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FAST MAGNETIC DYNAMOS AND CHAOTIC FLOWS

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These are lecture notes, intended for distribution to participants.

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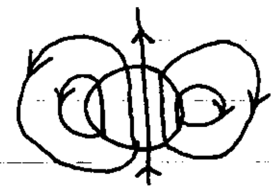
I. FAST MAGNETIC DYNAMOS AND CHAOTIC FLOWS

Large scale magnetic fields are observed in nature where there is flowing electrically conducting matter. E.g., The Earth and other planets, the Sun and stars, the galaxy. Why?

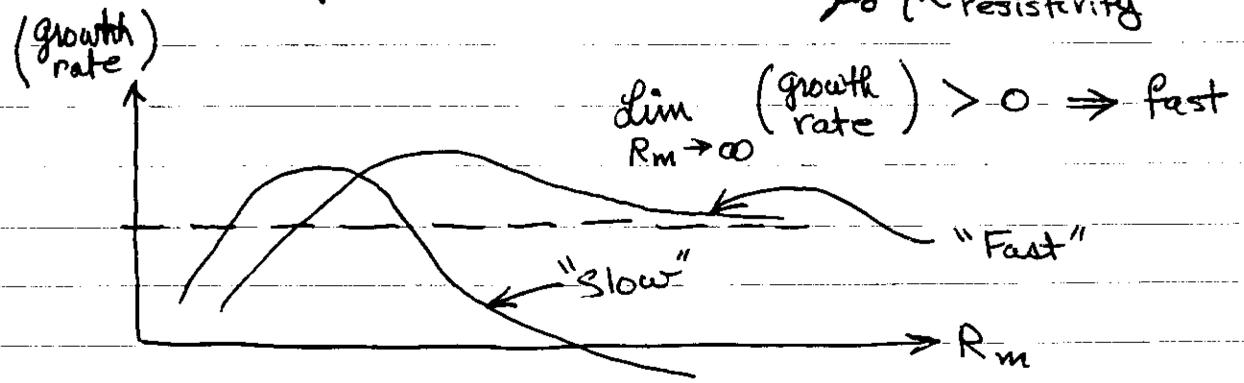
Resistive decay time too short to explain the Earth's field without dynamo action:
$B \sim e^{-t/\tau}$ $\tau \sim \mu_0 \sigma r^2$ (MKS units) $\sim 10^4$ yrs.
Age of the Earth $\sim 4.5 \times 10^9$ yrs.



<u>Kinematic Dynamo Problem</u>: Will a small seed magnetic field in an initially unmagnetized flowing electrically conducting fluid amplify exponentially in time?

"<u>fast</u>" and "<u>slow</u>" kinematic dynamos (Zeldovich & Vainshtine (1972)).

$$R_m = \text{magnetic Reynolds number} = \frac{v_0 L_0}{\mu_0 \eta} = \tau_{\text{resistive}}/\tau_{\text{flow}},$$
η = resistivity



$R_m \gtrsim 10^8$ in some astrophysical situations $\Rightarrow$ Only fast dynamos are of interest in such cases.

Basic Equations: Ampere's Law : $\nabla\times\underline{B} = \underline{J}$
Faraday's Law : $\nabla\times\underline{E} = -\partial\underline{B}/\partial t$
Ohm's Law : $\eta\,\underline{J} = (\underline{E}+\underline{v}\times\underline{B})$
Incompressible flow : $\nabla\cdot\underline{v} = 0$

$$\Rightarrow \boxed{\frac{\partial\underline{B}}{\partial t} + \underline{v}\cdot\nabla\underline{B} = \underline{B}\cdot\nabla\underline{v} + \frac{1}{R_m}\nabla^2\underline{B}} \quad \text{(normalized)}$$

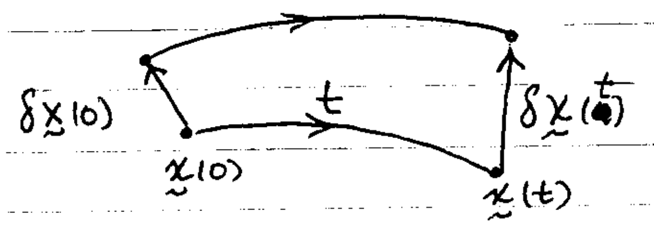
Note: We are solving a linear problem for perturbations about a basic $\underline{B}=0$ state. $\underline{J}\times\underline{B} = \frac{1}{\mu_0}(\nabla\times\underline{B})\times\underline{B}$ is second order in the perturbation $\underline{B}$. Thus, at linear order, $\underline{v}$ has no reaction (linear response) to $\underline{B}$.

"$R_M = \infty$" : Ideal MHD eq. for $\underline{B}$

$$\partial\underline{B}/\partial t + \underline{v}\cdot\nabla\underline{B} = \underline{B}\cdot\nabla\underline{v}$$

$$d\underline{B}/dt = \underline{B}\cdot\nabla\underline{v} \qquad d/dt \text{ follows } d\underline{x}/dt = \underline{v}(\underline{x},t)$$

Relevance of Chaos: $d\underline{x}/dt = \underline{v}(\underline{x},t)$



$\delta\underline{x}$ = differential displacement from $\underline{x}$

Lagrangian chaos = exponential growth of $\delta\underline{x}(t)$ for large t

$$|\delta\underline{x}(t)| \sim e^{ht}, \quad h>0$$

$$\frac{d\,\delta\underline{x}}{dt} = \delta\underline{x}\cdot\nabla\underline{v}(\underline{x},t) \qquad \text{(eq. giving } \delta\underline{x}(t))$$

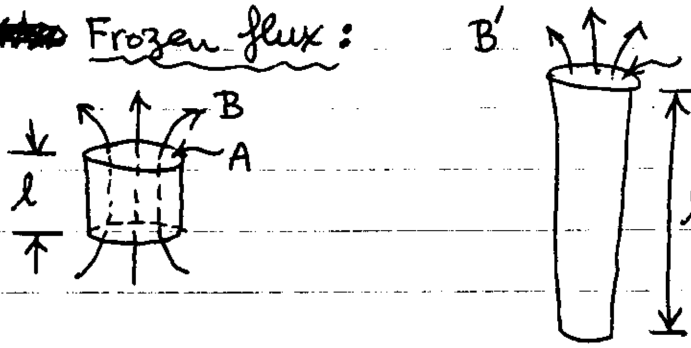
compare

$$\frac{d\underline{B}}{dt} = \underline{B}\cdot\nabla\underline{v}(\underline{x}(t),t) \qquad \text{(eq. giving } \underline{B})$$

Geometrical meaning: field lines are convected and magnetic field grows in proportion to how much a field line is stretched.

- Lagrangian chaos $\overset{??}{\Longrightarrow}$ Kinematic dynamo
  (Arnold, Zeldovich, Ruzmaikin & Sokolov JETP 1980)

Frozen flux:



$BA = B'A'$ Frozen

$lA = l'A'$ incompressible

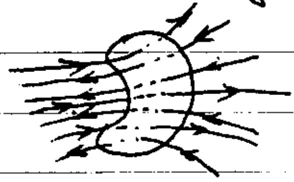
$$\boxed{B' = \left(\frac{l'}{l}\right) B}$$

← stretch

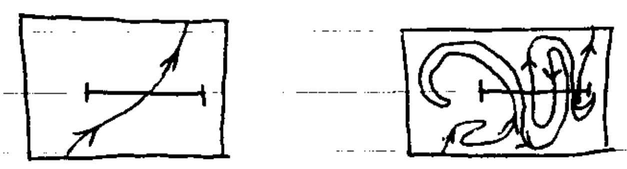
Main Points:

- Chaotic flows can lead to fast dynamos.
- $\mathbf{B}(\mathbf{x},t)$ typically concentrates on a fractal.
- Extreme tendency for cancellation.

} $R_m \to \infty$ (Finn & Ott Phys. Fluids 1988)



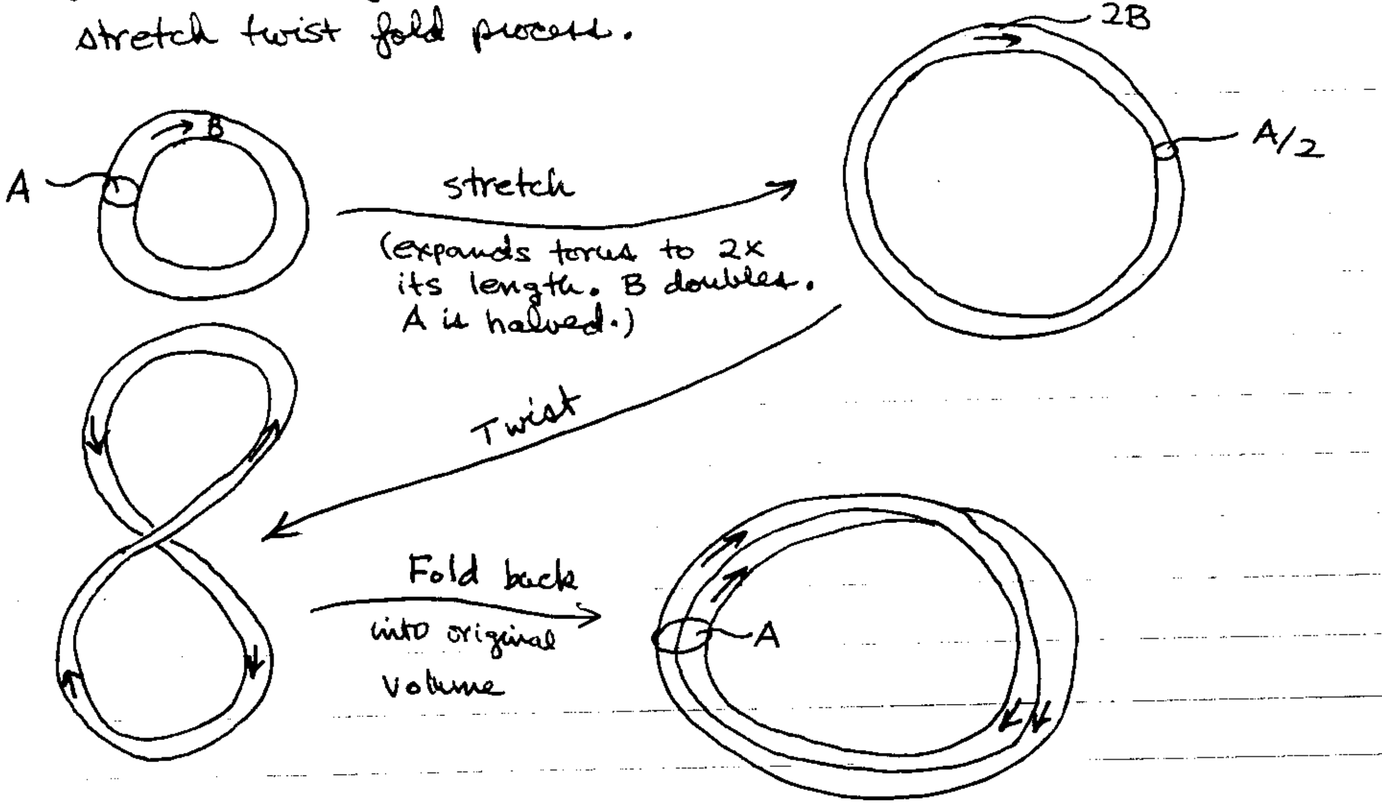
2D: Chaos is possible for a 2D flow but a kinematic dynamo is not possible.



$t=0$ $\qquad$ $t=t_1$

Stretch-twist-fold: (Zeldovich & Vainshtein 1972 JETP)

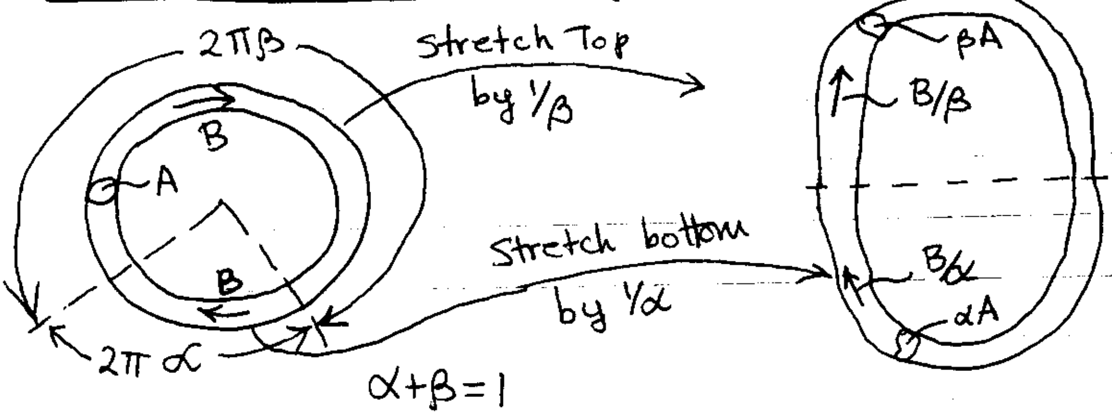
Think of ~~[illegible]~~ this as resulting from a time periodic flow that every T units of time cycles through a stretch twist fold process.



On each cycle the magnetic flux doubles.
Effective growth rate $= \frac{1}{T} \ln 2$.

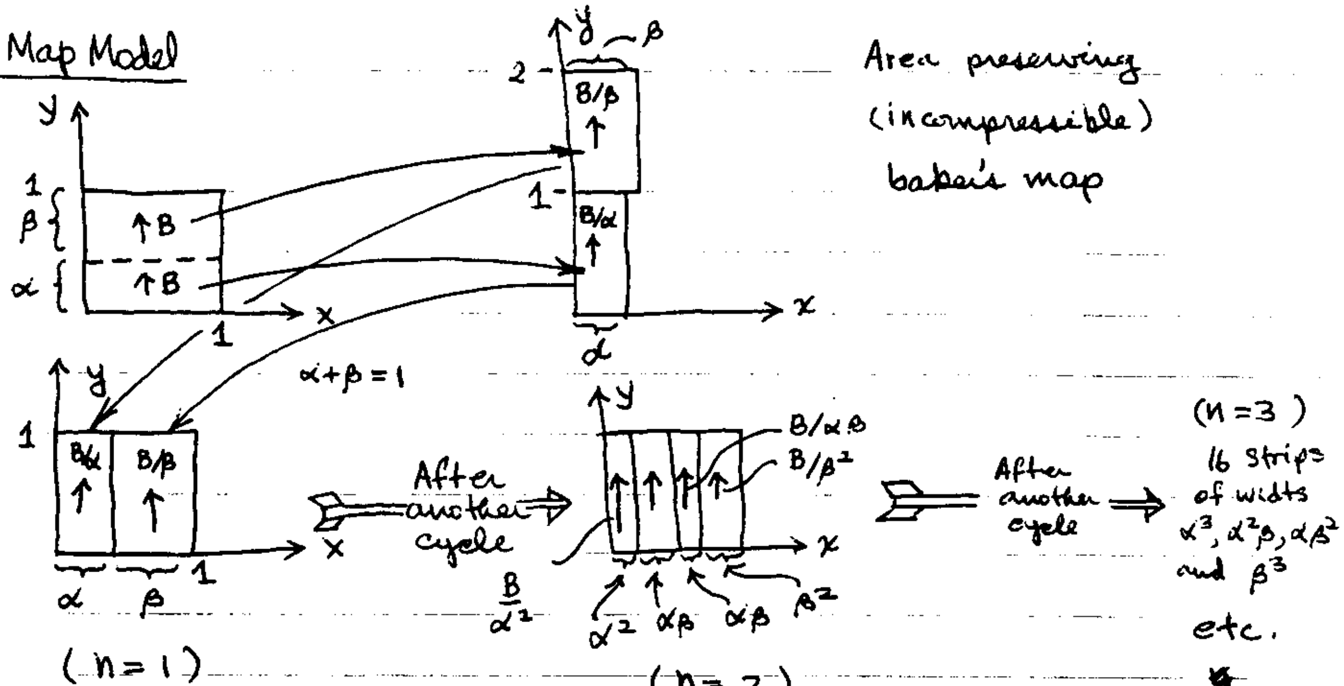
- This shows that fast dynamos are possible.
- Flow is chaotic

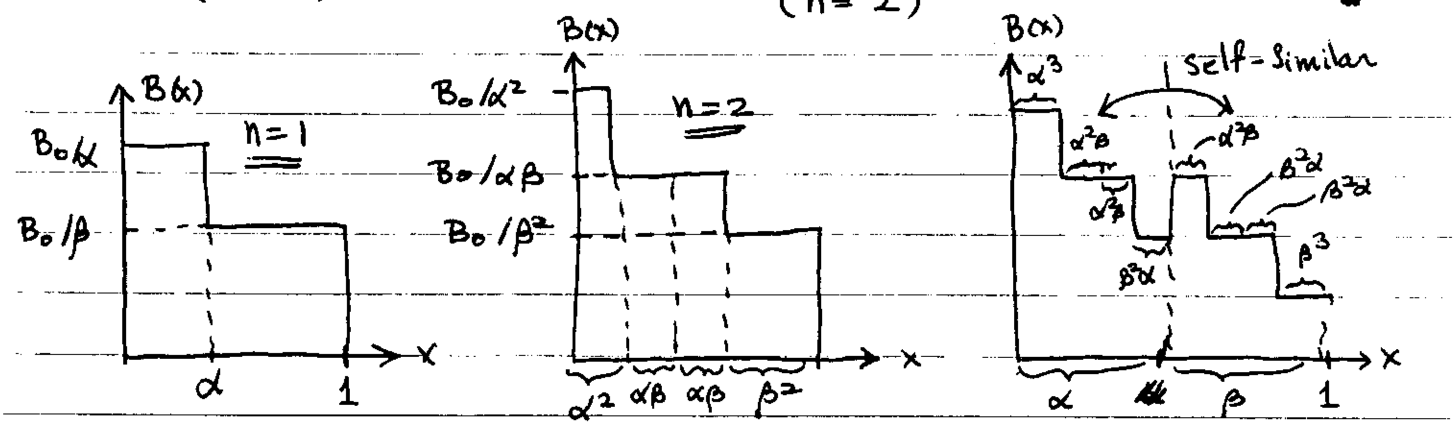
Nonuniform Stretch-twist-fold (Finn & Ott Phys. Fluids 1988)



Then Twist & fold and keep repeating

2D Map Model



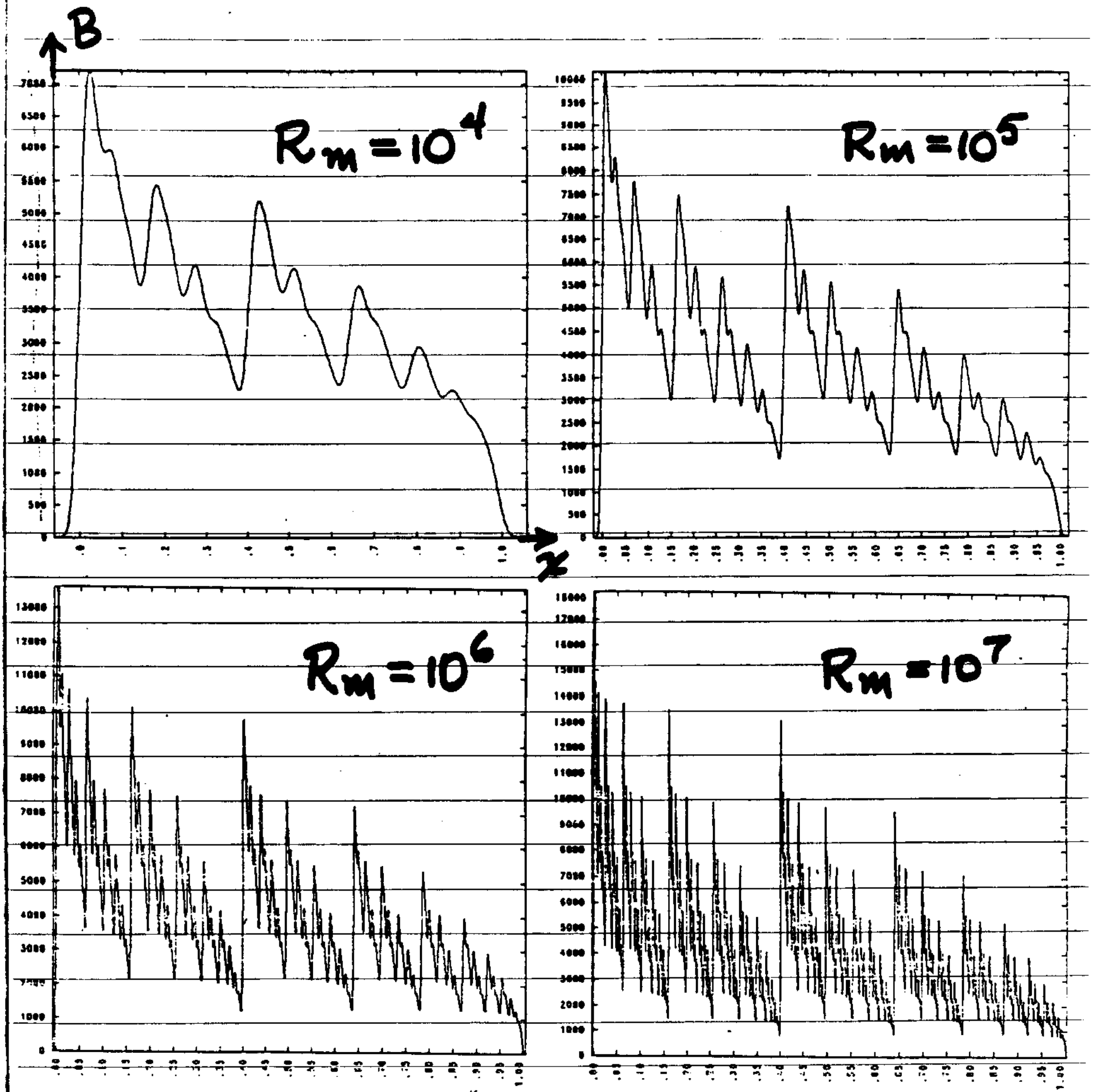


What effect does finite R_m have?

$$\frac{\partial \underline{B}}{\partial t} + \underline{v}\cdot\nabla\underset{\sim}{B} = \underline{B}\cdot\nabla\underline{v} + \frac{1}{R_m}\nabla^2\underline{B}$$

Apply map impulsively
Diffuse ($\underset{\sim}{v}=0$)
Apply map impulsively
Diffuse ($\underset{\sim}{v}=0$)
⋮
$\Rightarrow$ settles into eigenfunction and grows at rate $\cong \frac{1}{T}\ln 2$

EIGENFUNCTIONS WITH FINITE CONDUCTIVITY



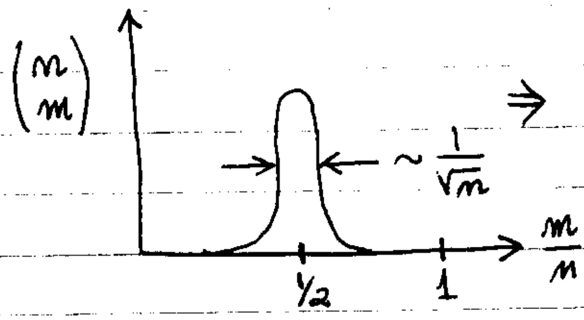
Claim: As $R_m \to \infty$ the magnetic flux tends to concentrate on a fractal set. Now show this.

After n applications of the map there are 2^n strips of varying widths,

$$\text{widths} = \alpha^{n-m}\beta^{m} \qquad (m = 0, 1, 2, \ldots, n)$$

$$(\#\text{ of strips of width } \alpha^{n-m}\beta^{m}) = \binom{n}{m} = \underset{\text{coeff.}}{\text{binomial}} = \frac{n!}{m!\,(n-m)!}$$

For large n



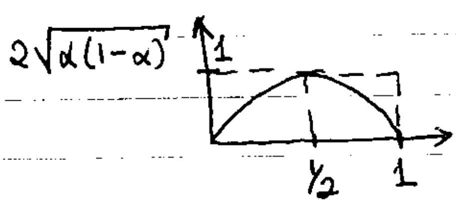
$\Rightarrow$ If $n \gg 1$, then $\frac{m}{n} \cong \frac{1}{2}$ for most strips.

$$\text{widths} = \alpha^{n-m}\beta^{m} = \left(\alpha^{1-\frac{m}{n}}\beta^{\frac{m}{n}}\right)^n \cong \left(\sqrt{\alpha\beta}\right)^n \text{ for most strips.}$$

Each strip has the same flux (independent of its width), and this flux is equal to the original initial flux (frozen in).

$\Rightarrow$ The total width of strips carrying most of the flux

$$\sim 2^n\left(\sqrt{\alpha\beta}\right)^n = \left[2\sqrt{\alpha(1-\alpha)}\right]^n$$



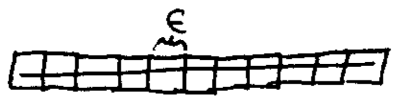
If $\alpha \neq \beta$ (i.e., $\alpha \neq \frac{1}{2}$) then $2\sqrt{\alpha\beta} < 1$

$\Rightarrow$ Flux concentrates on a set of zero total width as $R_m \to \infty$.

Fractal Dimension

$$d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}$$

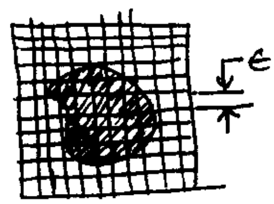
A point  $N(\epsilon) = 1$ $d = 0$

A line  $N(\epsilon) \sim l/\epsilon$

$\ln N(\epsilon) \sim \ln(1/\epsilon) + \ln l$

$\Rightarrow d = 1$

An area $N(\epsilon) \sim A/\epsilon^2$ $d = 2$



A fractal $N(\epsilon) \sim \text{const.}/\epsilon^d$ d typically not an integer

Fractal Dimension in x that our Baker's Map B field Concentrates On

Use $\epsilon \sim (\sqrt{\alpha\beta})^n$ (width of most strips)

$N(\epsilon) \sim 2^n$

$$\Rightarrow \quad d = \frac{\ln 2}{\ln\left(\frac{1}{\sqrt{\alpha(1-\alpha)}}\right)}$$

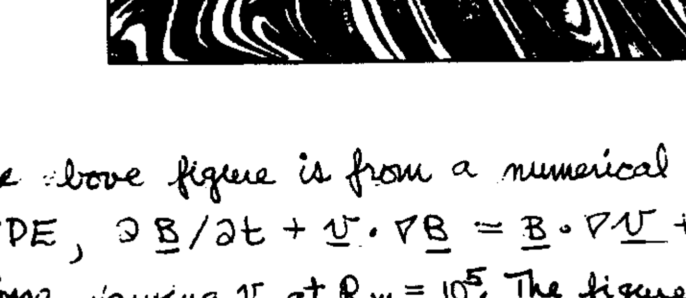
$d < 1$ for $\alpha \neq \frac{1}{2}$

$d = 1$ for $\alpha = 1/2$

"A strange attractor for the magnetic flux".

This is to be expected for solutions of $\frac{\partial \underline{B}}{\partial t} + \underline{v}\cdot\nabla\underline{B} = \underline{B}\cdot\nabla\underline{v} + \frac{1}{R_m}\nabla^2\underline{B}$ for typical Lagrangian chaotic fluid flows.

The above figure is from a numerical solution of the kinematic dynamo PDE, $\partial \underline{B}/\partial t + \underline{v} \cdot \nabla \underline{B} = \underline{B} \cdot \nabla \underline{v} + (\nabla^2 \underline{B})/R_m$ using a smooth time varying $\underline{v}$ at $R_m = 10^5$. The figure shows a grey scale plot of $B_z(x, y, z = \text{const.})$ versus x and y. (C. Reyl, T. Antonsen and E. Ott Phys. Plasmas 3 (1996).) For this figure suppose we put down a grid of ϵ by ϵ squares, and determine the smallest number of squares needed to contain some fraction f of the total $\int B_z^2\, dx\,dy$ integrated over the area of the figure. Let $\bar{N}(\epsilon)$ denote this number. $\bar{N}(\epsilon) \sim \text{const.}/\epsilon^d$



Slope = fractal dim $\cong 1.9 < 2 \Rightarrow$ fractal

$1 + 1.9 = 2.9$

For $R_m \to \infty$ d does not depend on f as long as $f < 1$. (For $f = 1$, $d = 2$ (not fractal).)