948, the anomalous skin-effect takes place which can not be described in the model of independent particles. This phenomena will be studied below in the next lectures.

Finally the most penalizing failure of the model of independent particles was exposed in the description of longitudinal oscillations of plasma. From the field equations (16) taking in consideration (21) we get the following dispersion equation for such oscillations

$$\varepsilon = 0 \tag{27}$$

The solution of this equation in high frequency range ($\omega >> \nu_e$) looks as ($\omega \rightarrow \omega + i\delta$)

(28)

$$\omega^2 = \omega_L{}^2 , \delta = -v_e{}^2$$

Plasma longitudinal oscillations firstly was investigated by I. Langmuir in 1926.Moreover he firstly obtained these formulas and gave their physical interpretation.
However I. Langmuir was also the first who noticed that the discussed model is limited. This model can not explain the existence of low frequency oscillations with spectrum $\omega = kv_s$ discussed above, which were called by I. Langmuir as ion sound waves. And finally this model leads to the obviously absurd result for the problem of static potential for a rest point charged particle in a plasma. In order to show this let us consider a point particle with varying charge $\rho_0 = qe^{-i\omega t}\delta(r)$. From field equations (10) we find the following expressions for static field and its potential

(29)

$$E = -\nabla \Phi, \quad \Phi(r) = qe^{-i\omega t} / r\varepsilon(\omega)$$

where $\varepsilon(\omega)$ is given by the expression (2...). If now we take the static limit $\omega \to 0$ we obtain obviously absurd result : $\Phi(r) \to 0$ because $\varepsilon(\omega) \to \infty$ when $\omega \to 0$. Thus in the low frequency limit the model of independent particles is absolutely incorrect for description of isotropic plasma.

* 4 The properties of magnetoactive plasmas in the model of independent particles

Despite above mentioned failures of the independent particles model for description of isotropic plasma, let us now apply this model to the magnetoactive plasmas. Remind that this model occurs to be very successful for the problems of radiowaves propagation in the earth ionosphere. But the ionospheric plasma is magnetoactive and therefore below we will consider the properties of such plasmas.
We suppose that external magnetic field B_0 is parallel to **OZ** axis. Then in the linear approximation for perturbations from the equations (10) one can easily find small v and use the relation $j = \Sigma env$ to calculate the induced current density which leads to the following expression for dielectric permittivity:

(30)

$$\varepsilon_{ij}(\omega) = \begin{pmatrix} \varepsilon_\perp & ig & 0 \\ -ig & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_{//} \end{pmatrix}$$

$\varepsilon_\perp = 1 + \Sigma[\omega_L{}^2(\omega+i\nu)/\omega(\Omega^2-(\omega+i\nu)^2)$, $g = \Sigma\omega_L{}^2\Omega/\omega(\Omega^2-(\omega+i\nu)^2)$, $\varepsilon_{//} = 1 - \Sigma\omega_L{}^2/(\omega(\omega+i\nu))$
where $\Omega = eB_0/mc$ is the Larmor frequency for charged particles rotation around the magnetic field B_0. By using this tensor of dielectric permittivity the above mentioned success in the analysis of radiowave propagation through the ionospheric plasma was achieved by V. Ginzburg. Below we will not discuss these triumphal results. On the contrary we will consider the problems for which this tensor and more generally the independent particles model is'nt correct.
Let us begin from dispersion equation which can be easily got from the field equations :

(31)

$$\left| k^2\delta_{ij} - k_i k_j - (\omega^2/c^2)\varepsilon_{ij}(\omega) \right| = 0$$

$$k^2(k_\perp{}^2\varepsilon_\perp + k_{//}{}^2\varepsilon_{//}) - \omega^2/c^2[(\varepsilon_\perp{}^2-g^2-\varepsilon_\perp\varepsilon_{//})k_\perp{}^2 + 2k^2\varepsilon_\perp\varepsilon_{//}] + (\omega^4/c^4)\varepsilon_{//}(\varepsilon_\perp{}^2-g^2) = 0$$

Here $k_\perp$ = ksinθ and k =kcosθ respectively represent the transversal and longitudinal components of the wave vector **k** and θ is the angle between **k** and $\mathbf{B_0}$

In general the equation (31) is very complicated and the solutions ω(k) are impossible to find analytically. At the same time, it can be solved very easily as an equation relative to

k(ω) = (ω/c)n(ω)

We find the following solution

$$n_{1,2}^2(\omega) = (-B \pm \sqrt{B^2 - 4AC})/2A,$$

(32)

$$A=\varepsilon_\perp \sin^2\theta + \varepsilon_\parallel \cos^2\theta, \; C = \varepsilon_\parallel(\varepsilon_\perp^2 - g^2) \; B = -\varepsilon_\perp \varepsilon_\parallel (1+\cos^2\theta)-(\varepsilon_\perp^2-g^2)\sin^2\theta$$

The quantities $n_{1,2}$ are called as the complex reflection coefficients $n_1(\omega)$ for ordinary waves and $n_2(\omega)$ for extraordinary waves

Just using the relations (31) and (32) and taking into account the expressions (30) the radiowave propagation (their reflection and absorption) in the F-layer of ionospheric plasma as a function of the angle θ were explained. On the figure 1 there are presented the dependences $n_{1,2}^2(\omega)$ for θ≠0,π/2 and $\omega_{Le}^2 \gg \Omega_e^2$ (as it takes place in F-layer where $\omega_{Le} \cong 10^8 s^{-1}$ and $\Omega_e \cong 10^7 s^{-1}$). Moreover (31) and (32) give the good quantitative explanation for not only high frequency ($\omega \gg \Omega_e$), but also low frequency ones in the range $\Omega_i \ll \omega \ll \Omega_e$. These formulas predict the existence of the transverse waves, in the low-frequency range with spectrum

(33)

$$\omega = (k^2c^2 |\Omega_e| \cos\theta)/\omega_{Le}^2$$

Such waves were really observed in the ionospheric plasma and they were called as whistlers. The model of independent particles appears to be significantly less successful in explaining the low frequency waves in the range $\omega < \Omega_i$. From dispersion equation (31) in this frequency range one can obtain the spectra of two branches of low frequency waves

(34)

$$\omega_1^2 = k_\parallel^2 v_A^2/(1+v_A^2/c^2), \quad \omega_2^2 = k^2 v_A^2/(1+v_A^2/c^2)$$

where $v_A = B_0/\sqrt{4\pi n M}$ is called Alfven velocity. The first branch corresponds to the purely transverse waves and is well known as Alfven waves. They are predicted theoretically by H. Alfven in the framework of MHD equations and were really observed in ionospheric plasma.

As for the second branch then the theoretical spectrum (34) differs from the experimental observed. In experiments there exist two branches of low frequency waves with significant longitudinal field components instead of one branch. Besides the phase velocities of both depend on the plasma temperature, which is completely ignored in the model of independent particles. This fact indicated to the serious difficulties of the model. However the main difficulty of the independent particles model was clarified when static potential of a point charged particle in the magnetoactive plasma was considered. The result of this consideration, using the field equations (16), leads to the following formula for the field potential

(35)

$$\Phi(r) = q/2\pi^2 \int dk(e^{ikr})(k_\perp^2\varepsilon_\perp(\omega) + k_\parallel^2 \varepsilon_\parallel(\omega))^{-1} \to 0$$
$$\lim \omega \to 0$$

where $\varepsilon_{\perp}(\omega)$ and $\varepsilon_{\parallel}(\omega)$ are given by (30). Thus in magnetoactive plasma the static potential of point charged particle tends to zero. This result coincides with the result of the isotropic plasma and also seems perfectly absurd.

Thus from the above consideration we can conclude that the model of independent particles is quite satisfactory for description of fast and high-frequency processes in a plasma, with typical (phase) velocity much higher than thermal velocity of charged particles. Specially this occurs true for transverse waves. For longitudinal waves and low-frequency processes the typical phase velocity is of the order of the particles thermal velocity and therefore their properties can't be described by this model. Specially, this occurs catastrophic for describing purely potential static electric fields in a plasma. For improving the model, firstly, all thermal motion effects of charged particles such as hydrodynamical effects pressure, viscosity, thermoconductivity and diffusion of particles as well as kinetic velocity effects and energy distributions of particles must be taken into account.

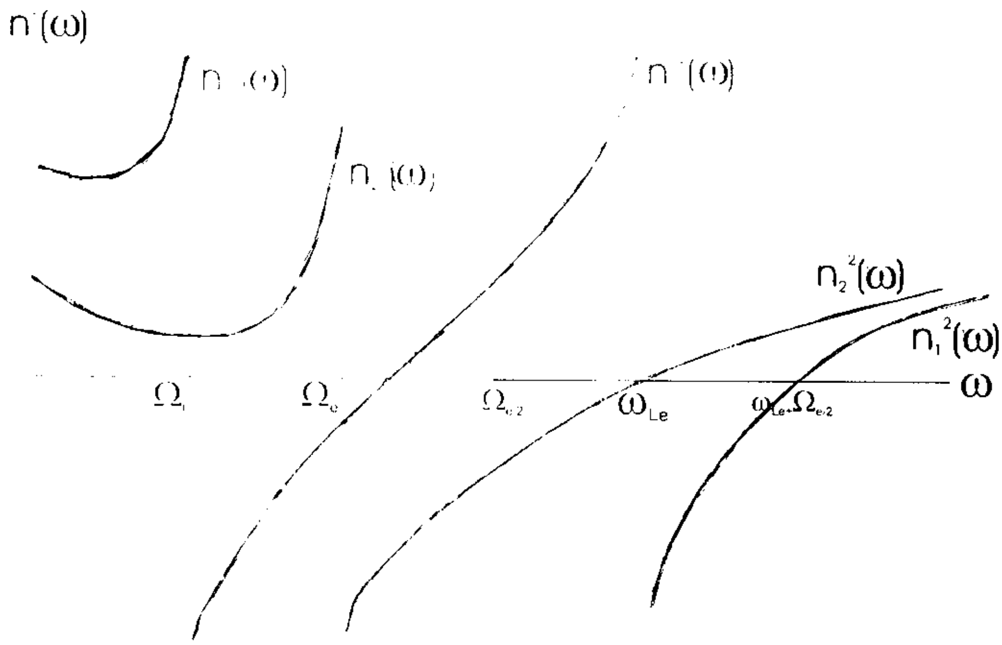

Figure 1 : Dependences $n_{1,2}^2(\omega)$ for $\theta \neq 0, \pi/2$ in the model of independent particles

LECTURE 3

Magnetohydrodynamical (MHD) Plasma Models

* 5 Two fluids hydrodynamics

In the previous section it was shown that the model of independent particles occurs incorrect in the low frequency range. For this reason the physicists decided at the end of 30-s that in this frequency range the hydrodynamical descriptions of plasma must be more suitable. At that time two different types of hydrodynamical models were developed. The first one was proposed by I. Langmuir and the second by H. Alfven. Below we will consider only the simplest versions of these models, which is quite sufficient for clarifying the reasons of their success and their failure.

In this section we begin with the I. Langmuir model, known as two fluids hydrodynamics . This model generalizes the independent particles model by taking into consideration the kinetic pressures of electrons and ions. Therefore the equations of motions look as (compare with equations (10)):

(36)

$$d\mathbf{v}_e/dt = \partial\mathbf{v}_e/\partial t + (\mathbf{v}_e\nabla)\mathbf{v}_e = -\nabla n_eT_e/mn_e + e/m\{\mathbf{E}+1/c[\mathbf{v}_e\times\mathbf{B}]\} - \nu_e\mathbf{v}_e$$

$$d\mathbf{v}_i/dt = \partial\mathbf{v}_i/\partial t + (\mathbf{v}_i\nabla)\mathbf{v}_i = -\nabla n_iT_i/Mn_i + e_i/M\{\mathbf{E}+1/c[\mathbf{v}_i\times\mathbf{B}]\} - \nu_i\mathbf{v}_i$$

As above these equations must be completed by the Maxwell equations (12) and continuity equations (13). Taking into account the temperatures T_e and T_i, below we will assume that they are constant. Such assumption simplifies the problem in a significant way and at the same time it doesn't influence the validity of the model. To determinate the validity limits of models is our main goal.

The basic equations of two fluids hydrodynamics (36) differ from the equations of motions (10) in the independent particles model by taking in consideration the thermal pressure. For this reason we may hope that low frequency processes for which the independent particles model occurs incorrect will be described quite sufficiently well. To prove this statement let us consider linear electromagnetic properties of spatially homogeneous isotropic plasmas in the model of two fluids hydrodynamics. We will investigate the small perturbations of an equilibrium state in which

$$E_0 = B_0 = 0, \quad v_{0e,i} = 0, \quad n_{0e,i} = \text{const}$$

Then from the linearized equations (35) and (13) one can easily obtain the following expression for the dielectric permittivity of plasma

(37)

$$\varepsilon_{i,j}(\omega,\mathbf{k}) = (\delta_{ij} - k_ik_j/k^2)\varepsilon^{tr}(\omega,\mathbf{k}) + k_ik_j/k^2\varepsilon^{l}(\omega,\mathbf{k})$$

where

(38)

$$\varepsilon^{tr}(\omega,\mathbf{k}) = 1 - \Sigma_\alpha(\omega_{L\alpha}{}^2)/(\omega(\omega+i\nu_\alpha))$$

$$\varepsilon^{l}(\omega,\mathbf{k}) = 1 - \Sigma_\alpha(\omega_{L\alpha}{}^2)/(\omega(\omega+i\nu_\alpha) - k^2v_{T\alpha}{}^2)$$

represent the transversal and longitudinal permittivities respectively. Thus we see that they are different. Besides, the transverse dielectric permittivity (38) coincides with (21). Consequently all the difficulties which take place in the independent particles model remain in the considered model of two fluids hydrodynamics. Particularly this model occurs to be incorrect for describing the transverse field penetration into the plasma (skin effect) in low frequency range when $v_e << \omega << v_{Te}/\lambda_{sk}$, where λ_{sk} is given by the relation (26).
At the same time, for the longitudinal dielectric permittivity from (38) it follows

(39)

$$\lim_{\omega\to 0}\varepsilon^l(\omega,\mathbf{k}) = 1 + \Sigma_\alpha(\omega_{L\alpha}{}^2/ k^2 v_{T\alpha}{}^2) = 1 + 1/k^2 r_D{}^2$$

This expression leads to the correct expression for the field potentiel of the static charged particles q located in a plasma at $r = 0$

(40)

$$\Phi(r) = (q/r)e^{-r/r_D}$$

Thus the Debye screening of the potential takes place as it must be in accordance to (8).
Moreover longitudinal dielectric permittivity (38) describes quite correctly the low frequency oscillations, when $\omega << v_\alpha$, namely the diffusion processes in a plasma, the monopolar diffusions of electrons and ions independently as well as the ambipolar one (In the considered approximation with T_α = const.). Really in this frequency range from (38) it follows

(41)

$$\varepsilon^l(\omega,\mathbf{k}) = 1 - \Sigma_\alpha (\omega_{L\alpha}{}^2/(i\omega v_\alpha - k^2 v_{T\alpha}{}^2))$$

For the short wave length perturbations, when $k^2 r_{D\alpha}{}^2 >> 1$, the solutions of $\varepsilon^l(\omega,\mathbf{k}) = 0$ coincides with the poles of (41)

(42)

$$i\omega v_\alpha - k^2 v_{T\alpha}{}^2 = 0$$

It is easy to understand that this relation corresponds to the diffusion equation for the particles of type α.

(43)

$$\partial n_\alpha/\partial t - (v_{T\alpha}2)/v_\alpha)\Delta n_\alpha = 0$$

Thus the coefficient of monopolar particle diffusion in the two fluids hydrodynamics occurs to be equal

(44)

$$D_\alpha = v_{T\alpha}{}^2/v_\alpha$$

In the opposite limit of long wave length perturbations when $k^2 r_{D\alpha}{}^2 << 1$, the solution of $\varepsilon l(\omega,\mathbf{k})=0$ can be presented as

(45)

$$i\omega v_i - k^2(v_s{}^2 + v_{Ti}{}^2) = 0,$$

where $v_s = \sqrt{T_e / M}$. As above, this relation corresponds to the equation

(46)

$$\partial n/\partial t - Da\Delta n = 0$$

which represents the diffusion equation with ambipolar diffusion coefficient

(47)

$$D_a = (v_s^2+v_{Ti}^2)/v_i = (T_e+T_i)/Mv_i$$

The both results concerning monopolar diffusion as well as ambipolar one were confirmed experimentally by I. Langmuir. These facts demonstrate the obvious success of two fluids hydrodynamics.This success firstly was noted by I. Langmuir.
However very quickly appear new difficulties of the model besides remarked above with regard to the low frequency transverse field (skin effect). Namely from the expression (38) for $\varepsilon^l(\omega,\mathbf{k})$ it follows that in a low density collisionless plasma ($\omega \gg v_\alpha$) when the wavelength of perturbations is sufficiently short, $kr_{D\alpha} \gg 1$, there exist the longitudinal oscillations with spectra (for electrons and ions)

$$\omega^2 = k^2 v_{T\alpha}^2 \qquad (48)$$

which correspond to the poles of $\varepsilon^l(\omega,\mathbf{k})$. Nobody observed such oscillations and moreover it will be shown below that even in a collisionless plasma they occur very strongly damped. This result was the serious failure of two fluids model.
Finally let us indicate to the very widespread mistake repeated up to day and which follows from two fluids model. We mean the long wave length ($k^2 r_{D\alpha}^2 \ll 1$) and low-frequency ($\omega^2 \ll \omega_{Le}^2$) longitudinal oscillations with phase velocity less than electron thermal velocity ($\omega v_e \ll k^2 v_{Te}^2$). Under these conditions from equations $\varepsilon^l(\omega,\mathbf{k})=0$ taking into account the expression (38) we obtain the following dispersion equation

$$\omega^2 - k^2 (v_s^2+v_{Ti}^2) + i\omega v_i = 0 \qquad (49)$$

In the limiting case $\omega \ll v_i$, this equation leads to (45) which describes the ambipolar diffusion, confirmed by numerous experiments. However in the opposite limit, when $\omega \gg v_i$ from (49) we obtain the spectrum of weakly damping oscillations ($\omega \to \omega + i\delta$)

$$\omega = k\sqrt{(T_e + T_i)/M}\,, \quad \delta = -v_i/2 \qquad (50)$$

I. Langmuir supposed that these oscillations represent the usual accoustic sound oscillations with spectrum $\omega = k\sqrt{\gamma P/P_0}$, $\gamma = 1$ and he called them " ion-accoustic waves". Moreover he really observed such a type of oscillations in a nonisothermal ($T_e \gg T_i$) gas discharge plasma. Only one question remained unclear : what is $\gamma = c_p/c_v$ and why $\gamma = 1$ for plasma. From experimental datas for nonisothermal plasma followed that $\gamma = 1$. But why? This question had remained unclear up to 1954 when G. Gardeev clarified it (see below).
Above we restrict ourselves by considering only low frequency processes of isotropic plasma intentionally. Firstly it must be noticed that for high frequency processes with characteristic velocity much higher than the thermal velocities of charged particles the two fluids model corresponds to the independent particles approximation which is quite satisfactory for such processes as it was shown in previous section.
Secondly, just namely this model was proposed for description of low frequency processes by I. Langmuir and namely for them we were convinced that it arised very serious difficulties. For the magnetoactive plasma, when the two fluids model was proposed by I. Langmuir, just at the same time the one fluid magnetohydrodynamic (MHD) was developped by H. Alfven. The MHD represents the generalization of usual hydrodynamics for the conducting liquids and it seems that MHD must be valid only for very high density plasma.

However H.Alfven applied this model for description of ionospheric plasma and it occured very successfully. In the next section we will discuss the one fluid MHD.

* 6 One fluid MHD equations

As it was noticed above, the MHD equations differ from usual hydrodynamics by the additional volumetric force, which affects the conducting media with current **j** by the magnetic field **B**

$$\mathbf{f} = 1/c[\mathbf{j}\times\mathbf{B}] = 1/4\pi[\mathrm{rot}\mathbf{B}\times\mathbf{B}] \qquad (51)$$

Taking into consideration this force one can easily make the generalization of usual hydrodynamics on the case of conducting liquid. Supposing the ideal conductivity and neglecting all dissipative processes the MHD equations can be presented as

$$\mathrm{div}\mathbf{B} = 0,\ \partial\mathbf{B}/\partial t = \mathrm{rot}\,[\mathbf{v}\times\mathbf{B}] \qquad (52)$$

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} = -\nabla P/\rho - (1/4\pi\rho)[\mathbf{B}\times\mathrm{rot}\mathbf{B}]$$

Here **v** is the velocity of liquid with density ρ and P is the pressure, which is connected with ρ and temperature T by the state equation

$$P = P(\rho, T) \qquad (53)$$

The first achievement of MHD was connected to the analysis of small perturbations of stationary homogeneous equilibrium with

$$\mathbf{v}_0 = 0,\ \rho_0 = \mathrm{const},\ P_0 = \mathrm{const},\ \mathbf{B}_0 = \mathrm{const}.$$

For the perturbations v_1, ρ_1, and b from (52) we obtain

$$\mathrm{div}\mathbf{b} = 0,\ \mathrm{rot}[\mathbf{v}_1\times\mathbf{B}_0] = \partial\mathbf{b}/\partial t \qquad (54)$$

$$\partial\mathbf{v}_1/\partial t = -(v_s^2/\rho_0)\nabla\rho_1 - (1/4\pi\rho_0)[\mathbf{B}_0\times\mathrm{rot}\mathbf{b}]$$

$$\partial\rho_1/\partial t + \mathrm{div}\rho_0\mathbf{v}_1 = 0$$

Here v_s is the sound velocity for isoentropic processes, which follows from (53)

$$P_1 \equiv -v_s^2\rho_1 \equiv (\partial P/\partial\rho)_s\rho_1 \qquad (55)$$

For the solutions of type $e^{(-i\omega t+ikr)}$ from the linear system (54), one can obtain the dispersion relations

$$\omega_1^2 = k_z^2 v_A^2, \qquad (56)$$

$$\omega_{2,3}^2 = k^2/2\left\{\left(v_A^2+v_S^2\right)\pm\sqrt{\left(v_A^2+v_S^2\right)^2-4v_A^2v_S^2\cos^2\theta}\right\}$$

where v_A is the Alfven velocity introduced above and θ is the angle between **k** and $\mathbf{B}_0$.
Thus in the framework of MHD exist 3 branches of small oscillations. The first describes purely transverse waves, **b** and $\mathbf{v}_1$ are perpendicular to $\mathbf{B}_0$ and **k**, is known as Alfven waves. The second and third are called as the fast and slow sound waves in a conducting liquid. It must be noticed that namely 3 types of oscillations were really observed in the ionospheric plasma. Moreover, in the ionospheric plasma the ratio $\beta = (8\pi p_0)B_0^{-2} \cong v_S^2/v_A^2 << 1$ and therefore the last two branches become separated

(57)

$$\omega_2^2 = k^2 v_A^2, \quad \omega_3^2 = k_z^2 v_S^2$$

The oscillations with spectrum $\omega_2(k)$ are transverse as well as $\omega_1(k)$ whereas the oscillations with spectrum $\omega_3(k)$ are purely longitudinal and correspond to the isoentropic oscillations of density and pressure in which $\mathbf{v}_1$ is parallel to **k**.
Despite of the observations of mentioned oscillations in ionospheric plasma, some problems have still remained. The first one is quantitative and about some parameters: what are $v_s = \sqrt{(\partial P/\partial\rho)_s} = \sqrt{\gamma T/M}$ and $\gamma = c_p/c_v$?. For ionospheric plasma in accordance to its consistent in the F-layer this quantity must be of the order of $\gamma=5/3$, whereas from the experiments it follows that $\gamma \approx 1$ and besides of this the oscillations occur to be very damping. What is the reason of their absorption ? The second problem is more principal: what is the reason of success for MHD in applications to the ionospheric plasma? MHD as usual hydrodynamics must be valid only for dense gaseous, where collisional effects are dominant, whereas ionospheric F-layer plasma seems to be collisionless. Neverthless the predictions of MHD occur in a good agreement with experimental observations.
The first attempt of derivation MHD equations was made starting from the equations of two fluids hydrodynamics (36) at the beginning of 50-s. Really let us suppose the inequalities

$$\omega<<kv_{Te}, \Omega_i \qquad \nu_e<<\Omega_e, \quad \nu_i<<\Omega_i<<\omega_{Li}$$

Under these conditions the displacement current may be neglected and the Maxwell equation for magnetic field takes the form

(58)

$$\mathrm{rot}\mathbf{B} = (4\pi/c)\,\mathbf{j}$$

Using this equation and taking into consideration that under the above restrictions, plasma can be considered as quasineutral (ne=ni=n), from the equations (36) by summing them, one can obtain:

(59)

$$Mn(\partial\mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v}) = -\nabla[n(T_e+T_i)] - 1/4\pi[\mathbf{B}\times\mathrm{rot}\,\mathbf{B}] - Mn\nu\mathbf{v}$$

where $\mathbf{v}=\mathbf{v}_i$. Besides of this from the first equation of (36) follows

(60)

$$\mathbf{E}_\perp = -1/c\,[\mathbf{v}\times\mathbf{B}]$$

As a result the Maxwell equation for electric field takes the form

$$\partial \mathbf{B}/\partial t = -c \, \mathrm{rot}\mathbf{E} = \mathrm{rot}[\mathbf{v}\times\mathbf{B}] \quad (61)$$

Finally completing the above equations by the continuity equation for ions we obtain the following system of one fluid MHD

$$\mathrm{div}\mathbf{B} = 0, \; \mathrm{rot}[\mathbf{v}\times\mathbf{B}] = -\partial\mathbf{B}/\partial t \quad (62)$$

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} = -\nabla P/\rho - 1/4\pi\rho[\mathrm{rot}\mathbf{B}\times\mathbf{B}] - \nu\mathbf{v}$$

$$\partial\rho/\partial t + \mathrm{div}\rho\mathbf{v} = 0$$

where $\rho = Mn$, $P = n(T_e + T_i)$, or $T = T_e + T_i$ = constant. The last relation for P represents the state equation for plasma, which corresponds to (53).

The system (62) coincides practically with the one fluid MHD equations (52). The only difference consists of the existence of the last term in the equation of motion (62). This term takes into account a friction of ions on neutral particles and it is obvious that in the purely one fluid hydrodynamics it does not exist. In this case the system (62) seems more general, it is valid for weakly ionized plasma as well.

Thus the derivation of MHD equations from the two fluids hydrodynamics was a significant success of plasma theory. Nevertheless all the above noted difficulties which are inherent in two fluids hydrodynamics, force the scientist to attempt avoiding them by using the kinetic consideration. More correctly at the end of 30-s the scientists attempted to generalize the Boltzmann's kinetic equation for the case of the systems of charged particles, or in other words, for plasma. We'll speak about this in the next sections.

LECTURE 4

Plasma Kinetic Descriptions

* 7 Boltzmann - Landau kinetic equation

The first attempts of generalization of Boltzmann kinetic equation for a gas of ionized particles were made before the second world war independently by S.Chapman and T.Couling and L.Landau. The basis of kinetic description of systems consisted of a large number of particles is the probability description. Therfore the distribution function of n particles can be introduced as

$$f_n(\mathbf{r}_1, \mathbf{p}_1, \mathbf{r}_2, \mathbf{p}_2, \ldots, \mathbf{r}_n, \mathbf{p}_n, t) \tag{63}$$

This function represents the probability that at the moment t the particles with momentums $\mathbf{p}_1, \mathbf{p}_2, \ldots \mathbf{p}_n$ are located at $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_n$ correspondingly. It is obvious that the distribution function (63) is very general and gives the complete description of the system. However it is very complicated because it depends on too much arguments. As a result it occurs to be practically useless.

Let us remind that the plasma is a gas and in * 1, the corresponding conditions for validity gas approximations were given. Namely these conditions were used by S.Chapman and T.Cooling for a weakly ionized plasma when they attempted to generalize Boltzmann kinetic theory. In * 1, it was shown that the neutral particles can be considered as the hard balls with radius a. Then, a weakly ionized plasma is a gas if (see(4))

$$\eta_0 = a n_o^{1/3} << 1 \tag{64}$$

where n_o is the density of neutrals. In the zero approximation in the condition (64), or in other words, when the particles interaction is completely neglected, then the function fn (63) can be presented as

$$f_n(\mathbf{r}_1, \ldots \mathbf{r}_n; \mathbf{p}_1 \ldots \mathbf{p}_n, t) = \prod_{i=1}^{n} f(\mathbf{r}_i, \mathbf{p}_i, t) \tag{65}$$

Here f(r,p,t) is a probability that a charged particle with momentum p at the moment t is located at r. It is obvious, that in this approximation this probability is constant and therefore it satisfied the Liouville's equation

$$df(\mathbf{r}, \mathbf{p}, t)/dt = \partial f/\partial t + \mathbf{r}\partial f/\partial \mathbf{r} + \mathbf{F}\partial f/\partial \mathbf{p} = 0 \tag{66}$$

Here v is the particles velocity and **F** the force which determines particles motion. For charged particles

$$v = d\mathbf{r}/dt, \; d\mathbf{P}/dt = \mathbf{F} + e\{\, \mathbf{E} + 1/c[v \times \mathbf{B}]\} \tag{67}$$

where **E** and **B** are the external electric and magnetic fields, and e the particles charge. Of course the Liouville's equation (66) must be written for each particles of type α.

Moreover, let underline that the Liouville's equation (66) doesn't take into account the particles interaction. In Boltzmann consideration the particles interaction leads to the appearance of the nonzero right side of equation (66). In the lowest approximation it takes into account the pair interactions of particles type α with all particles of type β and therefore

(68)

$$df_\alpha/dt = (\partial f_\alpha/\partial t)_{st} = \Sigma_\beta(\partial f_\alpha/\partial t)_{st}^{\alpha\beta} = \Sigma_\beta \Im_{\alpha\beta}(f_\alpha,f_\beta)$$

We will restric ourselves by taking into account only elastic interaction (scattering), as it shown on Figure 2.

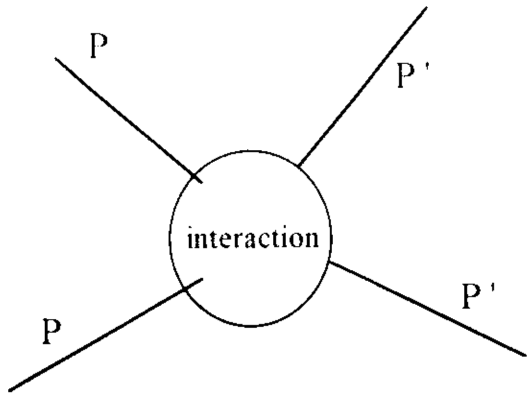

Figure 2

This leads to the following form of collision integral $\Im\alpha\beta$ ($f\alpha$,$f\beta$)

(69)

$$\Im_{\alpha\beta}(f_\alpha,f_\beta) =$$

$$\int d\mathbf{p}_\beta d\mathbf{p}_\beta' d\varepsilon_\beta' v_{\alpha\beta} d\sigma_{\alpha\beta}[f_\alpha(\mathbf{p}_\alpha')f_\beta(\mathbf{p}_\beta')-f_\alpha(\mathbf{p}_\alpha)f_\beta(\mathbf{p}_\beta)]\delta(\mathbf{p}_\alpha+\mathbf{p}_\beta-\mathbf{p}_\alpha'-\mathbf{p}_\beta')\delta(\varepsilon_\alpha+\varepsilon_\beta-\varepsilon_\alpha'-\varepsilon_\beta')$$

This expression takes into account the particles momentum **p** and energy ε conservations in the scattering processes. Moreover the probabilities of forward and backward scatterings are supposed to be equal. These probabilities are the product of the particles relative velocity $\mathbf{v}_{\alpha\beta} = |\mathbf{v}_\alpha - \mathbf{v}_\beta|$ and scattering crosssection $d\sigma_{\alpha\beta}$. The last quantity depends on

(70)

$$\mathbf{P}_{\alpha,\beta} = \pm \mu_{\alpha\beta}\mathbf{v}_{\alpha\beta} + (m_{\alpha,\beta}/(m_\alpha+m_\beta))(\mathbf{p}_\alpha+\mathbf{p}_\beta)$$

$$\mathbf{P}_{\alpha,\beta}' = \pm \mu_{\alpha\beta}\mathbf{v}_{\alpha\beta}\mathbf{n}+(m_{\alpha,\beta}/(m_\alpha+m_\beta))(\mathbf{p}'_\alpha+\mathbf{p}_\beta')$$

n is the vector in the direction of particle α velocity in the frame of centrum of inertia (in which $\mathbf{p}_\alpha+\mathbf{p}_\beta = 0$) and $\mu_{\alpha\beta} = m_\alpha \cdot m_\beta/(m_\alpha+m_\beta)$
For the scattering of charged particles on the neutrals (hard balls) we have

(71)

$$d\sigma_{\alpha\beta}\ a^2 d\Omega = a^2 \sin\theta d\theta d\varphi$$

where $d\Omega$ is the solid angle of scattering. The expression (69), when taking in consideration (71), describes the elastic scattering of charged particles on neutrals in a plasma. In this sense

Chapman and Couling supposed that the kinetic equation (68) can be applied to the weakly ionized plasma. It must be noticed however that in their interpretation the electric and magnetic fields in the left side of (68) (see (66) and (67)) are external and only external fields. They did not understand the idea of selfconsistent fields, which was clear much earlier for I. Langmuir in his model of independent particles.
The next important progress in developing of plasma kinetic theory was made in 1937 by L. Landau. Starting from the Boltzmann collision integral (69), he derived the kinetic equation for completely ionized plasma. For this aim L. Landau used the Rutherford formula for Coulomb scattering

$$d\sigma_{\alpha\beta}\, d\Omega = 4\pi e_\alpha{}^2 e_\beta{}^2 / (\mu_{\alpha\beta}{}^2\, v_{\alpha\beta}{}^4 \sin^4(\theta/2)) \tag{72}$$

However it is wellknown that this expression leads to the divergence of the total crosssection of scattering. How it can be avoided? At this question the answer was found by L.Landau and this answer was full of genius. He noticed that in a plasma takes place Debye screening of Coulomb potential which is a consequence of the validity of gas approximation

$$\eta_1 = e^2 n^{1/3} / \langle\varepsilon\rangle \sim e^2 n^{1/3} / T \ll 1 \tag{73}$$

Under this condition the potential energy of charged particles interaction in a plasma looks as

$$U(r) = (e^2/r) e^{-r/r_D} \tag{74}$$

It must be noticed that the condition (73) is equivalent to the requirement $U(r_{av}) \ll T$, where $r_{av} \cong n^{-1/3}$. At the same time, the expression (74) means that the characteristical radius of charged particles interaction in a plasma is r_D and this radius in accordance to (73) is much larger than the average distances between particles $\sim n^{-1/3}$

$$n^{1/3} r_D \cong r_D / n^{-1/3} \cong \sqrt{T / e^2 n^{1/3}} \cong 1/\eta_1{}^{1/2} >> 1 \tag{75}$$

In this sense the above condition is opposite to (64) if instead of a we substitute r_D. Despite this, L.Landau used the Boltzmann collision integral (69) substituting the expression (72) in it. This unsubstantiality was strongly justified by N. Bogolyubov in 1946 when he developed mathematically correct method of derivation of the kinetic equations.
Besides of the inequality (75) L.Landau supposed that for Coulomb scattering an other inequality takes place as well. Namely in this process

$$|p_{\alpha\beta}' - p_{\alpha\beta}| \ll p_\alpha, p_\beta, \tag{76}$$

which means that the change of particles momentum is small, or the scattering angle $\theta \ll 1$. This assumption together with Born approximation which is valid when

$$e^2/r_{min} \ll T \tag{77}$$

allow him to get the convergence collision integral. This integral is known as Landau collision integral and it looks as

$$\Im_{\alpha\beta}(f_\alpha, f_\beta) = (\partial/\partial P_{\alpha i}) \int d\mathbf{P}_\beta (2\pi e_\alpha{}^2 e_\beta{}^2\, L/u^3)(u^2\delta_{ij} - u_i u_j)((\partial f_\alpha/\partial P_{\alpha j}) f_\beta - (\partial f_\beta/\partial P_{\beta j}) f_\alpha) \tag{78}$$

Here $u=v_\alpha - v_\beta$ is the relative velocity of scattered particles and the quantity L

(79)

$$L = {}^{r_D}\int_{r\,min} dr/r = \ln r_D/r_{min} = \ln(T/(e^2 n^{1/3})) \sim 10 >> 1$$

is called as Coulomb logarithm.
The kinetic equation (68) with collision integral (78) is known as the equation Boltzmann-Landau. Below we will use it for describing electromagnetic properties of completely ionized plasma. It must be noticed here that L. Landau as S.Chapman and T.Couling was sure that in the left side of his equation electromagnetic fields **E** and **B** are only external. The interaction of particles is completely taken into account in the collision integral (77). But this belief, of course, was an annoying mistake of great scientist.

* 8 Relaxations of momentum and energy

Let us now follow S.Chapman and T.Couling and L.Landau to consider the relaxations of particles momentum and energy in a plasma slightly deviated from thermodynamical equilibrium. In this connection, it must be noticed that the above obtained collision integrals occur to be identically zero for the equilibrium Maxwell distribution

(80)

$$f_{0\alpha} = (n_\alpha/(2\pi m_\alpha T_\alpha)^{3/2})\exp(-m_\alpha v^2/2T_\alpha)$$

Of course, this statement is correct only for stationary and spatially homogeneous distribution (80) and when

$$T_e = T_i = T_0 = T,\ \Sigma_\alpha e_\alpha n_\alpha = 0$$

and only in the case of a plasma without any fields. Let us now consider the small deviations from equilibrium and calculate the time relaxations of nonequilibrium momentum and energy of particles. Suppose that at t = 0 the particles (electrons) distribution function differs from Maxwellian by the existence of a small velocity $u_0 << v_{Te}$, or

(81)

$$f_e = (n_e/(2\pi m T_e)^{3/2})e(-m(v-u(t))^2/2T_e)$$

and $u(t=0) = u_0$. The problem is to find the dynamic equation describing time relaxation of u(t). Substituting (81) in the kinetic equation (68) after integration over momentum of electrons one can obtain the following equation

(82)

$$\partial \mathbf{u}/\partial t = -\nu_e\, \mathbf{u}$$

$$\nu_e = \begin{cases} \nu_{e0} = \pi a^2 v_{Te} n_0 & \text{for weakly ionized plasma} \\ \nu_{eff} = 4/3\left(\sqrt{2\pi/m}\, e^2 e_i^2 n_i / T_e^{3/2}\right)L & \text{for completely ionized plasma.} \end{cases}$$

It must be noticed that for weakly ionized plasma the equation (82) is exact, whereas for completely ionized plasma the accuracy of this equation is of the order of a factor ~ 1
From (82) the following relation can be obtained

(83)

$$u(t) = u_0 \exp(-\nu_e t)$$

Thus the momentum relaxation time for electrons in a plasma is equal $\tau \sim 1/\nu_e$

Let us now consider the relaxation of energy. Suppose that at $t = 0$ the electron temperature T_{e0} differs from the temperature of neutrals Tn (for weakly ionized plasma) or ions T_{i0} (for completely ionized plasma). The problem is to derive the dynamic equation describing relaxation time of electrons temperature Te(t), when

(84)

$$f_\alpha = (n_\alpha (2\pi m_\alpha T_\alpha(t))^{3/2}) e(-m_\alpha v^2/2T_\alpha(t))$$

Substituting these expressions into the equation (68) and integrating over momentum of particles, as above, after simple calculations we obtain

(85)

$$\partial(T_e - T_n)/\partial t = -(\nu_{en} 2m/M_n)(T_e - T_n)$$

$$\partial(T_e - T_i)/\partial t = -(\nu_{eff} 2m/M)(1+|ei/e|)(T_e - T_i)$$

for weakly and completely ionized plasma correspondingly. Here for weakly ionized plasma it was supposed that T_n = const, which follows from obvious inequality $n_0 >> n_e$. At the same time, for completely ionized plasma in derivation (85) we take into account, that

$$\partial(T_e + T_i)/\partial t = 0$$

Thus from (85) follows that the energy relaxation time is much larger than that of momentum, or

(86)

$$\tau_\varepsilon \sim (M/2m)\, \tau_m >> \tau_m \sim 1/\nu_e$$

In conclusion let us consider the behaviour of the completely ionized plasma in external constant electric field $\mathbf{E}_0$. The solution of this problem shows the above mentioned incorrectness of the calculations of the momentum relaxation time of plasma offered (82). The Boltzmann-Landau equation for this problem looks as

(87)

$$(e\mathbf{E}_0/m)\partial f_e/\partial v = \Sigma_\beta \int d\mathbf{p}_\beta (2\pi e_\alpha{}^2 e_\beta{}^2 L/u^3)(u^2\delta_{ij} - u_i u_j)[f_\beta \partial f_e/\partial p_j - f_e \partial f_\beta/\partial p_j]$$

where $\mathbf{u} = \mathbf{v}_\alpha - \mathbf{v}_\beta$ and the summation carried out over β = e.i

If the field $\mathbf{E}_0$ is sufficiently weak we can represent $f_e = f_{0e} + \delta f_e$, where f_{0e} is the Maxwellien distribution (84)(not(81)) and δf_e a small correction. In this case the influence $\mathbf{E}_0$ field on ions is negligible and ion distribution function is identically Maxwellian. For calculating δf_e let us expand it in the series

(88)

$$\delta f_e = (\mathbf{v}\mathbf{E}_0/E_0)[a_0 + a_1(5/2 - v^2/2v_{Te}{}^2)] f_{0e}$$

For the determination of the constant coefficients a_0 and a_1 from (87) the following system can be obtained

$$eE_0/T_e = -\nu_{eff}(a_0+3/2a_1) \quad (89)$$

$$(3/2)a_0 + (13+4\sqrt{2})a_1/4 = 0$$

Here for simplicity we suppose $e_i = -e$.
After solving the system (89) we can calculate the current in a plasma

$$j = e\int vf_e dp = 1{,}96(e^2n_e/m\nu_{eff})\mathbf{E}_0 \equiv \sigma\mathbf{E}_0 \quad (90)$$

Or for the plasma conductivity we have

$$\sigma = 1{,}96(e^2n_e/m\nu_{eff}) \quad (91)$$

The factor 1,96 instead of 1 about which the above remark was done. With increasing the ratio $Z = |e_i/e|$ this factor tends to 1. When $Z > 10$ we can neglect the electron - electron collisions and this factor in (91) becomes equal to 1.
For a weakly ionized plasma quite similar calculation leads to the expression

$$j = e\int vfdp = (e^2n_e/m\nu_{e0})\mathbf{E}_0 = \sigma\mathbf{E}_0 \quad (92)$$

Thus the plasma conductivity is

$$\sigma = e^2n_e/m\nu_{e0} \quad (93)$$

where $\nu_{e0} = \pi a^2 v_{Te}\, n_0$ is the electron neutral collision frequency. For weakly ionized plasma this expression is exact and therefore the Lorentz approximation taking into account only electron-neutral collisions is correct.

LECTURE 5

Selfconsistent Field Approximation

* 9 Vlasov-Maxwell equations

Above we emphasized many times on the fact that in the Boltzmann equation for a weakly ionized plasma and in the Boltzmann-Landau equation for a completely ionized one the electric **E** and magnetic **B** fields are proposed to be external and only external. As a result of this assumption, all relaxation processes, considered in these equations, are aperodically damping in time and are determined by particles collisions (electrons collisions in considered cases).
The first who draw world scientists attention to the inconsistence of such a treatment of electric and magnetic fields in kinetic theory was A. Vlasov. In his famous work publicated in 1938, A. Vlasov showed that in the lowest approximation of the gaseous parameter η_1, the interaction between the charged particles can be taken into account if in the Liouville's equation electromagnetic fields are considered not only as external, but as full fields satisfying Maxwell equation with induced charges and current densities

$$\rho = \Sigma_\alpha e_\alpha \int f_\alpha d\mathbf{p}, \quad \mathbf{j} = \Sigma_\alpha e_\alpha \int \mathbf{v} f_\alpha d\mathbf{p} \tag{94}$$

Thus the interaction of plasma particles with each fields are to be taken into account because the distribution function f_α itself satisfies the kinetic equation of the lowest approximation

$$\partial f_\alpha/\partial t + \mathbf{v}\partial f_\alpha/\partial \mathbf{r} + e_\alpha \{ \mathbf{E} + (1/c)[\mathbf{v}\times\mathbf{B}]\}\partial f_\alpha/\partial \mathbf{p} = 0 \tag{95}$$

Now we can write the field equations in the form

$$\mathrm{div}\mathbf{E} = 4\pi\Sigma_\alpha e_\alpha \int f_\alpha e d\mathbf{p} + 4\pi\rho_0, \quad \mathrm{rot}\mathbf{E} = -(1/c)\partial\mathbf{B}/\partial t \tag{96}$$

$$\mathrm{div}\mathbf{B} = 0\ , \ \mathrm{rot}\ \mathbf{B} = (1/c)\partial\mathbf{E}/\partial t + (4\pi/c)(\Sigma_\alpha e_\alpha \int \mathbf{v} f_\alpha d\mathbf{p}) + (4\pi/c)\mathbf{j}_0$$

The system (95) - (96) represents the complete system of selfconsistent equations for **E**, **B** and f_α which describe the plasmas in the lowest approximation of the gaseous parameter.
Only in the following, more higher approximation arises the right side of the kinetic equation (95), taking into account the particles scattering (collisions).
In scientific literature the equation (95) is known as Vlasov equation whereas the complete system (94) - (95) is called Vlasov-Maxwell system of equations. Sometimes they are also called equations for collisionless plasmas taking into account particle interactions only via selfconsistent fields.
Here it must be noticed that the basis of Vlasov equation was not sufficiently strict. First of all it was not understood how in the Liouville's equation the particles correlations can be taken into account, although this equation describes a completely noncorrelated particles system. Moreover in this sense the Vlasov equation for many scientists remainded very doubtful. Among them were the great L. Landau, M. Leontovich, V. Fock and others. In 40-s between the scientists arised the wellknown disputes, the results of which led to new dicoveries and new excellent investigations. Let us briefly discuss these disputes.
Following A. Vlasov, let us consider a small perturbation of the equilibrium Maxwell distribution f_{0e}

$$f_e = f_{0e} + \delta f_e \quad (97)$$

The distribution f_{0e} satisfies the equation (95) in the absence of $\mathbf{E_0}$ and $\mathbf{B_0}$. Besides we suppose that $\rho_0 = \mathbf{j}_0 \equiv 0$, i.e. $\Sigma_\alpha e_\alpha n_{0\alpha} = 0$. Then from (95) can be obtained linear equation for δf_e

$$\partial \delta f_e/\partial t + \mathbf{v}\partial \delta f_e/\partial \mathbf{r} + e\mathbf{E}\partial f_{0e}/\partial \mathbf{p} = 0 \quad (98)$$

where the field perturbation **E** must be determined from the system of Maxwell equations (96). Below for simplicity we will restrict ourselves by considering the potential field $\mathbf{E} = -\nabla\Phi$ only and therefore

$$\Delta\phi = -4\pi e \int \delta f_e d\mathbf{p} \quad (99)$$

The system (98) and (99) represents the complete system of linear equations which allows to investigate the time development of initial perturbations $\delta f_e(0,\mathbf{r},\mathbf{p})$. Suppose that

$$\delta f_e(0,\mathbf{r},\mathbf{p}) = \delta f_0(\mathbf{p})e^{i\mathbf{kr}} \quad (100)$$

It should be noted that an arbitrary perturbation can be represented as a sum of Fourier harmonics such as (100).
Now we can find the solutions of system (98) and (99) as $(\delta f_e, E) \sim e^{(-i\omega t + i\mathbf{k}\cdot\mathbf{r})}$ and from the existence condition of nontrivial solutions determine $\omega(\mathbf{k})$. Namely this quantity gives the time development of initial perturbations of type (100). Really from (98) it follows that

$$\delta f_e(\mathbf{p}) = (-ie\mathbf{E}\partial f_{0e}/\partial \mathbf{p}/(\omega - \mathbf{k}\cdot\mathbf{v})) = (-e\mathbf{k}\partial f_0/\partial \mathbf{p}/(\omega - \mathbf{k}\cdot\mathbf{v}))\phi \quad (101)$$

After substituting this expression into the equation (99) the following dispersion equation can be obtained

$$1 - (4\pi e^2/k^2) \int (\mathbf{k}\, \partial f_0 e/\partial \mathbf{p}/(\omega - \mathbf{k}\,.\mathbf{v}))d\mathbf{p} = 0 \quad (102)$$

which represents the condition of existence the nontrivial solutions of the system (98) - (99).
The main disagreement between A. Vlasov and L. Landau is related to the analysis of the equation (102). A.Vlasov supposed that the pole $\omega = \mathbf{k.v}$ in integrand of the equation (102) must be understood in the sense of principal value. Then for the long range perturbations, $k^2 r_{De}^2 << 1$, he obtained nondamping frequency spectrum *

$$\omega^2 = \omega_{Le}^2 + 3k^2 v_{Te}^2 = \omega_{Le}^2(1 + 3k^2 r_{De}^2) \quad (103)$$

From (103) follows that the group velocity of such perturbations are small in comparison with the thermal velocity of electrons

$$v_g = \partial\omega/\partial\mathbf{k} = 3\, k\, r_{De}\, v_{Te} << v_{Te} \quad (104)$$

* It must be noticed that this spectrum differs from that founded by I. Langmuir in the two fluids model by the factor 3 instead of 1 in (48). This indicates the nonaccuracy of hydrodynamical description of plasma oscillations.
However it must be noted that A. Vlasov understood that the oscillations damping really exists and moreover he supposed that it arises in the second approximation of particles interaction, or as a result of electron - ion collisions. In this sense he thought, that his equation (98) describes "collisionless plasmas"
Quite another sense of this pole gave L.Landau in his famous paper from 1946, in which he criticized A. Vlasov. In agreement with the causality principle he proposed that

$$1/(\omega - \mathbf{k}\cdot\mathbf{v}) = (\wp\,(\omega - \mathbf{k}\cdot\mathbf{v})) - i\pi\delta(\omega - \mathbf{k}\cdot\mathbf{v}) \tag{105}$$

The first term corresponds to the A. Vlasov treatment, whereas the second leads to the oscillations damping ($\omega \rightarrow \omega - i\delta$)

$$\delta = -\sqrt{\pi/8}\left(\omega_{Le}/k^3 r_{De}^3\right)\exp\left(-\left(1/\left(2k^2 r_{De}^2\right)^{-3/2}\right)\right) \tag{106}$$

This damping was called as the Landau damping of plasma oscillations with frequency spectrum (103), which was obtained in the Vlasov approximation. Here it must be noted that L. Landau did not notice at that time that this damping contradicts the momentum relaxation time, obtained by him in 1936 and equal ~ 1/ vei. Only in 1946 everything was clarified by N. Bogolyubov in his famous book "Dynamical problems in statistical Physics". In this book the strong derivation Vlasov equation and Landau collision integral were given as an expansion on powers of gas parameter (73).
Thus now we can write the exact kinetic equation for completely ionized plasma (α=e,i)

$$\partial f_\alpha/\partial t + \mathbf{v}\partial f_\alpha/\partial \mathbf{r} + e_\alpha\,\{\mathbf{E} + (1/c)[\mathbf{v}\times\mathbf{B}]\}\,\partial f_\alpha/\partial \mathbf{p} = \Sigma_\beta\,(\partial f_\alpha/\partial t)^{\alpha\beta} \tag{107}$$

which can be called as Vlasov-Landau equation. If we add to the right side of this equation the integral of charged particles collision with neutrals this equation can be applied to the weakly ionized plasma also.
Physical meaning of Landau damping was clarified by R. Sagdeev in 1956 when he noticed that it is a result of Cherenkov emission and absorption of the plasma oscillations (103) by the plasma electrons at $\omega = \mathbf{k}\cdot\mathbf{v}$. As for Maxwell distribution $\partial f_0/\partial v < 0$ then the absorption exceeds on emission and we obtain oscillations damping (see Fig.3)

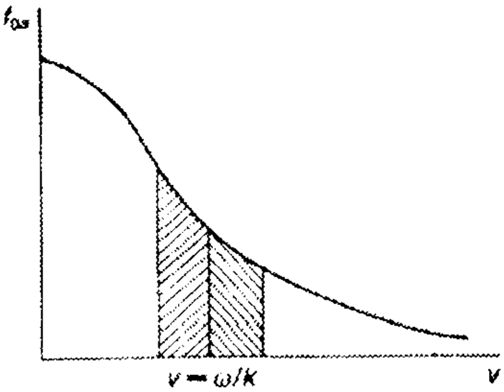

Figure 3

In conclusion let us repeat once more that for the system of charged particles under the condition of gas approximation the principal interaction between the particles is taken into account by the Vlasov kinetic equation, or in other words the principal interaction is the interaction via selfconsistent fields. Only in the second approximation at least for completely ionized plasma the particles collisions must be taken into account. In this sense the Landau - Boltzmann kinetic equation takes into account the effects of higher order than Vlasov's equation.The Vlasov-Landau equation (107) is that which takes into account not only particles interaction via selfconsistent field but also interaction via their direct collisions. The fundamental property of the system of charged particles consists in that the self consistent interaction surpasses the direct collisions of particles. Namely this property represents the beauty of plasma and makes it as a very interesting and important scientific object.

* 10 Bathnagar-Gross-Krook collision integral

The kinetic equation (107) is very complicated because of its right side which represents nonlinear integral operator. It is difficult to make use of this equation. Therefore in scientific literature very often the various phenomenological and approximate collision integrals are used. Despite phenomenological character of such collision integrals, sometimes it occurs to be not only qualitatively but quantitatively also correct.

Every model of collision integral must take into account the principal conservation laws such as conservation of particles number, their momentum, and anergy. Of course, we mean only elastic collision integrals. Thus the following relations must be satisfied

$$\int d\mathbf{p}_\alpha \, (\partial f_\alpha/\partial t)_{st}{}^{\alpha\beta} = 0, \tag{108}$$

$$\int \mathbf{p}_\alpha \, (\partial f_\alpha/\partial t)_{st}{}^{\alpha\beta} \, d\mathbf{p}_\alpha + \int \mathbf{p}_\beta \, (\partial f_\beta/\partial t)_{st}{}^{\beta\alpha} \, d\mathbf{p}_\beta = 0,$$

$$\int \varepsilon_\alpha \, (\partial f_\alpha/\partial t)_{st}{}^{\alpha\beta} \, d\mathbf{p}_\alpha + \int \varepsilon_\beta \, (\partial f_\beta/\partial t)_{st}{}^{\beta\alpha} \, d\mathbf{p}_\beta = 0,$$

Here ε is energy.

In addition, for the thermodynamically equilibrium (Maxwell) distributions of particles, the collision integrals must be zero. This follows from Boltzmann H-theorem.

Below we will use the most perfect, from our point of view, model of collision integral proposed in 1954 by P. Bathnagar, E. Gross and M. Krook. It looks as

(109)

$$(\partial f_\alpha/\partial t)_{st}^{\alpha\beta} = -\nu_{\alpha\beta}(f_\alpha - N_\alpha\phi_{\alpha\beta}),$$

where

(110)

$$\Phi_{\alpha\beta} = (1/(2\pi m_\alpha T_{\alpha\beta})^{3/2})\exp[-(m_\alpha(v-v_\alpha)^2)/2T_{\alpha\beta}]$$

$$v_\alpha = 1/N\int dp\, v\, f_\alpha \quad , N_\alpha = \int dpf_\alpha$$

$$T_{\alpha\beta} = (m_\alpha T_\beta + m_\beta T_\alpha)/(m_\alpha + m_\beta),\ T_\alpha = (m_\alpha/2N_\alpha)\int dp\,(v-v_\alpha)^2 f_\alpha$$

For satisfying the relations (108) it is necessary that

(111)

$$m_\alpha N_\alpha \nu_{\alpha\beta} = m_\beta N_\beta \nu_{\beta\alpha}$$

The physical meaning of quantities $\nu_{\alpha\beta}$ is clear from the analysis of relaxation processes which were considered above using Boltzmann and Landau collision integrals. Namely $\nu_{\alpha\beta}^{-1}$ represents the momentum relaxation time of α particles stipulated by their collisions with β particles. So

(112)

$$\nu_{e0} = \pi a^2 v_{Te} N_0,\ \nu_{i0} = \pi a^2 v_{Ti} N_0$$

$$\nu_{ee} = \left(4/3\sqrt{\pi/m}\right)\left(e^4 N_e L/T_e^{3/2}\right),\ \nu_{ei} = 4/3\sqrt{(2\pi/m)}e^2 e_i^2 N_i L/T_i^{3/2}$$

$$\nu_{ii} = 4/3\sqrt{\pi e_i^4 N_i L/\left(MT_i^{3/2}\right)},\ \nu_{ie} = (m/M)|e_i/e|\,\nu_{ei}$$

Below these expressions will be used in all estimations.

* 11 About hydrodynamical description of collisionless plasmas

Above it was shown that the Vlasov-Maxwell equations take into account the principal interactions of charged particles and in this sense they can describe all properties of plasmas quite sufficiently. However this system is complicated yet because the distribution function $f(\mathbf{p},\mathbf{r},t)$ is a function of 7 variables. Below we will show that under the definite conditions this system can be simplified and reduced to the the system of equation for hydrodynamical quantities

(113)

$$N_\alpha = \int dpf_\alpha(\mathbf{p},\mathbf{r},t)$$

$$N_\alpha v_\alpha = \int dpvf_\alpha(\mathbf{p},\mathbf{r},t)$$

$$N_\alpha T_\alpha = \int dp(m_\alpha v^2/2)f_\alpha(\mathbf{p},\mathbf{r},t)$$

Such simplification is possible in high frequency range, when the characteristic velocity is much larger than the thermal velocities of particles, and in low frequency range, when this velocity

exceeds the ion thermal velocity but is much less than the thermal velocity of electrons. In the second case plasma must be nonisothermal with $T_e >> T_i$.
For derivation of hydrodynamical equation we start from Vlasov equation

$$\partial f_\alpha/\partial t + \mathbf{v}\partial f_\alpha/\partial \mathbf{r} + e_\alpha\{\mathbf{E} + 1/c[\mathbf{v}\times\mathbf{B}]\}\partial f_\alpha/\partial \mathbf{p} = 0 \tag{114}$$

In this equation particles collisions are neglected which means that the characteristical time τ and characteristical scale L_0 of processes must satisfy the inequalities

$$1/\tau >> \Sigma_\beta \nu_{\alpha\beta}, \quad L_0 << v_{T\alpha}/\Sigma_\beta \nu_{\alpha\beta} \tag{115}$$

Multiplying equation (114) on 1 and v and integrating over momentum we obtain

$$\partial N_\alpha/\partial t + \mathrm{div}\, N_\alpha \mathbf{v}_\alpha = 0 \tag{116}$$

$$\partial N_\alpha v_{\alpha i}/\partial t + \partial \Pi_{\alpha ij}/\partial r_j = e_\alpha N_\alpha/m_\alpha\{\mathbf{E} + 1/c[\mathbf{v}_\alpha\times\mathbf{B}]\}$$

where

$$\Pi_{\alpha ij} = \int \mathbf{dp}(v_i v_j) \mathbf{f}_\alpha(\mathbf{p},\mathbf{r},\mathbf{t}) \tag{117}$$

The first equation coupled the first moment N_α to the second one $N_\alpha \mathbf{v}_\alpha$ is the continuity equation and it is closed in hydrodynamical sense. At the same time the second equation which connects the second moment N_α $\mathbf{v}_\alpha$ to the third $\Pi_{\alpha ij}$ occurs nonclosed. The problem of deriving hydrodynamical equations consists closing this equation.

In collisionless plasmas exist two possibilities of closing this equation. First concerns the high frequency and fast processes whereas the another concerns the low frequency and slow processes.

In the high frequency range when

$$L_0/\tau \sim \omega/k >> \mathbf{v}_{T\alpha} \tag{118}$$

the thermal motion of particles can be neglected and $f_\alpha \sim \delta(\mathbf{v}-\mathbf{v}_\alpha)$. Then from (117) it follows

$$\Pi_{\alpha ij} = N_\alpha v_{\alpha i} v_{\alpha j} \tag{119}$$

Substituting this expression into the equation (116) we obtain the Euler equation

$$\partial \mathbf{v}_\alpha/\partial t + (\mathbf{v}_\alpha.\nabla)\mathbf{v}_\alpha = (e_\alpha/m_\alpha)\{\mathbf{E} + (1/c)[\mathbf{v}_\alpha\times\mathbf{B}]\} \tag{120}$$

The system of equations (116) and (120) together with the definitions of charge and current densities

$$\rho = \Sigma_\alpha e_\alpha N_\alpha, \qquad \mathbf{j} = \Sigma_\alpha e_\alpha N_\alpha \mathbf{v}_\alpha \tag{121}$$

form the complete system of hydrodynamical equations. It is easy to notice that this system coincides with two-fluid hydrodynamical equations if $v_\alpha \to 0$, or in other words with the I. Langmuir hydrodynamics of collisionless plasmas.

The other limit, when the hydrodynamical description of collisionless plasma is valid, is the low frequency limit, when

(122)

$$v_{Ti} \ll \omega/\mathbf{k} \ll v_{Te}$$

The ions in this limit can be considered as a "cold" one, therefore for them the hydrodynamical description is valid ($\mathbf{v}_i = \mathbf{v}$, $N_i = N$)

(123)

$$\partial N/\partial t + \mathrm{div}\, N\mathbf{v} = 0$$

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} = (e_i/M)\{\mathbf{E} + (1/c)[\mathbf{v}\times\mathbf{B}]\}$$

As about electrons, in the limit (122), the Vlasov equation

(124)

$$\mathbf{v}\partial f_e/\partial \mathbf{r} + (e/m)\{\mathbf{E} + (1/c)[\mathbf{v}\times\mathbf{B}]\partial f_e/\partial \mathbf{p} = 0$$

can be solved exactly.

Let us begin from unmagnetized electrons and purely potential field $\mathbf{E} = -\nabla\phi$. Then the solution of (124) can be presented as

(125)

$$f_e = (N_{e0}/(2\pi m T_e)^{3/2}) \exp(-mv^2/2T_e - e_i\phi/T_e)$$

From this follow

(126)

$$N_e = N_{e0}\exp(-e\phi/T_e), \quad \nabla N_e/Ne = e\mathbf{E}/T_e = -e\nabla\phi/T_e$$

It must be noted that in this presentation we supposed T_e = const,which is the consequence of the right inequality (122).

Now we can write the system (123) in a purely hydrodynamical form (remind that B = 0)

(127)

$$\partial N/\partial t + \mathrm{div} N\mathbf{v} = 0$$

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} = -|e_i/e|\, \nabla N T_e/NM$$

If one introduces $\rho = MN$ and $P = NT_e$, where T_e = const (isothermic approximation) then this system coincides with the one fluid hydrodynamics of usual liquid.

Quite analogically can be derived the one fluid MHD equations for nonisothermal , $T_e >> T_i$, and magnetized collisionless plasmas under the conditions

(128)

$$\mathbf{k}v_{T\alpha} \sim v_{T\alpha}/L_0 \ll \Omega_\alpha, \; \Omega_i \ll \omega_{Li}$$

This system of equations coincides with Alfven hydrodynamics of ideal liquid

(129)

$$\partial N/\partial t + \mathrm{div}\, N\mathbf{v} = 0$$

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} = -|e_i/e|\nabla NT_e/MN + (1/4\pi NM)[\mathbf{B}\times \mathrm{rot}\mathbf{B}]$$

$$\partial \mathbf{B}/\partial t - \mathrm{rot}\,[\mathbf{v}\times\mathbf{B}] = 0,\ \mathrm{div}\,\mathbf{B} = 0$$

Here $\rho = NM$, $P = NT_e$, $T_e \gg T_i$ and $T_e = const$

This derivation of MHD equations was done only in 1956 by V. Silin and Y. Klimontovich. Only after the publication of their paper it becomes clear why the applications of these equations to the low frequency phenomena in the collisionless ionospheric plasma occured so successful. Quite analogically the above given derivation of equations for "cold" two fluids hydrodynamics clarifies the success of their application to the problems of fast radiowaves propagation in the ionospheric plasma.

LECTURE 6

Linear Electrodynamics of Isotropic Plasma

* 12 Linear electrodynamical properties of isotropic collisionless plasma

Below it will be shown that the Vlasov-Landau or Vlasov - Boltzman kinetic equations give the completely adequate descriptions of all properties of gaseous plasma. In this section we will begin from collisionless isotropic plasma and for this reason we will start from Vlasov equation

(130)

$$\partial f_\alpha/\partial t + \mathbf{v}\partial f_\alpha/\partial \mathbf{r} + e_\alpha/m_\alpha\{\mathbf{E} + 1/c[\mathbf{v}\times\mathbf{B}]\}\partial f_\alpha/\partial \mathbf{v} = 0$$

where α = e,i. For thermodynamically equilibrium plasma in the absence of external fields

(131)

$$f_{0\alpha} = (n_{0\alpha}/(2\pi m_\alpha T_\alpha)^{3/2})\exp(-m_\alpha v^2/2T_\alpha)$$

Moreover we suppose that plasma is quasineutral

(132)

$$\Sigma_\alpha en_{0\alpha} = en_{0e} + e_i n_{0i} = 0$$

Let now consider small deviation from $f_{0\alpha}$, or $f\alpha = f_{0\alpha} + \delta f\alpha$. Then for $\delta f\alpha$, we obtain

(133)

$$-i(\omega - \mathbf{kv})\,\delta f_\alpha = -e_\alpha\,\mathbf{E}\,\partial f_{0\alpha}/\partial\mathbf{p}$$

Here we suppose that in linear approximation $\delta f\alpha \sim \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$. As a result we have

(134)

$$\delta f_\alpha = -(i\,e_\alpha\mathbf{E}\,\partial f_{0\alpha}/\partial\mathbf{p})/(\omega - \mathbf{k}\cdot\mathbf{v})$$

By substituting this expression into the relation

(135)

$$j_i = {}_\alpha\Sigma e_\alpha \int v_i \delta f_\alpha\, d\mathbf{p} = \sigma_{ij}(\omega,\mathbf{k})\,E_j$$

we find the conductivity tensor $\sigma_{ij}(\omega,k)$ and then the tensor of dielectric permittivity

(136)

$$\varepsilon_{ij}(\omega,\mathbf{k}) = \delta_{ij} + (4\pi i/\omega)\sigma_{ij}(\omega,\mathbf{k}) = \delta_{ij} + \Sigma_\alpha (4\pi e_\alpha^2/\omega)\int d\mathbf{p}(\,v_i\,\partial f_{0\alpha}/\partial \mathbf{p}_j)/(\omega - \mathbf{k}\cdot\mathbf{v})$$

It is obvious that

(137)

$$\varepsilon_{ij}(\omega,k) = (\delta_{ij} - (k_ik_j/k^2)\varepsilon^{tr}(\omega,\mathbf{k}) + (k_ik_j/k^2)\varepsilon^{l}(\omega,\mathbf{k})$$

where

(138)

$$\varepsilon^{tr}(\omega,\mathbf{k}) = 1 - \Sigma_\alpha(\omega_{L\alpha}{}^2/\omega^2)\,\Im(\omega/\mathbf{k}\mathbf{v}_{T\alpha})$$

$$\varepsilon^{l}(\omega,\mathbf{k}) = 1 + \Sigma_\alpha(\omega_{L\alpha}{}^2/k^2 v_{T\alpha}{}^2)[\,1 - \Im(\omega/\mathbf{k}\mathbf{v}_{T\alpha})]$$

When we integrated (136) we took into account the causality principle and the pole $\omega = \mathbf{kv}$ was avoided in the sense of Landau (105). Namely as a result of such consideration it appears in (138) the function $\Im(X)$, which has not only real part for real X, but also imaginary part

(139)

$$\Im(X) = X\exp(-X^2/2)\int_{i\infty}^{X} d\tau\, \exp(\tau^2/2) = \begin{cases} 1 + 1/X^2 + \ldots\ldots - i\sqrt{\pi/2}\, e(-X^2/2)X & \text{when } |X| >> 1 \\ -i\sqrt{(\pi/2)}X + X^2 & \text{when } |X| << 1 \end{cases}$$

The imaginary parts for $\text{Im}\,\varepsilon^{l,tr} \neq 0$ corresponds to the wave dissipation of small oscillations in plasma. This dissipation is stipulated by Cherenkov absorption.
Let us now investigate different limiting cases of ω and $\mathbf{k}$ and clarify the principal meaning of $\varepsilon^{l}(\omega,\mathbf{k})$ and $\varepsilon^{tr}(\omega,\mathbf{k})$. First of all let us consider low frequency (static) limit $\omega \to 0$. Then

(140)

$$\varepsilon^{l}(0,\mathbf{k}) = 1 + 1/k^2 r_D{}^2)$$

$$\lim_{\omega\to 0} \varepsilon^{tr}(\omega,\mathbf{k}) = 1 + i\sqrt{(\pi/2)}\omega_{Le}{}^2/\omega\mathbf{k}\mathbf{v}_{Te} = 1 + i(4\pi\sigma^{tr}(0,\mathbf{k})/\omega)$$

The first expression coincides with that obtained in the static limit from the model of two fluids hydrodynamics and corresponds to Debye screening of the field for static point charged particle in a plasma. So we see that the Vlasov equation gives the correct description of electrostatic properties of a collisionless plasmas, or for the fields $\mathbf{E} = -\nabla\Phi$.
More interesting phenomenon is described by the second expression (140). From this expression we see that the collisionless plasma in the static limit has the finite conductivity in connection with the transverse electric field, div $\mathbf{E} = 0$, which is the function of $\mathbf{k}$

(141)

$$\sigma^{tr}(0,\mathbf{k}) = (\sqrt{\pi/2})e^2 n_{0e}/m\mathbf{k}\mathbf{v}_{Te}$$

This conductivity is stipulated by the Cherenkov dissipation and leads to the anomalous skin-effect for quasistatic transverse fields in a plasma, the new phenomenon which arises only in collisionless plasmas. To show this let us write the material equation (Ohm's law) corresponding to (141) in the form

(142)

$$\text{rot}\mathbf{j} = -(\sqrt{\pi/2})e^2 n_{0e}\mathbf{E}/mv_{Te}$$

Using this relation from the Maxwell equations can be obtained the following equation for $\mathbf{B}$

(143)

$$\text{rotrotrot}\mathbf{B} = (4\pi/c^2)(\sqrt{\pi/2})(e^2 n_{0e}/mv_{Te})(\partial\mathbf{B}/\partial t)$$

This equation leads to the wellknown formula for anomalous skin-effect- penetration of low frequency transverse field in collisionless plasmas (for $E \sim \exp(i\omega t + i\mathbf{k}\cdot\mathbf{r})$)

(144)

$$k^3 = i(\sqrt{\pi/2})\omega_{Le}{}^2\omega/\,c^2 v_{Te} \Rightarrow \lambda_{sk} = 1/\text{Im}k \sim (c^2 v_{Te}/\omega\omega_{Le}{}^2)^{1/3}$$

Namely this formula was firstly obtained by A. Pippard in 1949 who also gives the physical explanation of the phenomenon. But the mathematically correct consideration of the boundary problem of field penetration into the collisionless plasma was done by E. Reuter and E. Sondheimer in 1958; they show that the anomalous skin effect takes place if

$$\omega \quad \mathbf{k}v_{Te} \quad \omega_{Le}.$$

Let us consider now the problem of waves propagation in collisionless isotropic plasma, or in other words, find the conditions for existence of nontrivial solutions of type $\exp(-i\omega t+i\mathbf{k}\cdot\mathbf{r})$ in such a plasma in the absence of external field sources. These conditions follow from the field equations which in the space $(\omega,\mathbf{k})$ look as

$$\{k^2\delta_{ij} - k_ik_j - (\omega^2/c^2)\varepsilon_{ij}(\omega,\mathbf{k})\}E_j = 0 \tag{145}$$

For the isotropic media with ε_{ij} as (137) this system separates into the two independent equations

$$E^l\,\varepsilon^l(\omega,\mathbf{k}) = 0 \tag{146}$$

$$E^{tr}[k^2c^2 - \omega^2\varepsilon^{tr}(\omega,\mathbf{k})]^2 = 0$$

The first describes the longitudinal field with **E** // **k** and the condition for existence of non trivial solutions is

$$\varepsilon^l(\omega,\mathbf{k}) = 0 \tag{147}$$

This equation is called as dispersion equation for longitudinal waves in an isotropic plasma. Quite analogically from the second equation we obtain the dispersion equation for transverse $(\mathbf{E} \perp \mathbf{k})$ waves

$$k^2c^2 - \omega^2\varepsilon^{tr}(\omega,\mathbf{k}) = 0 \tag{148}$$

In the isotropic plasma this type of waves occurs twice degenerated.
Let us now consider very shortly the spectrum of electromagnetic waves in an isotropic plasma and analyze the solutions of the equations (147) and (148).

1- Longitudinal Waves :

a) Let us begin the analysis of (147) for the high frequency range, $\omega \gg \mathbf{k}\,\mathbf{v}_{Te,i}$, when

$$\varepsilon^l(\omega,k) = 1 - (\omega_{Le}^2/\omega^2)(1+3(k^2v_{Te}^2)/\omega^2)+i\sqrt{(\pi/2)}(\omega_{Le}^2\omega/k^3v_{Te}^3)\exp(-\omega^2/2k^2v_{Te}^2) \tag{149}$$

Then from (147) it follows $(\omega \rightarrow \omega + i\delta)$

$$\omega^2 = \omega_{Le}^2 + 3k^2v_{Te}^2 \tag{150}$$

$$\delta = -(\sqrt{\pi/8})(\omega_{Le}/k^3r_{De}^3)\exp(-3/2-1/2k^2r_{De}^2)$$

These coincide with (103) and (106) as it should be expected. They describe high frequency plasma oscillations and their absorption due to the Cherenkov mechanism of waves absorption by plasma electrons. Wave damping increases with the increase of k and in the short wave range when $kr_{De} >> 1$, these waves occur aperiodically damping. This spectrum differs from that obtained by I. Langmuir by the factor 3 instead of 1 in $\omega(k)$

b) In the intermediate frequency range, when $v_{Ti} << \omega/k << v_{Te}$, we have

$$\varepsilon^l = 1 - \omega_{Li}^2/\omega^2 + \omega_{Le}^2/k^2 v_{Te}^2 (1 + i\sqrt{(\pi/2)}\omega/kv_{Te}) \tag{151}$$

Substituting this expression into the equation (147) we obtain ($\omega \rightarrow \omega + i\delta$)

$$\omega^2 = \omega_{Li}^2/(1 + \omega_{Le}^2/k^2 v_{Te}^2) = \begin{cases} k^2 v_s^2 & \text{if } k^2 r_{De}^2 << 1 \\ \omega_{Li}^2 & \text{if } k^2 r_{De}^2 >> 1 \end{cases} \tag{152}$$

$$\delta = -\sqrt{(\pi/8)|ei/e|m/M}\,\omega^4/k^3 v_s^3$$

Namely the long wave range of these oscillations

$$\omega = kv_s\,,\ \delta = -\sqrt{(\pi/8)|ei/e|\,m/M}\,\omega \tag{153}$$

was investigated by G.Gardeev in 1954 who showed that the frequency spectrum $\omega(k)$ differs from that obtained by I. Langmuir in the model of two fluids hydrodynamics by the dependence only on the electrons temperature T_e and nondependence on T_i. Moreover as $\omega >> kv_{Ti}$ the inequality $T_e >> T_i$ must take place. But under these conditions, the I. Langmuir result occurs to be correct. Besides G.Gardeev showed that these oscillations are damping, the reason of which is the Cherenkov absorption of ion-accoustic (just same as that called by I. Langmuir) oscillations by the plasma electrons. The frequency spectra (150) and (152) are presented on the Figure 4.

c) Finally if $\omega << kv_{Ti}$ then the first expression of (140) is valid and the Debye screening takes place.

2. Transverse Waves :

Let us now consider the transverse waves and analyse the equation (147)

a) In the high frequency range $\omega >> kv_{Te}$, when

$$\varepsilon^{tr}(\omega) = 1 - \omega_{Le}^2/\omega^2 \tag{154}$$

from (148) follows

$$\omega^2 = \omega_{Le}^2 + k^2 c^2 \tag{155}$$

We see that the phase velocity of waves is higher than the light speed. Therefore interaction of such waves with charged particles (emission or absorption) is impossible. As a result in a considered case of collisionless plasma they don't damp and besides the spectrum (155) exactly coincides with spectrum of transverse waves obtained in the model of independent particles in collisionless plasma ($\lim \nu e \rightarrow 0$). The spectrum is presented on the Figure 4.

b) Concerning low frequency range when $\omega << kv_{Te} = v_{Te}\omega_{Le}/c$ the expression ε^{tr} coincides with (140) corresponding to the anomalous skin effect considered above.

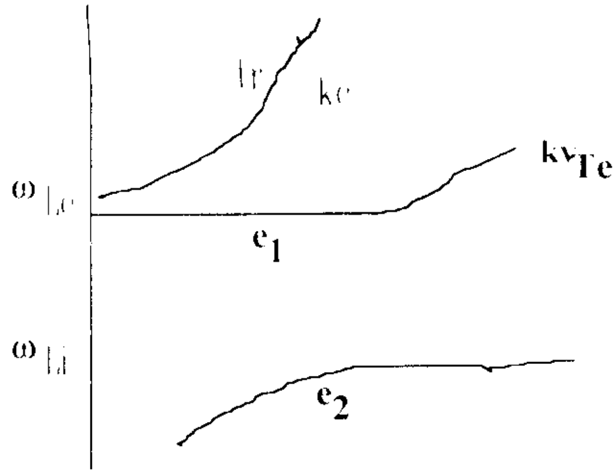

Figure 4

* 13 Collisions influence on the oscillations spectra of isotropic plasma

Below we will restrict ourselves by considering only the qualitative effects caused by the particles collisions in a plasma and BGK collision integral will be considered. Concerning the completely ionized plasma, only the corrections will be given. Thus we will start from the Vlasov-BGK equation

$$\partial f_\alpha/\partial t + \mathbf{v}\partial f_\alpha/\partial \mathbf{r} + e_\alpha\{\mathbf{E}+(1/c)[\mathbf{v}\times\mathbf{B}]\}\partial f_\alpha/\partial \mathbf{p} = -\nu_{\alpha 0}\,(f_\alpha - N_\alpha \phi_{\alpha 0}) \tag{156}$$

Here

$$f_{\alpha 0} = (1/(2m_\alpha T_\alpha)^{3/2})e(-m_\alpha v^2/2T_\alpha) \tag{157}$$

The equilibrium distribution as it is easily seen coincides with Maxwellien $f_{0\alpha} = N_{0\alpha}\phi_{\alpha 0}$. Therefore for a small perturbation of type $\delta f_\alpha \sim \exp(-i\omega t+i\mathbf{k}\cdot\mathbf{r})$ from (156) we obtain

$$-i(\omega - \mathbf{k}\cdot\mathbf{v})\delta f_\alpha + e_\alpha\,\mathbf{E}\partial f_\alpha/\partial \mathbf{p} = -\nu_{\alpha 0}\,(\,\partial f_\alpha - \phi_{\alpha 0}\textstyle\int \mathbf{dp}\,\delta f_\alpha) \tag{158}$$

Here for simplicity the isothermal model of BGK integral was used, that means T_α = const. The equation (158) is the Voltera type integral equation which can be easely solved. We omite the solution and give the final results - the expressions for $\varepsilon^l(\omega,k)$ and $\varepsilon^{tr}(\omega,k)$

$$\varepsilon^l(\omega,\mathbf{k}) = 1 + {}_\alpha\Sigma(\omega_{Le}{}^2/k^2 v_{T\alpha}{}^2) \times$$
$$((1-\Im((\omega+i\nu_{\alpha 0})/\mathbf{k}v_{T\alpha}))/(1-(i\nu_{\alpha 0}/(\omega+i\nu_{\alpha 0})\,\Im((\omega+i\nu_{\alpha 0})/\mathbf{k}v_{T\alpha})))$$
$$\varepsilon^{tr}(\omega,\mathbf{k}) = 1 - \Sigma_\alpha\,\omega_{Le}{}^2/(\omega(\omega+i\nu_{\alpha 0})\,\Im(\omega + i\nu_{\alpha 0})/(\mathbf{k}v_{T\alpha})) \tag{159}$$

As it should be expected for the collisionless plasma when $\nu_{\alpha 0} \to 0$ these expressions coincide with (138).

Let us begin the analysis of the expressions (159) in the static limit when $\omega \to 0$. It is easy to show that independently of the ratio $\nu_{\alpha 0}/kv_{T\alpha}$, we have

(160)

$$\varepsilon^l(0,\mathbf{k}) = 1 + 1/k^2 r_{De}^2$$

Thus in the static limit for collisional plasma as for collisionless one, we have Debye screening of potential fields.

Quite another situation arises for $\varepsilon^{tr}(\omega,k)$. The above result (140) corresponding to the anomalous skin-effect for low frequency transverse field in a plasma is correct for collisional plasma also if

(161)

$$\omega << \nu_e < v_{Te}\omega_{Le}/c$$

Remind that in the collisionless limit, $\nu_e << \omega$, the anomalous skin effect takes place in the frequency range $\omega < v_{Te}\,\omega_{Le}/c$.

In the opposite to (161) limit when $\nu_e > v_{Te}\omega_{Le}/c$ the anomalous skin effect is impossible. Then in the low frequency range, $\omega << \nu_e$, only wellknown normalous skin effect takes place and the formula

(162)

$$\varepsilon^{tr} = 1 + (i\omega_{Le}^2/\omega\nu_{e0}) = 1 + 4\pi i\sigma/\omega,\ \sigma = e^2 n_{0e}/m\nu_{e0}$$

obtained in our second lecture is valid*.

Let us now go over high frequency range and consider the particle collisions influence on high frequency phenomena. From the expression (159) for $\varepsilon^l(\omega,k)$ follows that if $\omega \sim \omega_{Le} >> kv_{Te}, \nu_{e0}$ we find the correction to the expression (149)

(163)

$$\Delta\varepsilon^l = i\,\omega_{Le}^2\nu_e/\omega^3$$

Here we take into account also electron-ion collisions and therefore in the formula $\nu_e = \nu_{e0} + \nu_{eff}$. The correction (163) leads to the correction of Landau damping (150) of plasma high frequency oscillations

(164)

$$\Delta\delta = -\nu_e/2$$

Remind that for completely ionized plasma e= 1,96 $e^2 n_{oe}$ / meff

Comparing this correction to (150) we can determine the condition when plasma should be considered as collisional in the high frequency range

(165)

$$\nu_e/\omega_{Le} > (\sqrt{\pi}\exp(-1/2k^2r_{De}^2))/(5{,}5k^3r_{De}^3)$$

In the opposite case when the oscillations are sufficiently short wave length, particles collisions can be neglected. Moreover the quantity (164) allows us to determine when plasma should be considered as completely ionized for high-frequency plasma oscillations

(166)

$$\nu_{eff}/\nu_{e0} = 2\cdot 10^{-5}\,(n_{0e}/n_0)/(ZL/a^2\,T_e^2) \sim Zn_{0e}10^{11}/n_0T_e^2 >> 1 \text{ where } Z = |e_i/e|$$

In the opposite case plasma is weakly ionized. For example if $T_e > 10^4$K then plasma is completely ionized if $n_{0e} > 10^{-3}\,n_0$. At the same time, if $T_e \sim 10^8$K (thermonuclear plasma)

then even at $n_0 > 10^{-5} n_{0e}$ plasma occurs weakly ionized, the electron-neutral collisions exceeds the collisions between charged particles.
Quantitavely another situation takes place for low-frequency longitudinal oscillations when, $kv_{Ti} < \omega < kv_{Te}$. if in addition to these inequalities $v_i >> \omega$ and $kv_{Te} > \omega$ are satisfied then the following collisional correction to the (151) arises

* Remind that for completely ionized plasma with $e = -e$, $\sigma = 1.96\, e^2 n_{0e}/mv_{eff}$

(167)

$$\Delta\varepsilon^l = i(\omega_{Li}^2/\omega^3)\cdot \begin{cases} v_{i0} & \text{for weakly ionized plasma} \\ 8v_{ii}k^2v_{Ti}^2/5\omega^3 & \text{for completely ionized plasma} \end{cases}$$

As a result we obtain the correction to the damping decrement (152)

(168)

$$\Delta\delta = - \begin{cases} v_{i0}/2 \\ 4v_{ii}k^2v_{Ti}^2/5\omega^2 \end{cases}$$

for weakly and completely ionized plasma correspondingly. Comparison of this expression with the damping decrement (152) leads to the following condition:
if $v_i/\omega > \sqrt{Zm/M}$, where

(169)

$$v_i = \begin{cases} - v_{i0} \\ - 8/5\, v_{ii}k^2v_{Ti}^2/\omega^2 \end{cases}$$

Then the particles collisions determine the low frequency waves absorption and if opposite inequalities take place the Cherenkov mechanism of absorption is dominative.
Besides from the ratio of two expressions (168) we determine the condition when plasma can be considered as weakly ionized for low frequency oscillations and vice versa. Thus in the case of long wave oscillations when the relations (153) are valid we conclude : if

(170)

$$n_0/n_{0e} > 10^{11}Z^2/T_eT_i$$

then the plasma should be considered as weakly ionized and as completely ionized in opposite case. For example if $T_e = 10^5$K, $T_i = 10^3$K, $Z=1$ then a plasma only with $n_0/n_{0e} > 10^3$ can be considered as weakly ionized.
In conclusion let us analyse the properties of strongly collisional plasma. But before we'll consider the expressions (159) in the limit $|\omega + iv_\alpha| >> kv_{T\alpha}$, where the model of two fluids hydrodynamics seems to be valid. From (159) under this condition follows

(171)

$$\varepsilon^l(\omega,\mathbf{k}) = 1-\Sigma_\alpha\omega_{L\alpha}^2/(\omega+iv_{\alpha0})[\omega-i(k^2v_{Te}^2v_{\alpha0})/(\omega+iv_{\alpha0})^2]$$

$$\varepsilon^{tr}(\omega,\mathbf{k}) = 1 - \Sigma_\alpha\omega_{L\alpha}^2/(\omega(\omega+iv_{\alpha0}))$$

Comparing these expressions with (38) we conclude that $\varepsilon^{tr}(\omega,k)$ is identical whereas $\varepsilon^{l}(\omega,k)$ differs by the factor $\nu_{\alpha n}/(\omega+i\nu_{\alpha n})$ of the term taking into account the thermal motion of particles.
This difference is very principal. Moreover the correctness of the expressions (171) are defined by the condition $|\omega - i\nu_{\alpha,\alpha}| >> kv_{Te}$ and therefore the expressions (38) are correct in two cases: when $\nu_{\alpha 0} >> \omega$ and the mentioned factor becomes equal to unity, or when $\omega >> \nu_{\alpha 0}$, $kv_{T\alpha}$ and the thermal motions of particles is a small correction. In the first case

$$\varepsilon^{l}(\omega,\mathbf{k}) = 1-\Sigma_{\alpha}\omega_{Li}^{2}/(i\omega\nu_{\alpha}-k^{2}v_{T\alpha}^{2}) \tag{172}$$

This expression coincides with (41) which describes the diffusion processes in a plasma (see lecture 3). In the opposite limit when $\omega >> \nu_{\alpha 0}$, $kv_{T\alpha}$ thermal motions of particles can be neglected and the model of independent particles considered in the lecture 2 is valid.

LECTURE 7

Linear Electrodynamics of Magnetoactive Plasma

* 14 Linear electromagnetic properties of collisionless magnetoactive plasma

A magnetoactive plasma represents a system with practically infinite number of degrees of freedom. In a magnetized plasma there exist different types and different branches of oscillations and waves. It is obvious that the investigation of all these oscillations in detail is impossible in one lecture.
Therefore we will restric ourselves by the consideration of only the most specific phenomena which characterized magnetoactive plasma and which every physicist must know.
First of all it must be noted that the charged particles rotate around the magnetic field lines. This rotation can be considered as individual oscillations of particles with frequency equal to the well known Larmor frequency $e_\alpha B_0/m_\alpha c = \Omega_\alpha$. It is easy to understand that if the electromagnetic field frequency is $\omega = n\Omega_\alpha$ then the resonance interaction between the field and charged particle must take place. As a result the dielectric permittivity ε_{ij} (ω,k), its hermitian as well as antihermitian parts (which describes energy absorption) have the poles at $\omega \sim n\Omega_\alpha$. In the lecture 2 we showed that such poles arise in the model of independent particles, but only for $n = \pm 1$. Below we will show that the kinetic consideration leads also to the appearence of the poles at $n \neq 1$.
The second obvious phenomenon which arises in a magnetoactive plasma is the magnetic pressure $B_0^2/8\pi$ which follows from the elasticity of magnetic field lines. We have already met this phenomenon in the model of independent particles and namely it is the reason of Alfven type oscillation. Of course this phenomenon exists in two fluids hydrodynamics as well as one fluid (Alfven) MHD, and below it plays a very important role also in a kinetic theory of plasma oscillations.
Finally it is easy to understand that the behaviour of magnetoactive plasma at $\omega = n\Omega_\alpha$ must be somewhat like to the behaviour of isotropic plasma at $\omega \to 0$, because the Larmor rotation remind the Doppler shift for electromagnetic fields. Below this will be shown by considering the field penetration (skin-effect) into the magnetoactive plasma.
As in the previous section we begin from collisionless plasma described by the Vlasov equation (130). From this equation we obtain the equation for equilibrium distribution function $f_0(\mathbf{p})$ (external magnetic field $\mathbf{B}_0$ is proposed to be parallel to **OZ** axis)

$$e[\mathbf{v}\times\mathbf{B}_0]\,\partial f_{0\alpha}/\partial\mathbf{p} = -\Omega_\alpha\,\partial f_{0\alpha}/\partial\varphi = 0 \qquad (173)$$

Here φ is the angle in the cylindrical frame : v_z , $v_x = v_\perp\cos\varphi$, $v_y = v_\perp\sin\varphi$. The solution of this equation we have choosed as

$$f_{0\alpha} = (N_{0\alpha}/(2\pi m_\alpha T_\alpha)^{3/2})\exp(-m_\alpha v^2/2T_\alpha) \qquad (174)$$

Besides, it was supposed that the plasma is quasi neutral, or $\sum_\alpha e_\alpha N_{0\alpha} = 0$.
*The detail consideration of linear electromagnetic phenomena in magnetoactive plasma, can be found in many text book on plasma physics.
For a small deviation from $f_{0\alpha}$ which is taken as $\delta f_\alpha \sim e^{-i\omega t+i\mathbf{kr}}$ we obtain

(175)

$$-i(\omega - \mathbf{k}\cdot\mathbf{v})\delta f_\alpha - \Omega_\alpha \partial\delta f_\alpha/\partial\varphi = -e_\alpha \mathbf{E}\partial f_{0\alpha}/\partial\mathbf{p}$$

Taking into account the obvious condition of periodicity

(176)

$$\delta f_\alpha(\varphi + 2\pi) = \delta f_\alpha(\varphi)$$

the general solution of (175) can be written as

(177)

$$\delta f_\alpha = e_\alpha \Omega_\alpha\ {}_{\infty}\!\int^{\varphi} d\varphi'\ \mathbf{E}(\partial f_{0\alpha}/\partial\mathbf{p}) \exp[i\,\Omega_\alpha\ {}_{\varphi}\!\int^{\varphi'} d\varphi''(\omega-\mathbf{k}\cdot\mathbf{v})_{\varphi''}]$$

Here we suppose that $\delta f_\alpha(\infty) = 0$

Substituting the expression (177) into the formula for induced current (135) we find plasma conductivity and then the dielectric permittivity

(178)

$$\varepsilon_{ij}(\omega,\mathbf{k}) = \delta_{ij} + (4\pi i/\omega)\,\sigma_{ij}(\omega,\mathbf{k})$$

$$= \delta_{ij} + \Sigma_\alpha (4\pi e_\alpha^2/\omega\Omega_\alpha) \int d\mathbf{p}\ \partial f_0/\partial\varepsilon_\alpha v_i\ {}_0\!\int^{\infty} d\varphi' v_j(\varphi') \exp(-i/\Omega_\alpha\ {}_{\varphi}\!\int^{\varphi'} d\varphi''(\omega - \mathbf{k}\ \mathbf{v}_{\varphi''}))$$

It can be easily shown that this tensor has 6 independent components: $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy} = -\varepsilon_{yx}$, $\varepsilon_{zy} = -\varepsilon_{yz}$, $\varepsilon_{xz} = \varepsilon_{zx}$ (remind that in the case of isotropic plasma we have only two components ε^{tr} and ε^{l}). However below in general we will consider only two of them. The first quantity

(179)

$$\varepsilon_\perp(\omega,\mathbf{k}) = \varepsilon_{xx} \pm i\varepsilon_{xy} = 1 + {}_\alpha\Sigma(2\pi e_\alpha^2/\omega)\int (\partial f_{0\alpha}/\partial\varepsilon_\alpha)(v_\perp^2/(\omega \pm \Omega_\alpha - k_z v_z))d\mathbf{p}$$

$$= 1 \pm {}_\alpha\Sigma(\omega_{L\alpha}^2/\omega(\omega \pm \Omega_\alpha))\Im((\omega \pm \Omega_\alpha)/k_z v_{T\alpha})$$

describes the purely transverse field ($\mathbf{E} \perp \mathbf{k}$) depending only on the parallel coordinates ($k_\perp = 0$, $k_{//} \neq 0$). Whereas the second one

(180)

$$\varepsilon(\omega,\mathbf{k}) = k_i k_j \varepsilon_{ij}(\omega,\mathbf{k})/k^2 = 1 - \Sigma_\alpha(4\pi e_\alpha^2/k^2)\int d\mathbf{p}\partial f_{0\alpha}/\partial\varepsilon_\alpha[1 - \Sigma_n\ \omega\Im_n^2(b_\alpha)/(\omega - n\Omega_\alpha - k_z v_z]$$

$$= 1 + \Sigma_\alpha\ \omega_{L\alpha}^2/k^2 v_{T\alpha}^2[1 - \Sigma_n\ (\omega/(\omega - n\Omega_\alpha))A_n(z_\alpha)\Im((\omega - n\Omega_\alpha)/k_z v_{T\alpha})]$$

describes purely longitudinal field ($\mathbf{E} = -\nabla\phi$, $\mathbf{E}\ //\ \mathbf{k}$) arbitrarily depending on coordinates ($k_\perp \neq 0$, $k_{//} \neq 0$). Here $b_\alpha = k_\perp v_\perp/\Omega_\alpha$, $Z_\alpha = k_\perp^2\ v_{T\alpha}^2/\Omega_\alpha^2$ and $A_n(z) = I_n(z)\exp(-z)$, $I_n(z)$ is the Bessel function.

First of all it must be noted that from the expressions (179) and (180) follows that their poles really correspond to the one particle cyclotron resonances at

(181)

$$\omega = n\Omega_\alpha$$

These poles describe one particle oscillations and therefore in the ranges of these frequencies the resonance waves absorption must arise. Indeed from the integrand (179) and (180) we see that under the conditions

(182)

$$\omega - n\Omega_\alpha - k_z v_z = 0$$

the resonance wave absorption takes place. It follows from the Landau prescription that

$$I(\omega - k_z v_z - n\Omega_\alpha) = (\wp(\omega - k_z v_z - n\Omega_\alpha)) - i\pi\delta(\omega - k_z v_z - n\Omega_\alpha) \tag{183}$$

At n=0 the absorption is coming from the Cherenkov mechanism considered in the previous section, whereas the absorption at n≠0 is known as cyclotron absorption. At the same time, the last one can be treated also as Cherenkov absorption taking into account the Doppler shift $n\Omega_\alpha$ stipulated to the one particle oscillations.
More obvious the Doppler shift is seen from the character of waves propagation described by the dispersion equation

$$|k^2\delta_{ij} - k_i k_j - \omega^2\varepsilon_{ij}(\omega, \mathbf{k})| = 0 \tag{184}$$

For purely longitudinal propagation when $k_\perp = 0$ this equation takes the form

$$k^2c^2 = \omega^2\varepsilon_\perp \tag{185}$$

which corresponds to the purely transverse waves. In the frequency range $\omega \pm \Omega_e << k_z V_{Te}$ from this equation in taking into account (179) we obtain *

$$k^2c^2 = i(\omega_{Le}{}^2\omega/kv_{Te})\sqrt{\pi/2} \tag{186}$$

This equation coincides exactly with (144) and describes the anomalous skin-effect. The penetration depth obviously coincides with (144)

$$\lambda_{sk} = (\Im m k)^{-1} = ((\sqrt{\pi/2})\omega_{Le}{}^2\omega/c^2v_{Te})^{-1/3} \tag{187}$$

At the same time we can rewrite these relations in the language of frequency spectrum, near the cyclotron frequencies when $(\omega \pm \Omega_e) << k_z v_{Te}$ from (185) we obtain

$$\omega = -i(k^3c^2v_{Te}/\omega_{Le}{}^2)\sqrt{2/\pi} \tag{188}$$

We can say that near the cyclotron frequencies, there exist cyclotron waves which intensively are absorbed by the plasma electrons. When the frequency is shifted, this absorption decreases and in the frequency range far from the resonance frequency, when $\omega >> |\omega \pm \Omega_e| >> k_z v_{Te}$ it becomes exponentially weak. Then from (179) we have

$$k^2c^2 = -(\omega_{Le}{}^2\omega/(\omega \pm \Omega_e))[1 - i\sqrt{\pi/2}((\omega \pm \Omega_e)/kv_{Te})e^{-((\omega \pm \Omega_e)^2/2k^2v_{Te}{}^2)} \tag{189}$$

From this equation we find the spectrum of cyclotron waves $(\omega + i\delta)$

$$\omega = \pm\Omega_e - \omega_{Le}{}^2\omega/k^2c^2, \ \delta = -\sqrt{\pi/2}(\omega_{Le}{}^2\omega/k^2c^2)(1/kv_{Te})\exp-((\omega \pm \Omega e)^2/2k^2v_{Te}{}^2) \tag{190}$$

*For simplicity we consider the waves only near electron cyclotron frequencies.

Sometimes these relations are presented in the optical language for the reflection index and absorption coefficient ($k=\omega n/c$, $n \to n+i\chi$). If $|\omega \pm \Omega_e| >> n\omega v_{Te}/c$, then

(191)

$$n^2 = \omega_{Le}^2 \, \omega(\omega \pm \Omega_e),\ \chi = (\sqrt{\pi}/8)(\omega_{Le}^2 c/\omega^2 v_{Te} n^2) \exp-((\omega \pm \Omega_e)^2 c^2/2n^2\omega^2 v_{Te}^2)$$

In the opposite limit, when $|\omega \pm \Omega_e| << n\omega V_{Te}/c$, this quantity occurs to be essentially complex

(192)

$$n^3 = i(\sqrt{\pi}/2)\omega_{Le}^2 c/\omega^2 v_{Te}$$

On the Figure 5 the dependence of complex reflection index n on the frequency shift $\omega-\Omega_e$ is presented

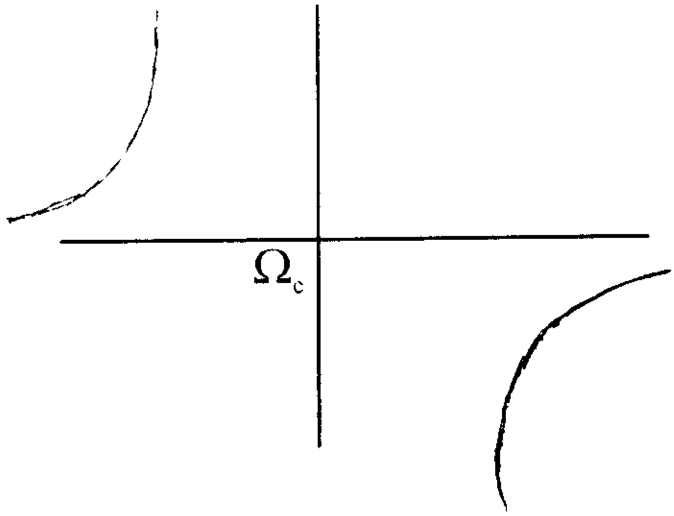

Figure 5

Let us now pass to the quasi longitudinal oscillations of magnetoactive plasma which exist under the conditions of $|\omega - n\Omega_\alpha| << kc$, or they can be considered as slow waves (in taking into account Doppler shift). In the statical case, $\omega \to 0$, from the expression (180) follows

(193)

$$\varepsilon(0,\mathbf{k}) = 1 + 1/(k^2 r_D^2),$$

which means that the electrostatic field of a point charged particle in magnetoactive plasma as well as in isotropic one is screened and the field penetration length is equal to the Debye length.

From the expression (180) it follows that the one particle cyclotron resonances at $\omega = n\Omega_\alpha$ take place for the longitudinal fields also. As a result near the cyclotron frequencies there exist the longitudinal cyclotron waves. Below we will show this for the purely electron plasma and transverse propagation of waves ($k_z = 0$). The dispersion equation for such waves looks as

(194)

$$\varepsilon = 1 + (\omega_{Le}2/k^2 v_{Te}^2)[1 - {}_n\Sigma(\omega/(\omega - n\Omega_e))A_n(k^2 v_{Te}^2/\Omega_e^2)] =$$

$$1 - 2 \, {}_{n=1}\Sigma^{\infty}(\omega_{Le}^2 n^2 \Omega_e^2/(k^2 v_{Te}^2(\omega^2 - k^2\Omega_e^2)))A_n(k^2 v_{Te}^2/\Omega_e^2) = 0$$

The solutions of this equation are known as Bernstein oscillations in honour of I. Bernstein who theoritically predicted them in 1959. They are presented on the Figure 6.

In conclusion of this section we will discuss very shortly the another branches of electromagnetic waves of magnetoactive plasma which every physicist must know. In the first turn let consider the oscillations of "cold" plasma, or in other words, the limiting case when

(195)

$$(\omega-n\Omega_{\alpha})\, k_z v_{T\alpha} \gg 1,\ k_{\perp}^2 v_{T\alpha}^2\, \Omega_{\alpha}^2 \ll 1$$

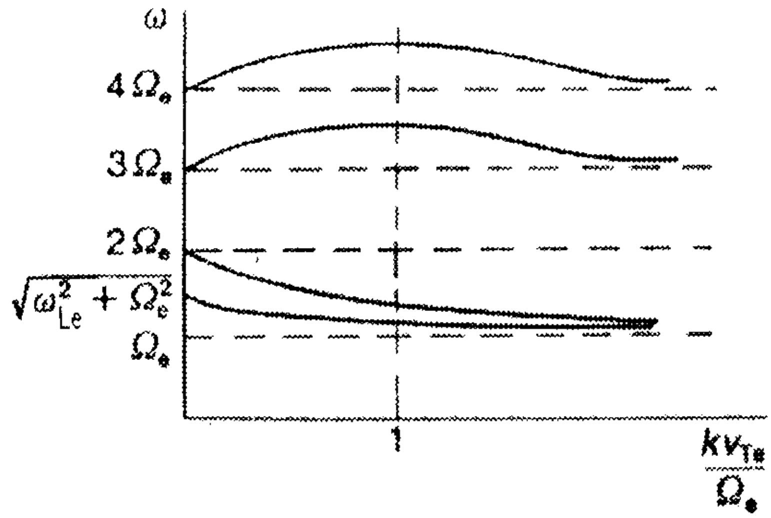

Figure 6

In accordance of these inequalities the phase velocities of waves in taking into account Doppler shift are supposed much larger than the thermal velocities of particles. The dielectric tensor (178) in this limit coincides with the well-known expression considered in the lecture 2 when the model of independent particles was discussed (of course for $v_\alpha \to 0$)

(196)

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_\perp & ig & 0 \\ -ig & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_{//} \end{pmatrix}$$

$$\varepsilon_\perp = 1 - {}_\alpha\Sigma\omega_{L\alpha}^2/(\omega^2 - \Omega_\alpha^2),\ g = \Sigma_\alpha\, \omega_{L\alpha}^2\, \Omega_\alpha/(\omega(\omega^2 - \Omega_\alpha^2)),\ \varepsilon_{//} = 1 - \Sigma\omega_{L\alpha}^2/\omega^2$$

In the lecture 2 the waves described by this tensor were investigated. By this reason here we will restrict only to the statement that inequalities (195) represent the validity of the results of these investigations and more generally the validity of the independent particles model for describing the properties of magnetoactive plasma.

The another simplest model which was considered in the lecture 3 is the Alfven one fluid MHD model for describing nonisothermal $T_e \gg T_i$ magnetoactive plasma. It can be easely shown that this model is working under the condition

(197)

$$\omega^2 \ll \Omega_i^2 \ll \omega_{Li}^2,\ k_\perp^2 v_{T\alpha}^2/\Omega_\alpha^2 \ll 1,\ v_{Ti} \ll \omega/k_z \ll v_{Te}$$

Then the tensor (178) looks as

(198)

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & \varepsilon_{yz} \\ 0 & \varepsilon_{yz} & \varepsilon_{zz} \end{pmatrix}$$

$\varepsilon_{yy} = \varepsilon_{xx} = \omega_{Li}^2\,\Omega i^2$, $\varepsilon_{zy} = -\varepsilon_{yz} = i\omega_{Le}^2 k_\perp k_z\,\omega\Omega_e$. $\varepsilon_{zz} = -\omega_{Li}^2/\omega^2 + \omega_{Le}^2/k_z^2 v_{Te}^2$

It must be noted that here we completely neglect the dissipative processes due to the Cherenkov mechanism of wave absorption. Only in this case the expressions (198) provide exact correspondence to the ideal MHD model considered in the Lecture 3.

* 15 Influence of particles collisions on the properties of magnetoactive plasma

Passing to the collisions of particles we wish to notice that as above we will restrict ourselves by considering only a weakly ionized plasma and therefore only the Vlasov equation with BGK collision integral will be solved. This equation for a small perturbation of distribution function looks as

(199)

$$+(\omega - \mathbf{k}\cdot\mathbf{v})\delta f_\alpha + ie_\alpha \mathbf{E}\partial f_{0\alpha}/\partial \mathbf{p} - i\Omega_\alpha \partial\delta f_\alpha/\partial\varphi = -i\nu_\alpha(\delta f_{\alpha 0} - f_{\alpha n}\int d\mathbf{p}\delta f_\alpha)$$

$$f_{\alpha 0} = (1/(2\pi m_\alpha T_\alpha)^{3/2})\exp(-m_\alpha v^2/2T_\alpha$$

This Voltera type integral equation can be easely solved. We will present here the result of the solution and calculation of the dielectric tensor. Moreover as above we will write only $\varepsilon_\perp$ and ε. The quantity $\varepsilon_\perp$ is equal

(200)

$$\varepsilon_\perp(\omega,\mathbf{k}) = 1 + (\Sigma_\alpha\,[(\omega+i\nu_\alpha)/\omega)[\varepsilon_{xx}{}^{(\alpha)} - 1 \pm i\varepsilon_{xy}{}^{(\alpha)}]=$$

$$1 - {}_\alpha\Sigma(\omega_{L\alpha}{}^2/\omega\,[(\omega+i\nu_{\alpha 0})\pm\Omega_\alpha]\Im((\omega\pm\Omega_\alpha+i\nu_{\alpha 0})/k_z v_{T\alpha})$$

Here $\varepsilon_{xx}{}^\alpha$ and $\varepsilon_{xy}{}^\alpha$ are the components of dielectric tensor of collisionless plasma with changing $\omega \to \omega + i\nu_{\alpha 0}$.

This quantity describes the purely transverse fields depending only on z (or $k_\perp = 0$, $k_z \neq 0$). For the quantity $\varepsilon(\omega,\mathbf{k})$ which describes purely potential field ($\mathbf{E} = -\nabla\phi$) we have ($k_\perp \neq 0$, $k_z \neq 0$)

(201)

$$\varepsilon(\omega,\mathbf{k}) = 1 + {}_\alpha\Sigma(\omega_{L\alpha}{}^2/k^2 v_{T\alpha}{}^2) \times$$
$$\{1 - {}_n\Sigma(\omega+i\nu_{\alpha 0}/(\omega+i\nu_{\alpha 0}-n\Omega_\alpha))A_n(k_\perp{}^2 v_{T\alpha}{}^2/\Omega_\alpha{}^2)\Im((\omega+i\nu_{\alpha 0}-n\Omega_\alpha)/k_z v_{Te})\} \times$$
$$\{1 - {}_n\Sigma i\nu_{\alpha 0}/(\omega+i\nu_{\alpha 0} - n\Omega_\alpha)\Im((\omega+i\nu_{\alpha 0}-n\Omega_\alpha)/k_z v_{Te})\}^{-1}$$

As in the previous section let us consider the transverse field behaviour near the electron cyclotron frequency, $|\omega - n\Omega_e| \ll \omega$. Substituting the expression (200) into the equation (185), we obtain that, if $\nu_e \ll k_z v_{Te} \sim v_{Te}/\lambda_{sk}$ (collisionless plasma), λ_{sk} is given by the relation (187).

However if the opposite inequality takes place (collisional plasma), from (185) we obtain the normalous skin-effect for field penetration

(202)

$$\lambda_{sk} = 1/\Im mk \sim (c^2 \nu_{e0}/\omega\omega_{Le}{}^2)^{1/2}$$

Quite analogical to (188) we can write this relation in spectral representation

$$\omega = -ik^2c^2\nu_{e0}/\omega_{Le}^2 \tag{203}$$

In conclusion of this section let us consider the behaviour of potential field in a collisional magnetoactive plasma, described by the expression (201). In the static limit, when $\omega \to 0$, independently from the ratio $\nu_{\alpha 0}/kv_{T\alpha}$ we have

$$\varepsilon(0,\mathbf{k}) = 1+1/k^2r_D^2 \tag{204}$$

This means that Debye screening of the potential field of static point charged particles takes place as it was shown for isotropic plasma above.
Analogical to the isotropic plasma can be considered the problem of particles diffusion also. For this aim the expression (201) must be written in the limit $\omega, kv_{T\alpha} << \nu_{\alpha 0}, \Omega_\alpha$

$$\varepsilon(\omega,\mathbf{k}) = 1+i_\alpha\Sigma\omega_{L\alpha}^2/k^2v_{T\alpha}^2(((k_\perp^2v_{T\alpha}^2\nu_{\alpha 0})/(\nu_{\alpha 0}^2+\Omega_\alpha^2))+k_z^2v_{T\alpha}^2/\nu_{\alpha 0}) \times$$
$$(\omega+i\nu_{\alpha 0}(k_\perp^2v_{T\alpha}^2/(\Omega_\alpha^2+\nu_{\alpha 0}^2)+k_z^2v_{T\alpha}^2/\nu_{\alpha 0}))^{-1} \tag{205}$$

The poles of this expression describe the monopolar diffusions of electrons and ions in a rare plasma ($\omega_{L\alpha}^2 << k^2v_{T\alpha}^2$)

$$\partial N_\alpha/\partial t - D_{\perp\alpha}\Delta_\perp N_\alpha - D_{//\alpha}\partial^2N_\alpha/\partial z^2=0 \tag{206}$$

Here $D_{\perp\alpha} = v_{T\alpha}^2\nu_{\alpha 0}/(\Omega_\alpha^2+\nu_\alpha^2)$, $D_{//\alpha} = v_{T\alpha}^2/\nu_{\alpha 0}$

represent the transverse and longitudinal monopolar diffusion coefficients correspondingly for α = e,i. At the same time in the opposite limit of dense plasma when $k^2v_{T\alpha}^2 << \omega_{L\alpha}^2$ the equation $\varepsilon(\omega,\mathbf{k})=0$ describes the ambipolar diffusion of particles in the magnetoactive plasma

$$\partial N/\partial t - D_{\perp a}\Delta_\perp N - D_{//a}\partial^2N/\partial z^2=0 \tag{208}$$

Therefore the coefficients of ambipolar diffusion are equal to

$$D_{\perp a} = (\nu_{e0}(v_{Ti}^2+v_s^2)/((\nu_{e0}\nu_{i0}+(\Omega_e\Omega_i)),\ D_{//a} = (v_{Ti}^2+v_s^2)/\nu_{i0} \tag{209}$$

It can be easily confirmed that there is the well-known Einstein relation between the static partial conductivities of plasma particles and diffusion coefficients (α = e,i)

$$D_{ij\alpha} = (T_\alpha/e_\alpha^2N_\alpha)\sigma_{ij\alpha}(0) \tag{210}$$

This relation is correct only in static limit and only in thermodynamic equilibrium. Then

$$\sigma_{\perp\alpha}(0) = e_\alpha^2N_\alpha\nu_{\alpha 0}/m_\alpha(\Omega_\alpha^2+\nu_{\alpha 0}^2),\ \sigma_{//\alpha}(0)=e_\alpha^2N_\alpha/m_\alpha\nu_{\alpha 0} \tag{211}$$

represents the transverse and longitudinal partial conductivities of charged particles of type α=e,i.

PART 2

Electromagnetic Properties and Stability Problems of Thermodynamically Nonequilibrium and Spatially Bounded Plasmas

LECTURE 8

Electromagnetic Properties of Spatially Bounded Plasmas

* 16 Surface waves in a cold collisionless plasma

For plasma bounded in space two new problems appear. the first one is the problem of new type of waves. propagating along the plasma surface and damping across the surface, and the second is the well-known Fresnel's problem - the problem of waves reflection and refraction from the plasma surface.

For solution of the first problem the knowledge of boundary conditions is needed. Here arise too much different possibilities : plasma confined by the strong external magnetic field, when the particles reflection from plasma surface has quite definite character, plasma confined by the infinitely sharp potential wall, from which the mirror reflection of particles takes place, plasma confined by the glass walls, on which ionization and recombination of particles take place and so on. Below we will consider the simplest cases when the problem of boundary conditions has the obvious solutions.

Let us begin our analyses from the case of cold plasma, described by the model of independent particles. The dielectric permittivity of such a plasma in the absence of external magnetic field is:

$$\varepsilon_{ij}(\omega,\mathbf{r}) = \varepsilon(\omega,\mathbf{r})\delta_{ij},\ \varepsilon(\omega,\mathbf{r}) = 1- \Sigma_\alpha \quad \omega^2{}_{L\alpha}(\mathbf{r})/\omega(\omega+i\nu_\alpha) \tag{212}$$

It is very important to be noticed that this expression is valid for the arbitrary inhomogeneous density distribution n_α (r). We will assume below a sharp boundary, or

$$n_\alpha(\mathbf{r}) = \begin{cases} n_{0\alpha} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{213}$$

Then we can write the field equations representing the solutions as A(x)exp(-(iωt+ik$_z$z)) (0z axis is orientated along the plasma surface) :

$$\begin{aligned} &k_zE_y +(\omega/c)B_x = 0, && k_zB_y - (\omega/c)\varepsilon(x)E_x (x) = 0 \\ &k_zE_x +i\partial E_z/\partial x - (\omega/c)B_y = 0 && k_zB_x +i\partial B_z/\partial x+(\omega/c)\varepsilon E_y=0 \\ &i\partial E_y/\partial x - (\omega/c)B_z = 0 && i\partial B_y/\partial x - (\omega/c)\varepsilon E_z = 0 \end{aligned} \tag{214}$$

The boundary conditions can be obtained by integrating these equations over x near the plasma surface 0-δ ≤ x ≤ 0+δ, where δ → 0. They look as

$$\{E_z\}_{x=0} =\{E_y=0\}_{x=0} = \{B_z\}_{x=0} = \{B_y\}_{x=0} =0 \tag{215}$$

Here we take into account that the fields **E** and **B** must be finite as physical quantities.

The equations (214) can be reduced to the two equations for Ez and Bz:

(216-a)

$$\varepsilon(\omega,x)\,[\,\partial^2E_z/\partial x^2 - k_z^2E_z + (\omega^2/c^2)\varepsilon E_z] = 0$$

(216-b)

$$\partial^2B_z/\partial x^2 - k_z^2B_z + (\omega^2/c^2)\varepsilon B_z = 0$$

The other components of **E** and **B** can be easily expressed in terms of E_z and B_z:

(217)

$Ex = -(ikz/\chi 2)\partial Ez/\partial x,$ $\quad By = -(i\omega/c\chi 2)\varepsilon\partial Ez/\partial x,$

$Ey = (i\omega/c\chi 2)\partial Bz/\partial x,$ $\quad Bx = -(ikz/\chi 2)\partial Bz/\partial x,$

where $\chi^2 = k_z^2-(\omega^2/c^2)\varepsilon(x)$. These equations are valid for $x \geq 0$ and $x \leq 0$ and match the solutions at $x = 0$. First of all it must be noticed that the equation (216-a) admits the solution

(218)

$$\varepsilon(\omega,x) = 0,$$

which corresponds to the localized bulk potential oscillations, studied above (lecture 6). The other equations are separated for components E_x, E_z, B_y, and B_x, B_z, E_y, corresponding to TM and to TE modes respectively. As it was mentioned above there exist two problems for this system: boundary problem or waves reflection and refraction and initial problem or own surface oscillations. In this section we will consider only the second one.

It is easy to show that only the equation (216-a) allows the existence of surface type solutions:

(219)

$$Ez = \begin{cases} C_1 \exp\left[-\sqrt{\left(k_z^2-\left(\omega^2/c^2\right)\varepsilon\right)}\,x\right] & \text{when } x \geq 0 \\ C_2 \exp\left[\sqrt{k_z^2-\left(\omega^2/c^2\right)}\,x\right] & \text{when } x \leq 0 \end{cases}$$

From validity of these solutions follows $\omega^2 < k_z^2c^2$ and $\varepsilon\omega^2 < k_z^2c^2$, or the waves are slow. After subtituting the solution (219) into the boundary conditions (215) for E_z and B_y we obtain the dispersion equation for surface waves (for TM modes)

(220)

$$\sqrt{k_z^2c^2-\omega^2\varepsilon} + \varepsilon\sqrt{k_z^2c^2-\omega^2} = 0,$$

where $\varepsilon(\omega)$ is given by the expression (212) for $x \geq 0$. The solution of this equation for purely electron plasma and in the limit $\nu e \to 0$ is presented in the Fig.7.

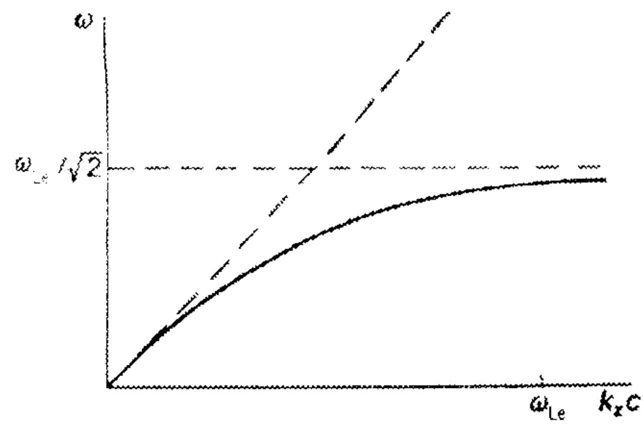

Figure 7

In the long-wave limit, when $\omega \approx k_z c << \omega_{Le}$, the phase velocity of waves tends to the light speed and they present the well-known Tsenike waves on the surface of metals. In the opposite limit (short-waves), $\omega \approx \omega_{Le}/\sqrt{2} << k_z c$, the waves become practically potential (longitudinal). Therefore in this limit these waves can be described by the Poisson's equation:

(221)

$$\mathrm{div}\mathbf{D} = (\partial/\partial r_i)(\varepsilon\partial\phi/\partial r_i) = \partial/\partial x(\varepsilon\partial\phi/\partial x) - k_z^2\, \varepsilon\phi = 0$$

with boundary conditions (222)

$$\{\phi\}_{x=0} = \{\varepsilon\partial\phi/\partial x\}_{x=0} = 0$$

From this system we obtain the following spectrum of oscillations

(223)

$$\varepsilon = -1 \Rightarrow \omega = \omega_{Le}/\sqrt{2} - i\nu e/2$$

Here we take into account the electron collisions also,which lead to the damping of surface waves in a cold isotropic plasma.

Let us now consider the surface waves in a cold magnetoactive plasma under the condition that the external magnetic field $\mathbf{B}_0$ is parallel to the plasma surface. For simplicity we will restrict ourselves by considering only potential surface waves. The dielectric tensor for a collisionless magnetoactive plasma looks as (see lecture 7)

(224)

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_\perp & ig & 0 \\ ig & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_{//} \end{pmatrix}$$

where

(225)

$$\varepsilon_\perp = 1 - {}_\alpha\Sigma(\omega_{L\alpha}^2(x)/(\omega^2 - \Omega_\alpha^2),\ g = {}_\alpha\Sigma(\omega_{L\alpha}^2(x)\Omega_\alpha)/\omega(\omega^2-\Omega_\alpha^2)$$
$$\varepsilon_{//} = {}_\alpha\Sigma\omega_{L\alpha}^2(x)/\omega^2$$

Representing all perturbations as $A(x)\exp(-i\omega t+ik_y y+ik_z z)$, from the Poisson's equation we obtain

$$\partial/\partial r_i \varepsilon_{ij}(\partial\phi/\partial r_j) = \partial/\partial x(\varepsilon_\perp(\partial\phi/\partial x)) - k_z^2\varepsilon_{//}\phi - k_y\phi\partial g/\partial x - k_y^2\varepsilon_\perp\phi = 0 \tag{226}$$

From this the following boundary conditions follow

$$\{\phi\}_{x=0} = 0, \; \{\varepsilon_\perp \; \partial\phi/\partial x - k_y g\phi\}_{x=0} = 0 \tag{227}$$

After substituting the solutions

$$\phi = \begin{cases} C_1 \exp\left[-\left(\sqrt{k_y^2 + k_z^2\varepsilon_{//}/\varepsilon_\perp}\right)x\right] & \text{if } x \geq 0 \\ C_2 \exp\left[\sqrt{k_y^2 + k_z^2}\; x\right] & \text{if } x \leq 0 \end{cases} \tag{228}$$

into the boundary conditions (227) we obtain the dispersion equation for the surface waves

$$\varepsilon_\perp\sqrt{k_y^2 + k_z^2\varepsilon_{//}/\varepsilon_\perp} + k_y g + \sqrt{k_y^2 + k_z^2} = 0 \tag{229}$$

Two very important consequences follow from this equation. Firstly we see that the surface waves with $\omega < \Omega_e$ can exist in a magnetoactive plasma only if the magnetic field is finite (we consider only purely electron oscillations), and secondly these waves seem to be one direction because

$$\omega(k_y) \neq \omega(-k_y) \tag{230}$$

Specially in the frequency range $\omega \ll \Omega_e$ (but $\omega \gg \Omega_i, \omega_{Li}$) we have

$$\omega = (k_y/|k_y|)(\omega_{Le}^2/(\Omega_e(2+\omega_{Le}^2/\Omega_e^2)) \tag{231}$$

Such type of waves were firstly observed in the solid state plasmas. They play very important role for diagnostics of solid state surface. In the next lecture it will be shown that they are very important for gaseous plasmas also - for the problem of plasma confinement.

* 17 Kinetic theory of plasma surface waves

For developing the kinetic theory of plasma surface waves let us use the system of Vlasov - Maxwell equations. Supposing that the equilibrium distribution is Maxwellian

$$f_{0\alpha} = n_{0\alpha}(x)/(2\pi m T_\alpha)^{3/2} \; \exp(-m_\alpha v^2/2T_\alpha) \tag{232}$$

where $n_{0\alpha}(x)$ is given by (213). The equation for the small perturbations of type $\delta f_\alpha(x) \exp[-(i\omega t + ik_z z)]$ can be presented as

(233)

$$-i(\omega - k_z v_z)\delta f_\alpha + v_x \partial(\delta f_\alpha)/\partial x + e_\alpha \mathbf{E} \partial f_{0\alpha}/\partial \mathbf{p} = 0$$

As a boundary condition for $\delta f\alpha(x)$ we take the simplest one corresponding to the mirror reflections of particles from the plasma surface

(234)

$$\delta f_\alpha(0, v_x > 0) = \delta f_\alpha(0, v_x < 0)$$

Then representing $\delta f_\alpha(x, v_z)$ as

(235)

$$\delta f_\alpha(x, v_x) = \delta f_\alpha^{+}(x, v_x) + \delta f_\alpha^{-}(x, v_x)$$

where

$$\delta f_\alpha^{\pm}(x, v_x) = \begin{cases} \delta f_\alpha(x, v_x > 0) \\ \delta f_\alpha(x, v_x < 0) \end{cases}$$

for $\delta f_\alpha^{\pm}(x, v_x)$ we obtain the obvious conditions

(236)

$$\delta f_\alpha^{+}(0, v_x) = \delta f_\alpha^{-}(x, v_x)$$

$$\delta f_\alpha^{-}(\infty, v_x) = 0$$

The last relation follows from the requirement $E(\infty) \to 0$, taken into account for the surface waves. Thus the boundary problem is formulated: for $\delta f^{-}(x, v_x)$ we have the equation (233) with zero boundary condition at $x \to \infty$ (the second relation (236)) whereas $\delta f^{+}(x, v_x)$ satisfies the equation (233) with the first relation (236) as the boundary condition. Omitting the details of calculations we will give the final result for the current density:

(237)

$$J_i(x) = {}_\alpha\Sigma e_\alpha \int d\mathbf{p} v_i \delta f(x, v_x) = \int dx' [k_{ij}(|x-x'|) + k_{ij}(|x+x'|)] E_j(x')$$

where

(238)

$$k_{ij}(|x|) = -\Sigma_\alpha e_\alpha^2 \int dp\, (v_i/v_x)(\partial f_{0\alpha}/\partial p_j) \exp[\, i(|x|/v_x)(\omega - k_z v_z)]$$

It is obvious that $j(x) \neq 0$ only in a plasma. Substituting the expression (237) into the field equations and usual boundary conditions (continuity of tangential field components) after some calculations the following dispersion equation for the surface waves of TM type can be obtained

(239)

$$\surd((k_z^2c^2/\omega^2)-1) + (2\omega/\pi c)_0\int^\infty (dk_x/k^2)[(k_z^2c^2/\omega^2\varepsilon^l(\omega,k)) - (k_x^2c^2/(k^2c^2-\omega^2\varepsilon^{tr}(\omega,k)))] = 0$$

Here $k^2 = k_x^2 + k_z^2$ and $\varepsilon^{tr,l}(\omega,k)$ are the transverse and longitudinal dielectric permittivities (see lecture 6).

It must be noticed that the equation (239) is valid for a collisional plasma too. If the spatial dispersion is neglected, $\varepsilon^{tr}(\omega,\mathbf{k}) = \varepsilon^l(\omega,\mathbf{k}) \to \varepsilon(\omega)$, it passes to the equation (229) considered above.

Below we will investigate the equation (239) only for slowing potential surface waves and therefore will take the limit $c \to \infty$. Then this equation takes the form

(240)

$$1 + 2/\pi \int_0^\infty ((|k_z| dk_x) /(k^2\varepsilon^l(\omega,k))) = 0$$

Let us begin the analysis of this equation assuming that $\omega >> k_z v_{Te}$ and therefore

$$\varepsilon^l(\omega,k) = 1-\omega_{Le}^2/\omega^2 + i\sqrt{(\pi/2)}(\omega\omega_{Le}^2/k^3 v_{Te}^3) \exp[-\omega^2/2k^2 v_{Te}^2] \qquad (241)$$

Substituting this expression into the (240), we obtain

$$2-(\omega_{Le}^2/\omega^2)(1 - i\sqrt{(8/\pi)}(|k_z| v_{Te})/\omega) = 0 \qquad (242)$$

The obvious solution of (242) is ($\omega \rightarrow \omega + i\delta$)

$$\omega = \omega_{Le}/\sqrt{2}, \quad \delta = -\sqrt{(2/\pi)}|k_z| v_{Te} \qquad (243)$$

On the contrary to the bulk waves with exponentially weak damping the surface waves occur much more intensively damping. This fact is a result of integration in (240) over all k_x , or in other words, the transverse coordinate components consists of all dependence and therefore the slowing waves also, which leads to the increasing of their damping.
For nonisothermal plasmas with $T_e >> T_i$ the possibility of existence of ion-accoustic surface waves appears. Really if the expression

$$\varepsilon^l = 1-\omega_{Le}^2/\omega^2 + (\omega_{Le}^2/k^2 v_{Te}^2)[1 + i\sqrt{(\pi/2)}(\omega/k_z v_{Te})] \qquad (244)$$

will be substituted in (240) after some calculations the following spectrum for surface waves in the phase velocity range $v_{Ti} << \omega/k_z << v_{Te}$ can be obtained

$$\omega^2 = \begin{cases} k_z^2 v_s^2 << & \omega_{Li}^2 \\ \omega_{Li}^2/2 >> & k_z^2 v_s^2 \end{cases} \qquad (245\text{-a})$$

$$\delta/\omega = \begin{cases} \sqrt{\pi m / 8M} & \text{if} \quad k_z^2 v_s^2 << \omega_{Li}^2 \\ \frac{1}{6}\sqrt{m/\pi M}\, \omega_{Li}^3 / k_z^3 v_s^3 & \text{if} \quad k_z^2 v_s^2 >> \omega_{Li}^2 \end{cases} \qquad (245\text{-b})$$

Here also we see the increasing of surface waves damping in comparison with their bulk analogous.

* 18 Plasma waveguides

The semibounded plasmas and the surface waves were considered above. Let us now consider the completely bounded plasmas and bulk waves in them. We will restrict ourselves by considering only fast waves when thermal motion of particles can be neglected. In other words, we will consider only cold plasma in a cylindrical geometry, which is interesting from many applications point of view.

Let us begin from isotropic collisionless plasmas, supposing that

(246)

$$n_{0\alpha}(r) = \begin{cases} n_{0\alpha} = const & \text{if} \quad r \leq R \\ 0 & \text{if} \quad r \geq R \end{cases}$$

where R is the radius of a metallic waveguide. The field equations in this case can be reduced to the following equations

(247-a)

$$\varepsilon(\omega,r)[\Delta_{\perp}E_z - k_z^2E_z + (\omega^2/c^2)\varepsilon(\omega,r)E_z] = 0$$

(247-b)

$$\Delta_{\perp}B_z - k_z^2B_z + (\omega^2/c^2)\varepsilon(\omega,r)B_z = 0$$

where $\varepsilon = 1 - \Sigma\omega_{L\alpha}^2/\omega^2$. The boundary conditions for this system look as

(248)

$$E_z \,|_{r=R} = 0, \quad E_\varphi = \partial B_z/\partial r \,|_{r=R} = 0$$

First of all from (247-a) follows that in the plasma waveguide as in semibounded plasma there exist the bulk Langmuir oscillations with dispersion equation (218). Besides there exist also two branches of TM and TE waves for which the solutions of the equations (247) look as

(249)

$$E_z = C_1\Im_e(i\chi r), \qquad B_z = C_2\Im_e(i\chi r)$$

where $\chi^2 = k_z^2 - (\omega^2/c^2)\,\varepsilon$. Substituting these solutions into the boundary conditions (248) one can easily obtain for TM and TE waves correspondingly:

(250)

$$\omega^2 = k_z^2c^2 + \omega_{Le}^2 + \mu_{es}^2c^2/R^2, \qquad \omega^2 = k_z^2c^2 + \omega_{Le}^2 + \mu'_{es}{}^2c^2/R^2$$

where $\Im_e(\mu_{es}) = 0$ and $\Im'_e(\mu'_{es}) = 0$. In the Fig. 8 the spectra (250) are presented. We see that these waves can propagate only if $\omega > \omega_{cr}$. The minimal values of critical frequencies are determined by $(\mu_{es})min = 2.4$ and $(\mu'_{es})min = 1.7$.

Quite another situation takes place in a magnetoactive plasma. When the magnetic field increases the critical frequency of TE modes decreases and tends to the vacuum critical frequencies in waveguides, $\omega_{cr0} = \mu'_{es}c/R$.

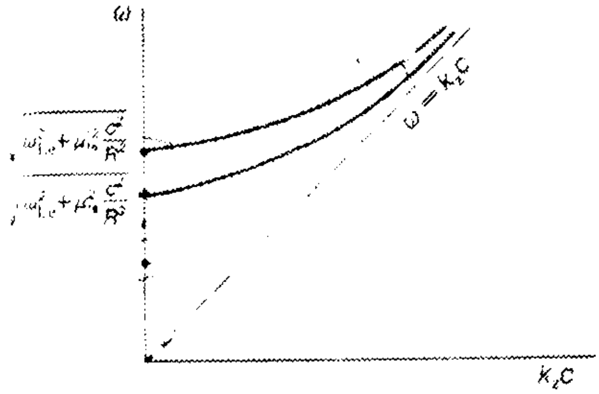

Figure 8

$$\omega^2_{aE} = \omega_{Le}^2 + \mu_{es}^2 c^2/R^2$$

$$\omega^2_{aB} = \omega_{Le}^2 + \mu'_{es}{}^2 c^2/R^2$$

As about the TM modes (E_z, E_r, B_ϕ) they occur to be coupled with longitudinal oscillations and satisfy the following equation for E_z:

$$\Delta_\perp E_z - (k_z^2 - \omega^2/c^2)\varepsilon E_z = 0 \qquad (251)$$

This equation in taking into account boundary conditions (248) leads to the following dispersion equation for TM-mode in a magnetoactive purely electron plasma:

$$\mu^2_{es} c^2/R^2 + (k_z^2 c^2 - \omega^2)(1 - \omega_{Le}^2/\omega^2) = 0 \qquad (252)$$

The solutions of this equation are presented in the Fig.9, which shows that there exist two branches of waves: fast one with $\omega/k_z > c$ and slow one with $\omega/k_z < c$

$$\omega_{cr} = \sqrt{\omega_{Le}^2 + \frac{\mu_{es}^2 c^2}{R^2}}$$

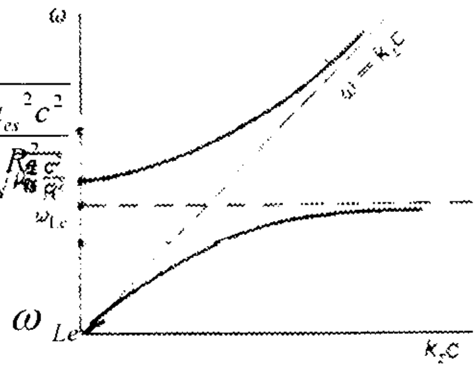

Figure 9

For fast waves the critical frequency $\omega_{cr} > \omega_{Le}$ and $\omega \geq \omega_{cr}$, whereas for slow waves there exists the maximal frequency equal to ω_{Le}.

* 19 The Fresnel's problem

Let us turn to the semibounded plasma and consider the Fresnel's problem, or the problem of waves reflection and refraction from the plasma surface. We will restrict ourselves by consideration of S-polarization of the incident waves, because namely in this case there arises an interesting phenomenon of waves transformation to the plasma surface. In this case E_z, E_r, and B_ϕ are nonzero. For E_z component we can write

(253)

$$E_z = \exp(-i\omega t + ik_z z) \times \begin{cases} E_{z0}\exp(ik_{x0}x) + E_{z1}\exp(-ik_{x0}x) & \text{when } x \le 0 \\ E_{z2}\exp(ik_{x1}x) & \text{when } x \ge 0 \end{cases}$$

where $k_{x0} = \surd((\omega^2/c^2)-k_z^2)$ and $k_{x1} = \surd((\omega^2\varepsilon/c^2)-k_z^2)$. The incident waves amplitude E_{z0} is supposed to be given, whereas the amplitude of reflection E_{z1} and refraction E_{z2} waves must be found. Using the boundary conditions (215) and taking into account the relations (217), after some calculations we obtain:

(254)

$$E_{z2}/E_{z0} = \left(2\sqrt{(\omega^2\varepsilon/c^2)-k_z^2}\right)/\left(\sqrt{(\omega^2\varepsilon/c^2)-k_z^2} + \varepsilon\sqrt{\omega^2/c^2-k_z^2}\right),$$

$$r = E_{z1}/E_{z0} = (E_{z2}/E_{z0}) - 1$$

The quantity $|r|^2$ is known as reflection coefficient, whereas the quantity $A = 1-|r|^2$ is called as refraction one. It must be noted that the denominator of (254) all times is nonzero. Thus the excitation of surface waves is impossible. Moreover, when $\varepsilon(\omega)<0$, then the complete reflection of incident waves takes place. Really, if we introduce the surface impedance Z by the relation

(255)

$$r = -(1-\omega Z/4\pi)/(1+\omega Z/4\pi), \quad Z = (4\pi/c)E_z(0)/B_v(0) = (4\pi i/\omega\varepsilon)\sqrt{k_z^2 - \omega^2\varepsilon/c^2}$$

then it is easy to see that $\varepsilon<0$ corresponds to the purely imaginary Z. As a result $|r|^2=1$ and we have complete reflection of incident waves.

The complete reflection of waves, when $\varepsilon(\omega)<0$, takes place only when the plasma boundary is very sharp. But if we take into account the finite size of boundary inhomogeneities the reflection occurs to be noncomplete and the surface wave absorption takes place. For convincing this let us consider a thin inhomogeneous layer near the plasma surface $0 \le x \le a$ and suppose that $\omega < \omega_{Le}(a) = \text{const}$. Then in this layer there exists the point x_0 where

(256)

$$\varepsilon(x_0) = 1 - \omega_{Le}^2(x_0)/\omega^2 = 0$$

Below it will be shown that in this point the incident wave will excite the resonance plasma oscillations and as a result significant incident waves absorption will take place. The field equation in the case of inhomogeneous plasma looks as (compare with (216a))

(257)

$$\partial/\partial x[(\varepsilon(x)/((\omega^2\varepsilon(x)/c^2)-k_z^2))\partial E_z/\partial x] + \varepsilon(x)E_z = 0$$

Outside of layer ($x\le 0$, $\varepsilon=1$; $x\ge a$, ε=const.) the solutions of this equation can be written as (253). At the same time, in the layer taking into account that $a/\lambda_0 = a\omega/c \ll 1$ we obtain from (257):

(258)

$$E_z(x) = C_2 + C_1 \int_0^x (\chi^2(x')dx'/\varepsilon(x'))$$

where $\chi^2(x)=k_z^2-\omega^2\varepsilon(x)/c^2$. After integration taking into account that $\varepsilon(x_0)=0$ we have

$$E_z(a)=E_z(0)+(\pi c/\omega)B_y(0)(k_z^2/|d\varepsilon(x_0)/dx|),\ B_y(a)=B_y(0) \tag{259}$$

On the other hand , according to (217)

$$E_z(0)/B_y(0)=cZ/4\pi,\ Z=(4\pi/\omega)[(i\chi(a)/\varepsilon(a))+\pi k_z^2/|d\varepsilon(x_0)/dx|] \tag{260}$$

The surface impedance Z occurs to be complex with nonzero real part. As a result the reflection coefficient $|r|^2\neq 1$ and the wave absorption is of the order of

$$A=1-|r|^2=(4\pi k_z^2/\chi_0|d\varepsilon(x_0)/dx|)\approx(4\pi c/\omega)(1/|d\varepsilon(x_0)/dx|)\approx 2a/\lambda<<1 \tag{261}$$

LECTURE 9

Stability Problem of Plasma Magnetic Confinement

* 20 Dielectric permittivity of inhomogeneous plasma confined by magnetic field

Plasma magnetic confinement is one of the most important applied problem in plasma physics and nuclear fusion. By this reason we will consider this problem in detail. Real plasma confinement systems have the toroidal forms. However the linear sizes of experimental devices are much larger than all the characteristic sizes of plasmas, such as Debye and Larmor lengths, characteristic sizes of plasma inhomogeneities and so on. Therefore with sufficiently accuracy we can consider the flat geometry instead of cylindrical or torodoidal. Thus, we suppose that external magnetic field is parallel to the OZ axis, whereas the inhomogeneity of plasma is directed along the OX axis. Then for equilibrium distribution function $f_{0\alpha}$ we can write

$$v_x \partial f_{0\alpha}/\partial x - \Omega_\alpha \partial f_{0\alpha}/\partial \varphi = 0 \qquad (262)$$

The solution of this equation is an arbitrary function of the characteristic

$$dx/(v_\perp \cos\varphi) = -d\varphi/\Omega_\alpha(x) \Rightarrow v_\perp \sin\varphi + {}^x\!\int dx'\Omega_\alpha(x') = \text{const} = C \qquad (263)$$

Thus, the solution of equation (262) is

$$f_{0\alpha}(\varepsilon,C) = (1+(v_\perp \sin\varphi/\Omega_\alpha)\ \partial/\partial x)\, f_{\mu\alpha}(x)$$
$$f_{\mu\alpha}(x) = (n_{0\alpha}(x)/(2\pi m_\alpha T_\alpha)^{3/2})(\exp(-m_\alpha v^2/2T_\alpha(x)) \qquad (264)$$

Here we assumed that the particle Larmor radii are small in comparison with plasma inhomogeneity size L_0 .

$$(v_{T\alpha}/\Omega_\alpha)1/L_0 << 1 \qquad (265)$$

Substituting (264) into the Maxwell equation

$$\mathrm{rot}\mathbf{B}_0 = (4\pi/c)\mathbf{j}_0 = (4\pi/c)_\alpha\Sigma e_\alpha \int d\mathbf{p}\mathbf{v}_\alpha f_{0\alpha}(x) \qquad (266)$$

we easely obtain the plasma equilibrium condition

$$\partial/\partial x(B_0^2/8\pi + {}_\alpha\Sigma n_{0\alpha}T_\alpha) = 0 \qquad (267)$$

which must be completed by the condition of plasma neutrality ${}_\alpha\Sigma e_\alpha n_{0\alpha} = 0$
The physical sense of considered equilibrium is clear from the Fig.10.

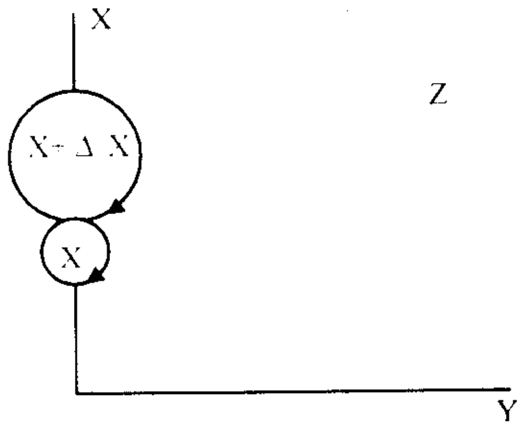

Figure 10

From this figure it follows

$$j_y = {}_\alpha\Sigma j_{y\alpha} = {}_\alpha\Sigma e_\alpha[n_{0\alpha}(x+\Delta x)v_{T\alpha} - n_{0\alpha}(x)v_{T\alpha}] = {}_\alpha\Sigma e_\alpha \Delta x(\partial n_{0\alpha} v_{T\alpha}/\partial x) = \Sigma(e_\alpha v_{T\alpha}^2/\Omega_\alpha)(\partial n_{0\alpha}/\partial x) \equiv {}_\alpha\Sigma e_\alpha n_{0\alpha}^2\, v_{dr\alpha} \tag{268}$$

Here $v_{dr\alpha}$ is the effective drift velocity

$$v_{dr\alpha} \approx (v_{T\alpha}^2/\Omega_\alpha)1/L_0 << v_{T\alpha} \tag{269}$$

Thus in the plasma confined by the external magnetic field the transverse current arises. However this current is not the result of real motions of particles, it is result of non-compensation of the particles Larmor currents in an inhomogeneous plasma. Therefore this current is called Larmor current and drift velocity (269) Larmor drift velocity.
Plasma with current may occur to be unstable because the particles distribution function (264) is thermodynamically nonequilibrium. The deviation of distribution function from equilibrium one is small but namely due to this deviation, it arises the current in a plasma. The characteristic time of developing instability may be estimated by the following way: using the expression (269) one can compose the quantity ω_{dr} with frequency dimension $\omega_{dr} \sim k_y v_{dr}$ and as a result with time dimension $\tau_{dr} \sim 1/\omega_{dr} \sim \Omega L_0^2/v_T^2$. Namely this quantity occurs to be the characteristic time of instability for the plasma magnetic confinement and this will be shown below. But before let's carry out some numerical estimations. In real thermonuclear devices $B_0 \sim 50\text{-}100$KG, $T_e \sim T_i \sim 10$kev, $L_0 \sim 10$cm, $\rho_e << \rho_i \sim v_{Ti}/\Omega_0 \approx 0.1\text{-}0.2$ cm and $\omega_{dr} \sim 10^5 s^{-1}$. Thus the characteristic time of drift instabilities is $\sim 10^{-5}$ s. The instabilities create low frequency fields which can lead to the anomalous diffusion of particles across the magnetic field. Just for this reason the drift instabilities seem to be very dangerous for the problem of magnetic plasma confinement.
For investigation of drift instabilities it is necessary to calculate the dielectric permittivity and analyze the spectra of small perturbations. But before let us discuss the restrictions which arise here.
First of all we must notice that the Larmor drifts and therefore drift instabilities are localized in the region of plasma inhomogeneity, or their spatial scale is of the order of L_0. If we will consider the oscillation with $\lambda_x \sim 1/k_x \sim 1/k_y << L_0$, then the geometrical optics approximation

occurs to be valid. In this approximation all calculations can be carried out as in a homogeneous plasma using the local description.
The second restriction concerns the ratio of plasma pressure to the magnetic field pressure. In the thermonuclear devices with magnetic confinement $n_e \sim n_i \sim 10^{14}$ to 10^{15} cm^{-3}. Therefore

(270)

$$\beta = 8\pi\Sigma n_\alpha T_\alpha / B_0^2 \leq 10^{-2} << 1$$

As a result

(271)

$$(d \ln nT/dx)/d/dx(\ln B_0^2/8\pi) \sim L_\beta / L_0 \sim B_0^2/8\pi\Sigma n_\alpha T_\alpha >> 1$$

This means that magnetic field is practically homogenous and below we will accept this condition. At the same time this condition means that the electric field of perturbations is practically potential - magnetic field of perturbations is neglegible. But the potential perturbations may be described by the much more simple equation, which is

(272)

$$\varepsilon(\omega,\mathbf{k},x) = (k_i k_j/k^2)\varepsilon_{ij}(\omega,\mathbf{k},x) = 0$$

Such approximation was used in homogeneous plasma and it is valid for inhomogeneous plasmas also but only under condition of geometrical optics approximation when $\lambda_x \sim 1/k_x << L_0$.
In the geometrical optics approximation the solution of Vlasov's equation for perturbation of type $\delta f_\alpha(x)\exp(-i\omega t+ik_y y+ik_z z)$

(273)

$$(\omega-k_y v_y-k_z v_z)\delta f_\alpha + iv_x\partial\delta f_\alpha/\partial x - i\Omega_\alpha\partial\delta f_\alpha/\partial\varphi = -ie_\alpha\{\mathbf{E}+(1/c)[\mathbf{v}*\mathbf{B}]\}\partial f_{0\alpha}/\partial \mathbf{p}$$

can be presented as $\exp(+i\int^x k_x dx)$ and the spatial derivatives of $k_x(\omega,x)$ are neglected. Then for $\delta f_\alpha(k,x)$ we obtain from (273)

(274)

$$\delta f_\alpha(k,x) = (e_\alpha/\Omega_\alpha)\left({}_{\infty}\int^{\varphi} d\varphi'[(1-\mathbf{kv}/\omega)\delta_{ij}+k_i v_j/\omega]_{\varphi'})\partial f_{0\alpha}(\varphi',x)/\partial p_j \times E_i(\omega,k)\exp[1/\Omega_\alpha \, {}_{\varphi}\int^{\varphi'} d\varphi''(\omega-\mathbf{kv})_{\varphi''}]\right)$$

Using this expression and substituting it in the distribution (264) we can calculate the induced current and conductivity of plasma and then dielectric permittivity. Here we will give only the low frequency limit of dielectric permittivity, when $\omega << \Omega_\alpha$:

(275)

$$\varepsilon(\omega,\mathbf{k},x) = 1+{}_\alpha\Sigma(\omega_{L\alpha}^2/k^2 v_{T\alpha}^2)\{1-[1-(k_y v_{T\alpha}^2/\omega\Omega_\alpha)((\partial \ln N_\alpha/\partial x)+(\partial T_\alpha/\partial x)\partial/\partial T_\alpha)]\times A_0(k_\perp^2 v_{T\alpha}^2/\Omega_\alpha^2)\Im(\omega/k_z v_{T\alpha})\}$$

It must be noticed that the drift instability takes place only if $\omega \leq \omega_{dr\alpha} \sim k_y v_{dr\alpha}$ and as $v_{dr\alpha} \sim v_{T\alpha}^2/\Omega_\alpha L_0 << v_{T\alpha}$, then $\omega_{dr\alpha} \sim v_{T\alpha}^2/\Omega_\alpha L_0^2 << \Omega_\alpha$. By this reason we will restrict ourselves by consideration of frequency range $\omega << \Omega_\alpha$. In other words $\omega \leq 10^{-5} s^{-1}$.
Below these inequalities will be taken in mind.
Let us finally generalize the expression (275) by taking into account the particles collisions. First of all it must be noticed that distribution (264) which consists of the drift motions of particles is valid only if $\nu_\alpha << \Omega_\alpha$. As about ratio ν_α/ω then it may be arbitrary.

Under these conditions BGK collision integral leads to the following result

(276)

$$\varepsilon(\omega,\mathbf{k},x)=1+{}_\alpha\Sigma(\omega_{L\alpha}^2/k^2v_{T\alpha}^2)\{1-[1-(k_yv_{T\alpha}^2/(iv_\alpha+\omega)\Omega_\alpha)((\partial\ln N_\alpha/\partial x)+(\partial T_\alpha/\partial x)\partial/\partial T_\alpha)]\times A_0(k_\perp^2v_{T\alpha}^2/\Omega_\alpha^2)\Im((\omega+iv_\alpha)/k_zv_{T\alpha})\}\times [1-iv_\alpha/(\omega+iv_\alpha)A_0(k_j^2v_\alpha^2/\Omega_\alpha^2)\Im((\omega+iv_\alpha)/k_zv_{T\alpha})]^{-1}$$

This expression generalizes the formula (275)

* 21 Drift instabilities of magnetic confined plasmas

Before beginning the analyses of drift instabilities let us notice that in the static limit, $\omega\to 0$, despite the nonequilibrium of plasma the expressions (275) and (276) take one form

(277)

$$\varepsilon(0,\mathbf{k})=1+{}_\alpha\Sigma\omega_{L\alpha}^2/k^2v_{T\alpha}^2=1+1/k^2r_d^2$$

which corrèponds to the Debye screening of static potential field in such a plasma. Therefore below we will consider oscillations in the frequency ranges

(a)

$$v_i,\ k_zv_{Ti}<<\omega<<k_zv_{Te},\qquad v_e<<k_zv_{Te}$$

(b)

$$v_i,\ k_zv_{Ti}<<\omega<<(k_z^2v_{Te}^2)/v_e,\qquad k_zv_{Te}<<v_e$$

The first corresponds to the collisionless plasma whereas the second represents the collisional plasma. Besides we will restrict ourselves by considerating only long wavelength oscillations with $k_\perp v_{Ti}<<\Omega_i$. They seem to be very dangerous for the problem of plasma magnetic confinement. The dielectric permittivity and dispersion equation under these conditions look as:

(278)

$$\varepsilon(\omega,k,x)=1+(\omega_{Li}^2/k^2v_s^2)\,[1+(k_yv_s^2/\omega\Omega_i)\partial\ln N/\partial x-(k_z^2v_s^2/\omega^2)\times(1-(k_yv_{Ti}^2/\omega\Omega_i)\partial\ln NT_i/\partial x)+i\omega\chi(\omega,k_z)(1+(k_yv_s^2/\omega\Omega_i)\partial/\partial x(\ln(N/T_\beta))]=0$$

where

(279)

$$\chi=\begin{cases}\sqrt{(\pi/2)}1/|k_z|v_{Te},\ \beta=1/2 & \text{in the case (a)}\\ v_e/k_z^2v_{Te}^2,\ \beta=0 & \text{in the case (b)}\end{cases}$$

Dissipation is stipulated by the Cherenkov absorption and emission of waves in the case a) and by the electron diffusion in the case b). However in contrary of homogeneous plasmas in considered inhomogeneous plasmas they may occur to be negative and as a consequence they can lead to the fields excitation in the frequency range $\omega<\omega_{dr\alpha}$. Really, from (278) follows that if $\omega>\omega_{dr\alpha}$ the imaginary part of $\varepsilon(\omega,\mathbf{k},x)$ is positive and wave absorption takes place all time. But if $\omega\le\omega_{dr\alpha}$, then the $\Im_m\varepsilon(\omega,\mathbf{k},x)<0$ and wave excitation is possible. Thus, in the isothermal plasma with $T_e>>T_i$ in the frequency range $\omega_{dr\alpha}>>\omega>>k_zv_s$ from (278) we find two branches of unstable oscillations ($\omega\to\omega+i\delta$):

(280)

$$\omega_1=-(k_yv_s^2/\Omega_i)\partial\ln N/\partial x,\ \delta_1=(\omega_1^2/|k_z|v_{Te})\{k^2r_{De}^2-\beta\partial\ln T_e/\partial\ln N\}\times\begin{cases}\sqrt{\pi}/2\\ v_e/|k_z|v_{Te}\end{cases}$$

unstable if $k^2r_{De}^2>\beta\partial\ln T_e/\partial\ln N$, and

$$\omega_2^2 = -k_z^2 v_{Ti}^2 \partial \ln T_i/\partial \ln N \qquad (281)$$

unstable if $\partial \ln T_i/\partial \ln N > 1$

Thus, above was shown that the inhomogeneous plasma confined by the strong external magnetic field is unstable - in the frequency range $\omega \sim \omega_{dr\alpha}$ the electrostatic waves can be excited. Waves excitation takes place in collisionless plasma as well as in collisional one. Only one restriction which must be required for existence of drift instabilities is

$$\omega_{dr\alpha} > \begin{cases} |k_z| v_{Ti} & \text{in collisionless plasmas} \\ k_z^2 v_{Ti}^2/\nu_i & \text{in collisionlal plasmas} \end{cases} \qquad (282)$$

Taking $k_y \sim 1/L_\perp \sim 1/L_0$ and $k_{z min} \sim 1/L_{//}$, where $L_\perp$ is transverse and $L_{//}$ longitudinal sizes of system (devices) from (282) we conclude, that the drift instabilities are possible if

$$L_{//}/L_\perp > \begin{cases} L_\perp \Omega_i/v_{Ti} >> 1 & \text{in collisionless plasmas} \\ \sqrt{\Omega_i/\nu_i} >> 1 & \text{in collisional plasmas} \end{cases} \qquad (283)$$

In opposite cases the plasmas confined by magnetic field occur to be stable.

Now we can estimate the possibilities of drift instabilities existence in thermonuclear plasma. In toroidal devices (tokomak or stellerator) $T_e \sim T_i \sim 10$kev, $B_0 \sim 50$-100KG, $L_{//} \approx 2\pi R \sim 10^3$cm , $L_\perp \sim L_0 \sim 10$-100cm and $n_e \sim n_i \sim 10^{14}$cm^{-3}.Therefore we have $k_z v_{Te} \sim 6\ 10^6 s^{-1} >> \nu_e \sim 10^4 s^{-1}$ and $k_z v_{Ti} \sim 10^5 s^{-1} >> \nu_i \sim 2\ 10^2 s^{-1}$ that means plasma is collisionless. Then from the first condition (283) follows: that if $L_\perp \leq 10$ cm then the drift instabilities can be developed but if $L_\perp > 10^2$cm then cannot be developed. Namely on the basis of such estimations B. Kadomtsev in 1965 proposed tokamak-reactor as a thick torus with R ~1,5m and a ~ 1m. Just such parameters are realized in the INTOR tokamak.

In conclusion of this section let us discuss very shortly the so-called flute instability which takes place in the mirror machins. They were very popular at the beginning of thermonuclear investigations. In the Fig. 11 the magnetic field lines schematically are presented for the mirror systems.

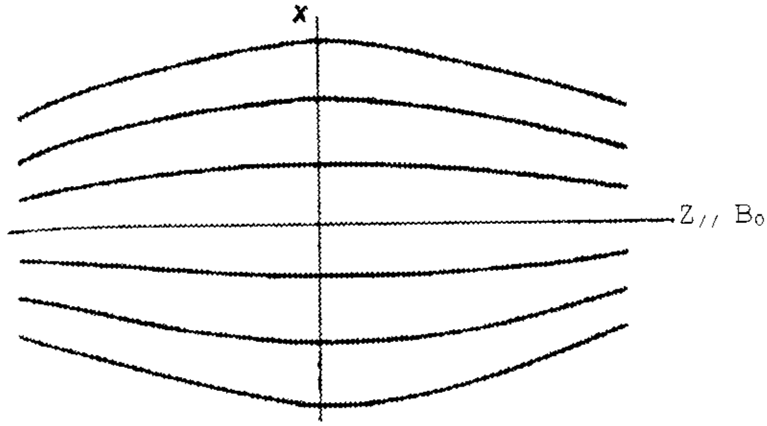

Figure 11

They have significantly large curvature with R ~1m in the real mirror systems. As a consequence of the magnetic lines curvature there arises a gravitational force acting on the charged particles and directed outside of the axis OX. The order of gravitational acceleration is :

(284)

$$g_\alpha \sim v_{T\alpha}{}^2/R$$

and parallel to the axis **OX**. The gravitational force leads to the real drift of electrons and ions with velocity

(285)

$$u_\alpha = -g_\alpha/\Omega_\alpha$$

parallel to the axis **OY**. As a result in the dielectric permittivity (275) ω must be changed $\omega \to \omega_\alpha' = \omega - k_y u_\alpha$, taking into account the gravitational Doppler shift.
Now we can investigate the effects which arise under the action of gravitational force. Let us consider only the fluid modes with $k_z=0$. Besides we suppose $k_\perp{}^2 v_{T\alpha}{}^2 << \Omega_\alpha{}^2$ and $\omega'_\alpha >> k_z v_{T\alpha}$. Then from (275) the following dispersion equation can be obtained.

(286)

$$1+_\alpha\Sigma(\omega_{L\alpha}{}^2/k^2)(k_y{}^2/\Omega_\alpha{}^2+(k_y/\Omega_\alpha\omega'_\alpha)\partial \ln N/\partial x) =$$
$$=1+(\omega_{Li}{}^2/k^2)[k_\perp{}^2/\Omega_i{}^2+(k_y/(\Omega_i(\omega-k_y u_i))-k_y/\,\Omega_i\,(\omega-k_y u_e)))\partial \ln N/\partial x] = 0$$

From this equation for $\omega >> k_y u_\alpha$ we obtain

(287)

$$\omega^2 = (k_y{}^2 g_{eff} \partial \ln N/\partial x)/(1+v_A{}^2/c^2)$$

where $g_{eff} = (v_{Ti}{}^2 + v_s{}^2)/R$. In the mirror systems with positive curvature or when $g_{eff}\partial \ln N/\partial x < 0$ in accordance with (287) $\omega^2<0$, or we have instability. This instability is known as gravitational or flute instability.
The growth rate of flute instability is sufficiently large: for thermonuclear plasmas with $T_e \sim T_i$ ~10kev and R~100cm we have $\Im m\ \omega \sim \sqrt{g_{eff}\partial \ln N/\partial x} \sim \sqrt{((v_{Ti}{}^2+v_s{}^2)/R^2)} \sim 10^6 s^{-1}$, which is two order higher than growth rate of drift instabilities. However fortunately this instability is not dangerous for toroidal systems as tokamak because it can be developed only if $L_{//}/L_\perp > \sqrt{M/m})$, $L_\perp \Omega_i/v_{Ti} >> \sqrt{(M/m)}$. At the same time , namely due to this instability the mirror machines were closed as thermonuclear reactors.

LECTURE 10

Plasma in an External Homogeneous Electric Field

* 22 Plasma instabilities in a constant electric field

In this lecture we will consider the electromagnetic properties of a plasma in the strong constant and variable electric fields. Let us begin from the constant electric field in which, as it was shown in the lecture 4, arises a current with electron drift velocity equal to

$$\mathbf{u} = e\mathbf{E}_0/m\nu_e \tag{288}$$

This expression is correct if $u < v_{Te}$ or

$$E_0 < E_{cr} \equiv m\nu_e v_{Te}/e \tag{289}$$

However when $E_0 > E_{cr}$ then the plasma occurs in a strong electric field and becomes nonstationary, because electron drift velocity is very high and their collisions with ions and neutral particles tend to zero. Then we have collisionless plasma and therefore we can use for its describtion the Vlasov's equation

$$\partial f_0/\partial t + e\mathbf{E}_0 \partial f_0/\partial \mathbf{p} = 0 \tag{290}$$

The solution of this equation is an arbitrary function of characteristic

$$d\mathbf{p}/dt = e\mathbf{E}_0 \quad \Rightarrow \quad \mathbf{p} = e\mathbf{E}_0 t \tag{291}$$

Consequently

$$f_0 = f_0(\mathbf{p} - e\mathbf{E}_0 t) \tag{292}$$

Thus we see that if $E_0 > E_{cr}$ the electrons momentum will increase infinitly, or in other words, they run away.
The critical field was firstly introduced in 1959 by H. Dreicer and is known as Dreicer's field. So we can conclude, that if the field is less than Dreicer's critical field then the current is constant and plasma can be considered as stationary, whereas in the case of overcritical fields plasma occurs nonstationary. Neverthless below we will consider sufficiently fast processes of instabilities and this allows us to suppose that plasma is stationary.
Thus we suppose that electron drift velocity **u** is constant and parallel to the external field $\mathbf{E}_0$. Moreover we accept that the external magnetic field $\mathbf{B}_0$, if it exists, also is parallel to $\mathbf{E}_0$. Under such conditions the plasma stability problem can be considered by the dispersion equation

$$\left| k^2\delta_{ij} - k_i k_j - (\omega^2/c^2)\varepsilon_{ij}(\omega,\mathbf{k}) \right| = 0 \tag{293}$$

where $\varepsilon_{ij}(\omega,\mathbf{k})$ is dielectric permittivity of the plasma with moving electrons and resting ions, or in other words, of current driven plasma. For calculating this quantity we will use the Lorentz transformation for current and charge densities as it was proposed by A. Rukhadze in 1960. As a result we obtain for current driven plasmas

(294)

$$\varepsilon_{ij}(\omega,k)=\delta_{ij}+\sum_{\alpha}^{\beta}\mu i(u_{\alpha})[\varepsilon_{\mu\nu}{}^{\alpha}(\omega'_{\alpha},k'_{\alpha})-\delta_{\mu\nu}]\beta_{\nu j}(u_{\alpha})$$

where

(295)

$$\omega'_{\alpha}=(\omega-\mathbf{k}\mathbf{u}_{\alpha})\gamma_{\alpha},\ \mathbf{k}'_{\alpha}=\mathbf{k}+\mathbf{u}_{\alpha}\gamma_{\alpha}[(\mathbf{k}\mathbf{u}_{\alpha}/u_{\alpha}{}^{2})(1-1/\gamma_{\alpha})-\omega/c^{2}]$$
$$\beta_{ij}(\mathbf{u}_{\alpha})=(\omega'_{\alpha}/\omega)\delta_{ij}+\gamma_{\alpha}[(u_{\alpha i}u_{\alpha j}/u_{\alpha}{}^{2})((1/\gamma_{\alpha})-1)+k_{i}u_{\alpha j}/\omega],\ \gamma_{\alpha}=(1-u_{\alpha}{}^{2}/c^{2})^{-1/2}$$

Here $\mathbf{u}_e=\mathbf{u}$, and $\mathbf{u}_i=0$, and $\varepsilon_{ij}{}^{\alpha}(\omega,\mathbf{k})$ is the partial dielectric permittivity of particles of type α in own (moving) frame, which is known from previous lectures.

Let us now consider some examples of using of the above presented theory.

a) Buneman's instability. First instability of current driven plasmas was discovered by O. Buneman in 1959. It concerns the case when $u>>v_{Te}$ and the thermal motions of particles can be neglected . Then

(296)

$$\varepsilon_{ij}{}^{e}(\omega',\mathbf{k}')=(1-(\omega_{Le}{}^{2}\gamma^{-3})/(\omega-\mathbf{ku})^{2})\delta_{ij},\ \varepsilon_{ij}{}^{i}=(1-\omega_{Li}{}^{2}/\omega^{2})\delta_{ij}$$

For simplicity we have neglect the particles collisions also.

Substituting (296) into the (295) and after in (294) we obtain the following dispersion equation for purely longitudinal oscillations (with **k//E0**)

(297

$$1-(\omega_{Le}{}^{2}\gamma^{-3}/(\omega-\mathbf{ku})^{2})-\omega_{Li}{}^{2}/\omega^{2}=0$$

From this equation immediatly follows that unstable solutions (with $\Im m\omega<0$) may appear only if

(298)

$$\omega_{Le}{}^{2}\geq(\mathbf{ku})^{2}\gamma^{3}$$

and correspondingly solutions look as

(299)

$$\omega=\begin{cases}(m/2M)^{1/3}((-1+i\sqrt{3})/2)\omega_{Le}/\gamma^{1/2} & \text{if } \omega_{Le}{}^{2}=(\mathbf{ku})^{2}\gamma^{3}\\ i\sqrt{(m/M)}\mathbf{ku}\gamma^{3/2} & \text{if } \omega_{Le}{}^{2}>(\mathbf{ku})^{2}\gamma^{3}\end{cases}$$

It is inquisitive that the condition (298) can be written as

(300)

$$j>j_{cr}=(\pi/4)(mc^{3}/e)(u^{3}/c^{2})\gamma^{3}(1/L_{//}{}^{2})\sim(u^{3}/c^{3})(\gamma^{3}/L_{//}{}^{2})13\ \text{kA/cm}^{2}$$

In the nonrelativistic case when $\gamma\rightarrow 1$ this relation reminds the wellknown Child-Langmuir formula for critical current density above which the beam current in a diode becomes unstable. Let us notice also that considered instability known as Buneman's instability represents the stimulated Cherenkov emission of low frequency electrostatic waves in a current driven plasma. This follows from the condition of existence of instability $\omega<\mathbf{ku}$. Above we considered purely longitudinal propagation of perturbations, $\mathbf{k}_{\perp}=0$, or **k//B0//OZ** and therefore **E//B0**. As a result the external magnetic field in this case doesn't influence the character of instability.

Let us now take into account the finite $\mathbf{k}_\perp$ and suppose that the magnetic field is very strong and magnetizes completely electrons but not ions. So, we suppose

$$\Omega_e^2 \gg \omega_{Le}^2, \quad \omega_{Li}^2 \gg \Omega_i^2 \tag{301}$$

Under these conditions the Poisson's equation takes the form

$$(1-\omega_{Li}^2/\omega^2)\Delta\phi - (\omega_{Le}^2\gamma^{-3}/(\omega-k_z u)^2)k_z^2\phi = 0 \tag{302}$$

And the simplest boundary condition we used

$$\phi_{r=R} = 0 \tag{303}$$

In other words, we consider the current driven plasma in metallic waveguide. The general solution of (302)

$$\phi(r) = \exp(ie\varphi)\Im_e(r\mu_{es}/R)\phi_0 \tag{304}$$

substituting in (303) leads to the following dispersion equation

$$(1-\omega_{Li}^2/\omega^2)(k_z^2+\mu_{es}^2/R^2) - \omega_{Le}^2\gamma^{-3}k_z^2/(\omega-k_z u)^2 = 0 \tag{305}$$

from which follows the instability condition (compare to (298))

$$\omega_{Le}^2>\gamma^3u^2(k_z^2+\mu_{es}^2/R^2) \tag{306}$$

Taking into account that min $\mu_{es} = \mu_0 = 2.4$ for sufficiently long length systems, $L_{//} >> R$, from (306) the expression for critical current follows

$$\Im_{cr} = (mc^3/4e)(2,4)^2(\gamma^2-1)^{3/2} \sim 24\gamma^3 \text{ kA} \tag{307}$$

As for the growth rates , we notice that equation (305) is similar to (307) and therefore the growth rates are similar to (299)

$$\omega = \begin{cases} (m/2M)^{1/3}((-1+i\sqrt{3})/2)\omega_{Le}/\gamma^{3/2}(\pi R/2.4L_{//})^{1/3} & \text{if } \omega_{Le}^2 = (2.4/R)^2u^2\gamma^3 \\ i(m/M)^{1/2}(2,4u/R)\gamma^{3/2} & \text{if } \omega_{Le}^2 >(2.4/R)^2u^2\gamma^3 \end{cases} \tag{308}$$

b) Fillamentation Buneman's instability as it was emphasized above represents the stimulated Cherenkov radiation of low frequency fields in the current driven plasmas. It takes place when $\mathbf{ku} = k_z u > \omega$. Let us consider now purely transverse instability with $k_z=0$. This instability was firstly investigated by R. Vaibel in 1961. If $k_z=0$ and $k_\perp = k$ then from (293) taking into account (294)-(296) follows

$$(k^2-\omega^2/c^2+(\omega_{Le}^2/\gamma c^2))(1-(\omega_{Le}^2\gamma^{-3}/\omega^2)) = (\omega_{Le}^2\omega_{Li}^2k^2u^2)/(\gamma\omega^4c^2) \tag{309}$$

It is easy to show that this equation in the low frequency limit when $\omega^2<<\omega_{Le}^2\gamma^{-3}$ has the solution corresponding to the aperiodically unstable oscillations with spectrum:

$$\omega^2 = -(\omega_{Li}^2k^2u^2\gamma^2)/(k^2c^2+\omega_{Le}^2\gamma^{-1}) \leq -\omega_{Li}^2(u^2/c^2)\gamma^2 \tag{310}$$

As we have neglected the thermal motions of particles then this expression is valid if $|\omega| >> kv_s,kv_{Ti}$, what means that this instability can be developed in the systems with sufficiently large transverse sizes, when

$$\omega_{Li}^2(u^2/c^2)\gamma^2 > (1/L_\perp^2)(v_s^2+v_{Ti}^2)$$

This requirement means that magnetic pressure in current driven plasma is higher than kinetic one and as a consequence the selfpressing (pinch-effect) is possible. Just this effect takes place when the considered instability develops. As a result the fillamentation of current plasma arises. Such mechanism of instability is proved by the fact that it takes place in collisional plasmas also when $\omega<<v_i$. In this case the growth rate reduces to

$$\omega = i(\omega_{Li}^2/v_i)(u^2/c^2)\gamma^2 \tag{311}$$

Moreover in the external sufficiently strong magnetic field the fillamentation is impossible, which proves also the above mentionned mechanism of magnetic selffocusing.

c)Ion-accoustic Instability The last instability of current driven plasmas which is very important for the plasma confinement problem is so called ion-accoustic instability which takes place when $v_{Te} > u > v_{Ti}$. This instability as Buneman's instability is connected to the stimulated Cherenkov radiation and therefore it takes place in the frequency range where accoustic oscillations exist, or when $kv_{Ti} << \omega << kv_{Te}$. In this frequency range the equation (293) takes the form

$$1+(\omega_{Le}^2/k^2v_{Te}^2)(1+(i\sqrt{\pi}/2)((\omega-\mathbf{ku})/kv_{Te})-\omega_{Li}^2/\omega^2 = 0 \tag{312}$$

Taking into account the smallness of imaginary part of this equation we find the solution $(\omega\rightarrow\omega+i\delta)$

$$\omega^2 = \omega_{Li}^2/(1+\omega_{Li}^2/k^2v_s^2),\ \delta/\omega = -\sqrt{\pi}/8(\omega^3/k^3v_{Te}^3)(1-\mathbf{ku}/\omega) \tag{313}$$

This spectrum corresponds to the ion-accoustic oscillations which in the absence of current $(u\rightarrow 0)$ are damping. However with increasing of drift velocity the decrement decreases and when $ku>\omega$ it changes sign and becomes increment (or growth rate). From these conditions it is clear that the mechanism of instability is the stimulated Cherenkov radiation by moving electrons ion-accoustic oscillations in nonisothermal plasma with $T_e > T_i$.

* 23 Plasma in a strong microwave field

Let us now consider the electromagnetic properties of plasma in a strong homogeneous microwave field $E_0(t) = E_0\sin\omega_0 t$, where $\omega_0 >> \omega_{Le},\Omega_e,v_e$, or in other words the field fequency is much higher than all characteristic frequencies of plasma. Besides, we will neglect the spatial inhomogeneity of the electric field that means

$$k_0^2 \sim (\omega_0^2/c^2)\varepsilon(\omega_0) \sim \omega_0^2/c^2 << k^2 \tag{314}$$

In other words, we will consider the perturbations with wavelength much shorter than the field inhomogeneity $\sim 1/k_0$. Such approximation is satisfied by the obvious inequality $k/k_0 \sim c/v_{Te} >> 1$.

The equilibrium distribution function of electrons is determined from the Vlasov's equation

(315)

$$\partial f_{0e}/\partial t + e\mathbf{E}_0 \sin\omega t\, \partial f_{\alpha}/\partial p = 0$$

From this it follows

(316)

$$f_{0e} = f_0(\mathbf{p}-\mathbf{p}_0(t)) = (N/(2\pi m Te)^{3/2})\exp((-m(\mathbf{v}-\mathbf{v}_0)^2)/2Te)$$

where $p_0 = mv_0(t)$, $v_0(t) = (-eE_0/m\omega_0)\cos\omega_0 t$. As for ions we neglect the influence of microwave field on them.
In order to investigate the stability of plasma with Maxwellian ions and with electrons distribution(316) let us consider small perturbations δf_e and δf_i, satisfying the equations:

(317)

$$\partial\delta fe/\partial t + i\mathbf{kv}\delta fe + e\mathbf{E}_0 \sin\omega_0 t\,\partial\delta fe/\partial p + (e/c)[\mathbf{v}\times\mathbf{B}]\partial\delta f_{\alpha}/\partial p + e\{\mathbf{E} + 1/c[\mathbf{v}\times\mathbf{B}]\}\partial f_0 e/\partial\mathbf{p} = 0$$

$$\partial\delta f_i/\partial t + i\mathbf{kv}\delta f_i + (e_i/c)[\mathbf{v}\times\mathbf{B_0}]\partial\delta f_0/\partial\mathbf{p} + e_i\mathbf{E}\partial f_{0i}/\partial\mathbf{p} = 0$$

Here we suppose that $\delta f_{e,i} \sim \exp(i\mathbf{k.r})\delta f(t,v)$. The system (317) is completed by the Maxwell equations.

(318)

$$\mathbf{rotE} = (-1/c)\partial\mathbf{B}/\partial t,\ \mathbf{rotB} = (1/c)\partial\mathbf{E}/\partial t + (4\pi/c)\Sigma e\int\delta f\mathbf{v}dp$$

For solution of the system (317) - (318) we use the following representations

(319)

$$\Psi_e = \exp((-ie/m)(\mathbf{kE}_0 \sin\omega_0 t/\omega_0^2))\delta f_e(\mathbf{p} + (e\mathbf{E}_0/\omega_0)\cos\omega_0 t)$$

$$(\Psi_e, \delta f_i) = \exp(-i\omega t)_n\Sigma\exp(-in\omega_0 t)(\Psi_{en}, \delta f_{in})$$

Substituting these expressions into the system (317) we obtain the infinite system of algebric equations, which can be easily reduced if $\omega_0 >> \omega_{Le}, \Omega_e, \nu_e$. As a result one can obtain the condition of nontrivial solutions of this system, or the dispersion equation of small perturbations

(320)

$$\varepsilon^l(\omega,\mathbf{k})/(\delta\varepsilon_i^l(\omega,\mathbf{k})[1+\delta\varepsilon_e^l(\omega,\mathbf{k})]) + 1/2(\mathbf{k.v}_E)^2 / \omega_0^2)$$

$$-1/2[\mathbf{k}\times\mathbf{v}_E]^2(\delta\varepsilon_e^l(\omega,\mathbf{k})/k^2c^2(1+\delta\varepsilon_e^l(\omega,\mathbf{k})) = 0$$

Here $\varepsilon^l = 1 + \delta\varepsilon_e^l + \delta\varepsilon_i^l$ and $\delta\varepsilon_{e,i}^l$ are the partial longitudinal dielectric permittivities of electrons and ions, $v_E = eE_0/m\omega_0$. Besides in derivation of (320) the following inequality was supposed $\mathbf{k.v}_E/\omega_0 \equiv (\mathbf{k.r_E}) << 1$ and therefore only the second order terms of external field taken into account.
If $(\mathbf{k.v}_E) \neq 0$ then the second term in (320) is much larger than the third and therefore the last can be neglected. In this limit the oscillations is purely potential and stable. But in this limit a new type of oscillations appears. Thus in the frequency range $\mathbf{k}v_{Ti} << \omega << \mathbf{k}v_{Te}$ from (320) follows $(\omega \rightarrow \omega + i\delta)$

(321)

$$\omega^2=(kv_E)^2(\omega_{Li}^2/2\omega_0^2)+k^2v_s^2,\ \delta = -\omega\sqrt{(\pi m/8M)}$$

In the absence of external field, $v_E \to 0$, this spectrum corresponds to the wellknown ion-accoustic oscillations. However if $v_E\omega_{Li}/\omega_0 > v_s$ the new branch of oscillations which is known as electrical sound appears.
Very interesting phenomenon is observed when $k.v_E=0$ and we have purely transverse wave propagation. Then from (320) follows

(322)

$$\varepsilon^l(\omega,\mathbf{k})-1/2(v_E^2/c^2)\delta\varepsilon_e^l(\omega,\mathbf{k})\delta\varepsilon_i^l(\omega,\mathbf{k})=0$$

In the low frequency range when $\omega<<kv_{Te}$ but $\omega>>kv_{Ti}$ the equation has the nonstable solutions

(323)

$$\omega^2 = -\omega_{Li}^2 v_E^2/2c^2 \qquad \text{if } \omega>\nu_i$$

$$\omega = i\omega_{Li}^2 v_E^2/\nu_i c^2 \qquad \text{if } \omega<\nu_i$$

The threshold of instability follows from the condition $\omega>>\mathbf{k}v_{Ti}$ and looks as

(324)

$$v_E^2/2c^2 > (1/L_\perp^2)(v_{Ti}^2+v_s^2)/\omega_{Li}^2$$

where $L_\perp$ is the transverse size of system.
The considered instability is analogous to the fillamentation instability of current driven plasmas, or anisotropic R. Vaibel's instability. It is easy to understand this. In this case the high frequency current creates high-frequency magnetic field, which compresses plasma, or usual pinch-effect takes place. The multiplying factor 1/2 proves this. Thus in a plasma placed in the sufficiently strong microwave field as in a current driven plasma the transverse fillamentation takes place.

LECTURE 11

Beam Plasma Instabilities

* 24 Stimulated Cherenkov radiation of electron beams

In this section we will consider nonequilibrium plasma which consists of a rest dense plasma and a moving monoenergetic electron beam with much less density. Such a plasma beam system is unstable and as a result develops the instability, the monochromatic and coherent electromagnetic radiation can be created. Let us consider this phenomenon with more details. It is well-known that a fast charged particle (electron) moving in a medium can radiate. However the radiation of one particle is a spontaneous radiation and therefore its intensity will increase proportionally to t during the time of particles radiation. If we have not one but a sufficiently large number of moving particles then the radiation of first particle can influence the radiation probability of the second one. Moreover if the particles are identical, or in other words, if we have a monoenergetic beam of charged particles, then their radiation becomes stimulative and it occurs a monochromatic and coherent one. As a result the radiation intensity will increase exponentially in time, as $\exp(\delta t)$. Namely such dependence indicates that the radiative instability takes place. Below this phenomenon will be studied on the example of interaction of monoenergetic and stright electron beam with a rest and cold plasma. The instability arising in this case is the result of stimulated Cherenkov radiation of electron beams. Stimulated Cherenkov radiation or beam-plasma instability plays very important role in many applications and first of all just this phenomenon is the basis of plasma electronics, namely due to this radiation very high power microwave sources are working out today.

1.1 Let us begin our consideration from the theory of spontaneous radiation of an electron moving with a constant velocity **u** // **OZ**. The current and charge densities we can be written as

(325)

$$\mathbf{j} = eu\mathbf{i}_z\delta(\mathbf{r}_\perp)\delta(z-ut),\quad \rho = e\delta(z-ut)\delta(\mathbf{r}_\perp)$$

Then we can calculate the work of the field of a nonmonochromatic wave on the electron with current density (325). Representing the field of the wave as

(326)

$$\mathbf{E}(z,t) = \mathbf{E}\sin(\omega t - k_{//}z + \varphi)$$

and assuming that the directions of the wave propagation and the electron motion coincide we obtain the following expression for the work during $T \gg 1/\omega$:

(327)

$$A_p = \int_{-T/2}^{T/2} dt \int d\mathbf{r}\mathbf{j}\, \mathbf{E}(\mathbf{r},t) = e\int_{-T/2}^{T/2} dt\, uE_{//}(z,t)_{z=ut} \Rightarrow$$
$$= \pi euE_{//}\sin\varphi\, \delta(\omega - k_{//}u) \equiv A_\infty \quad (T \to \infty)$$

Here φ is the initial phase of the wave, $E_{//}$ is the field component in the direction of particles motion.

From (327) follows that: firstly, the work is nonzero only if $\mathbf{E}.\mathbf{u} \neq 0$ and Cherenkov condition

(328)

$$\omega = k_{//}u$$

is satisfied; And, secondly, the sign of work dependes on the phase φ. If $\sin\varphi>0$ then $A_\infty>0$ and one can say about the wave emission, whereas if $\sin\varphi<0$ then $A_\infty<0$ and we have the wave absorption.

The considered phenomenon is known as spontaneous Cherenkov radiation or absorption of an electron which takes place under the condition (328). If we have not one but the homogenous in space and monoenergetic beam of electrons then the expression (327) must be summarized over all electrons of the beam by unity length. As a result we obtain

$$A_\infty = \pi euE_{//}\delta(\omega-k_{//}u)_j\Sigma\sin\varphi_j = 0 \tag{329}$$

In other words this sum occurs to be zero if phases φ_j are arbitrary, since the coherent waves from electrons cancel each other as a result of their interference.

1.2 In order to obtain nonvanishing coherent radiation it is necessary to refuse the assumption that the motions of particles are independent, i.e. it is necessary to take into account the reaction of the radiation wave field on the motion of each beam electron. It is the self-consistent approach in which the stimulated coherent radiation appears.

Below we will consider the stimulated radiation in the case of a rectilinear beam of electrons made one dimensional motion along a very strong external magnetic field. We will assume in addition that the frequency of the radiated wave field

$$E_{//}(z,t) = 1/2\ [E_{//}\exp(-i\omega t+ik_{//}z+i\varphi) + c.c] \tag{330}$$

to be close to one of the own frequency of the system, or $\omega \approx \omega\ (k//)$. Under this condition the perturbation of the electron trajectories is determined by the equations

$$dz/dt = v_{//},\ dv_{//}/dt = e/m\beta(1-v_{//}^2/c^2)^{3/2}\ E_{//}(z,t) \tag{331}$$

where β is a parameter which characterizes the strength of beam coupling with the radiation field, or in other words the quantity β takes into account the radial structures of electric field $E_{//}(z,t)$ and electron beam.

We solve the system (331) in using the perturbation method. Assuming

$$z = z_0 + \delta z(E_{//}),\ z_0 = u(t-t_0) \tag{332}$$

and supposing $E_{//} \rightarrow 0$ when $t_0 \rightarrow -\infty$ after some calculations with the accuracy $\sim E_{//}^2$ we obtain

$$dP/dt = (-\beta^2\omega\ \delta\omega\ \omega_b^2(\omega-k_{//}u)/((\alpha\gamma^3)[(\omega-k_{//}u)^2 + \delta\omega^2]^2))P \equiv 2\delta\omega P \tag{333}$$

$$P=\alpha\ k_{//}|E_{//}|^2/(\omega 8\pi)$$

Here we take into account that the frequency ω contains an imaginary correction $i\delta\omega$ (increment) due to the exponential increasing of the field in the linear stage of the radiation process.

The quantity $\gamma = (1-u^2/c^2)^{-1/2}$ and α depend on the radial distribution of field $E_{//}(z,t)$ in the waveguide, it takes into account the connection of the field energy density P to the $|E_{//}|^2$. In accordance to the conservation laws

(334)

$$d/dt(P-P_e) = 0,\ P_e = n_b m\langle v(1-v^2/c^2)^{-1/2}\rangle$$

where n_b is the beam electrons density, $\omega_b = \sqrt{(4\pi e^2 n_b/m)}$ their Langmuir frequency, and the bracket $\langle\ \rangle$ means the averaging over all electrons of the beam.
From (334) we can draw a very important conclusion: the energy of the wave increases, $\delta\omega > 0$, only when $k_{//}u > \omega$, or the Cherenkov radiation condition holds. We will maximize the right hand side of (334) by adjustement of ω-$k_{//}u$. As a result we obtain

(335)

$$\omega = k_{//}u - \delta\omega/\sqrt{3},\ \delta\omega = (\sqrt{3})/2((\beta^2\omega\omega_b^2)/(2\alpha\gamma^3))^{1/3}$$

Moreover the optimal condition of stimulated radiation is known as the single particle Cherenkov stimulated radiation, or, as the Thomson regime of stimulated radiation.

1.3 Let us now consider the Raman regime of stimulated Cherenkov radiation. This means that the frequency of beams own oscillations is comparable to the growth rate of instability. Then instead of equation (331) we must write the equations which take into account the beams collective oscillations with its own frequency Ω_b:

(336)

$$dz/dt = v_{//}$$
$$dv_{//}/dt + \Omega_b^2(z-z_0) = (e/c)\beta(1-v_{//}^2/c^2)^{3/2}E_{//}(z,t)$$

where $z = z_0 + u_{//}(t-t_0)$. The calculations similar to these carried out above give the following result (compare to (333)):

(337)

$$dP/dt = (-\beta^2\omega_b^2(\omega-k_{//}u)\delta\omega.\omega.P)/(\alpha\gamma^3[(\omega-k_{//}u)^2-\Omega_b^2-\delta\omega^2]^2+4(\omega-k_{//}u)^2\delta\omega^2) = 2\delta\omega P$$
$$P=\alpha k_{//}|E_{//}|^2/(\omega 8\pi)$$

When $\Omega_b^2 \to 0$ this equation coincides with (333). However when $\Omega_b^2 >> \delta\omega^2$ the new regime of instability arises. The maximum value of the increment is realized for the condition

(338)

$$\omega = k_{//}u - \Omega_b,\ \delta\omega = 1/2(\omega\omega_b^2\beta^2/(2\alpha\gamma^3\Omega_b))^{1/2}$$

This regime is known as collective regime, or Raman regime.
In conclusion let us notice that for magnetized unbounded electron beam the dispersion equation for small oscillations is

(339)

$$1 - \omega_b^2/(\gamma^3(\omega-k_{//}u)^2) = 0$$

From this equation follows that $\Omega_b = \omega_b/(\gamma^{3/2})$. Therefore if

(340)

$$\omega_b/(\gamma^{3/2}) >> (\sqrt{3})/2(\beta^2\omega\omega_b^2/(\alpha 2\gamma^3))^{1/3}$$

the Raman regime of stimulated radiation takes place, whereas in the opposite limit the Thomson regime is preferable.

* 25 The dispersion equation for plasma beam interaction in infinitely strong magnetic field

1.1 Stimulated Cherenkov radiation corresponds to the radiative instabilities of the plasma beam system. This means that the above considered phenemenon can be described in the frame of macroscopic electromagnetics of non equilibrium media. Below we will show this for a thin relativistic electron beam interacting with homogeneous strongly magnetized plasma. The dielectric permittivity of such a plasma is

(341)

$$\varepsilon_{ij}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & \varepsilon_p \end{pmatrix}, \ \varepsilon_p = 1-\omega_p^2/\omega^2$$

Let us consider first of all the own oscillation of such a plasma without the beam. Substituting (341) into the general dispersion equation we obtain

(342)

$$| k^2\delta_{ij}-k_ik_j-(\omega^2/c^2)\varepsilon_{ij}(\omega,k) | \Rightarrow k_\perp^2c^2+(k_{//}^2c^2-\omega^2)(1-\omega_p^2/\omega^2) = 0$$

Here $k_\perp$ is perpendicular (across B_0) wave number, and $k_{//}$ is the parallel one. This equation determines two branches of plasma oscillations, the spectra of which are presented on the Fig. 9 (lecturs 8) as a function of $k_\perp = \mu_{es}/R$ (in the case of magnetized plasma waveguide)

(343)

$$\omega_{1,2}^2 = (1/2)\left\{ \omega_p^2 + k^2c^2 \pm \sqrt{\left(k^2c^2 + \omega_p^2\right)^2 - 4k_{//}^2c^2\omega_p} \right\},$$

where $k^2 = k_\perp^2 + k_{//}^2$. The first solution corresponds to the fast wave, $\omega_1 > k_{//}c$, whereas the second one - to the slow wave with $\omega_2 < k_{//}c$.
It is obvious that only slowing wave can be excited by electron beam as a result of stimulated Cherenkov radiation. In fact, taking into account that the thin electron beam is located at $r_\perp = r_{\perp b}$ one can easely obtain the beam correction to the equation (342):

(344)

$$(\omega^2-\omega_1^2(k))(\omega^2-\omega_2^2(k))[(\omega-k_{//}u)^2 - \omega_b^2/\gamma^{3/2}] = \beta^2\omega^4\omega_b^2/\alpha\gamma^3$$

The quantities β^2 and α take into account the wave amplitudes radial distribution in the wave guide and the beams geometry. The equation (344) describes four branches of oscillations. The left side of this equation describes four branches of noncoupled oscillations, from which two correspond to (343) and another two are beam's waves

(345)

$$\omega_{3,4}(k) = k_{//}u \pm \omega_b / \gamma^{3/2}$$

The right hand side of (344) describes the coupling of these four branches. However it is easy to show that only slow plasma $\omega_2(k)$ and beam $\omega_4(k)$ waves can interact to each other, or in other words, only these two branches can intersect $\omega_2(k) = \omega_4(k)$. (see Fig. 12 and Fig. 9) Such intersection of dispersion curves corresponds to their interaction and as a result the extra energy from the beam converts into the electromagnetic energy of plasma wave. Namely this conversion is the stimulated Cherenkov radiation of plasma waves by the beam.
Therefore representing

(346)

$$\omega = \omega_2(k)+i\,\delta\omega$$

and substituting this into the dispersion equation (344) we obtain $\delta\omega$ equal to (335), if opposite to inequality (340) takes place and to (338) when (340) is satisfied. Thus the method of dispersion equation is completely equivalent to the microcospic description which was used above.

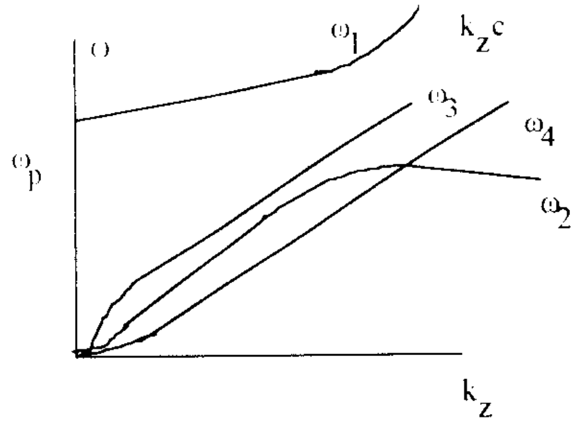

Figure 12

1.2 Up to time we considered the interaction of monoenergetic electron beam with high frequency and fast plasma waves. Let us show now that the stimulated Cherenkov radiation is possible even when the energy spread of beam electrons is sufficiently wide. The corresponding instability in this case is the sign irreversible Landau damping. The growth rate of the instability in this case , which is known as kinetic one in opposite to the above considered (hydrodynamical instability), is much less and therefore this instability has much less practical applications. However, at the same time it occurs very interesting from purely scientific point of view, because it had leaded to development of quasilinear theory, which will be studied below. For simplicity we will consider the stimulated Cherenkov radiation of a spatially unbounded hot plasma electron beam with energetic spread for the case of nonrelativistic beam supposing $u \ll c$. Then the potential approximation is valid and dispersion equation for " cold" plasma and "hot" beam system looks as

$$\varepsilon(\omega,k) = 1 - \omega_p^2/\omega^2 + (\omega_b^2/k^2 v_{Tb}^2)[1-\Im((\omega-\mathbf{ku})/kv_{Tb})] = 0 \qquad (347)$$

Here v_{Tb} is the thermal velocity of beam electrons in the own frame.
If $(\omega - \mathbf{ku}) \gg kv_{Tb}$ from (347) we obtain

$$1 - \omega_p^2/\omega^2 - \omega_b^2/(\omega - \mathbf{ku})^2 = 0 \qquad (348)$$

This equation corresponds to the above considered stimulated Cherenkov radiation of "cold" beam (for the case $\alpha = \beta = 1$) in the case of Thomson (single particle) regime; Thus the above considered approximation of cold beam occurs to be valid if

$$\delta\omega \gg kv_{Tb} \Rightarrow u/v_{Tb} \gg (n_b/2n_p)^{1/3} \qquad (349)$$

In the opposite limit the thermal motion of beam electrons are essential and equation (347) takes the form

$$1 - \omega_p^2/\omega^2 + i\sqrt{(\pi/2)}(\omega_b^2(\omega-\mathbf{ku})) / k^3 v_{Tb}^3 = 0 \qquad (350)$$

From this equation we obtain ($\omega \rightarrow \omega + i\delta$)

$$\omega = \omega_{Le}, \; \delta/\omega = (-\sqrt{\pi}/8)(\omega_b^2(\omega - \mathbf{ku}))/k^3 v_{Tb}^3 \tag{351}$$

The instability ($\delta > 0$) is possible only when $\mathbf{ku} > \omega$ and this means that it corresponds to the stimulated Cherenkov radiation of plasma waves by the "hot" electron beam.

LECTURE 12

Nonlinear Wave Interactions in a Plasma

* 26 Multiindex dielectric permittivities. Shortened equation for nonlinear wave interaction

In a plasma exist two types of nonlinear phenomena - nondissipative and dissipative. To the nondissipative phenomena must be concerned the creation of nonlinear combinative harmonics of waves, or, more general, wave-wave interactions, whereas to the dissipative phenomena are concerned the wave absorptions which lead to the changing of such plasma parameters as particles temperature and their distribution function. Below we will consider these phenomena separately and compare the time scales of their developments.

In this section we will begin from the wave-wave interactions. The theory of these phenomena is based on the nonlinear dielectric permittivities, or, more exactly, on the nonlinear material equations. Namely for such a description nonlinear optics and nonlinear theory of solid states is used. The advantage of the plasma theory consists in the existence of correct nonlinear equation for its description, that is the Vlasov kinetic equation. This allows us to obtain exact nonlinear material equations and nonlinear dielectric permittivities. In order to obtain the nonlinear material equations, the Vlasov equation

$$\partial f/\partial t + \mathbf{v}\partial f/\partial \mathbf{r} + e\{\mathbf{E} + 1/c[\mathbf{v}\times\mathbf{B}]\}\partial f/\partial \mathbf{p} = 0 \qquad (352)$$

must be solved by using the expanding procedure

$$f = f_0 + f_1 + f_2 + \dots\dots\dots f_n + \dots\dots \qquad (353)$$

where f_0 is the Maxwellian distribution function and $f_n \sim E^n$. For simplicity below we restrict ourselves by consideration of nonmagnetized plasma, or $B_0 = 0$. Then from (352) follows

$$\partial f_n/\partial t + \mathbf{v}\partial f_n/\partial \mathbf{r} + e\{\mathbf{E} + (1/c)[\mathbf{v}\times\mathbf{B}]\}\partial f_{n-1}/\partial \mathbf{p} = 0 \qquad (354)$$

Representing all quantities as Fourier transform

$$A(r,t) = \int d\omega \int d\mathbf{k} A(\omega,\mathbf{k})\exp(-i\omega t + i\mathbf{k}\mathbf{r}) \qquad (355)$$

we can express $f_n(\omega,k)$ in terms of f_{n-1} and so on. After some calculations we obtain the expansion

$$j(\omega,k) = \Sigma e\int f\mathbf{v}d\mathbf{p} = \mathbf{j}_1 + \mathbf{j}_2 + \dots\dots \mathbf{j}_n + \dots\dots \qquad (356)$$

where $\mathbf{j}_n \sim \mathbf{E}^n$. Finally we have

$$D_i(\omega,\mathbf{k}) = \varepsilon_{ij}(\omega,\mathbf{k})E_j(\omega,\mathbf{k}) + \sum_{n=2}^{\infty}\int d\omega_1,\dots\dots\dots d\omega_{n-1}d\mathbf{k}_1\dots\dots d\mathbf{k}_{n-1}\times$$
$$\varepsilon_{ij1\dots\dots jn}(\omega,\mathbf{k},\omega_1,\mathbf{k}_1,\dots\dots\omega_{n-1},\mathbf{k}_{n-1})E_{j1}(\omega-\omega_1,\mathbf{k}-\mathbf{k}_1)\dots\dots E_{jn}(\omega_{n-1},\mathbf{k}_{n-1}) \qquad (357)$$

Here $\varepsilon_{ij}(\omega,\mathbf{k})$ is the wellknown dielectric permittivity second index tensor, and $\varepsilon_{ij1\dots jn}$ is the multiindex dielectric permittivity

(358)

$$\varepsilon_{ij1\ldots jn}(\omega,\mathbf{k},\omega_1,\mathbf{k}_1\ldots\omega_{n-1},\mathbf{k}_{n-1}) = \delta_{n1}\delta_{ij1}-4\pi(-ie)^{n-1}\int d\mathbf{p}(v_i/\omega)g_1\Gamma_{j1}\ldots g_n\Gamma_{jn}f_0$$

where

$$g=1/(\omega-\mathbf{k}.\mathbf{v}), \qquad g_n=1/(\omega_n-\mathbf{k}_n.\mathbf{v}) \tag{359}$$

$$\Gamma_{jn} = (1/(\omega_{n-1}-\omega_n))[(\mathbf{k}_n-\mathbf{k}_{n-1})_{jn}v_j+\delta_{jnj}(\omega_n-\omega_{n-1}-\mathbf{v}(\mathbf{k}_n-\mathbf{k}_{n-1}))]\partial/\partial p_j$$

The nonlinear material equation (357) is exact and naturally the Maxwell equations in taking into account this relation is very complicated. The simplification of these equations can be reached by neglecting high order nonlinear terms. If we will restrict in taking into account the nonlinearity to no more than cubic terms then the three waves interactions can be considered only. Below we will consider this approximation. But before that it must be noticed that for the analysis of nonlinear interactions the linear wave behaviour is very important. They are described by the dispersion equation

$$|k^2\delta_{ij}-k_ik_j-(\omega^2/c^2)\varepsilon_{ij}(\omega,k)|=0 \tag{360}$$

which determine the relation $\omega(\mathbf{k})$ for linear oscillations. In the linear approximation the wave amplitudes are constant. However when the nonlinear wave interactions are considered they become slowly varying

$$\mathbf{E}(r,t) = \mathbf{E}(\omega,\mathbf{k},t)\exp(-i\omega t+i\mathbf{kr})+c.c \tag{361}$$

The equations for the amplitudes one can be obtained from the relation

$$\mathbf{E}\partial\mathbf{D}/\partial t + \mathbf{B}\partial\mathbf{B}/\partial t + (1/4\pi)\mathrm{div}[\mathbf{E}\times\mathbf{B}] = 0 \tag{362}$$

by substituting the expression (357) into this relation and taking into account only the terms with $n\leq 3$ and averaging over the phases of waves. Taking into account that we get

$$\begin{gathered}\langle E(\omega,\mathbf{k})\rangle = 0\\ \langle E_i(\omega,\mathbf{k})E_j(\omega',\mathbf{k}')\rangle = \langle E_iE_j\rangle_{\omega,\mathbf{k}}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}')\\ W(\omega,\mathbf{k}) \equiv (1/4\pi)\partial/\partial\omega[M_{ij}^{h}(\omega,\mathbf{k})\omega]\langle E_iE_j\rangle_{\omega,\mathbf{k}}\\ M_{ij}\equiv\varepsilon_{ij}(\omega,\mathbf{k})-(k^2c^2/\omega^2)(\delta_{ij}-k_ik_j/k^2)\end{gathered} \tag{363}$$

and expressing the more higher order collecrators in terms of second order one we obtain the following nonlinear equation for the quantity $\langle E_iE_j\rangle_{\omega,k}$ which is known as shortened equation:

$$\begin{gathered}\omega\partial/\partial\omega[\omega M_{ij}(\omega,\mathbf{k})\partial/\partial t\langle E_iE_j\rangle_{\omega,\mathbf{k}}]=2i\varepsilon_{ij}^{a}(\omega,\mathbf{k})\langle E_iE_j\rangle_{\omega,\mathbf{k}}+\\ \int d\omega' d\mathbf{k}'\{Q_{1ij\mu\nu}(\omega,\omega',\mathbf{k},\mathbf{k}')\langle E_iE_j\rangle_{\omega',\mathbf{k}'}\langle E_\mu E_\nu\rangle_{\omega-\omega',\mathbf{k}-\mathbf{k}'}+\\ Q_{2ij\mu\nu}(\omega,\omega'-\omega,\mathbf{k},\mathbf{k}'-\mathbf{k})\langle E_iE_j\rangle_{\omega,\mathbf{k}}\langle E_\mu E_\nu\rangle_{\omega'\mathbf{k}'}\end{gathered} \tag{364}$$

The tensor quantities Q_1 and Q_2 are very complicate functions and we will give below only their explicit expressions for the concrete cases. But it must be noted that they are proportional to $M_{ij}^{-1}(\omega,\mathbf{k})$, that means that they have the resonance dominators, corresponding to plasma own oscillations. Besides of this the first term in the right side of (357) describes the linear mechanisms of waves absorption.

The equation (364) takes into account all types of 3 waves interactions. It is obvious that the complete consideration of them is impossible. Therefore we will consider only the simple examples. Moreover we will suppose that the amplitude of one of the waves is given and constant. Then the problem of 3 waves interactions reduces to the problem of plasma wave excitation under the action of external electromagnetic field as a result of wave decay processes.

* 27 Parametric plasma wave excitation in a variable homogeneous electric field

Let us consider a plasma in a variable homogeneous electric field

$$\mathbf{E}_0(t) = \mathbf{E}_0 \sin\omega_0 t \qquad (365)$$

This problem was considered in the section *23 (lecture 10) under the assumption $\omega_0 >> \omega_{Le}, \Omega_e$. Now we will refuse this restriction and therefore will take into account the resonance wave excitations when $\omega_0 \sim \omega_{Le}, \Omega_e$. Only one assumption remains unchanged. E_0 is sufficiently small and only the quadratic terms of E_0^2 will be taken into account. This assumption exists in the equation (364) which leads to the dispersion equation for wave excitation for the considered problem

$$(\varepsilon(\omega,\mathbf{k})/(\delta\varepsilon_i(\omega,\mathbf{k})[1+\delta\varepsilon_e(\omega,\mathbf{k})])+(kr_E)^2/4[\ 1/\varepsilon(\omega+\omega_0\ ,\mathbf{k})+1/(1/\varepsilon(\omega-\omega_0\ ,\mathbf{k})] = 0 \qquad (366)$$

where $\mathbf{r}_E = \mathbf{v}_E/\omega_0 = e\mathbf{E}_0/m\omega_0^2$ - is the amplitude of electrons oscillations in the field E_0, $\varepsilon(\omega,\mathbf{k}) = 1 + {}_{e,i}\Sigma\ \delta\varepsilon_\alpha(\omega,\mathbf{k})$, where $\delta\varepsilon_\alpha(\omega,\mathbf{k})$ are the partial longitudinal dielectric permittivities for electrons and ions, α=e,i.

From (366) it is easy to see that the most strong interaction of external high-frequency waves with plasma oscillation takes place when the resonance conditions

$$\varepsilon(\omega,\mathbf{k}) = 0 \quad \text{and} \quad \varepsilon(\omega\pm\omega_0\ ,\mathbf{k}) = 0 \qquad (367)$$

are satisfied simultaneously, or in other words, when ω and $\omega\pm\omega_0$ coincide with the own frequencies. This means that

$$\omega_0 = \omega_1 \pm \omega_2 \qquad (368)$$

and $\omega_0, \omega_1, \omega_2$ are the plasma oscillations frequencies. The conditions (366) are known as decay conditions.

We will begin analysis of decay instabilities with help of equation (368), namely from an isotropic plasma in which two branches of longitudinal waves exist: $\omega_1 \approx \omega_{Le}$ and low-frequency ion accoustic waves with $\omega_2 \approx kv_s$. For this decay processes the equation (366) takes the form

$$\omega^2/\omega_{Li}^2 - k^2v_s^2/\omega_{Li}^2(1-i\sqrt{(\pi/2)}\ (\omega/kv_{Te})) + (\mathbf{k}.\mathbf{r}_E)^2/2(\Delta/(\Delta^2-4\omega^2/\omega_0^2)) = 0 \qquad (369)$$

where $\Delta = (\omega_{Le}^2/\omega_0^2 - 1)$ is the so called resonance shift. It is easy to see that if

$$\omega_0 = \omega_{Le} + kv_s = \omega_1 + \omega_2 \qquad (370)$$

then the solution of (369) is ($\omega \to \omega + i\delta$)

$$\omega = kv_s + i\delta = \omega_2 + i\delta$$
$$\delta = (\omega_0^2 - \omega_{Le}^2)/16\omega^2(\sqrt{2/\pi})(\mathbf{k}r_E)^2 kv_{Te}/k^2 r_{De}^2 \qquad (371)$$

We see that the instability takes place in the frequency range $\omega_0 > \omega_{Le}$, where plasma is transparent. This instability corresponds to the decay of external microwave into the Langmuir and ionaccoustic waves. Obviously it takes place if $\delta > \nu_e/2$, or when

$$\eta = E_0^2/8\pi n T_e > \sqrt{(8\pi m/M)}\nu_e/\omega_0 \qquad (372)$$

This inequality determines the field threshold developing the instabilities. Really it is sufficiently small. Thus, of $\omega_0 \sim 10^{11} s^{-1}$ and $\nu_e \sim 10^7 s^{-1}$ (or $n \sim 3 10^{12} cm^{-3}$ and $T_e \sim 10^5 K$) from (372) follows $E_0 > 30 V/cm$.

In a magnetized plasma the decay processes have much more diversities, because there exist much more number of various oscillation branches. We will consider only one example of decay processes when external microwave ω_0 decays into the two purely electron oscillations ω_1 and ω_2

$$\omega_1 \sim \omega_{Le}, \ \omega_2 \sim \Omega_e |k_z/k| \qquad (373)$$

Such oscillations exist in a sufficiently dense plasma where $\omega_{Le}^2 >> \Omega_e^2$. The first branch corresponds to the Langmuir waves and the second one - to the oblique Langmuir waves. We suppose $\omega_0 \approx \omega_1 + \omega_2$ and then the dispersion equation (366) takes the form

$$\omega_1^2/\omega_{Li}^2 + i\omega_1/2\delta - (\mathbf{k}.\mathbf{r}_E)^2/8(\omega_2(\omega_2^2 - \Omega_e^2)/(\Delta + i\delta)\omega_0^2) = 0 \qquad (374)$$

where $\omega = \omega_1 + i\delta$ and $\Delta \equiv \omega_0 - \omega_1 - \omega_2$. Supposing

$$\Delta = -m/M((\mathbf{k}.\mathbf{r}_E)^2/8)(\Omega_e^3 \cos\theta \sin^2\theta)/\omega_{Le}^2) >> \delta \qquad (375)$$

where θ is the angle between k and B_0, we obtain

$$\delta^2 = m/M(\Delta\omega_{Le}/2) \qquad (376)$$

We see that in the frequency range $\Delta > 0$ the quantity $\delta^2 > 0$ which corresponds to plasma instability. However there exists a threshold, which follows from the condition $\delta > \nu_e/2$.

* 28 Plasma in a strong electromagnetic wave. Stimulated scattering of transverse waves in a plasma

Above in the previous section we considered nonlinear wave-wave interactions for homogeneous waves in space and therefore only potential waves can interact. Below we will take into account the finite wave lengths of waves. This means that under the condition when the amplitude of one of the 3-waves is given (as above we will consider only this case), we have the problem of plasma stability in the field of external electromagnetic wave $\mathbf{E}_0(\mathbf{r},t) = \mathbf{E}_0 \sin(\omega_0 t - \mathbf{k}.\mathbf{r})$. Then from (364) we obtain the following dispersion equation for the longitudinal plasma oscillations

(377)

$$\varepsilon^l(\omega,\mathbf{k})\,((\omega-\omega_0)^2\varepsilon^{tr}(\omega-\omega_0\,,\mathbf{k})-c^2(\mathbf{k}-\mathbf{k_0})^2)+$$
$$\delta\varepsilon_e^l(\omega,\mathbf{k})(k^2(\omega_0-\omega)^2[(\mathbf{k}-\mathbf{k_0})\times v_E]^2)/4\omega_0^2(\mathbf{k}-\mathbf{k_0})^2=0$$

Here we restrict ourselves to the consideration of unmagnetized isotropic plasma. Besides of this, we suppose that $\omega_0 \gg \omega,\ \omega_{Le} \gg kv_{Te}, \omega_{Li}$. Moreover the given wave ω_0 and the combination wave $\omega_s = \omega_0 - \omega$, are perpendicular waves whereas the wave ω is supposed to be parallel, or $\mathbf{E_0} \perp \mathbf{k_0}$ and $\mathbf{E_s} \perp (\mathbf{k}-\mathbf{k_0})$. ω and $\mathbf{k}$ are the frequency and wave vector of the parallel plasma wave. Thus we have the 3-waves interaction which corresponds to the decay of perpendicular wave into the perpidencular wave and longitudinal plasma oscillation

(378)

$$\omega_0=\omega_s+\omega,\ tr \Rightarrow tr+l$$

Under these conditions

(379)

$$\varepsilon^l(\omega,\mathbf{k}) \sim 1 - \omega_{Le}^2/\omega^2\ , \qquad \delta\varepsilon_e^l \sim -\omega_{Le}^2/\omega^2\ , \qquad \varepsilon^{tr}(\omega_0-\varpi\,,\mathbf{k})\sim 1$$

As a result the equation (377) takes the form

(380)

$$(\omega^2-\omega_{Le}^2)(\omega_s^2-c^2k_s^2)=\omega_{Le}^2k^2v_E^2/4$$

where $\omega_s=\omega_0-\omega$, $\mathbf{k_s}=\mathbf{k_0}-\mathbf{k}$, $\omega_0\sim ck_0$, $\omega_s\sim ck_s$.

Now we can solve the equation (380) and investigate the process of wave scattering in the plasma. There exist two regimes of scattering: a) when $\omega>>\omega_0$, which is known as Thomson regime and b) when $\omega=\omega_{Le}+i\delta$, $\delta<<\omega_{Le}$, which is called as Raman regime.

For Thomson regime

(381)

$$\Im m\omega=\sqrt{3}\ \mathrm{Re}\omega=\sqrt{3}/2(k^2v_E^2\omega_{Le}^2/8\omega_0)^{1/3}\le(\sqrt{3})/2(\omega_0\omega_{Le}^2v_E^2/2c^2)^{1/3}$$

The maximum value of growth rate corresponds to $k\sim 2k_0\sim 2\omega_0/c$, or the back scattering of the incident wave. This regime takes place if

(382)

$$v_E^2/c^2 >> \omega_{Le}/\omega_0$$

In the opposite case of sufficiently small incident wave amplitude we have Raman scattering

(383)

$$\omega=\omega_{Le}+i\delta,\ \delta=(k^2v_E^2\omega_{Le}/16\omega_0)^{1/3} \le (v_E^2\omega_0\omega_{Le}/4c^2)^{1/2}$$

In this case also the maximal value of growth rate corresponds to the back scattering of incident wave, when $|\mathbf{k}|=|\mathbf{k}-\mathbf{k_0}|\sim 2k_0\sim 2\omega_0/c$. The threshold of this regime is given by

(384)

$$(v_E^2/c^2)\omega_0\ \ \omega_{Le} > \nu_e^2/2$$

In conclusion let us notice that the above results can be easily generalized to the problem of stimulated wave scattering on a relativistic electron beams. This phenemenon is the basis for the very popular high power microwave sources as free electron lasers (FEL). In such devices the relativistic electron beams represent the purely electron nonequilibrium plasmas and therefore the energy of beams can be transformed into the energy of scattering wave.

In other words, the electromagnetic energy generation or amplification takes place. The growth rate of instability and scattered waves frequency in this process can be easily calculated from the above relations by using the Lorentz transformations. Then for the scattered wave frequency one obtains

$$\omega_s = 4\gamma^2\omega_0 \quad (385)$$

where $\gamma=(1-u^2/c^2)^{-1/2}$. This formula indicates the significant shortening of incident wave length, in the order of $4\gamma^2$ times. For example if $\omega_0 \sim 10^{11}$ s^{-1} ($\lambda_0 \sim 3$ cm) and $\gamma \sim 30$ (or beam energy ~ 15 MeV) we have $\omega_s \sim 4*10^{14}$ s^{-1}(or $\lambda_s \sim 10^{-3}$cm~10μ), which corresponds to the optical frequency range.

For growth rate of the scattering processes the Lorentz transformation leads to the following result

$$\delta \sim 2\delta' \quad (386)$$

where δ' are the growth rates in the beam frame and are determined by the relations (380) and (382).

LECTURE 13

Nonlinear Waves and Solitons. Quasi Linear Relaxation of Plasma Oscillations

* 29 Quasi linear relaxation of plasma oscillations

Let us begin from the quasilinear theory of plasma oscillations. Above it was noted that besides of non-linear wave interactions, purely nondissipative processes, exist dissipative nonlinearities in the plasma. As a result of wave absorption the temperature and other equilibrium parameters of a plasma will change, or more exactly, the equilibrium distribution functions of particles will change. This leads to the changing of wave absorption in fact. And finally new distributions and wave amplitudes will be established. The goal of quasilinear theory is to determine of the such new states in the plasma under the action of oscillation fields and their absorption.
Here we will consider for simplicity only potential fields ($\mathbf{E}= -\Delta\phi$) and only simplest problems of quasilinear theory in the absence of external magnetic field. Such systems can be described by the following Maxwell-Vlasov equations:

(387)

$$\partial f/\partial t + \mathbf{v}\partial f/\partial \mathbf{r} + e\mathbf{E}\partial f/\partial \mathbf{p} = 0$$
$$\mathrm{div}\mathbf{E} = 4\pi\Sigma e\int f d\mathbf{p} \Rightarrow \partial \mathbf{E}/\partial t + 4\pi\Sigma e\int f\mathbf{v}d\mathbf{p} = 0$$

The last two equations are completely equivalent. We suppose that the oscillation fields are sufficiently weak and therefore the following condition takes place

(388)

$$(e^2n^{1/3}/T_e)^{3/2}\sim\nu_{ei}/\omega_{Le} << E^2/8\pi nT_e << 1$$

The last of these conditions allows us to neglect the particle collisions, whereas the right side allows us to use the linear approximation for field perturbations:

(389)

$$f(\mathbf{p},\mathbf{r},t) = f_0(\mathbf{p},t) + f_1(\mathbf{p},\mathbf{r},t) = f_0(\mathbf{p},t) + {}_k\Sigma \mathrm{Re} f_{1k}\exp(-i\omega t+i\mathbf{k}.\mathbf{r})$$

Here $f_1(\mathbf{p},\mathbf{r},t) << f_0(\mathbf{p},t)$ is a small perturbation of the equilibrium distribution function $f_0(\mathbf{p},t)$. In the linear approximation $\mathbf{E}(\mathbf{r},t)$ also can be presented as

(390)

$$\mathbf{E}(\mathbf{r},t) = {}_k\Sigma \mathrm{Re} E_k\exp(-i\omega t+i\mathbf{k}\mathbf{r})$$

Substituting (389) and (390) into the equation (387) and averaging in time we obtain two equations for $f_0(\mathbf{p},t)$ and $f_1(\mathbf{p},\mathbf{r},t)$:

(391)

$$\partial f_0/\partial t - e\langle\mathbf{E}\partial f_1/\partial \mathbf{p}\rangle = 0$$
$$\partial f_1/\partial t + \mathbf{v}\partial f_1/\partial \mathbf{r} + e\mathbf{E}\partial f_0/\partial \mathbf{p} = 0$$

Taking into account that $f_0(\mathbf{p},t)$ is slowly varying function, the solution of the second equation (390) can be presented as

(392)

$$f_{1k} = (-ie\mathbf{E}/(\omega-\mathbf{k}\mathbf{v}))(\partial f_0/\partial \mathbf{p})$$

This expression leads to the following formula for dielectric permittivity of a plasma and dispersion equation

(393)

$$\varepsilon(\omega,\mathbf{k}) = 1 + \Sigma(4\pi e^2/k^2)\int(\mathbf{k}\partial f_0/\partial \mathbf{p}/(\omega-\mathbf{kv}))d\mathbf{p} = 0$$

From this equation follows the results of linear theory:

(394)

$$\partial|E_{\mathbf{k}}|^2/\partial t = 2\,\delta_{\mathbf{k}}\,|E_{\mathbf{k}}|^2$$
$$\delta k = (-1/2\,\Sigma 4\pi e^2\omega/k^2)\int d\mathbf{p}\,\Im m(\mathbf{k}\partial f_0/\partial \mathbf{p})/(\omega-\mathbf{kv}))$$

Now we can use the relations (391) and (393) and substituting them into the equation (390) we obtain the equation for $f_0(p,t)$

(395)

$$\partial f_0/\partial t = (\partial/\partial p_i)D_{ij}\partial f_0/\partial p_j$$
$$D_{ij} = -e^2/2(\,_k\Sigma\, k_i k_j / k^2 |E_{\mathbf{k}}|^2\, \Im m\, 1/(\omega-\mathbf{k.v}))$$

The equations (392) - (394) represent the complete system of quasilinear theory for longitudinal oscillations of an isotropic plasma. It can be easily generalized for the anisotropic plasma and arbitrary electromagnetic oscillations. They provide the general conservation laws as are conservation of particles number, their momentum and energy:

(396)

$$d/dt\int f d\mathbf{p} = 0$$
$$d/dt[\Sigma\int \mathbf{p} f d\mathbf{p} + {}_{\mathbf{k}}\Sigma\mathbf{k}|\mathbf{E_k}|^2/8\pi\omega] = 0$$
$$d/dt[\Sigma\int d\mathbf{p}(p^2/2m)f + {}_{\mathbf{k}}\Sigma|\mathbf{E_k}|^2/8\pi] = 0$$

Let us now apply the system of quasilinear theory to the concrete problems. Suppose that at t = 0 in a thermodynamical equilibrium plasma the initial oscillations are given in the definite narrow phase velocity range

(397)

$$W_{\mathbf{k}}(0) \equiv |\mathbf{E_k}(0)|^2/8\pi = \begin{cases} 0 & \text{if } \omega/k \le v_1 \\ W_0 = \text{const} & \text{if } v_1 \le \omega/k \le v_2 \\ 0 & \text{if } \omega/k > v_2 \end{cases}$$

On the Fig. 13 are presented the initial Maxwell distribution $F_0(v,0)$ (one dimensional case) and the velocity range $\omega/k \in (v_1, v_2)$, where $\Delta v = v_2 - v_1 << v_{Te}$

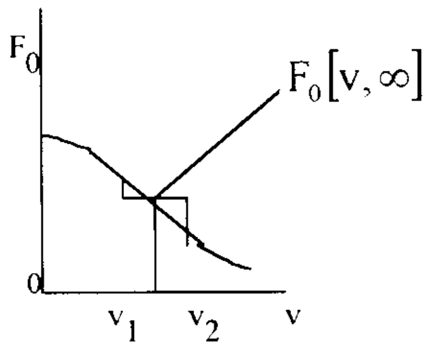

Figure 13

(398)

$$F_0(v,0) = \sqrt{m/2\pi Te}\,\exp(-mv^2/2Te)$$

As a result of oscillations absorption the function $F_0(v,t)$ will be changed and oscillations intensity $W_k(t)$ decreases. The quasilinear equations describing the time dynamics of this process are

(399)

$$\partial F_0/\partial t = \partial/\partial v D \partial F_0/\partial v$$
$$D = -e^2/2m^2 {}_k\Sigma |E_k|^2 \Im m(1/(\omega-\mathbf{kv})) = (\pi/2)e^2|E_k|^2/m^2\mathbf{kv}$$
$$\partial|E_k|^2/\partial t = 2\delta_k |E_k|^2$$
$$\delta_k = -\omega_{Le}^3/4k^2\int dv k\partial f_0/\partial v \Im m(1/(\omega-\mathbf{kv})) = (\pi/2)\omega_{Le}^3/k^2 \partial F_0/\partial \mathbf{v}|_{v=\omega/k}$$

From this system follows that it admits the stationary solutions, if in the velocity region $v \in (v_1,v_2)$ takes place $\partial F_0/\partial v = 0$. As a result $\delta_k = 0$ and $|E_k|^2 \neq 0$. This solution is shown in the fig. 9 and it is known as a "ploto" solution in the final stationary state. From the conservation of the number of particles we find

(400)

$$F_0(v,\infty) = 1/(v_2 - v_1) \; {}_{v_1}\int^{v_2} F_0(v,0)dv$$

Now from (396) we can determine the final state of wave energy

(401)

$$W_k(\infty) - W_k(0) = m^2\omega_{Le}^4/4\pi e^2k^2 \; {}_{v_1}\int^{v_2} dv[1/(v_1-v_2) \; {}_{v_1}\int^{v_2} dvF_0(v,0) - F_0(v,0)]$$

It must be noticed that " ploto" state can be reached only if the initial wave energy is sufficiently large. If not, then the oscillations are completely absorbed before the "ploto" arises on the distribution function.

Finally let us estimate the quasilinear relaxation time. From (398) follows that this time is order of

(402)

$$t_r \sim v^2/D \sim v^2 nT_e/v_{Te}\omega_{Le}W_k \sim (\omega_{Le}^2)/k^2v_{Te}^2(nT_e/W_k)(1/\omega_{Le}) >> 1/\omega_{Le}$$

Moreever this time is more than $1/\delta_k$

Let us now consider another example of the quasilinear relaxation. Namely we will consider the relaxation of "cold" monoenergetic beam interacting with "cold" plasma. For simplicity we will consider nonrelativistic beam, when purely potential plasma waves can be excited (see lecture 10) with spectrum

(403)

$$Re(\omega) = \mathbf{k.u} = \omega_{Le}, \quad \Im m(\omega) = \delta_{\mathbf{k}} = (\sqrt{3})/2(n_b/2n_p)^{1/3}\omega_{Le}$$

These oscillations acted on the beam electrons which leads to change its distribution function, the beam electrons decelerated and their velocity spread increased. Really from the quasilinear equations (399) in the case of one dimensional instability we have

(404)

$$\partial F_0/\partial t = (\partial/\partial v)D\partial F_0/\partial v$$
$$D = -e^2/2m^2({}_k\Sigma|E_k|^2\Im m(1/(\omega-kv)) = (-e^2/2m^2)\delta_k\Sigma|E_k|^2$$
$$\partial|E_k|^2/\partial t = 2\delta|E_k|^2$$

Here $\delta_k=\delta$, where δ is given by (403) and it is independent from k.

The system (404) must be solved in taking into account initial conditions

(405)

$$F_0(v,0) = \delta(v-u), \ |E_k(0)|^2 = W_k(0)$$

After introducing the variables

$$d\tau/dt = D(t), \ d/dt=(d\tau/dt)d/d\tau = Dd/d\tau \tag{406}$$

the problem can be reduced to

$$\partial F_0/\partial\tau = \partial^2 F0/\partial v^2, \ \partial D/\partial\tau = 2\delta, \ D= (-4\pi e^2/m^2\delta)W \tag{407}$$

where $W = {}_k\Sigma W_k$ is the total energy of oscillations and the conditions (404) takes the form

$$F_0(v,0) = \delta(v-u), \ D(v,0) = D_0=(-4\pi e^2/M^2\delta)W(0) \tag{408}$$

The solution of (407) in taking into account (408) has the form

$$F_0(v,\tau) = \sqrt{(m/2\pi\tau)}\exp(-(m(v-u)^2/2\tau) \tag{409}$$

We see that the beam distribution function becomes more and more widespread in a time. This is result of increasing in time of the diffusion coefficient D(t), that follows from (406)

$$D(v,\tau) = D_0+2\delta\tau \Rightarrow 2\delta\tau \tag{410}$$

The solution (409) is valid until

$$(n_b/2n_p)^{1/3} > v_{Te}/u = \sqrt{(2\tau/mu^2)} \tag{411}$$

After that the instability becomes kinetic which can be considered as above was investigated the "ploto" creation on the distribution function. Here also the "ploto" creation takes place. From this condition we can estimate the relaxation time

$$T_{max} = 2\tau_{max} = mu^2(n_b/n_p)^{1/3}, \ \tau_p \sim 1/\delta=(2n_p/n_b)^{1/3}(1/\omega_{Le}) \tag{412}$$

We see that on the hydrodynamical stage of instability the $2(n_b/2n_p)^{1/3}$ part of energy transfers into the thermal energy of beams electrons.

* 30 Solitons and nonlinear waves in plasmas

In the last section of our lectures we will consider some exact solutions of the field equations in the nonlinear plasma media. As above we will restrict ourselves by consideration of a collisionless plasma neglecting all the dissipative processes.

30.1) Up to day the most developed theory of solitons is in the plasma. The basic theory of solitons will be discuss below. Before let us remind that in the plasma exist two types of dispersion dependence $\omega(k)$: accoustical type with

$$\omega(k) = kv_0-\beta k^3, \ \beta = v_s r_{De}^2, \ v_0=v_s \tag{413}$$

and optical type for which

(414)

$$\omega(k) = \omega_0 + \beta k^2,\ \omega_0=\omega_{Le},\ \beta = 3/2(v_{Te}r_{De})$$

The quantities v_0, β and ω_0 are given for ion-accoustic and Langmuir oscillations in the isotropic plasma (see lecture 6)
The field equations corresponding to the relations (413) and (414) may be written as

(415)

$$\partial\phi/\partial t + v_0\partial\phi/\partial x + \beta\partial^3\phi/\partial x^3 = 0$$

(416)

$$i\partial\phi/\partial t - \omega_0\phi + \beta\partial^2\phi/\partial x^2 = 0$$

Let us now generalize the equations (415) and (416) by taking into account the nonlinear effects. In the case of ion-accoustic oscillations this is the electron heating under the action of oscillations which leads to the increasing of v_0

(417)

$$v_0 = v_s + \alpha\phi^{m-1}$$

where $m>1$ and α is the wave absorption intensity. At the same time substituting (417) into the (415) we obtain the nonlinear equation

(418)

$$\partial\phi/\partial t + v_s\partial\phi/\partial x +(\alpha/m)\partial\phi^m/\partial x + \beta\partial^3\phi/\partial x^3= 0$$

which is known as kdv equation. This equation admits the exact solution. Really representing $\phi = \alpha^{(1-m)}f(x-v_s t,t)$ can be written as

(419)

$$\partial f/\partial t +(1/m)\partial f^m/\partial x + \beta\partial^3 f/\partial x^3 = 0$$

Below we will consider the solutions of the equation (418) depending only on ξ =x-vt, where v = const. Then from (418) follows

(420)

$$\beta\partial^2 f/\partial\xi^2 -vf + f^m/m = C_1 = \text{const}$$

The solution of this equation under the condition $f(\xi)\to 0$ if $\xi\to\infty$ can be easily found. From this follows that C_1=0 and the solution of kdv equation looks as a soliton

(421)

$$f(\xi) = f\max/ch^{2n}(\xi/\Delta)$$

where

(422)

$$n=1/(m-1) >0,\quad f^{m-1}_{max} = 2(1+n)(1+2n)\beta/\Delta^2,\ v = 4n\beta/\Delta^2$$

On the Fig. 14 it is shown the solution (421) for $\beta >0$

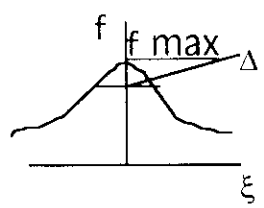

Figure 14

If $\beta<0$, then the solution occurs to be negative and symmetrical, or $f \to -f$, $\xi \to -\xi$ and $v \to -v$. Thus the soliton type solution exists for arbitrary sign of β. In the case when m=2 this solution corresponds to the ion-accoustic soliton with $\beta = v_s r_{De}^2$.
Let us now consider the optical soliton. In order of this let us return to the equation (420) and take into account the nonlinear dependence of ω_0 on $|\phi|$. If this dependence looks as (the result of field dependence on plasma density)

(423)

$$\omega_0 = \omega_{00} - \alpha|\phi|^{2m},$$

where $m>0$, from (420) then we obtain

(424)

$$i\partial\phi/\partial\tau - \omega_{00} - \alpha|\phi|^{2m}\phi + \beta\partial^2\phi/\partial x^2 = 0$$

This equation is known as nonlinear Schrœdinger equation (NS). In the "stationary" case, when $\phi = \psi\exp(-i\omega t)$ from (424) follows

(425)

$$\partial^2\psi/\partial x^2 + ((U(x,t)/6\beta)+\varepsilon)\psi = 0$$

Here $U(x,t)=-6\alpha|\psi|^{2m}$ is the potential energy slightly varying in time and $\varepsilon=(\omega-\omega_0)/\beta$ is the own value of this equation. If $U(x,t)>0$ in the finite region of x and $U(x,t) \Rightarrow 0$ when $x \Rightarrow \pm\infty$, then in this region there exists the localized solution of (425). Moreover if U(x,t) represents the solution of kdv equation then the own value ε = const. In this case we have optical soliton which is rounded by accoustic soliton. For ion accoustic soliton with n=1 we have

(426)

$$U(x,t) = (6\beta U_0)/ch^2(\xi/A) = -6\alpha|\psi|^{2m}$$

where $\xi=x-vt$. Then the equation (424) with m=1 has the solution as a hypergeometrical function ψ_k which corresponds to the eigenvalue with $k < s$

(427)

$$s=1/2(\sqrt{(1+4U_0\,\Delta^2)}-1), \qquad \xi_k = -(s-k)^2/\Delta^2$$

Taking into account that m=1, from (422) we have $U_0\Delta^2=2$ and therefore s=1 and n=0, or

(428)

$$\varepsilon=-1/\Delta^2 = -U_0/2$$

Such soliton is called a rounded soliton.

30.2) The nonlinear transverse wave in a cold plasma with nonzero components $B_x(z,t)$ and $E_y(z,t)$ and frequency near to the electron Langmuir frequency can be described by the following system of equations

(429)

$$\partial B_x/\partial z - 1/c\partial E_y/\partial t = (4\pi/c)eN_e v_y, \quad \partial v_y/\partial t = (e/m)E_y, \qquad \partial E_y/\partial z - 1/c\partial B_x/\partial t = 0$$

This system of equations must be completed by the density distribution in the average potential of high-frequency transverse field. If Te = const then

(430)

$$N_e = N_0\exp(-e^2|Ey|^2/4m\omega^2Te) \sim N_0(1-(e^2|Ey|^2/4m\omega^2Te))$$

Taking into account this relation from (429) we obtain one equation for E_y:

$$i\partial E_y/\partial t - \omega_{Le} E_y - c^2/\omega_{Le} \partial^2 E_y/\partial z^2 + (1/8m\omega_{Le}) e^2/Te |E_y|^2 E_y = 0 \tag{431}$$

This equation represents nonlinear Schrodinger equation with potential $U(z,t) = 3/4(e^2|E_y|^2/m\omega_{Le}Te$. Therefore if $|E_y|^2 \to 0$ when $z \to \pm\infty$, then this equation describes an optical soliton which is surrounded by the function $|E_y|^2(z,t)$. This phenomenon is known also as self focusing of nonlinear transverse wave.

PART 3

Problems and their Solutions

LECTURE 14

The Problems for thermodynamically equilibrium plasmas

Problem 1 (Lecture 1)

For the purely electron plasmas draw the n(T) diagrams and indicate the regions of degeneration and gaseous approximation

Solution

The region of degeneration is determined by the inequality

$$\varepsilon_F = ((3\pi^2)^{3/2}\eta^2 n^{2/3})/2m \geq T \quad (1)$$

On the figure 1 it corresponds to the line 1

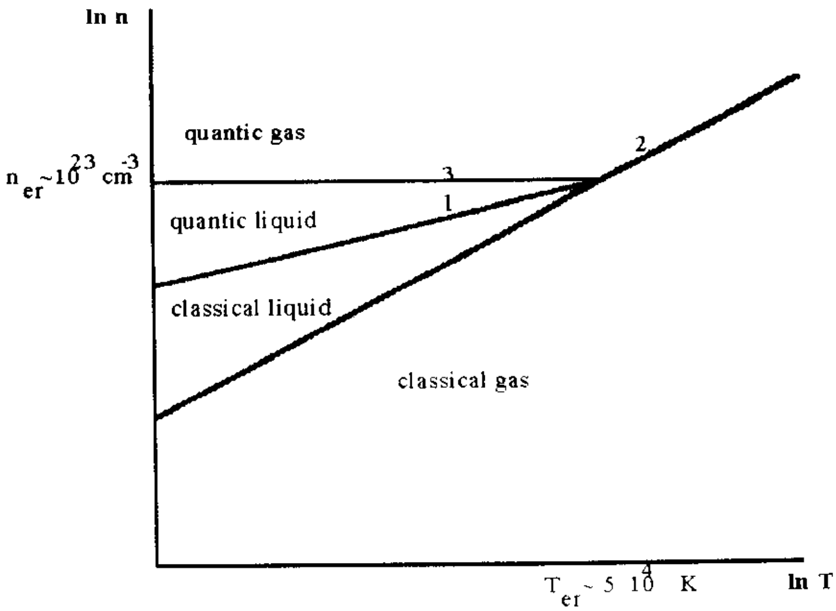

Figure 1

The region of gaseous approximation for the classical electron plasmas is defined from the inequality

$$\eta_{cl} = (e^2 n^{1/3})/T \leq 1 \quad (2)$$

and on the Fig. 1 is given by line 2. Finally, the region of gaseous approximation for the quantum degenerate plasma is determined by the line 3 on the Fig. 1, which is

(3)

$$\eta_{quant} = (e^2 n^{1/3})/\varepsilon_F \leq 1$$

All these three lines intersected at the point A, corresponding to $n_{er} \sim 10^{23}$ cm^{-3} and $T \sim \varepsilon_F \sim$ 5eV (or $5 \cdot 10^4$K). For real metals at $T \sim 300$K and $n \sim 10^{22}$ cm^{-3} we have $\varepsilon_F \sim 1eV \sim 10^4 K > T$; therefore the metallic plasmas occur degenerate. At the same time $\eta_{quant} \geq 1$, which means that they are sufficiently nonideals, or quantum liquids. Quite opposite situation takes place for the ionospheric plasma, where $n \sim 10^6$-10^7 cm^{-3} and $T \sim 10^4$K. As a result, $\varepsilon_F << T$ and $\eta_{el} \sim 10^{-5} << 1$, or the ionospheric plasma represents quite ideal classical gas.

Problem 2 (Lecture 2)

Using the model of independent particles calculate the energy loss of a fast nonrelativist charged particle in an isotropic plasma.

Solution

The energy loss of charged particles is determined by the work of Lorentz force on this particle

(1)

$$W = \mathbf{F.v}/v = q(\mathbf{vE})/v \,|_{\mathbf{r} = \mathbf{v}t}$$

Here q is a charge of particle and v its velocity, E (r,t) is the field induced in a plasma by the moving particle. As v << c the potential approximation is valid and therefore $\mathbf{E} = -\nabla\phi$, and ϕ satisfies the Poisson's equation

(2)

$$\text{div}\, \mathbf{D} = 4\pi q\delta(\mathbf{r} - \mathbf{v}t)$$

Using the Fourier expansion

(3)

$$D(\mathbf{r}) = \int d\mathbf{k} \exp(i\mathbf{kr})D(\mathbf{k}),\ \delta(\mathbf{r} - \mathbf{v}t) = (1/(2\pi)^3)\int d\mathbf{k} \exp(i\mathbf{k}(\mathbf{r}-\mathbf{v}t))$$

and taking into account the relation for isotropic plasma

(4)

$$D(\omega,\mathbf{k}) = \varepsilon(\omega)E(\omega,\mathbf{k}) = -i\mathbf{k}\varepsilon(\omega)\phi(\omega,\mathbf{k})$$

where in the model of independent particles

(5)

$$\varepsilon(\omega) = 1 - (\omega_{Le}^2/(\omega(\omega + i\nu e))$$

After very easy calculations we obtain

(6)

$$\phi(\mathbf{r},t) = (q/2\pi^2)\int d\mathbf{k} \exp(i\mathbf{k}(\mathbf{r}-\mathbf{v}t)/k^2\varepsilon(\mathbf{kv}))$$

Now we can substitute this expression into (1) and taking into account (5) calculate the searched quantity. These calculations occur to be easy in the collisionless limit when $\nu_e \lozenge 0$. Then from (1) follows

(7)

$$W = (-2q^2/\pi v^2)\ {}_0\!\int^{\infty}\omega d\omega\ {}_0\!\int^{\infty} (\xi d\xi/(\xi^2 + \omega^2/v^2))\Im m(1/\varepsilon(\omega))$$

$$= (2q^2/v^2)\ {}_0\!\int^{\infty}\omega d\omega\delta(\varepsilon(\omega))\ {}_0\!\int^{\infty}(\xi d\xi/(\xi^2 + \omega^2/v^2))$$

$$= (q^2\omega_{Le}^2/v^2)\ {}_0\!\int^{\infty}\xi d\xi/(\xi^2 + \omega_{Le}^2/v^2)$$

$$\approx (q^2\omega_{Le}^2/2v^2)\ln(v^2\xi^2_{max}/\omega_{Le}^2) = (q^2\omega_{Le}^2/2v^2)\ln(v^2/v_{Te}^2)$$

Here we take into account that the independent particles model is valid when $\omega >> kv_{Te}$ and introduce the maximal value of $\xi_{max} = \omega_{Le}/v_{Te}$

Problem 3 (Lecture 3)

Show that the Alfven wave with spectrum $\omega = k_{//} v_A$ is the exact solution of nonlinear M.H.D. equations for noncompressing and ideal conductive liquid.

Solution

For incompressible liquid $\rho = \rho_0$ = const, and therefore div **v** = 0. This means that the fluid velocity **v** as well as the magnetic field occur to be transverse. Then if all the quantities are depending only on the longitudinal space component z which supposed parallel to $\mathbf{B}_0$ from the M.H.D equations follows:

(1)

$$\partial\mathbf{B}/\partial t = \mathbf{B}_0\partial\mathbf{v}/\partial z,\ \partial\mathbf{v}/\partial t = (\mathbf{B}_0/4\pi\rho_0)\partial\mathbf{B}/\partial z$$

Here **v** and **B** are perturbations and ρ_0 and $\mathbf{B}_0$ represent the equilibrium quantities. It is easy to see that the system (1) has the exact solution of type $\exp(-i\omega t+ik_z z)$ and for ω and k_z the following coupling exists

(2)

$$\omega^2 = k_z^2 v_A^2$$

which is valid not only in linear approximation but in nonlinear too.

Problem 4 (Lecture 4)

Calculate the plasma static conductivity in the Lorentz gaseous approximation

Solution

In the Lorentz gaseous approximation only electron-ion collisions are taken into account. This approximation is valid when $z=|e_i/e| >> 1$. Then the Landau kinetic equation for the plasma electrons in a static external electric field $\mathbf{E}_0$ takes the form

$$e\mathbf{E}_0\partial f_e/\partial \mathbf{p} = 2\pi z^2 e^4 n_i L(\partial/\partial p_i)((v^2\delta_{ij}-v_iv_j)/v^3)\partial f_e/\partial p_j) \quad (1)$$

The solution of this equation is represented as

$$f_e = f_\mu + \mathbf{v}\mathbf{f}_1/v \quad (2)$$

where f_μ is the Maxwellian distribution and $|f_1| << f_\mu$. Then from (1) follows

$$e\mathbf{E}_0\partial f_\mu/\partial \mathbf{p} = (4\pi z^2 e^4 n_i L/m^2)\mathbf{v}\mathbf{f}_1/v^4 \quad (3)$$

As a result of solution of this equation we obtain

$$f_e = f_\mu + (e\mathbf{E}_0/m\nu(v))\partial f_\mu/\partial \mathbf{v},\ \nu(v) = (4\pi e^4 z^2 n_i L)/m^2 v^3 \quad (4)$$

Substituting this expression into the formula of current density we have

$$\mathbf{j} = e\int \mathbf{v} f_e\, d\mathbf{p} = (32/3\pi)(e^2 n_e/m\nu_{eff})\mathbf{E}_0 \equiv \sigma\, \mathbf{E}_0 \quad (5)$$

where

$$\sigma = (32\, e^2\, n_e)/3\pi m\nu_{eff},\quad \nu_{eff} = (4/3)(\sqrt{2\pi}/m)(e^4 z^2 n_i L/T_e^{3/2}) \quad (6)$$

is the static conductivity of plasma in the Lorentz approximation, which differs from the Spitzers formula (91) (see Lecture 5) by the factor of the order of ≈1,5.
From the condition $|f_1| << f_\mu$ follows that the above obtained results are valid only if

$$\langle v\rangle \approx eE_0/m\nu_{eff} << v_{Te} \quad (7)$$

Or in other words if

$$E_0 < E_{cr} = m\nu_{eff}v_{Te}/e \quad (8)$$

This critical field E_{cr} coincides with the Dreicer's field introduced in the lecture (10) for completely ionized plasma.

Problem 5 (Lecture 5)

Starting from the equations of two fluids hydrodynamics of " cold" collisionless plasma calculate the averaged force acting on the plasma in the external inhomogeneous microwave field.

Solution

The equation of motion for electrons in this case can be presented as

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} = (e/m)\mathbf{E}(r,t) + (e/mc)[\mathbf{v} * (\mathbf{B}_0+\mathbf{B}(r,t))] \tag{1}$$

Here B_0 is the external homogeneous magnetic field and $\mathbf{E}(r,t)$, $\mathbf{B}(r,t)$ are the components of microwave field with frequency ω_0:

$$\mathbf{E}(r,t) = \mathbf{E}(r)\sin(\omega_0 t),\ \mathbf{B}(r,t) = (c/\omega_0)\mathrm{rot}\mathbf{E}(r)\cos\omega_0 t \tag{2}$$

In the first approximation from (1) follows

$$\partial \mathbf{V}_0/\partial t = (e\mathbf{E}(r,t)/m) + e/mc[\mathbf{V}_0 * \mathbf{B}_0] \tag{3}$$

The solution of this equation is

$$\mathbf{V}_0 = \mathbf{V}_1\sin\omega_0 t + \mathbf{V}_2\cos\omega_0 t = (e^2/m^2c)([\mathbf{E}(r)*\mathbf{B}_0]/(\omega_0^2-\Omega^2))\sin\omega_0 t - \tag{4}$$

$$(e\omega_0/m)(1/(\omega_0^2-\Omega^2))\{\mathbf{E}(r)-(e^2/m^2c^2)(1/\omega_0^2)\mathbf{B}_0(\mathbf{B}_0\mathbf{E}(r))\}\cos\omega_0 t$$

where $\Omega = eB_0/mc$ is the Larmor frequency for electrons.
Substituting the expression (4) into the nonlinear terms of the equation (1) and averaging over ω_0 we obtain the averaged force acting on the electron component of plasma

$$\mathbf{F}_{av} = -m <(\mathbf{V}_0\nabla)\mathbf{V}_0> + (e/\omega_0) <[\mathbf{V}_0*\mathrm{rot}\mathbf{E}(r)]\cos\omega_0 t> \tag{5}$$

$$=(-m/2)\{(\mathbf{V}_1\nabla)\mathbf{V}_1 + (\mathbf{V}_2\nabla)\mathbf{V}_2 -(e/\omega_0)[\mathbf{V}_2*\mathrm{rot}\mathbf{E}(r)]\}$$

In the absence of external magnetic field, $B_0 = 0$, this expression becomes very simple

$$\mathbf{F}_{av} = -(e^2/4m\omega_0^2)\nabla E^2(\mathbf{r}) \tag{6}$$

This force is known as Miller's force

Problem 6 Lecture (6)

Investigate transverse electromagnetic field penetration into the isotropic purely electron plasma as a function of its frequency ω.

Solution

The transverse field penetration into the isotropic plasma is described by the solutions $k(\omega)$ of equation

$$k^2c^2 - \omega^2\varepsilon^{tr}(\omega,\mathbf{k}) = 0 \tag{1}$$

Penetration depth is determined by the relation

$$\lambda_{sk} = 1/\Im m k(\omega) \tag{2}$$

Let us consider the different limiting cases.

a) In the frequency range $\omega >> \nu_e, kv_{Te}$

$$\varepsilon^{tr}(\omega,\mathbf{k}) = 1-(\omega_{Le}^2/\omega^2)(1-i\nu_e/\omega) \tag{3}$$

Substituting this expression into the (1) we obtain

$$\lambda_{sk} = \begin{cases} 2c\omega^2/\omega_{Le}^2\nu_e & \text{if } \omega > \omega_{Le} > \nu_e \\ c/\omega_{Le} & \text{if } \nu_e << \omega << \omega_{Le} \end{cases} \tag{4}$$

b) In the frequency range $kv_{Te} >> \omega, \nu e$

$$\varepsilon^{tr}(\omega,\mathbf{k}) = 1 + i\sqrt{(\pi/2)}(\omega_{Le}^2/\omega k v_{Te}) \tag{5}$$

After substitution of this expression into the (1) we obtain

$$\lambda_{sk} = 2(\sqrt{(2/\pi)}(c^2 v_{Te}/\omega_{Le}^2\omega))^{1/3} \tag{6}$$

This expression coincides with the penetration depth of transverse field in the case of anomalous skin-effect, which takes place under the conditions

$$(v_{Te}/c)\omega_{Le} >> \omega >> \omega^* = \nu_e(\nu_e^2 c^2/\omega_{Le}^2 v_{Te}^2), \lambda e \tag{7}$$

c) And finally, when $\nu_e >> kv_{Te}, \omega$

$$\varepsilon^{tr}(\omega,\mathbf{k}) = 1 + i\alpha(\omega_{Le}^2/\omega\nu_e), \tag{8}$$

which leads to the expression for penetration depth in the case of normalous skin-effect

(9)

$$\lambda_{sk} = (2\nu_e c^2/\alpha\omega\omega_{Le}^2)^{1/2}$$

It is obvious that in this case ω ω^*,ν_e, and $\alpha = 1$ corresponds to weakly ionized plasma whereas $\alpha = 1.96$ to completely ionized one.
The results of the analyses is presented in the Fig. 2.

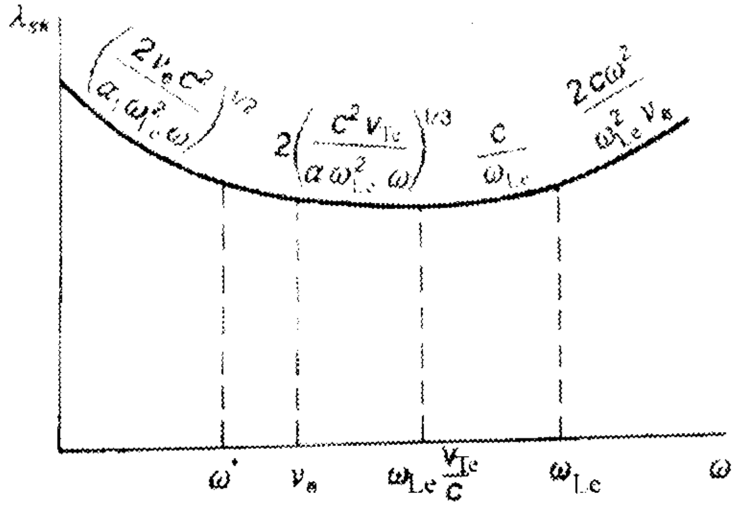

Figure 2

Problem 7 (Lecture 7)

Show that for purely longitudinal propagation of transverse electromagnetic waves in a magnetized plasma it appears the collisionless absorption if the relativistic effect of electrons mass dependence on their velocity is taken into account, even when the spatial dispersion of dielectric permittivities is completely neglected.

Solution

From the general expression of dielectric permittivity of magnetized plasma (179) follows that when spatial dispersion is neglected (k α0), the nonrelativistic wave absorption completely vanishes. However if we take into account the relativistic dependence of electrons mass on its velocity the wave absorption appears again even when k α 0. Really if in the (179) we suppose

$$m = m_0\gamma = m_0\,(1-v^2/c^2)^{-1/2} \approx m_0(1+v^2/2c^2) \qquad (1)$$

and in replacing Ω_e by Ω_e/γ, we obtain

$$\varepsilon_\perp = 1 + (2\pi e^2/\omega)\int d\mathbf{p}(\partial f_0 e/\partial\varepsilon)(v_\perp^2/\omega \pm \Omega_e/\gamma) = 1 + \qquad (2)$$

$$(4\pi e^2/3\omega)\int dp v^2(\partial f_0 e/\partial\varepsilon)[\wp/(\omega \pm \Omega e/\gamma) - i\pi\delta(\omega \pm \Omega e/\gamma)]$$

For the Maxwellian distribution f_{0e} this leads to

(3)

$$\varepsilon_\perp(\omega) = 1 - \omega_{Le}^2 / (\omega(\omega - \Omega_e)) +$$

$$i(4\pi\omega_{Le}^2 c^5 / 3\omega\Omega_e v_{Te}^5)((\Omega_e - \omega)/\omega)^{3/2} = \begin{cases} 1 & \text{if } \omega < \Omega_e \\ 0 & \text{if } \omega > \Omega_e \end{cases}$$

Here we restrict ourselves by $\omega/\Omega_e < 1$, $|\omega - \Omega_e| << \Omega_e$. After substituting this expression into the dispersion equation we obtain

(4)

$$k^2c^2 = \omega^2\varepsilon_\perp(\omega) \approx -(\omega\omega_{Le}^2/(\omega - \Omega_e))[1 + (i4\pi c^5/3v_{Te}^5)((\Omega_e - \omega)/\Omega_e)^{5/2}]$$

Comparing this equation with (180) we conclude that its solution ($\omega\alpha\omega + i\delta$)

(5)

$$\omega = \Omega_e - \omega\omega_{Le}^2/k^2c^2, \ \delta = -4\pi c^5/3v_{Te}^5(\omega_{Le}^2/k^2c^2)^{7/2}$$

differs from (190) by the existence of non exponentially small purely relativistic wave absorption ($\delta \neq 0$).

LECTURE 15

THE PROBLEMS ON NONEQUILIBRIUM PLASMAS

Problem 8 Lecture(8)

Investigate the electromagnetic waves in a planar layer of "cold" collisionless plasma

Solution

Let us orientate the **OX** axis perpendicular to the surface of layer. Then the electromagnetic field equations for the components Ex, Ez, By (TM-mode) can be reduced to the one equation for Ez:

(1)

$$\partial^2 E_z/\partial X^2 - k_z^2 E_z - (\omega^2/c^2)\varepsilon(x)E_z = 0$$

Here

(2)

$$\varepsilon(x) = 1 - \omega_{Le}^2(x)/\omega^2$$

and k_z is the wave vector along the surface.
Equation (1) is valid not only in the plasma-layer, $0 \le x \le a$, but outside of it, $x \le 0$, and $x \ge a$. In the layer we suppose $\varepsilon = \varepsilon(\omega) = \text{const}$ because $n_e = n_0 = \text{const}$, whereas outside the layer $\varepsilon = 1$. Taking into account this from the equation (1) one can easily obtain the boundary conditions

(3)

$$\{E_z\}_{x=0,a} = 0, \quad \{B_y\}_{x=0,a} = 0$$

Here

(4)

$$By = -(i\omega/c)(\varepsilon(x)/(k_z^2 - \omega^2\varepsilon(x)/c^2)$$

Now we can find the solution of (1) in the form

(5)

$E_z(x) =$	$\{ C_1\exp(\chi_0 x)$	$x \le 0$
	$\{C_2\exp(\chi x)+C_3\exp(-\chi x)$,	$0 \le x \le a$
	$\{C_4\exp(-\chi_0 x)$,	$x \ge a$

where $\chi^2 = k_z^2 - (\omega^2/c^2)\varepsilon(\omega)$, $\chi_0^2 = k_z^2 - \omega^2/c^2$. . Besides it was supposed that $\chi_0^2 > 0$ and $\chi^2(\omega) > 0$ for surface type waves and $\chi^2 < 0$ for bulk waves.
After substituting the solution (5) into the boundary conditions (3) (taking into account (4) we obtain the following dispersion equation for the small electromagnetic oscillations of a plasma layer

(6)

$$(\varepsilon^2(\omega)/\chi^2 - 1/\chi_0^2)^2\exp(-\chi a) = (\varepsilon(\omega)/\chi + 1/\chi_0)^2\exp(\chi a)$$

For the surface wave ($\chi^2 > 0$) in the short wave length limit from (6) follows

(7)

$$\varepsilon(\omega)\chi_0 + \chi = 0$$

This equation coincides with (233) for the surface waves in semibounded plasma. At the same time, in the long wave limit when $\chi a << 1$ independently of sign χ^2 we have

(8)

$$1 + \varepsilon(\omega)a\chi_0 = 0$$

The solutions of this equation exist only when $\varepsilon(\omega) < 0$, or $\omega < \omega_{Le}$ and then

(9)

$$\omega \approx \begin{cases} k_z c, & \text{if } \omega << \omega_{Le}\sqrt{(|k_z| a} \\ \omega_{Le}\sqrt{(|k_z| a)}, & \text{if } \omega << |k_z| c \end{cases}$$

Finally if $\chi^2 < 0$ (bulk waves) and $|\chi| a \geq 1$, we obtain from (6)

(10)

$$\chi^2 a^2 = \pi^2 n^2$$

$n = 1,2,...$ which leads to the bulk waves dispersion relation in a plasma layer.

(11)

$$\omega^2 = k_z^2 c^2 + \pi^2 n^2 c^2/a^2 + \omega_{Le}^2$$

<u>Problem 9 (Lecture 9)</u>

Using the equation(285) investigate the fluid instability in a strongly collisional plasma in the mirror geometrical magnetic field confined plasma

Solution

As it was shown in lecture 9 in the mirror magnetic field confined plasma arises the real drift of particles with velocity $u_\alpha = -g_\alpha/\Omega_\alpha$. For takin into account this drift in the dispersion equation of strongly collisionnal inhomogeneous plasma (285) the following change must be done $\omega \alpha\ \omega'_\alpha = \omega - k_y u_\alpha$. As a result this equation for " cold" collisionnal plasma ($\Gamma_\alpha\ \alpha 0$) takes the form

(1)

$$\varepsilon(\omega,\mathbf{k},x) = 1 + {}_\alpha\Sigma(\omega_{L\alpha}{}^2/k^2)(\omega'_\alpha + i\nu_\alpha)/\omega'_\alpha\{k_\perp{}^2/\Omega_\alpha{}^2 - k_z{}^2/(\omega'_\alpha + i\nu_\alpha)^2 +$$

$$(k_y/\Omega_\alpha(\omega'_\alpha + i\nu_\alpha))\partial \ln N_\alpha/\partial x\} = 0$$

Let us now suppose that

(2)

$$\omega'_e << \nu_e,\ \ \omega'_i >> \nu_i,\ \ \omega >> k_y u_\alpha$$

Then from (1) follows

(3)

$$(1 + c^2/v_A{}^2)k_\perp{}^2 + i4\pi\sigma k_z{}^2/\omega - (c^2\, g_{eff} k_y{}^2/v_A{}^2\omega^2)\partial \ln N/\partial x = 0$$

where $\sigma = \omega_{Le}{}^2/4\pi\nu_e$ is the plasma conductivity and $g_{eff} = -(v_s{}^2 + v_{Te}{}^2)/\Omega_i R$. For fluid type oscillations with $k_z\ \alpha 0$, from (3) we find unstable solution

(4)

$$\omega^2 = -(g_{eff}k_y^2/(1+v_A^2/c^2))\partial lnN/\partial x(1/k_\perp^2) \sim -g_{eff}\,\partial lnN/\partial x$$

Instability takes place if $g_{eff}\,\partial lnN/\partial x > 0$, or, in other words, when the curvature of field is positive. At the same time, in accordance of (3) the finite value of k_z leads to the stabilisation of fluid instability when

(5)

$$k_z^2 k_\perp^2 \quad L_\perp^2 L^{-2} > (1+4\pi\sigma)\nu\,|g_{eff}\partial lnN/\partial x|$$

Problem 10 (Lecture 10)

Show that in a purely electron plasma in the external microwave electric field it arises the parametric instability when the relativistic effects in the motion of electrons are taken into account.

Solution

Supposing that the microwave field has the form

(1)

$$\mathbf{E_0}(t) = \mathbf{E_0}\sin\omega t$$

the relativistic equation of motion leads to

(2)

$$\mathbf{u}(t)/\sqrt{1-u^2(t)/c^2} = -(e\mathbf{E_0}/m\omega_0)\cos\omega t$$

Taking into account this relation the hydrodynamics equations for the small perturbation of motion can be written as

(3)

$$\partial\delta N/\partial t + div(N_0\delta N\mathbf{u}(t)) = 0$$

$$(\partial/\partial t + \mathbf{u}(t)\partial/\partial\mathbf{r})(\delta v + \mathbf{u}(\mathbf{u}\partial\mathbf{V})/c^2)/(1-u^2/c^2)^{3/2}) = -e/m\nabla\phi$$

$$\Delta\phi = -4\pi e\delta N$$

Here we suppose that the electric field of perturbation is potential, $\mathbf{E} = -\nabla\phi$

Below we will restrict ourself by consideration of the perturbations $\sim \exp(i\mathbf{kr})$ in which k//u. Then the system (3) can be reduced to one equation for δN:

(4)

$$(\partial/\partial t+\mathbf{ku})(1/(1-u^2/c^2)^{3/2})(\partial/\partial t + i\mathbf{ku})\delta N = -\omega_{Le}^2\delta N$$

where $\omega_{Le} = \sqrt{4\pi e^2 N_0/m}$ is the Langmuir frequency. Using the substitution

(5)

$$y = (1/(1-u^2/c^2)^{3/2})\partial\delta N/\partial t\exp(i\int^t \mathbf{ku}(t')dt')$$

from (4) we obtain

(6)

$$\partial^2 y/\partial t^2 + \omega_{Le}^2(1-u^2(t)/c^2)^{3/2}\,y = 0$$

In a weakly relativistic case when $u^2/c^2 << 1$ from (6) follows

$$d^2y/dt^2 + (a-2q\cos 2\omega_0 t)y = 0 \tag{7}$$

where

$$a = (\omega_{Le}^2/\omega_0^2)(1-3/4(u_0^2/c^2)); \quad q = 3/8(au_0^2/c^2) << 1, \ u_0 = eE_0/m\omega_0 \tag{8}$$

In the Fig. 3 was shown phase diagram a (q) and to the parametric instabilities correspond stroking regions.

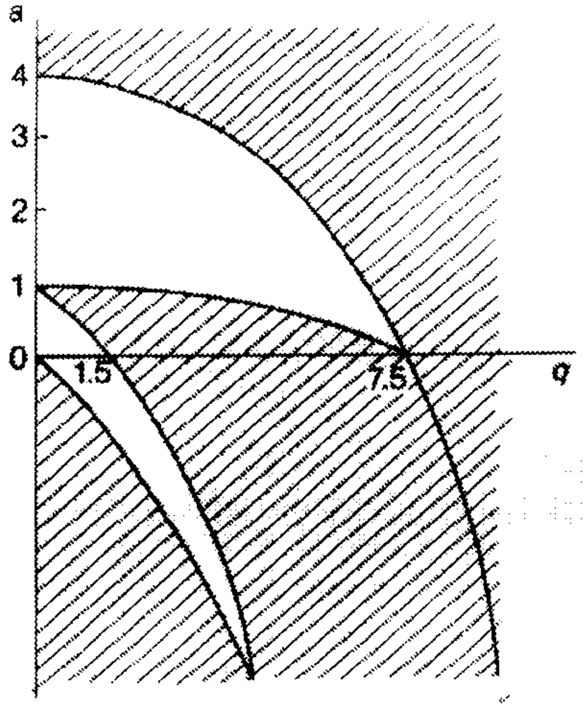

Figure 3

When q << 1 these regions are determined by the condition $a = n^2$, or

$$n^2\omega_0^2 = \omega_{Le}^2 (1-3/4u_0^2/c^2) \tag{9}$$

and growth rate of y ($y \sim \exp(\delta t)$) is equal to

$$\delta = 3/16(\omega_{Le}u_0^2/c^2) \tag{10}$$

Problem 11 (Lecture 11)

Show that the beam- plasma corresponding to the stimulated Cherenkov radiation takes place in a "cold" collisionnal plasma also

Solution

Let us consider only nonrelativistic case, when $u \ll c$ and potential approximation is valid. The dispersion equation for potential perturbations looks as (see equation (354))

(1)

$$1 - \omega_{Le}^2/\omega^2(1-i\nu_e/\omega) - \omega_b^2/(\omega-ku)^2 = 0$$

Here we suppose that $\omega \approx \omega_{Le} \gg \nu_e$. Besides we had neglected the beam electron collisions which practically always is possible.

In the absence of beam ($\omega_b \alpha 0$) equation (1) describes the wellknown plasma oscillations, which are damping due to the plasma electron collisions ($\omega \alpha \omega + i\delta_0$)

(2)

$$\omega = \omega_{Le}, \quad \delta_0 = -\nu_e/2$$

In the presence of beam these oscillations can be excited by the stimulated Cherenkov radiation of beam electrons. Really representing the solution of (1) as

(3)

$$\omega = \omega_{Le} + i\delta = ku + i\delta$$

From (1) we obtain

(4)

$$\delta = \begin{cases} \left(i+\sqrt{3}\right)/2\left(n_b/2n_p\right)^{1/3}\omega_{Le} & |\delta| \gg \nu_e \\ \left(1+i\right)/\sqrt{2}\left(n_b\omega_{Le}/2n_p\nu_e\right)^{1/2}\omega_{Le} & |\delta| \ll \nu_e \end{cases}$$

The first expression corresponds to the collisionless plasma and was obtained in Lecture 11 for nonrelativistic beam interaction with plasma. The second is new and corresponds to dissipative instability-dissipative stimulated radiation of beam electrons in collisional plasma. It takes place when $\nu_e \gg |\delta| \gg \nu_b$, where ν_b is collision frequency of beam electrons.

Problem 12 Lecture (12)

Calculate the limiting current of the monoenergetic relativistic electron beam in a drift chamber in the external infinitely strong longitudinal magnetic field.

Solution

Under the action of electrons space charge in a drift chamber there arises a space potential ϕ, which can be calculated from the Poisson's equation

(1)

$$\Delta\phi = 4\pi j/v = -4\pi j/c[1-(\gamma-e\phi/mc^2)^{-2}]$$

Here j is the beam current density, $\gamma = (1 - u^2/c^2)^{-1/2}$, where u is the injected velocity of beam electrons and v is determined from the energy conservation law

(2)

$$mc^2(1-v^2/c^2)^{-1/2} + e\phi = mc^2\gamma$$

The equation must be completed by the boundary conditions. For the plane drift chamber these conditions look as

(3)

$$\phi|_{x=\pm d} = 0,\ \phi|_{x=0} = \phi_0$$

where $x = \pm d$ corresponds to the surfaces of chamber and ϕ_0 must be determined from the condition

(4)

$$j(\phi_0) = j_{max} \equiv j_0$$

quite analogical for the cylindrical drift chamber with the radius R we have the following boundary conditions

(5)

$$\phi|_{r=R} = 0,\ \phi|_{r=0} = \phi_0$$

Here also ϕ_0 is determined from the condition (4).

Below we will restrict ourself by consideration only the result of the solutions of the mathematical problems. In the plane case the nonlinear equation can be exactly solved only in the limits of nonrelativistic, $\gamma \sim 1$, or ultrarelativistic, $\gamma >> 1$, beams. The result is

(6)

$$j_0 = (1mc^3/2\pi e)(1/d^2) \begin{cases} (2\sqrt{2})/9)(\gamma-1)^{3/2} & \gamma = 1 + u^2/2c^2 \sim 1 \\ \gamma & \gamma >> 1 \end{cases}$$

Analogically for the cylindrical drift chamber completely fullfilled by the beam we have

(7)

$$I_0 = \pi R^2 j_0 = \begin{cases} (2\sqrt{2}/9)(mc^3/e)(\gamma-1)^{3/2} & \gamma \sim 1 + k^2/2c^2 \sim 1 \\ (mc^3/e)\gamma & \gamma >> 1 \end{cases}$$

The formulas (6) and (7) can be written in uniform by using the interpolation

(6)'

$$j_0 = (1/2\pi)(mc^3/e)((\gamma^{2/3} - 1)/d^2)^{3/2}$$

(7)'

$$I_0 = (me^3/c)(\gamma^{2/3} - 1)^{3/2}$$

Let us notice that $mc^3/e \sim 17$ kA

Problem 13 Lecture (12)

Using (381) investigate stimulated scattering of incident transverse waves on the ion-accoustic oscillation of non-isothermal ($T_e >> T_0$) isotropic plasma Mandelstam-Brilluen stimulated scattering

Solution

In the frequency range of ion-accoustic oscillations, $\omega_{Le}, kv_{Te} >> \omega >> kv_{Ti}, \nu_i$, we have

$$\delta\varepsilon_e^l = (\omega_{Le}^2/k^2v_{Te}^2)(1 + i\sqrt{(\pi/2)}\omega/kv_{Te}), \qquad \delta\varepsilon_i^l = -\omega_{Li}^2/\omega^2 \tag{1}$$

Substituting these expressions into the equation (381) we obtain

$$k^2r_{De}^2 - \omega^2/\omega_{Li}^2 - i\sqrt{(\pi/2)}(\omega/kv_{Te})k^2r_{De}^2 = (k^2/4(\mathbf{k}-\mathbf{k_0})^2)([(\mathbf{k}-\mathbf{k_0})v_E]^2/(2\omega\omega_0+c^2(\mathbf{k}-\mathbf{k_0})^2) \tag{2}$$

The denominator in the right side corresponds to the scattered wave with frequency $\omega_0\alpha\omega$ and wave vector $k_0\alpha k$, or taking into account $\omega << \omega_0$,

$$(\omega_0-\omega)^2\varepsilon^{tr}(\omega_0-\omega)-c^2(\mathbf{k_0}-\mathbf{k})^2 \approx 2\omega\omega_0 + c^2(k^2-2\mathbf{kk_0}) \tag{3}$$

In the equation (381) we must take into account also the scattered wave with $\omega_0 + \omega$ and $k_0 + k$, which can be done quite analogically as for $\omega_0-\omega$ and k_0-k in the right side of (2).
Now we can solve the equation (2) supposing that $\omega = kv_s+i\delta$. The maximal value of δ corresponds to the resonance for scattered wave (3) and it is equal

$$\delta_{max} = 1/2\left(\sqrt{(\pi m/8\mu)(k^2v_s^2) + (\omega_{Li}^2/4\omega_0 kv_s)([k_0 - kv_E]^2/(k-k_0)^2)} - \sqrt{\pi/8}(m/M)kv_s\right) \tag{4}$$

This expression is positive and therefore the stimulated Mandelstame-Brilluen scattering occurs to be unthreshold. Only one requirement must be satisfied

$$\delta_{max} > \nu_e\omega_{Le}^2/2\omega_0^2 \tag{5}$$

which corresponds to the nondamping of incident wave during the process of stimulated scattering.

Problem 14 (Lecture 13)

Investigate nonlinear dynamic of the "cold" plasma beam instability assuming that stimulated radiation of a monochromatic longitudinal wave takes place

Solution

For monoenergetic and nonrelativistic beam the potential wave excitation in a "cold" plasma takes place. Supposing that the plasma density is much higher than beam density and therefore the linear approximation for the plasma oscillations is valid. Then the nonlinear system of beam-plasma interaction can be written as:

$$\partial u/\partial t + \partial nv/\partial x = 0, \qquad \partial v/\partial t + v\partial v/\partial x = -e/m(\partial\phi/\partial x) \tag{1}$$

$$(1-\omega_p^2/\omega^2)\partial^2\phi/\partial x^2 = -4\pi e(n-n_0)$$

Here $\omega_p = \sqrt{4\pi e^2 n_0/m}$, where n_0 is unperturbed plasma density. Moreover, we restrict ourself by consideration of only one dimensional case, or all the quantities are the function only x and t. Supposing that this dependence on $\xi = t - xk/\omega$ from the system (1) follows first integrals

$$n/(\omega/k - v) = n_{b0}/(\omega/k - u) = C_1, \quad m/2(\omega/k - u)^2 = m/2(\omega/k - v) + e\phi = C_2 \tag{2}$$

Taking into account these integrals the third equation of (1) takes the form

$$(1-\omega_p^2/\omega^2)d^2\phi/d\xi^2 = (\omega_0^2\omega^2 m/ek^2)[((u-\omega/k)/\sqrt{((u-\omega/k)^2-2e\phi/m)})-1] \tag{3}$$

where $\omega_b = \sqrt{4\pi e^2 n_{b0}/m}$

In the linear approximation on ϕ we obtain well-known dispersion equation (354)

$$1 - \omega_p^2/\omega^2 - \omega_b^2/(\omega-ku)^2 = 0 \tag{4}$$

which has unstable solution $\omega = \omega_b + \delta = ku + \delta$ and

$$\delta = ((-1+i\sqrt{3})/2)(n_{b0}/2n_p)^{1/3}\omega_p \tag{5}$$

The field potential then will increase until in the frame of wave its amplitude is less than energy electrons. Therefore the maximal value of ϕ is equal

$$e\phi_{max} = m(u-\omega/k)^2 \sim (n/k^2)\delta^2 \sim mu^2/2(2n_{b0}/n_p)^{2/3} \tag{6}$$

This value corresponds to the saturation of unstability. At this stage we can estimate also the field energy and as a result the efficiency of stimulated Cherenkov radiation

$$\eta = E^2/(8\pi n_{b0} mu^2/2) = 1/2(n_{b0}/n_p)^{1/3} << 1 \tag{7}$$

It is easy to understand that the saturation of instability is a result of beam electrons captured in the well of wave potential.

Problem 15 (Lecture 13)

Investigate the form and velocity of one dimensional ion-accoustic soliton in a nonisothermal ($T_e >> T_i$) isotropic plasma in taking into account the capture of electrons in the field of soliton .

Solution

Let us use hydrodynamics equations for describing ions motion

(1)

$$\partial v/\partial t + v\partial v/\partial x = -(e/M)\partial\phi/\partial x, \qquad \partial Ni/\partial t + \partial Niv/\partial x = 0$$

$$\partial^2\phi/\partial x^2 = -4\pi e\,(N_i - N_0\exp(e\phi/T_e) - \Delta N]$$

Here N_0 is the electrons and ions density at $\phi = 0$. ΔN is the density of captured electrons in the field at ion - accoustic waves

(2)

$$\Delta N = (4N_0/3\sqrt{\pi})(e\phi/T_e)^{3/2}$$

Moreover for simplicity we consider only the case when ei = -e

The system (1) admits the existence of solution which depends only on ξ-ut, where u = const. For such solutions vanishing at $\xi\ \alpha\pm\infty$ we have from (1)

(3)

$$(v-u)v' = (-e/M)\phi, \quad (N_iv)' - uN_i' = 0$$

Taking into account that at $\xi\alpha\pm\infty$, $\phi\alpha 0$, $v\alpha 0$ and $N_i\alpha N_0$ we have

(4)

$$\phi = (Mu^2/2e)[1-(v-u)^2/u^2], \qquad N_i = N_0/\sqrt{(1-2e\phi/\mu u^2)}$$

Substituting (2) and (4) into the equation for ϕ in (1) and taking into account that , $e\phi << T_e$ ((2) was estimated under this condition), we obtain

(5)

$$\phi''-4\pi e^2N_0/T_e[(1-v_s^2/u^2)\phi + 4/3(1/\sqrt{\pi})(\sqrt{e}(\phi^{3/2})/T_e^{1/2})] = 0$$

Now we can use the general theory of solitons developed in the lecture 13 which leads to the following soliton type solution

(6)

$$\Phi(\xi) = \Phi_m/(Ch^4[(\xi/\sqrt{15})\, r_{De}(e\Phi_m/\pi T_e)^{1/4}])$$

$$u = v_s[! + 8/15(e\phi_m/\pi T_e)^{1/2}]$$

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APPENDIX

On the History of Fundamental Papers on the Kinetic Plasma Theory

This appendix is printed from Plasma Physics Reports Vol 23 N°5 1997 pp 442 - 447 following to the paper of A. Alexandrov and A. Rukhadze.

1. 1996 is marked by a number of important dates related to milestones in the history of the development of the kinetic plasma theory, i.e., the theory of a gas consisting of particles interacting via an electromagnetic field. Sixty years ago, in 1936, one of the most frequently cited papers by Landau, " Kinetic Equation for the Coulomb Interaction" [1], was published. In that paper, the well-known integral describing of elastic Coulomb collisions between charged particles (the Landau collision integral), which plays an important role in the kinetic plasma theory, was derived. Ten years later, in 1946, Landau published the very popular paper based on the kinetic Vlasov equation, "Oscillations of an Electron Plasma"[2], in which he described a new discovered phenomenon-collisionless damping of electron Langmuir oscillations; this damping was later referred to as Landau damping. Between the appearance of these two papers by Landau, in 1938, Vlasov published his fundamental paper "Vibrational Properties of an electron gas"[3], in which the kinetic equation for plasma was derived in the approximation treating Coulomb collisions as the interaction through the self consistent field. This equation was later called the Vlasov equation. Although, initially, this equation was not justified precisely, results obtained by using this equation (including, first of all, the results obtained by Vlasov himself) became a foundation for the present-day kinetic plasma theory.

N.N. Bogolyubov was the first to accurately justify the Vlasov equation in his monograph "Problems of Dynamic theory in Statistical Physics" [4]; this brillant monograph was published fifty years ago in 1946. Bogolyubov not only justified the Vlasov equation as an equation corresponding to the leading order approximation in describing gases that consist of particles interacting via Coulomb forces but also showed that the Landau collision integral describes the next-order effects in the Coulomb interaction of particles in a plasma. The Vlasov equation supplemented by the Landau collision integral forms the general kinetic equation for a plasma and is referred to as the Vasov-Landau equation. Hence, Vlasov and Landau laid the foundations of the kinetic plasma theory.

Below, we will briefly discuss the papers by Landau [1,2] and Vlasov [3] in the context of the present-day views, which coincide conceptually with the interpretation proposed by Bogolyuubov in his monograph[4]. In conclusion, we will present our own opinion about the critical paper by four authors [5] and the answer by Vlasov[6], which, unfortunately, was published in an almost unknown (at that time) departmental journal.

2. By the early 1930s, the development of the kinetic plasma theory of neutral (on the whole) gases consisting of electrons and ions became relevant. This stems primarily from the experimental investigations by I.Langmuir on the relaxation processes in a gas-discharge plasma in wide density and temperature ranges. Landau was the first to make significant progress in this field. In 1936, he derived a kinetic equation for gas consisting of particles interacting via Coulomb forces. To derive the kinetic equation for the distribution function f(**p**, **r**, t) determining the probability for a particle with momentum **p** to be observed at the point **r** and the time t, he used the Boltzmann equation

(1)

$$df/dt = \partial f/\partial t + (d\mathbf{r}/dt)\partial f/\partial \mathbf{r} + (d\mathbf{p}/dt)\partial f/\partial \mathbf{p}) = I(f, f).$$

in which the change in f(p,r,t) is governed by pair collisions.[1] Here ,

(2)

$$dr/dt = \mathbf{v},\ dp/dt = \mathbf{F} = e\{\mathbf{E}_0 + 1/c(\mathbf{v}\times\mathbf{B}_0]\}$$

and I(f ,f) is the integral that describes pair elastic collisions and is a bilinear functional of f(p,r,t). In the Boltzmann approximation, the force F is external, so that the fields $\mathbf{E}_0$ and $\mathbf{B}_0$ are external fields defined by Maxwell's equations in which the prescribed charge ρ_0 and current $\mathbf{j}_0$ densities are treated as sources of these fields.

It is notable that, in deriving equation (1) for conventional gases consisting of neutral particles, Boltzmann regarded the particles as solid spheres of radius a_0, which is the radius of the interaction region. For such gases, the condition for the applicability of the kinetic equation (1) is

(3)

$$n_0^{1/3} a_0 << 1$$

where n_0 is the particle density. This inequality, which assumes that the particle size a_0, i.e., the radius of the interaction between particles, is small compared to the mean distance ~ $n_0^{1/3}$ between them, is the condition for the applicability of the gas approximation for a system of neutral particles; as long as this condition is satisfied, the particles move freely most of the time and collisions are rare. Although the potential of interaction is infinitely high and the interaction can be treated as strong, this relates only to short intervals of time corresponding the infrequent collisions between particles.

In deriving equation (1) for a gas consisting of particles interacting through Coulomb forces, Landau could not use condition (3), because, for such a gas, the characteristic radius of interaction is infinitely large. He used the condition that the mean potential energy (which is proportional to ~ $e^2 n^{1/3}$) of the interaction between particles is much lower than the mean kinetic energy χT of their thermal motion and adopted the following condition for the plasma to be a gaseous medium:

(4)

$$e^2\ n^{1/3} / \chi T << 1,$$

where χ is Boltzmann's constant. This assumption allowed him to obtain the converging integral of pair collisions and to write the kinetic equation (1) in the form

(5)

$$\partial f_\alpha/\partial t + v\partial f_\alpha/\partial r + e_\alpha\{\mathbf{E}_0 + 1/c[\mathbf{v}\times\mathbf{B}_0]\} = I(f_\alpha, f_\beta)$$

$$= \pi e^2_\alpha L\ {}_\beta\Sigma\partial/\partial p_i(\int dp' e_\beta^2 (u^2\delta_{ij} - u_i u_j)/u^3)$$

$$\times[\partial f_\alpha/\partial p_i(f_\beta(\ p',r,t\ )) - f_\alpha(\partial f_\beta(p',r,t)/\partial p'_j]$$

[1] As in the papers by Landau and Vlasov, we restrict ourselves to considering only the electron plasma component assuming that the electrons are scattered elastically by infinitely heavy plasma ions.

Here, $\mathbf{u} = \mathbf{v}-\mathbf{v}'$ is the relative velocity of the interacting particles, and L is the Coulomb logarithm

$$L = \ln\left(\chi T / e^2 n^{1/3}\right)^3 >> 1 \tag{6}$$

The summation in (5) is carried out over electrons and ions.

Note that, under condition (4), the electric field of the test static charge q is screened in a plasma, and the field potential has the form

$$\Phi(r) = q/r \exp(-r/r_D) \tag{7}$$

where $r_D = \sqrt{\chi T / 4\pi e^2 n}$ is the Debye radius, which can be regarded as the characteristic radius of the interaction between charged particles in a plasma. It is this circumstance that allowed Landau to derive equation (5) and obtain the converging collision integral by cutting off the Coulomb interaction at the Debye radius. On the other hand, comparing the Debye radius r_D with the mean distance between particles, we can see that their ratio is large,

$$r_D n^{1/3} = \sqrt{\chi T / 4\pi e^2 n^{1/3}} >> 1 \tag{8}$$

This indicates that there are many other plasma particles within the Debye sphere, and, in this connection, the question arises of the extent to which the allowance for only pair collisions is justified and, hence, the Landau kinetic equation (5) is valid.

3. Vlasov was the first to point out that the Boltzmann approximation cannot be used to describe the plasma. In paper [3], he wrote, " For a system of charged particles, the method of the derivation of the kinetic equation by taking into account only pair collisions between particles interaction by impact is not, strictly speaking, correct. In the theory of such systems, long-range interaction should play an important role, and, consequently, the system of charged particles should be, in essence, treated as a peculiar system governed by long-range forces rather than as a conventional gas."[2] Vlasov justified this assertion by inequality (8), which follows from (4) and showed that there are many particles inside the Debye sphere, whereas, in the Boltzmann approximation (3), the opposite condition should be satisfied. This gave Vlasov a lead, and he supposed that the main interaction between plasma particles is the interaction of each particle with all the other particles through the electromagnetic fields produced by them. In this case, pair collisions should be incorporated as small corrections.

As a result, the kinetic equation for electrons takes the form

$$\partial f/\partial t + \mathbf{v}\partial f/\partial \mathbf{r} + e\{\mathbf{E} + 1/c[\mathbf{v} \times \mathbf{B}]\}\partial f/\partial \mathbf{p} = 0 \tag{9}$$

Here, unlike the Landau equation (5), **E** and **B** are the total fields generated by plasma particles and by external sources.

Hence, these fields should satisfy the Maxwell equations

$$\mathrm{div}\,\mathbf{E} = 4\pi(\rho+\rho_0),\ \ \mathrm{div}\,\mathbf{B} = 0, \tag{10}$$

$$\mathrm{rot}\,\mathbf{E} = -1/c\,\partial\mathbf{B}/\partial t,\ \mathrm{rot}\,B = 1/c\,\partial\mathbf{E}/\partial t + 4\pi/c(\mathbf{j} + \mathbf{j}0),$$

[2] This idea is coherent to the definition given by D.A. Frank Kamenetskii, who called the plasma " the fourth aggregative state of matter". Also, Vlasov was engrossed in developing this idea: to the end of his life he did not stop attempts to describe also the crystalline state of matter by using the model of the self-consistent interaction mechanism.

which include, along with the external sources ρ_0 and $\mathbf{j}_0$, the sources ρ and $\mathbf{j}$ induced in the plasma:

$$\rho = \sum_{e,i} \int_e fdp, \ j = \sum_{e,i} \int_e vfdp. \tag{11}$$

Here, as before, the summation is carried out over all species of the charged particles.

As for the collision integral (which we have not written out here) in (9), Vlasov assumed it to be small and, in place of it, used the Landau collision integral (5). The only difference was that, in terms of the Boltzmann approximation, he proposed that the Coulomb interaction should cut off at a length on the order of the mean distance between electrons rather than at the Debye radius. Consequently, he used the Coulomb logarithm L, which was reduced by a factor of 3/2 in comparison with (6). Although, at first glance, this dicrepancy seems to be insignificant, it is of fundamental importance. In this connection, we must point out the intuition of Landau whose predictions concerning the cutting point turned out to be completely true. However, the validity of the Landau collision integral was completely confirmed only in the late 1950s by A. Lenard and R. Balescu, who not only obtained the integral describing pair collisions with allowance for the plasma polarization but also justified the fact that the Coulomb interaction should be cut off precisely at the Debye radius. As mentioned, based on the expansion in parameter (4), the mathematically correct derivation of equation (9) was made by Bogolyubov in his monograph [4]. In the literature, the set of equations (9)-(11), in which pair collisions are neglected, are usually called the Vlasov-Maxwell equations, and the kinetic equation (9) is referred to as the Vlasov equation or the kinetic equation for collisionless plasma. However, the latter term seems to be inappropriate, since equation (9), even without allowance for the right-hand side, takes into account the long-range interaction or more precisely, the particule interaction through the self-consistent fields[3]. Based on above set of equations neglecting pair collisions, Vlasov studied small linear plasma oscillations in the absence of external sources and external fields. He showed that, in such an isotropic plasma, purely longitudinal ($\mathbf{E} = -\nabla\phi$) and purely transverse (div $\mathbf{E} = 0$) waves can occur and obtained general dispersion relations connecting the frequency ω and the wave vector $\mathbf{k}$ for perturbations of the form $\exp(-i\omega t + i\mathbf{kr})$. Here, we restrict ourselves to analyzing only longitudinal oscillations, since the results correponding to these oscillations can be compared with the results obtained by Landau in the cited papers.

The analysis of the dispersion relation was performed by Vlasov for small longitudinal oscillations for an isotropic electron plasma with an equilibrium Maxwellian velocity electron distribution function.

[3] Here, we must draw attention to the fundamental properties of the system consisting of like-charged particles interacting via Coulomb forces, i.e., systems characterized exclusively by either repulsing or attracting forces (also, the latter case corresponds to systems of gravitating bodies). In such systems, no Debye screening of the electric field of the charge exists, and it is impossible to obtain the converging collision integral. For this reason, thermodynamically stable gases consisting of such particles cannot exist. In the absence of external forces, these gases either collapse (in the case of attracting forces) or expand (in the case of repulsing forces). For the gas to be stable, it should contain oppositely charged particles. However, in this case, it is impossible to construct the theory of a nonideal plasma based on the inequality that is opposite to (8) or (4) and coincides with the condition for the applicability of the Boltzmann approximation. This seems to be suitable for constructing the kinetic theory. Unfortunately, in such gases, oppositely charged particles are trapped and recombine into atoms. However, these problems are beyond the scope of our study, which is aimed at a historical overviewing of the development of the kinetic theory of an ideal plasma.

This analysis showed that, without allowance for pair collisions between particles, these oscillations (i.e. waves) do not damp when their phase velocity is higher than the electron thermal velocity and can be described by the dispersion relation

(12)

$$\omega^2 = \omega_p^2 + 3k^2 v_{Te}^2,$$

where $\omega_p = \sqrt{4\pi e^2 n/m}$ is the plasma frequency introduced by I. Langmuir, and $v_{Te} = \sqrt{\chi Te/m}$ is the electron thermal velocity. The presence of high-frequency electron oscillations with low group velocities,

(13)

$$v_g = \partial\omega/\partial k = (3kv_{Te}/\omega_p)v_{Te} \ll v_{Te},$$

agrees well with the results of familiar experiments carried out by Langmuir and Tonks [8]. The fact that slow longitudinal oscillations cannot occur in a purely electron plasma also confirms the validity of Vlasov's theory: in the range $\omega/k \ll v_{Te}$, the field of these oscillations is screened at distances on the order of the Debye radius. Note that this is in agreement with the length of the Debye screening of the field (7) of a static charge in a plasma obtained by Landau on the basis of entirely thermodynamic considerations. [4]

On the other hand, the fact that there was no damping of oscillations, although particle interaction was taken into account in the approximation of self consistent fields, was somewhat unsatisfactory. However, Vlasov did not consider this fact to be discouraging. Moreover, he believed that the damping of oscillations via pair collisions, which, according to the Landau theory, is governed by the electron-ion collision frequency,

(14)

$$\nu_{eff} = 4/3\sqrt{2\pi e^2 e_i^2 n_i L / m(\chi Te)^{3/2}}$$

is negligibly weak by virtue of the condition

(15)

$$\nu_{eff}/\omega_p \approx (e^2 n^{1/3}/\chi Te)^{2/3} << 1$$

He considered the dispersive spreading to be a more important process. In fact, by using the dispersion relation (12) to estimate the time τ_g required for an inhomogeneity of the length ~1/k to spread, we obtain

(16)

$$\omega_p \tau_s \approx 1/k(\partial\omega/\partial k)^{-1} \approx \omega_p^2/k^2 v_{Te}^2 = 1 / k^2 r_{De}^2 >> 1,$$

i.e., this time is long in comparison with the oscillation period. In this case, the effect of collisions is governed by the quantity $\tau_g \upsilon_{eff}$, which is a product of the small parameter (15) and the large parameter (16).

4. In [2], Landau rejected the concept of Vlasov in which the dissipation of small oscillations was absent when pair collisions were neglected. He acknowledged that the Vlasov equation could be applied to describe electron plasma oscillations but, nevertheless, wrote,

(4) Note that Langmuir and Tonsk[8] applied the hydrodynamic theory in order to develop the concept of the self consistent field; moreover, they obtained a dispersion relation similar to (12), which differs slightly from (12) in coefficient 1 (in place of 3) in the correction term; also not that the low frequency longitudinal field is screened at the Debye length. They also showed that small oscillations in a plasma have the frequency ω_p, do not damp aperiodically as time elapses, and that their weak damping is governed by only electron-electron collisions.

"Vlasov searched for solutions of the form exp(-iωt + ikr) and determined the dependence of the frequency ω on the wave vector k. In fact there is no definite dependence of ω on **k**, and arbitraries values of ω can exist for the prescribed k." Using the same approach as that applied by Vlasov, Landau solved the initial problem for small oscillations and obtained the same dispersion relation[5]

(17)

$$1 - 4\pi e^2 mk\int \partial f_0/\partial v(dv/(\omega - \mathbf{kv})) = 0,$$

which was analysed by Vlasov. Here, $f_0(v)$ is the equilibrium velocity distribution function, which was assumed to be Maxwellian and is normalized to the electron density n_e,

(18)

$$f_0(v) = ne\,(2\pi m\chi T_e)^{3/2}\exp(-mv^2\, 2\chi T_e)$$

Equation (17) contains the Cauchy improper integral having a pole of the integrand at the point $v = \omega/k$ on the real axis. The different approaches of Vlasov and Landau to taking this integral led to misunderstanding and disagreement between them. Vlasov suggested to find the principal value of this integral and, as a result, obtained the solution $\omega(k)$ to equation (17) in the form of undamping oscillations with the dispersion relation (12), whereas Landau proposed to integrate along a contour (the Landau path- tracing rule) corresponding to the following representation of a pole:

(19)

$$1/(\omega-\mathbf{kv}) = P/(\omega-\mathbf{kv})-i\pi\delta(\omega-\mathbf{kv}),$$

where P denotes the principal value of the integral. This gives rise to a small imaginary correction to the frequency ($\omega \alpha\ \omega + i\gamma$),

(20)

$$\gamma = -\sqrt{\pi/8}(\omega_p/(kr_{De})^3\exp(-(1/2k^2r_{De}^2)-3/2)$$

which describes a weak damping of oscillations corresponding to the dispersion relation(12). This damping was subsequently called the Landau "collisionless" damping. We use here quotation marks, because, in fact, the Vlasov equation takes into account multiparticle (or collective) collisions and does not incorporate only pair collisions, for which the right-hand side of the Vlasov equation should be supplemented by the collision integral. Taking into account pair collisions yields the appearance of additional damping, $\gamma\ \alpha\ \gamma + \delta\gamma$, where

(21)

$$\delta\gamma = -\nu_{eff}/2$$

and ν_{eff} is defined by expression (14).

Because of conditions (15) and (16), both the collisional (21) and collisionless (20) damping rates are weak in comparison with the oscillation frequency described by (12). However, the question arises of which of these processes, i.e., the collisional damping or the Landau collisionless damping, is dominant.

When

(22)

$$\gamma/\delta\gamma \approx 1/6L(\chi T_e/e^2n^{1/3})^{3/2}1/(kr_{De})^3\exp(1/2(kr_{De})^2 > 1$$

(5) Landau rejected even the notion of the dispersion relation, which is especially obvious from this paper [5]

The Landau collisionless damping dominates over the collisional damping, and when the opposite inequality is satisfied the collisional damping dominates. Hence, the Landau collisionless damping should be taken into account for $r_{De} << \lambda < r_{De}\sqrt{L}$. Since, for real plasmas, the Coulomb logarithm is $L \approx 10$, we can see that, for purely electron longitudinal oscillations, the range in which the Landau collisionless damping is important is very narrow. Moreover, the time scale ~ $1/\gamma$ for the collisionless damping and the time scale~ $1/\delta\gamma$ for the collisional damping are always much larger than that for the dispersive spreading defined by relationship (16). Due to this, it is difficult to distinguish the collisional and collisionless damping occuring against the background of the dispersive spreading.

However, the above fact that the range in which Landau damping is important is narrow and that the time scale for this damping is large in comparison with time scale for the dispersive spreading is valid only for a thermodynamically equilibrium plasma with a Maxwellian velocity distribution of charged particles and for purely electron longitudinal oscillations. In the general case of arbitrary oscillations of anisotropic and, especially, nonequilibrium plasmas, Landau damping, or, more precisely, the collisionless dissipation associated with the poles of the integrands, which arise in solving the Vlasov equation and calculating the induced charges and currents in the plasma, not only can be very important but also can change the sign and almost entirely determine the absorption and emission of the electromagnetic field by the plasma. Moreover, in a completely ionized plasma, collisionless dissipation is always dominant over collisional dissipation, except for particular cases when collisionless dissipation turns out to be weak for one reason or another, as in the above case of longitudinal electron oscillations. This is a distinctive feature of the plasma that is regarded as a system of charged particles interacting via the Coulomb forces, and this makes the Vlasov approach to describing the plasma very effective.

The above considerations have become physically obvious only when the nature of Landau damping and, consequently, of collisionless dissipation, was completely clarified. This nature can be clearly seen from the Landau path-tracing rule $\omega = \mathbf{kv}$, which was proposed by Landau in the form of relationship (19). This relationship indicates that the energy dissipation in a plasma is governed by the particles for which the condition $\omega = \mathbf{kv}$ holds; i.e., the emission and absorption of electromagnetic waves by charged particles are related to the Cherenkov resonance. Obviously, the probabilities for a charged particle to emit or absorb quanta of electromagnetic radiation are the same, and, consequently, the answer to the question which of the processes, emission (corresponding to the amplification of the field) or absorption (corresponding to the damping of the field), is dominant can be obtained by analyzing the velocity distribution function in the vicinity of the point $v = \omega/k$. If $\partial f_0/\partial v < 0$ (as in the case of a Maxwellian equilibrium distribution), then the dispersion relation (17) shows that the electromagnetic wave energy is transferred to charged particles; and this gives rise to the Landau damping; Otherwise, if $\partial f_0/\partial v > 0$ in the vicinity of $v \approx \omega/k$, the amplification of electromagnetic waves can occur in a plasma.[6]

The Vlasov equation, involving the self consistent field, describes the direct interaction of the charged particles with the field i.e., the emission and absorption of the field as first-order effects in the small parameter defined by (4).

[6] In particular, this interpretation of the physical nature of collisionless dissipation assumes that, in the approximation under consideration, undamping oscillations can exist in a plasma. Obviously, Landau damping should vanish if the equilibrium distribution is such that, for velocities close to the phase velocity of the wave, either the derivative is $\partial f_0/\partial v = 0$ or there are no particles with such velocities, as in the case of, e.g., a Fermi degenerate distribution when $\omega/k > v_F$ and also when $\omega/k > c$ (see, for detail, monograph [9])

The second-order effects in this parameter correspond to the interaction between the particles that can be treated as the emission of quanta of the electromagnetic field by one particle and the absorption of these quanta by another particle. This interaction relates to pair collisions between particles, which are incorporated by the Landau collision integral. Hence, the Vlasov-Landau generalized kinetic equation for the plasma takes into account the particle interaction correct to both first-order and second-order terms in parameter (4).

5. The above considerations were, in fact, given by Vlasov in paper [3], which, in turn, was initiated by paper [1] by Landau. Vlasov [3] not only physically justified the kinetic equation with the self-consistent field (the Vlasov equation), which takes into account for the main distribution of Coulomb forces into the interaction between plasma particles, but also showed clearly that the Landau collision integral is correct to the next order terms in the Coulomb interaction. Moreover, Vlasov supposed that the kinetic equation with the self consistent field should be supplemented by the Landau collision integral in order to adequately describe the damping of oscillations with time. He believed that nontrivial solutions of the form exp(-iωt+ikr) to the set of Vlasov-Maxwell uniform equations can exist for the specific relationship between the real ω and **k** defined by the dispersion law. Thus, Vlasov was the first to use the dispersion relation in the kinetic plasma theory and to find its solution ω(**k**) for longitudinal oscillations.

In turn, Landau showed that the analysis of small oscillations carried by Vlasov was incomplete. He showed that, even when collisions are neglected, small initial perturbations should damp as time elapses, and the nature of this damping is associated with the emission and absorption of electromagnetic waves by charged particles under Cherenkov resonance conditions. Later on, the damping of electron longitudinal oscillations obtained by Landau for the case of a Maxwellian equilibrium plasma was, by right, called Landau damping.

Thus, paper [2] by Landau completed the formulation of the physical principles of the kinetic theory developed by Vlasov and pointed out the pecularities of the solution to the kinetic equation introduced by Vlasov. As mentioned, the Vlasov kinetic theory was mathematically justified by Bogolyubov in his monograph [4]. He developed the methods for deriving the kinetic equations for two cases, namely, for a system of neutral particles that can interact strongly approaching each other, but the mean distance between these particles is much larger than the characteristic radius of interaction (the Boltzmann equation), and also for a system of charged particles interacting through Coulomb forces when the characteristic radius of interaction is much larger than the mean distance between the particles, and, consequently, the mean potential of interaction is much lower than the mean kinetic energy of the particles (the Vlasov-Landau equation). In other words, he justified both the Boltzmann equation and the Vlasov equation with the Landau collision integral. We have not given the mathematical essence of this justification, because already described in detail in a large number of monographs and even in textbooks on statistical physics of gases and plasmas. Note also that, Yu. L. Klimonotovich publicated a review on this topic in Ups Fiz. Nauk[10] which is devoted to the fiftieth anniversary of the paper [2] by Landau and is naturally focused on the problem of a thorough justification of the kinetic plasma theory.

Thus, as early as in 1946, the monograph by Bogolyubov, in which the kinetic plasma theory was mathematically justified, in fact clarified the relevant problems.

In this connection, the appearance of paper [5] in 1949 seems to be puzzling-because of is sharp and unjustified criticism of Vlasov's studies and, especially, Vlasov's approach to the kinetic plasma theory. In that paper, the monograph by Bogolyubov [4] was not mentioned at all. This is even stranger as, by that time, this fundamental monograph, which was closely related to the kinetic plasma theory, was generally recognized and widely cited in the literature. It is even surprising that the editorial board of Zh. Eksp. Teo. Fiz. refused to publish the answer of Vlasov, who was forced to publish his answer in a departmental journal [6], which, at that time, was neither widely known nor read. This was done in spite that the fact that the answer dealt with very deep problems, in particular, a description of plasma as a continuous medium in which the mean radius of interaction between particles is much larger than the mean distance between them, in which sense plasma is similar to liquids or solid bodies.[7] These ideas are not yet completely understood by the scientific community. At the same time, from remarks of the editorial board, we can see that the authors of paper [5] read the answer by Vlasov and, moreover, took into account this answer in preparing the final version of their manuscript.

As for Vlasov's contribution to creating the kinetic plasma theory, it is widely recognized by the physical community all over the world. In the scientific literature, the kinetic equation with a self-consistent field was referred to as Vlasov equation. Every year, hundreds of papers on plasma theory are published in scientific journals, and, at least in each second publication, the name of Vlasov is mentioned. In 1970, he was awarded the Lenin Prize "For a series of papers on plasma theory".

7) In this answer , in connection with the problem of describing the plasma as a continuous medium, Vlasov discussed the problem of the spatial averaging of microfields and a description of macrofields, i.e., a problem that has still not been completely resolved. In our opinion , this problem could be resolved by constructing the model of the medium and deriving the relations determining the induced charges and currents in the case of a plasma, and there is no need for additional averaging. Note also that this problem is discussed in detail in the above mentioned review by Klimontovich in Usp. Fiz Nauk.[10]

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